PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 103(117) (2018), 69–75 DOI: https://doi.org/10.2298/PIM1817069G

# ON WARPED PRODUCT MANIFOLDS SATISFYING RICCI–HESSIAN CLASS TYPE EQUATIONS

**Sinem Güler and Sezgin Altay Demirbağ**

Abstract. We deal with a study of warped product manifold which is also a generalized quasi Einstein manifold. Then, we investigate the relationships between such warped products and certain manifolds that provide some Ricci–Hessian type equations, such as $\mathrm{Ric}_f^m = \lambda g$ for some smooth function $\lambda$, where $\mathrm{Ric}_f^m$ denotes the $m$-Bakery–Emery Ricci tensor. Finally, we obtain some rigidity conditions for such manifolds.

## 1. Introduction

Let $(M^n, g)$ be a Riemannian manifold of dimension $n$ with $n > 2$ and let $f : M \to \mathbb{R}$ be a smooth function. A natural extension of the Ricci tensor is the $m$-Bakry–Emery–Ricci tensor [**3**] defined by $\mathrm{Ric}_f^m = \mathrm{Ric} + \mathrm{Hess}\, f - \frac{1}{m} df \otimes df$, for some positive integer $m$. According to [**5**], a smooth manifold $(M^n, g)$ is said to be generalized quasi Einstein manifold if there exist three smooth functions $f, \alpha$ and $\lambda$ satisfying the equation

$$\mathrm{Ric} + \mathrm{Hess}\, f - \alpha df \otimes df = \lambda g \tag{1.1}$$

In such a manifold, $f$ is called the potential function. This generalized class reduces to a *gradient Ricci soliton* [**2**] when $\alpha = 0$ and $\lambda \in \mathbb{R}$; *m-quasi Einstein manifold* [**4**] when $\alpha = \frac{1}{m}$, $m \in \mathbb{N}$ and $\lambda \in \mathbb{R}$ and also *m-generalized quasi Einstein manifold* whenever $\lambda \in C^\infty(M)$.

**1.1. Backgrounds and Basic Formulas.** In this section, we recall some basic definitions and formulas about warped products. Let $(M^q, g)$ and $(N^{n-q}, \bar{g})$; $(1 \leqslant \dim M = q \leqslant n, \dim N = n - q)$ be two Riemannian manifolds and let $f$ be a positive, smooth function of $M$. The *warped product* $M \times_f N$ of $(M, g)$ and $(N, \bar{g})$ is the product manifold $M \times N$ with the metric $\tilde{g} = g \times_f \bar{g}$, [**1, 12**]. Here, $(M, g)$ is base, $(N, \bar{g})$ is fiber, $f$ is the warping function and any geometric objects denoted by "bar" (resp. "tilde") symbol will be assumed to lie on the fiber $N$ (resp. on the

2010 *Mathematics Subject Classification:* 53C15; 53C25, 53B15.

*Key words and phrases:* $m$-Bakery–Emery Ricci tensor, generalized quasi Einstein manifold, warped product manifold, conformal vector field.

This work is partially supported by the project BAP-38385 of Istanbul Technical University.

product $M \times N$). Also, for such a warped product, any geometric objects without symbol will be assumed to lie on the base $M$. The local components of the metric $\tilde{g}$ can be expressed as

$$\tilde{g}_{ij} = \begin{cases} g_{ab}, & \text{if } i = a,\ j = b \\ f\bar{g}_{\alpha\beta}, & \text{if } i = \alpha,\ j = \beta \\ 0, & \text{otherwise} \end{cases}$$

where $a, b, c \cdots \in \{1, \cdots, q\}$, $\alpha, \beta, \gamma \cdots \in \{q+1, \cdots, n\}$ and $i, j, k \cdots \in \{1, \cdots, n\}$. The warped product manifold with constant warping function is simply called Riemannian product, (for more details, see [**9, 12, 14**]). Then, all components of the Ricci tensor of $M \times_f N$ are given by:

LEMMA 1.1. [**12**] *On $M^q \times_f N^{(n-q)}$, for all $X, Y \in \chi(M)$ and $V, W \in \chi(N)$,*

