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A NEW PERSPECTIVE FOR MULTIVALUED WEAKLY PICARD OPERATORS

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Abstract. This research contains some recent developments about multivalued weakly Picard operators on complete metric spaces. In addition, taking into account both multivalued θ-contraction and almost contraction on complete metric spaces, we present a new perspective for multivalued weakly Picard operators. Finally, we give a nontrivial example showing that the investigation of this paper is significant.

1. Introduction and Preliminaries

The concept of multivalued weakly Picard operator, which is introduced by Rus et al [**16**], is closely related to metric fixed point theory. Let (X, d) be a metric space and $T\colon X \to \mathcal{P}(X)$ be a mapping, where $\mathcal{P}(X)$ is the family of all nonempty subsets of X. Then T is said to be a multivalued weakly Picard (for short MWP) operator if there exists a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for any initial point x_0, which is convergent and its limit is a fixed point of T. Berinde and Berinde [**2**] show that the type multivalued contractions on complete metric spaces considered by Nadler [**12**], Petruşel [**13**], Reich [**14**] and Rus [**15**] are MWP operators.

For the sake of completeness we recall some important concepts and results about multivalued mappings. In 1969, Nadler [**12**] initiated the idea for multivalued contraction mapping and extended the Banach contraction principle to multivalued mappings and proved the following fundamental result:

Theorem 1.1 (Nadler [**12**]). *Let (X, d) be a complete metric space and $T\colon X \to \mathcal{CB}(X)$ a multivalued mapping, where $\mathcal{CB}(X)$ is the family of all nonempty closed and bounded subsets of X. If T is a multivalued contraction, that is, there exists $L \in [0, 1)$ such that $H(Tx, Ty) \leqslant Ld(x, y)$ for all $x, y \in X$, where H is the Pompeiu–Hausdorff metric on $\mathcal{CB}(X)$ defined by*

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\},$$

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and $d(x,B)=\inf\{d(x,y):y\in B\}$*, then there exists* $z\in X$ *such that* $z\in Tz$.

Inspired by his result, there has been vigorous and dense research activity for fixed point results concerning multivalued contraction, and by now, there are a number of results that generalize this result in many different directions and many researchers have given fantastic contributions to these areas (see [**3**–**5**, **9**–**11**]).

Recently, Berinde and Berinde [**2**] introduced the concepts of multivalued almost contraction (the original name was multivalued (δ,L)-weak contraction) and proved the following attracted result for MWP operators:

THEOREM 1.2 (Berinde and Berinde [**2**]). *Let* (X,d) *be a complete metric space and* $T\colon X\to\mathcal{CB}(X)$ *a given mapping. If* T *is a multivalued almost contraction, that is, there exist two constants* $\delta\in(0,1)$ *and* $L\geqslant 0$ *such that*

$$H(Tx,Ty)\leqslant\delta d(x,y)+Ld(y,Tx) \tag{1.1}$$

for all $x,y\in X$*, then* T *is an MWP operator.*

On the other hand, introducing a new type of contractive mapping, Jleli and Samet [**7**] presented an attracted generalization of the Banach contraction principle. Throughout this study we shall call the contraction defined in [**7**] the θ-contraction. Now, we recall basic definitions, relevant notions and some related results concerning the θ-contraction.

Let Θ be the set of all functions $\theta\colon(0,\infty)\to(1,\infty)$ satisfying the conditions:

- (θ_1) θ is nondecreasing;
- (θ_2) For each sequence $\{t_n\}\subset(0,\infty)$, $\lim_{n\to\infty}\theta(t_n)=1$ and $\lim_{n\to\infty}t_n=0^+$ are equivalent;
- (θ_3) There exist $r\in(0,1)$ and $l\in(0,\infty]$ such that $\lim_{t\to0^+}\frac{\theta(t)-1}{t^r}=l$.

Let (X,d) be a metric space and $\theta\in\Theta$. A mapping $T\colon X\to X$ is said to be a θ-contraction if there exists a constant $k\in[0,1)$ such that

$$\theta(d(Tx,Ty))\leqslant[\theta(d(x,y))]^k \tag{1.2}$$

for all $x,y\in X$ with $d(Tx,Ty)>0$.

