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Bougerol's identity in law and extensions*

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Abstract: We present a list of equivalent expressions and extensions of Bougerol's celebrated identity in law, obtained by several authors. We recall well-known results and the latest progress of the research associated with this celebrated identity in many directions, we give some new results and possible extensions and we try to point out open questions.

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1. Introduction

Bougerol's celebrated identity in law has been the subject of research for several authors since first formulated in 1983 [9]. A reason for this study is on the one hand its interest from the mathematical point of view and on the other hand its numerous applications, namely in Finance (pricing of Asian options etc.)-see e.g. [35, 11, 37]. However, one still feels that some better understanding remains to be discovered.

This paper is essentially an attempt to collect all the known results (up to now) and to give a (full) survey of the several different equivalent expressions and extensions (to other processes, multidimensional versions, etc.) in a concise way. We also provide a bibliography, as complete as possible. For the extended proofs, we address the reader to the original articles.

Bougerol's remarkable identity states that [9, 1] and [37] (p. 200), with $(B_u, u \geq 0)$ and $(\beta_u, u \geq 0)$ denoting two independent linear Brownian motions[§], we have:

$$\text{for fixed } t, \quad \sinh(B_t) \overset{(law)}{=} \beta_{A_t(B)} \ , \tag{1}$$

where $A_u(B) = \int_0^u ds \exp(2B_s)$ is independent of $(\beta_u, u \geq 0)$. For a first approach of (1), see e.g. the corresponding Chapters in [25] and in [10]. In what follows, sometimes, for simplicity, we will use the notation A_u instead of $A_u(\cdot)$. Alili, Dufresne and Yor [1] obtained the following simple proof of Bougerol's identity (1):

[§]When we simply write: Brownian motion, we always mean real-valued Brownian motion, starting from 0. For 2-dimensional Brownian motion, we indicate planar or complex BM.

Proof. On the one hand, we define $S_t \equiv \sinh(B_t)$; then, applying Itô's formula, we have:

$$S_t = \int_0^t \sqrt{1+S_s^2}\ dB_s + \frac{1}{2}\int_0^t S_s\ ds\ . \tag{2}$$

On the other hand, a time-reversal argument for Brownian motion yields: for fixed $t \geq 0$,

$$\beta_{A_t(B)} = \int_0^t e^{B_s} d\gamma_s \stackrel{(law)}{=} e^{B_t}\int_0^t e^{-B_s} d\gamma_s \equiv Q_t\ , \tag{3}$$

where $(\gamma_s, s \geq 0)$ denotes another 1-dimensional Brownian motion, independent from $(B_s, s \geq 0)$.
Applying once more Itô's formula to Q_t, we have:

$$dQ_t = \frac{1}{2}Q_t dt + (Q_t dB_t + d\gamma_t) = \frac{1}{2}Q_t dt + \sqrt{Q_t^2+1}\ d\delta_t, \tag{4}$$

where δ is another 1-dimensional Brownian motion, depending on B and on γ. From (2) and (4) ,we deduce that S and Q satisfy the same Stochastic Differential Equation with Lipschitz coefficients, hence, we obtain (1). □

With some elementary computations, from (1) (e.g. identifying the densities of both sides, for further details see [27, 4]), we may obtain the Gauss-Laplace transform of the clock A_t: for every $x \in \mathbb{R}$, with $a(x) \equiv \arg\sinh(x) \equiv \log\left(x+\sqrt{1+x^2}\right)$

$$E\left[\frac{1}{\sqrt{A_t}}\exp\left(-\frac{x^2}{2A_t}\right)\right] = \frac{a'(x)}{\sqrt{t}}\exp\left(-\frac{a^2(x)}{2t}\right). \tag{5}$$

where $a'(x) = (1+x^2)^{-1/2}$.
For further use, we note that Bougerol's identity may be equivalently stated as:

$$\sinh(|B_u|) \stackrel{(law)}{=} |\beta|_{A_u(B)}. \tag{6}$$

Using now the symmetry principle (see [3] for the original Note and [14] for a detailed discussion):

$$\sinh(\bar{B}_u) \stackrel{(law)}{=} \bar{\beta}_{A_u(B)}, \tag{7}$$

where, e.g. $\bar{B}_u \equiv \sup_{0\leq s\leq u} B_s$.

In the remainder of this article, we give several versions and generalizations of Bougerol's identity (1). In particular, in Section 2 we give extensions of this identity to other processes (i.e. Brownian motion with drift, hyperbolic Brownian motion, etc.). Section 3 is devoted to some results that we obtain from subordination and some applications to the study of Bougerol's identity in terms of planar Brownian motion and of complex-valued Ornstein-Uhlenbeck processes.