(1) $\tilde{\text{Ric}}(X,Y) = \text{Ric}(X,Y) - \dfrac{n-q}{f}\text{Hess}(f)(X,Y)$, (2) $\tilde{\text{Ric}}(X,V) = 0$

(3) $\tilde{\text{Ric}}(V,W) = \bar{\text{Ric}}(V,W) - \Big[\dfrac{\Delta f}{f} + (n-q-1)\dfrac{\|\nabla f\|^2}{f^2}\Big]\tilde{g}(V,W)$

DEFINITION 1.1. A Riemannian manifold $(M^n, g)$ $(n > 2)$ is called a mixed super quasi Einstein manifold [**13**] (briefly $(\text{MSQE})_n$) if its Ricci tensor is nonzero and satisfies the following condition

$$\begin{aligned}\text{Ric}(X,Y) = {} & ag(X,Y) + bA(X)A(Y) + cB(X)B(Y) \\ & + d[A(X)B(Y) + A(Y)B(X)] + eD(X,Y)\end{aligned}$$

where $a, b, c, d, e$ are real valued, nonzero scalar functions on $(M^n, g)$, $A, B$ are dual 1-forms of the vector fields $U$ and $V$ (which are called the generators), i.e., for all $X \in \chi(M)$, $A(X) = g(X,U)$ and $B(X) = g(X,V)$, $D$ is a symmetric $(0,2)$-tensor field such that $D(X,U) = 0$ and $tr(D) = 0$. If $e = c = 0$, then the manifold is said to be a generalized quasi Einstein manifold $(G(QE)_n)$ [**6**]; if $e = d = c = 0$, then the manifold reduces to a quasi Einstein manifold $((QE)_n)$ [**7**]. These manifolds have been studied extensively by several authors (see [**8, 10**]).

REMARK 1.1. These two definitions of generalized quasi Einstein manifolds given by [**5**] and [**6**] are not equivalent, but our main goal in this work is to establish a link between these two concepts.

## 2. Main Results

In this section, we express the Ricci tensor of warped product according to its base and fiber when the warped product is also a generalized quasi Einstein manifold in the sense of Chaki.

THEOREM 2.1. *Let $M^q \times_f N^{n-q}$, $(q = \dim M,\ n-q = \dim N > 1)$ be a $G(QE)_n$ warped product manifold whose generators are $\xi_1$ and $\xi_2$. Then for all $X, Y \in \chi(M)$ and $U, V \in \chi(N)$, the following statements hold:*

(i) *If* $\xi_1, \xi_2 \in \chi(M)$, *then*

$$\mathrm{Ric}(X,Y) = ag(X,Y) + \frac{n-q}{f}\operatorname{Hess} f(X,Y) + bg(X,\xi_1)g(Y,\xi_1) + c[g(X,\xi_1)g(Y,\xi_2) + g(X,\xi_2)g(Y,\xi_1)] \tag{2.1}$$

$$\bar{\mathrm{Ric}}(U,V) = f^2\Big(a + \frac{\Delta f}{f} + \frac{n-q-1}{f^2}\|\operatorname{grad} f\|^2\Big)\bar{g}(U,V) \tag{2.2}$$

(ii) *If* $\xi_1, \xi_2 \in \chi(N)$, *then*

$$\mathrm{Ric}(X,Y) = ag(X,Y) + \frac{n-q}{f}\operatorname{Hess} f(X,Y)$$

$$\bar{\mathrm{Ric}}(U,V) = f^2\Big(a + \frac{\Delta f}{f} + \frac{n-q-1}{f^2}\|\operatorname{grad} f\|^2\Big)\bar{g}(U,V) + bf^4\bar{g}(U,\xi_1)\bar{g}(V,\xi_1) + cf^4[\bar{g}(U,\xi_1)\bar{g}(V,\xi_2) + \bar{g}(U,\xi_2)\bar{g}(V,\xi_1)]$$

(iii) *If* $\xi_1 \in \chi(M)$ *and* $\xi_2 \in \chi(N)$, *then*

$$\mathrm{Ric}(X,Y) = ag(X,Y) + \frac{n-q}{f}\operatorname{Hess} f(X,Y) + bg(X,\xi_1)g(Y,\xi_1)$$

$$\bar{\mathrm{Ric}}(U,V) = f^2\Big(a + \frac{\Delta f}{f} + \frac{n-q-1}{f^2}\|\operatorname{grad} f\|^2\Big)\bar{g}(U,V)$$