Choosing some appropriate functions for θ, such as $\theta_1(t)=e^{\sqrt{t}}$ and $\theta_2(t)=e^{\sqrt{te^t}}$, we can obtain some different types of nonequivalent contractions from (1.2). Considering this new concept, Jleli and Samet proved that every θ-contraction on a complete metric space has a unique fixed point. In the literature some interesting papers concerning θ-contractions can be found (see [**1**, **8**]).

Naturally, the concept of θ-contraction extended to multivalued mappings by Hançer et al [**6**] and they introduced the concept of multivalued θ-contraction: Let (X,d) be a metric space, $T\colon X\to\mathcal{CB}(X)$ be a mapping and $\theta\in\Theta$. Then T is said to be a multivalued θ-contraction if there exists a constant $k\in[0,1)$ such that

$$\theta(H(Tx,Ty))\leqslant[\theta(d(x,y))]^k \tag{1.3}$$

for all $x,y\in X$ with $H(Tx,Ty)>0$.

Consequently, they established some fixed point results for multivalued θ-contraction mappings on complete metric spaces as follows:

THEOREM 1.3. [**6**] *Let (X,d) be a complete metric space and $T\colon X \to \mathcal{K}(X)$ be given a multivalued mapping, where $\mathcal{K}(X)$ is the family of all nonempty compact subsets of X. If T is a multivalued θ-contraction, then T has a fixed point.*

Since the compactness of Tx for all $x \in X$ in Theorem 1.3 is a strong condition, it is intended to replace $\mathcal{CB}(X)$ instead of $\mathcal{K}(X)$. However, in the same paper they also gave an example [**6**, Example 2.4] showing that this is not impossible. Even so, this replacement is possible by adding the following weak condition on θ:

(θ_4) $\theta(\inf A) = \inf \theta(A)$ for all $A \subset (0,\infty)$ with $\inf A > 0$.

Let Ω be the family of all functions θ satisfying (θ_1)-(θ_4). It is clear that $\Omega \subset \Theta$. If we define $\theta(t) = e^{\sqrt{t}}$ for $t < 1$ and $\theta(t) = 9$ for $t \geqslant 1$, then $\theta \in \Theta\backslash\Omega$. Note that, if θ is right continuous and satisfies (θ_1), then (θ_4) hold. Conversely, if (θ_4) hold, then θ is right continuous.

THEOREM 1.4. [**6**] *Let (X,d) be a complete metric space and $T\colon X \to \mathcal{CB}(X)$ be given a multivalued mapping. If T is a multivalued θ-contraction with $\theta \in \Omega$, then T has a fixed point.*

If we examine the proofs of Theorem 1.3 and Theorem 1.4, we can see that the mentioned multivalued mappings are MWP operators.

The aim of this paper is to give a new and general class of multivalued weakly Picard operators on complete metric space. For this, we will introduce a new type contraction for multivalued mappings taking into account both multivalued almost contraction and multivalued θ-contraction. Later, we give some fixed point results for mappings of this type on complete metric spaces.

2. Results

Our main results are based on the following new concept.

Let (X,d) be a metric space, $T\colon X \to \mathcal{CB}(X)$ be a given mapping and $\theta \in \Theta$. Then, we say that T is a multivalued almost θ-contraction if there exist two constants $k \in (0,1)$ and $\lambda \geqslant 0$ such that

$$\theta(H(Tx,Ty)) \leqslant [\theta(d(x,y) + \lambda d(y,Tx))]^k, \tag{2.1}$$

for all $x, y \in X$ with $H(Tx,Ty) > 0$.

Note that, taking into account the symmetry property of the metric, the multivalued almost θ-contractive condition includes the following dual one

$$\theta(H(Tx,Ty)) \leqslant [\theta(d(x,y) + \lambda d(x,Ty))]^k, \tag{2.2}$$

for all $x, y \in X$ with $H(Tx,Ty) > 0$. So, in order to check the multivalued almost θ-contractiveness of a multivalued mapping T, it is necessary to check both (2.1) and (2.2) or the following inequality:

$$\theta(H(Tx,Ty)) \leqslant [\theta(d(x,y) + \lambda \min\{d(y,Tx), d(x,Ty)\})]^k,$$

for all $x, y \in X$ with $H(Tx,Ty) > 0$.

REMARK 2.1. Taking $\theta(t) = e^{\sqrt{t}}$ in inequality (2.1), then it turns to (1.1) with $\delta = k^2$ and $L = k^2\lambda$. Thus, every multivalued almost contraction is also multivalued almost θ-contraction. On the other hand, taking $\lambda = 0$ in inequality (2.1), then it turns to (1.3). Thus, every multivalued θ-contraction is also multivalued almost θ-contraction. Therefore, Theorems 1.1, 1.2 and 1.4 are special cases of the following of first result of ours.