In Section 4, we give some 2 and 3 dimensional extensions of Bougerol's identity, first involving the local time at 0 of the Brownian motion B, and second by studying the joint law of 2 and 3 specific processes. In particular, in Subsection 4.2 we give a new 2-dimensional extension. In Section 5, we generalize Bougerol's identity for the case of diffusions, named *"Bougerol's diffusions"*, followed by some studies in terms of Jacobi processes. Section 6 deals with Bougerol's identity from the point of view of *"peacocks"* (see this Section for the precise definition, as introduced in e.g. [15]). In Section 7 we propose some possible directions for further investigation of this "mysterious" identity in law with its versions and extensions and we give an as full as possible list of references (to the best of author's knowledge) up to now. Finally, in the Appendix, we present several tables of Bougerol's identity and all the equivalent forms and extensions that we present in this survey. These tables can be read independently from the rest of the text.

We also note that (sometimes) the notation used from Section to Section may be independent.

2. Extensions of Bougerol's identity to other processes

2.1. Brownian motions with drifts

Alili, Dufresne and Yor, in [1], showed the following result:

Proposition 2.1. *With μ, ν two real numbers, for every x fixed, the Markov process:*

$$X_t^{(\mu,\nu)} \equiv (\exp(B_t + \mu t)) \left(x + \int_0^t \exp\left(-(B_s + \mu s)\right) d(\beta_s + \nu s) \right), \tag{8}$$

for every $t \geq 0$, has the same law as $(\sinh(Y_t^{(\mu,\nu)}),\ t \geq 0)$, where $(Y_t^{(\mu,\nu)}, t \geq 0)$ is a diffusion with infinitesimal generator:

$$\frac{1}{2} \frac{d^2}{dy^2} + \left(\mu \tanh(y) + \frac{\nu}{\cosh(y)} \right) \frac{d}{dy} \ , \tag{9}$$

starting from $y = \arg\sinh(x)$.

Proof. It suffices to apply Itô's formula to both processes $X^{(\mu,\nu)}$ and $\sinh(Y^{(\mu,\nu)})$. □

It follows now:

Corollary 2.2. *For every t fixed,*

$$\sinh(Y_t^{(\mu,\nu)}) \overset{(law)}{=} \int_0^t \exp(B_s + \mu s) d(\beta_s + \nu s). \tag{10}$$

In particular, in the case $\mu = 1$ and $\nu = 0$:

$$\sinh(B_t + \varepsilon t) \overset{(law)}{=} \int_0^t \exp(B_s + s) d\beta_s, \tag{11}$$

with ε denoting a symmetric Bernoulli variable taking values in $\{-1,1\}$.

Remark 2.3. *With $\mu=-1/2$ and $\nu=0$, we have that, $\sinh\left(Y_t^{(-1/2,0)}\right)$ is a martingale. Indeed, with $Y_t \equiv Y_t^{(-1/2,0)}$, Itô's formula yields:*

$$\begin{aligned}
\sinh(Y_t) &= \int_0^t \cosh(Y_s)\ dY_s + \frac{1}{2}\int_0^t \sinh(Y_s)\ ds \\
&= \int_0^t \cosh(Y_s)\left[dB_s - \frac{1}{2}\tanh(Y_s)\ ds\right] + \frac{1}{2}\int_0^t \sinh(Y_s)\ ds \\
&= \int_0^t \cosh(Y_s) dB_s\ .
\end{aligned}$$

Hence:

$$M_t \equiv \sinh(Y_t) = \beta_{\int_0^t ds(\cosh^2(Y_s))} \equiv \beta_{\int_0^t ds(1+\sinh^2(Y_s))}\ , \tag{12}$$

and for this Markovian martingale, we have:

$$M_t = \int_0^t \sqrt{1+M_s^2}d\gamma_s\ , \tag{13}$$

where γ is another Brownian motion, independent from B and from β.
It can also be seen directly from (8) that $\left(X_t^{(-1/2,0)}, t\geq 0\right)$ is the product of two orthogonal martingales. This property is true because:

$$X_t^{(-1/2,0)} = \frac{B_u}{R_u}\Big|_{u=A_t^{(1/2)}}\ , \tag{14}$$

with $A_t^{(\nu)} = \int_0^t ds\ \exp(2B_s^{(\nu)})$, $(B_t^{(\nu)}, t\geq 0)$ denoting a Brownian motion with drift, and $(R_t, t\geq 0)$ a 2-dimensional Bessel process started at 0. Further details about this ratio are discussed in Sections 5 and 7. We also remark that, with the notation of Section 1, $A_t^{(0)} \equiv A_t$.