(iv) *If* $\xi_1 \in \chi(N)$ *and* $\xi_2 \in \chi(M)$, *then*

$$\mathrm{Ric}(X,Y) = ag(X,Y) + \frac{n-q}{f}\operatorname{Hess} f(X,Y)$$

$$\bar{\mathrm{Ric}}(U,V) = f^2\Big(a + \frac{\Delta f}{f} + \frac{n-q-1}{f^2}\|\operatorname{grad} f\|^2\Big)\bar{g}(U,V) + bf^4\bar{g}(U,\xi_1)\bar{g}(V,\xi_1)$$

Proof. Since $M \times_f N$ is a $G(QE)_n$, we have

$$\widetilde{\mathrm{Ric}} = a\tilde{g} + b\,\tilde{A}\otimes A + c[\tilde{A}\otimes\tilde{B} + \tilde{B}\otimes\tilde{A}] \tag{2.3}$$

(i) Let $\xi_1, \xi_2 \in \chi(M)$. Then for all $X \in \chi(M)$, we get $\tilde{A}(X) = \tilde{g}(X,\xi_1) = g(X,\xi_1)$ and $\tilde{B}(X) = \tilde{g}(X,\xi_2) = g(X,\xi_2)$. Thus, equation (2.3) yields

$$\tilde{\mathrm{Ric}}(X,Y) = ag(X,Y) + bg(X,\xi_1)g(Y,\xi_1) + c[g(X,\xi_1)g(Y,\xi_2) + g(X,\xi_2)g(Y,\xi_1)] \tag{2.4}$$

Using (2.4) in Lemma 1.1(1), the Ricci tensor of the base manifold is of the form (2.1). Similarly, for all $U \in \chi(N)$, we get $\tilde{A}(U) = \tilde{g}(U,\xi_1) = 0$ and $\tilde{B}(U) = \tilde{g}(U,\xi_2) = 0$ and so again (2.3) yields

$$\tilde{\mathrm{Ric}}(U,V) = af^2\bar{g}(U,V) \tag{2.5}$$

Using (2.5) in Lemma 1.1(3), the Ricci tensor of the fiber is of the form (2.2).

Analogously, (ii)–(iv) can be proved in a similar way with the case (*i*). □

Theorem 2.2. *Let* $M^q \times_f N^{n-q}$ $(q = \dim M,\ n-q = \dim N > 1)$ *be a* $G(QE)_n$ *warped product manifold with generators* $\xi_1$, $\xi_2$ *and let* $h = -(n-q)\ln f$. *Then:*

(i) *If* $\xi_1, \xi_2 \in \chi(M)$ *such that* $\xi_2 = \nabla h$*, then* $M$ *is* $(MSQE)_q$ *with generators* $\xi_1, \nabla h$ *and associated tensor* $\mathrm{Hess}(h)$*, provided that* $g(\nabla_X \xi_1, \nabla f) = 0$*, for any* $X \in \chi(M)$ *and* $\Delta f = \frac{1}{f}\|\nabla f\|^2$*. Also,* $N$ *is Einstein.*

(ii) *If* $\xi_1, \xi_2 \in \chi(N)$*, then* $(M, g, h, a)$ *is a* $(n-q)$*-quasi Einstein manifold in the sense of Case and* $N$ *is a* $G(QE)_{n-q}$ *in the sense of Chaki with generators* $\bar{\xi} = f\xi_1$ *and* $\bar{\mu} = f\xi_2$*.*

(iii) *If* $\xi_1 \in \chi(M)$ *and* $\xi_2 \in \chi(N)$ *such that* $\xi_1 = \nabla h$*, then* $(M, g, h, a)$ *is a* $G(QE)_q$ *in the sense of Catino and* $N$ *is Einstein.*