In fact, our first result also presents a new class of multivalued weakly Picard operators on a complete metric space.

THEOREM 2.1. *Let (X, d) be a complete metric space and $T\colon X \to \mathcal{CB}(X)$ be given a mapping. If T is an multivalued almost θ-contraction with $\theta \in \Omega$, then T is a MWP.*

PROOF. Define a set $X^* = \{x \in X : d(x, Tx) > 0\}$. Let $x_0 \in X^*$ be an arbitrary point and choose $x_1 \in Tx_0$. If $x_1 \notin X^*$, then x_1 is a fixed point of T. Suppose $x_1 \in X^*$, then $0 < d(x_1, Tx_1) \leqslant H(Tx_0, Tx_1)$ and so from (θ_1), we obtain $\theta(d(x_1, Tx_1)) \leqslant \theta(H(Tx_0, Tx_1))$. From (2.1), we can write

$$\begin{aligned}\theta(d(x_1, Tx_1)) &\leqslant \theta(H(Tx_0, Tx_1)) \\ &\leqslant [\theta(d(x_1, x_0) + \lambda d(x_1, Tx_0))]^k \leqslant [\theta(d(x_1, x_0))]^k.\end{aligned} \tag{2.3}$$

From (θ_4), we know that $\theta(d(x_1, Tx_1)) = \inf_{y \in Tx_1} \theta(d(x_1, y))$, and so, from (2.3), we have

$$\inf_{y \in Tx_1} \theta(d(x_1, y)) \leqslant [\theta(d(x_0, x_1))]^k < [\theta(d(x_0, x_1))]^s, \tag{2.4}$$

where $s \in (k, 1)$. Then, from (2.4) there exists $x_2 \in Tx_1$ such that

$$\theta(d(x_1, x_2)) \leqslant [\theta(d(x_0, x_1))]^s.$$

If $x_2 \notin X^*$, then x_2 is a fixed point of T. Otherwise, by the same way, we can find $x_3 \in Tx_2$ such that $\theta(d(x_2, x_3)) \leqslant [\theta(d(x_1, x_2))]^s$. Therefore, continuing recursively, we can obtain a sequence $\{x_n\}$ in X^* such that $x_{n+1} \in Tx_n$ and

$$\theta(d(x_n, x_{n+1})) \leqslant [\theta(d(x_{n-1}, x_n))]^s, \tag{2.5}$$

for all $n \in \mathbb{N}$ (Otherwise T has a fixed point). Denote $c_n = d(x_n, x_{n+1})$, for $n \in \mathbb{N}$. Then $c_n > 0$ for all $n \in \mathbb{N}$ and, using (2.5), we have

$$\theta(c_n) \leqslant [\theta(c_{n-1})]^s \leqslant [\theta(c_{n-2})]^{s^2} \leqslant \cdots \leqslant [\theta(c_1)]^{s^{n-1}}.$$

Thus, we obtain

$$1 < \theta(c_n) \leqslant [\theta(c_1)]^{s^{n-1}} \tag{2.6}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.6), we obtain $\lim_{n\to\infty} \theta(c_n) = 1$. From (θ_2), $\lim_{n\to\infty} c_n = 0^+$ and so, from (θ_3), there exist $r \in (0, 1)$ and $l \in (0, \infty]$ such that

$$\lim_{n\to\infty} \frac{\theta(c_n) - 1}{(c_n)^r} = l.$$

Suppose that $l < \infty$. In this case, let $B = \frac{l}{2} > 0$. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,

$$\Big|\frac{\theta(c_n) - 1}{(c_n)^r} - l\Big| \leqslant B.$$

This implies that, for all $n \geqslant n_0$,

$$\frac{\theta(c_n) - 1}{(c_n)^r} \geqslant l - B = B.$$

Then, for all $n \geqslant n_0$, we have $n(c_n)^r \leqslant An[\theta(c_n) - 1]$, where $A = 1/B$.

Suppose now that $l = \infty$. Let $B > 0$ be an arbitrary positive number. From the definition of the limit, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geqslant n_0$,

$$\frac{\theta(c_n) - 1}{(c_n)^r} \geqslant B.$$

This implies that, for all $n \geqslant n_0$, we have $n(c_n)^r \leqslant An[\theta(c_n) - 1]$, where $A = 1/B$.