(iv) *If* $\xi_1 \in \chi(N)$ *and* $\xi_2 \in \chi(M)$*, then* $(M, g, h, a)$ *is an* $(n-q)$*-quasi Einstein manifold in the sense of Case and* $N$ *is a* $(QE)_{n-q}$ *in the sense of Chaki with generator* $\bar{\xi} = f\xi_1$*.*

PROOF. (i) Suppose that $\xi_1, \xi_2 \in \chi(M)$. Now, let us define

$$h = -(n-q)\ln f \quad \textit{and} \quad \xi_2 = \nabla h \tag{2.6}$$

Then we calculate

$$\mathrm{Hess}(h)(X,Y) = -\frac{n-q}{f}\mathrm{Hess}(f)(X,Y) + \frac{n-q}{f^2}\,df(X)\,df(Y) \tag{2.7}$$

In view of (2.6) and (2.7), (2.1) yields

$$\begin{aligned}\mathrm{Ric}(X,Y) = ag(X,Y) - \mathrm{Hess}(h)(X,Y) + \frac{1}{n-q}\,dh(X)\,dh(Y)\\ + bA(X)A(Y) + [A(X)\,dh(Y) + A(Y)\,dh(X)]\end{aligned}$$

Moreover, since $\xi_1$ and $\xi_2$ are orthonormal, $g(\xi_1, \xi_2) = g(\xi, \nabla h) = 0$ so $\xi_1(h) = 0$ and $\xi_1(f) = 0$. Thus, if $g(\nabla_X \xi_1, \nabla f) = 0$ holds for any $X \in \chi(M)$, we get $\mathrm{Hess}(f)(X, \xi) = 0$. That is, choosing $D =: \mathrm{Hess}(h)$, we get $D(X, \xi_1) = 0$, for all $X \in \chi(M)$. Also, we have $\Delta h = \frac{n-q}{f}[-\Delta f + \frac{1}{f}\|\nabla f\|^2]$. Thus, if we assume that the relation $\Delta f = \frac{1}{f}\|\nabla f\|^2$ holds, then the $(0,2)$-tensor field $D$ also satisfies $tr(D) = 0$. Therefore, the base manifold becomes $(MSQE)_q$. On the other hand, from (2.2) the fiber manifold becomes Einstein.

Analogously, (ii)–(iv) can be proved in a similar way with the case (i). □

EXAMPLE 2.1. Let us consider the function $f : \mathbb{R}^3 \to \mathbb{R}^+$ defined by $f(t) = e^t$ and $M = \{(t, x, y, z) \in \mathbb{R}^4 : \ t \in (-1-\sqrt{7}, 1-\sqrt{3}) \cup (-1+\sqrt{7}, 1+\sqrt{3})\}$ be an open subset of $\mathbb{R}^4$ endowed with the metric

$$ds^2 = g_{ij}dx^i dx^j = (dt)^2 + t^4(dx)^2 + t^4(dy)^2 + e^{2t}(dz)^2$$

In [**11**], the authors proved that $M$ is a $G(QE)_4$ with a nonconstant scalar curvature. This line element on $M$ also provides the warped product structure $\mathbb{R}^3 \times_{e^t} \mathbb{R}$, where $\mathbb{R}^3$ and $\mathbb{R}$ denote the base and respectively fiber.

Now, recall that a smooth vector field $\phi$ on a Riemannian manifold $(M^n, g)$ is called *conformal* if it satisfies

$$\mathcal{L}_\phi g = 2\Omega g \tag{2.8}$$

where $\mathcal{L}_\phi$ denotes the Lie derivative in the direction of $\phi$ and $\Omega$ is some smooth function on $M$, which is called a conformal factor. Particularly, if the vector field $\phi$

satisfies the condition $\nabla_X \phi = \Omega X$, for all $X \in \chi(M)$ and a smooth function $\Omega$, then it is called *closed conformal vector field*. Moreover, if $\Omega = 0$, it is called parallel. It can be easily proved that the Ricci tensor of a Riemannian manifold admitting a nontrivial closed and conformal vector field $\phi$ is of the form

$$\mathrm{Ric}(\phi, X) = -(n-1)g(\nabla\Omega, X). \tag{2.9}$$

for any $X \in \chi(M)$ where $\Omega$ denotes the conformal factor of $\phi$. The following algebraic characterization of a Riemannian manifold admitting closed and conformal vector fields will be used in the proof of our main theorem.

LEMMA 2.1. *Let $(M^n, g, f, \lambda)$ be an $m$-generalized quasi Einstein manifold admitting a closed conformal vector field $\phi$ with conformal factor $\Omega$. Then the following statements hold:*

(1) $\nabla\phi(f) = \frac{1}{m}\phi(f)\nabla f + (n-1)\nabla\Omega + \lambda\phi + \Omega\nabla f$.
(2) $\mathrm{Hess}(\phi(f)) = \frac{1}{m}[d(\phi(f)) \otimes df + \phi(f)\,\mathrm{Hess}\, f] + (n-1)\,\mathrm{Hess}\,\Omega$
$\quad + (d\lambda \otimes \phi^\flat) + \lambda\Omega g + \Omega\,\mathrm{Hess}\, f + d\Omega \otimes df$.
(3) $\frac{1}{m}d(\phi(f)) \otimes df + (d\lambda \otimes \phi^\flat) + d\Omega \otimes df$ *is symmetric.*

PROOF. In view of (2.9), the fundamental equation $m$-generalized quasi Einstein manifold yields

$$\mathrm{Hess}\, f(\phi, X) = \frac{1}{m}df(\phi)df(X) + (n-1)d\Omega(X) + \lambda g(\phi, X) \tag{2.10}$$

By using $\mathrm{Hess}(f)(\phi, X) = X(\phi(f)) - (\nabla_X\phi)f$ and (2.9) in (2.10), we get

$$X(\phi(f)) = \frac{1}{m}df(\phi)df(X) + (n-1)d\Omega(X) + \lambda g(\phi, X) + \Omega df(X) \tag{2.11}$$

If we express the last equation without the inner product of $X$, we obtain the first assertion. Taking the second covariant derivative of (2.11) in the direction of arbitrary vector field $Z$, we get

(2.12)

$$\begin{aligned}\nabla_Z X(\phi(f)) = \frac{1}{m}[Z(\phi(f))df(X) + \phi(f)ZX(f)] + (n-1)\nabla_Z(X(\Omega))\\ + d\lambda(Z)\phi^\flat(X) + \lambda(\nabla_Z\phi^\flat)(X) + d\Omega(Z)df(X) + \Omega\nabla_Z(X(f))\end{aligned}$$

Again using the definition of Hessian and equation (2.9) in (2.12), we obtain

$$\begin{aligned}\mathrm{Hess}(\phi(f))(Z, X) = \frac{1}{m}[d(\phi(f))(Z)df(X) + \phi(f)\,\mathrm{Hess}(f)(Z, X)]\\ + (n-1)\,\mathrm{Hess}(\Omega)(Z, X) + d\lambda \otimes \phi^\flat(Z, X) + \lambda\Omega g(Z, X)\\ + d\Omega \otimes df(Z, X) + \Omega\,\mathrm{Hess}(f)(Z, X)), \quad \forall\, X, Z \in \chi(M)\end{aligned}$$

which proves the second assertion. Since the Hessian and the metric tensor are symmetric, we can easily verify the third assertion. □

Now, by virtue of Lemma 2.1, we shall establish the following results.

THEOREM 2.3. *Let $(M^n, g, f, \lambda)$ be an $m$-generalized quasi Einstein manifold admitting a nonzero closed conformal vector field $\phi$ with conformal factor $\Omega$. Then, its Ricci tensor is of the form*

$$\text{Ric} = \lambda g + \Big(\frac{1-n}{\|\phi\|} d\Omega(U) - \lambda\Big) U^\flat \otimes U^\flat \tag{2.13}$$

*where $U^\flat$ is a 1-form corresponding to the unit vector field $U$ in the direction of $\phi$. If, in particular, $\phi$ is parallel, then the Ricci tensor reduces to the form* $\text{Ric} = \lambda(g - U^\flat \otimes U^\flat)$.