Thus, in all cases, there exist $A > 0$ and $n_0 \in \mathbb{N}$ such that $n(c_n)^r \leqslant An[\theta(c_n)-1]$, for all $n \geqslant n_0$. Using (2.6), we obtain $n(c_n)^r \leqslant An\big[[\theta(c_1)]^{s^{n-1}} - 1\big]$, for all $n \geqslant n_0$. Letting $n \to \infty$ in the above inequality, we obtain $\lim_{n\to\infty} n(c_n)^r = 0$. Thus, there exits $n_1 \in \mathbb{N}$ such that $n(c_n)^r \leqslant 1$ for all $n \geqslant n_1$. So, we have for all $n \geqslant n_1$

$$c_n \leqslant \frac{1}{n^{1/r}}. \tag{2.7}$$

In order to show that $\{x_n\}$ is a Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geqslant n_1$. Using the triangular inequality for the metric and from (2.7), we have

$$\begin{aligned} d(x_n, x_m) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &= c_n + c_{n+1} + \cdots + c_{m-1} = \sum_{i=n}^{m-1} c_i \leqslant \sum_{i=n}^{\infty} c_i \leqslant \sum_{i=n}^{\infty} \frac{1}{i^{1/r}}. \end{aligned}$$

By the convergence of the series $\sum_{i=1}^{\infty} \frac{1}{i^{1/r}}$, letting to limit $n \to \infty$, we get $d(x_n, x_m) \to 0$. This yields that $\{x_n\}$ is a Cauchy sequence in (X, d). Since (X, d) is a complete metric space, the sequence $\{x_n\}$ converges to some point $z \in X$, that is, $\lim_{n\to\infty} x_n = z$.

Now, from (θ_1) and (2.1), for all $x, y \in X$ with $H(Tx, Ty) > 0$, we get

$$H(Tx, Ty) < d(x, y) + \lambda d(y, Tx)$$

and so, for all $x, y \in X$, we get $H(Tx, Ty) \leqslant d(x, y) + \lambda d(y, Tx)$. Therefore,

$$d(x_{n+1}, Tz) \leqslant H(Tx_n, Tz) \leqslant d(x_n, z) + \lambda d(z, Tx_n) \leqslant d(x_n, z) + \lambda d(z, x_{n-1}).$$

Letting to limit $n \to \infty$ in the above inequality, we obtain $d(z, Tz) = 0$. Thus, we get $z \in Tz$. Therefore, it can be seen that, we can construct a sequence $\{x_n\}$ in X such that $x_{n+1} \in Tx_n$ for any initial point x_0, which is convergent and its limit is a fixed point of T. That is, T is a weakly Picard operator. Therefore T is a MWP. □

Now, we give a nontrivial example showing that T is a MWP because of it is multivalued almost θ-contraction on a complete metric space. Nevertheless, taking into account Theorem 1.2 (or Theorem 1.4), we can not guarantee that T is a MWP since it is not both multivalued almost contraction and multivalued θ-contraction.

EXAMPLE 2.1. Let $X = [0,1] \cup \{2,3,\cdots\}$ and

$$d(x,y) = \begin{cases} 0, & \text{if } x = y \\ |x-y|, & \text{if } x,y \in [0,1] \\ x+y, & \text{if one of } x,y \notin [0,1] \end{cases}$$

Then (X,d) is a complete metric space. Define a mapping $T\colon X \to \mathcal{CB}(X)$ by

$$Tx = \begin{cases} \{x\}, & x \in [0,1] \\ \{1, x-1\}, & x \in \{2,3,\dots\} \end{cases}.$$

First, suppose that T is a multivalued almost contraction. Then there exists two constants $\delta \in (0,1)$ and $L \geqslant 0$ satisfying $H(Tx,Ty) \leqslant \delta d(x,y) + Ld(y,Tx)$ for all $x,y \in X$. Now, for $y = 1$ and $x > 2$, since $d(y,Tx) = 0$, we get

$$x = H(Tx,Ty) \leqslant \delta d(x,y) + Ld(y,Tx) = \delta(x+1)$$

and so $\frac{x}{x+1} \leqslant \delta$ for all $x \in X$, which is impossible.