PROOF. From Lemma 2.1(1)(2), we get for any $X, Y \in \chi(M)$

$$\begin{aligned}\Big[\frac{1}{m}\lambda df(X) - d\lambda(X)\Big] g(\phi, Y) + \frac{n+m-1}{m}[d\Omega(Y)df(X) - d\Omega(X)df(Y)] \\ = \Big[\frac{1}{m}\lambda df(Y) - d\lambda(Y)\Big] g(\phi, X)\end{aligned} \tag{2.14}$$

Putting $X = \phi$ in (2.14), we get

$$\begin{aligned}\Big[\frac{1}{m}\lambda df(\phi) - d\lambda(\phi)\Big] g(\phi, Y) + \frac{n+m-1}{m}[d\Omega(Y)df(\phi) - d\Omega(\phi)df(Y)] \\ = \Big[\frac{1}{m}\lambda df(Y) - d\lambda(Y)\Big] \|\phi\|^2\end{aligned} \tag{2.15}$$

Let $U$ be a unit vector field in the direction of $\phi$. Then by (2.15), we obtain

$$\nabla f\Big(1 + \frac{n+m-1}{\lambda\|\phi\|} U(\Omega)\Big) = U(f)U + \frac{m}{\lambda}[\nabla\lambda - U(\lambda)U] + \frac{n+m-1}{\lambda\|\phi\|}\nabla\Omega U(f) \tag{2.16}$$

and so by virtue of (2.16), (2.10) yields

$$\text{Hess}\, f = \frac{1}{m} df \otimes df + \lambda U^\flat \otimes U^\flat + \frac{n-1}{\|\phi\|} d\Omega \otimes U^\flat \tag{2.17}$$

Next, substituting (2.17) in the fundamental equation of $m$-generalized quasi Einstein manifold, we obtain the Ricci tensor as in (2.13). If we take $\Omega = 0$, the rest of the proof is completed. □

Also from (2.8), the unit vector field $U$ in the direction of the closed conformal vector field $\phi$ satisfies $\nabla_X U = \frac{\Omega}{\|\phi\|}[X - U^\flat(X)U]$, for any $X \in \chi(M)$. This means that $U$ is a unit concircular vector field. According to K. Yano [**15**], a Riemannian manifold admits a concircular vector field if and only if there exists a coordinate system with respect to which the fundamental quadratic differential form may be written in the form $ds^2 = (dx^1)^2 + e^q g^*_{\alpha\beta} dx^\alpha dx^\beta$, where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are the functions of $x^\gamma$ only $(\alpha, \beta, \gamma, \delta = 2, 3, \ldots, n)$ and $q = q(x^1) \neq$ *constant* is a function of $x^1$ only. Hence we can state the following:

THEOREM 2.4. *Let $(M^n, g, f, \lambda)$ be an $m$-generalized quasi Einstein manifold admitting a nonzero closed conformal vector field $\phi$. Then $(M^n, g)$ is a locally warped product manifold $I \times_{e^{q/2}} M^*$, where $I$ is a real interval, $(M^*, g^*)$ is an $(n-1)$-dimensional Riemannian manifold and $q$ is a smooth function on $I$.*

EXAMPLE 2.2. Here, we construct an example for the particular case of Theorem 2.3 and 2.4: Let $M = I \times_f M^*$ be the warped product with the warping function $f = f(t)$. Then there exists a closed conformal vector field $\phi = f\partial_t$ on $M$ with conformal factor $\Omega = f'$. By using the fundamental equation (1.1), it can be shown that $M$ is an $m$-generalized quasi Einstein manifold with the potential function $h = h(t)$, provided that the fiber manifold $M^*$ is Einstein and the warping function $f$ and the potential funtion $h$ are related to each other with the equation $(1-n)\frac{f''}{f} + h'' - \frac{1}{m}(h')^2 = \lambda$, where "$'$" denotes the ordinary differentiation with respect to $t$. This shows the existence of the closed conformal vector field on an $m$-generalized quasi Einstein manifold. Moreover, if the conformal factor $\Omega = f'$ vanishes (i.e., $\phi$ is a parallel vector field and so $M$ is a simply direct product), then equation (2.13) also holds on $M$, where $U = \partial_t$.

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Department of Mathematics
Istanbul Technical University
34469 Maslak, Istanbul, Turkey
singuler@itu.edu.tr
saltay@itu.edu.tr