Second, T is not also multivalued θ-contraction, since $H(T0,T1) = 1 = d(0,1)$, then for all $\theta \in \Omega$ and any $k \in (0,1)$, we have

$$\theta(H(Tx,Ty)) = \theta(1) > [\theta(1)]^k = [\theta(d(x,y))]^k.$$

Finally, we claim that T is multivalued almost θ-contraction with $\theta(t) = e^{\sqrt{te^t}}$, $k = \frac{1}{\sqrt{2}}$ and $\lambda = 1$. To see this, we have to show that

$$\frac{H(Tx,Ty)e^{H(Tx,Ty)-d(x,y)-\min\{d(y,Tx),d(x,Ty)\}}}{d(x,y)+\min\{d(y,Tx),d(x,Ty)\}} \leqslant \frac{1}{2}, \tag{2.8}$$

for all $x,y \in X$ with $H(Tx,Ty) > 0$. Note that, $H(Tx,Ty) > 0$ if and only if $(x,y) \notin \Delta \cup \{(1,2),(2,1)\}$, where $\Delta = \{(x,x) : x \in X\}$. Now, for shortness we will assign the left side of (2.8) as $A(x,y)$. Without loss of generality, we may assume $x > y$ in the following three cases:

Case 1. For $x,y \in [0,1]$, since

$$H(Tx,Ty) = d(x,y) = \min\{d(y,Tx), d(x,Ty)\} = x - y,$$

we have

$$A(x,y) = \frac{x-y}{2(x-y)}e^{-(x-y)} < \frac{x-y}{2(x-y)} = \frac{1}{2}.$$

Case 2. For $y \in [0,1]$ and $x \in \{2,3,\dots\}$, since

$$H(Tx,Ty) = x+y-1, \quad d(x,y) = x+y, \quad \min\{d(y,Tx), d(x,Ty)\} = 1-y,$$

we have

$$A(x,y) = \frac{x+y-1}{x+1}e^{y-2} < e^{-1} < \frac{1}{2}.$$

Case 3. For $x, y \in \{2, 3, \dots\}$, since

$$H(Tx, Ty) = x + y - 2, \quad d(x, y) = x + y, \quad \min\{d(y, Tx), d(x, Ty)\} = 1 + y,$$

we have

$$A(x, y) = \frac{x + y - 2}{x + 2y + 1} e^{-3-y} < e^{-1} < \frac{1}{2}.$$

This shows that T is multivalued almost θ-contraction. Thus, all conditions of Theorem 2.1 are satisfied and so T is a MWP.

By taking $\theta(t) = e^{\sqrt{t^2+t}}$ in Theorem 2.1, we obtain the following corollary:

COROLLARY 2.1. *Let (X, d) be a complete metric space and $T\colon X \to \mathcal{CB}(X)$ be a mapping. Suppose that, there exists two constants $l \in (0, 1)$ and $\lambda \geqslant 0$ such that*

$$\frac{H(Tx, Ty)[H(Tx, Ty) + 1]}{[d(x, y) + \lambda d(y, Tx)][d(x, y) + \lambda d(y, Tx) + 1]} \leqslant l,$$

for all $x, y \in X$ with $H(Tx, Ty) > 0$. Then, T has a fixed point.

The following result is interested in the mapping $T\colon X \to \mathcal{K}(X)$. Here, we can remove the condition (θ_4) on the function θ.

THEOREM 2.2. *Let (X, d) be a complete metric space and $T\colon X \to \mathcal{K}(X)$ be given a mapping. If T is a multivalued almost θ-contraction, then T is a MWP.*

PROOF. As in proof of Theorem 2.1, we get

$$\theta(d(x_1, Tx_1)) \leqslant \theta(H(Tx_0, Tx_1)) \leqslant [\theta(d(x_1, x_0))]^k. \tag{2.9}$$

Since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that $d(x_1, x_2) = d(x_1, Tx_1)$. From (2.9),

$$\theta(d(x_1, x_2)) \leqslant \theta(H(Tx_0, Tx_1)) \leqslant [\theta(d(x_1, x_0))]^k.$$

By induction, we obtain a sequence $\{x_n\}$ in X^* with the property that $x_{n+1} \in Tx_n$ and

$$\theta(d(x_n, x_{n+1})) \leqslant [\theta(d(x_n, x_{n-1}))]^k$$

for all $n \in \mathbb{N}$. The rest of the proof can be completed as in the proof of Theorem 2.1. □

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