2.2. Hyperbolic Brownian motion

Alili and Gruet in [2] generalized Bougerol's identity in terms of hyperbolic Brownian motion:

Proposition 2.4. *We use the notation introduced in the previous Subsection and we denote by Ξ an arcsine variable such that $B^{(\nu)}$, R and Ξ are independent. Let ϕ be a function defined by:*

$$\phi(x,z) = \sqrt{2e^x\cosh(z) - e^{2x} - 1}, \quad for \quad z\geq |x|. \tag{15}$$

Then, for fixed t, we have:

$$\beta_{A_t^{(\nu)}} \overset{(law)}{=} (2\Xi - 1)\phi\left(B_t^{(\nu)}, \sqrt{R_t^2 + (B_t^{(\nu)})^2}\right). \tag{16}$$

In particular, with $\nu = 0$, we recover Bougerol's identity:

$$\beta_{A_t} \overset{(law)}{=} (2\Xi - 1)\phi\left(B_t, \sqrt{R_t^2 + B_t^2}\right) \overset{(law)}{=} \sinh(B_t). \tag{17}$$

This is an immediate consequence of the following:

Lemma 2.5. *(i) The law of the functional $A_t^{(\nu)}$ is characterized by: for all $u \geq 0$,*

$$E\left[\exp\left(-\frac{u^2}{2}\ A_t^{(\nu)}\right)\right] = e^{-\nu^2 t/2} \int_{\mathbb{R}} dx\ e^{\nu x} \int_{|x|}^{+\infty} dz\ \frac{z}{\sqrt{2\pi t^3}}\ e^{-z^2 t/2} J_0(\phi(x,z)), \tag{18}$$

where J_0 stands for the Bessel function of the first kind with parameter 0 [20].

(ii) In particular, taking $\nu = 0$, for $u \geq 0$ and $x \in \mathbb{R}$, we have:

$$\exp\left(-\frac{x^2}{2t}\right) E\left[\exp\left(-\frac{u^2}{2}\ A_t\right) \big| B_t = x\right] = \int_{|x|}^{+\infty} dz\ \frac{z}{t}\ e^{-z^2 t/2} J_0(\phi(x,z)). \tag{19}$$

Proposition 2.4 follows now immediately from Lemma 2.5 by using the classical representation of the Bessel function of the first kind with parameter 0 (see e.g. [20]):

$$J_0(z) = \frac{1}{\pi} \int_{-1}^{+1} \frac{dr}{\sqrt{1-r^2}}\ \cos(zr), \tag{20}$$

and remarking that (with Ξ denoting again an arcsine variable), for all real ξ:

$$J_0(\xi) = E\left[\exp\left(i\xi(2\Xi - 1)\right)\right]. \tag{21}$$

Proof. (Lemma 2.5)
With I_μ and K_μ denoting the modified Bessel functions of the first and the second kind respectively with parameter $\mu = \sqrt{\rho^2 + \nu^2}$ (for ρ and ν reals), we define the function $G_\mu : \mathbb{R}^2 \to \mathbb{R}^+$ by:

$$G_\mu(u,v) = \begin{cases} 2I_\mu(u)K_\mu(v) & u \leq v; \\ 2I_\mu(v)K_\mu(u) & u \geq v. \end{cases} \tag{22}$$

First, using the skew product representation of planar Brownian motion, the following formula holds (for further details, we address the interested reader to [2]):

$$\int_0^\infty dt\ \exp\left(-\frac{\rho^2}{2}\ t\right) E\left[\exp\left(-\frac{u^2}{2}\ A_t^{(\nu)}\right)\right] = \int_{-\infty}^{+\infty} dy\ e^{\nu y} G_\mu(u, ue^y). \tag{23}$$

Using the integral representation (see e.g. [20], problem 8, p. 140):

$$I_\mu(x)K_\mu(y) = \frac{1}{2}\int_{\log(y/x)}^{\infty} dr\ e^{-\mu r} J_0\left(\sqrt{2\cosh(r)xy - x^2 - y^2}\right), \quad y \geq x. \tag{24}$$

we can invert (23) in order to obtain part $i)$ of Lemma 2.5.
Part $ii)$ follows with the help of Cameron-Martin relation. □

3. Bougerol's identity and subordination

In this Section, we consider $(Z_t = X_t + iY_t, t \geq 0)$ a standard planar Brownian motion (BM), starting from $x_0 + i0, x_0 > 0$ (for simplicity and without loss of generality, we suppose that $x_0 = 1$). Then, a.s., $(Z_t, t \geq 0)$ does not visit 0 but it winds around it infinitely often, hence $\theta_t = \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ is well defined [17]. There is the well-known skew-product representation:

$$\log|Z_t| + i\theta_t \equiv \int_0^t \frac{dZ_s}{Z_s} = (B_u + i\gamma_u)\Big|_{u=H_t=\int_0^t \frac{ds}{|Z_s|^2}}, \tag{25}$$

where $(B_u + i\gamma_u, u \geq 0)$ is another planar Brownian motion, starting from $\log 1 + i0$. Thus:

$$H_u^{-1} \equiv \inf\{t : H_t > u\} = \int_0^u ds \exp(2B_s) := A_u(B).$$

For further study of the Bessel clock H, see e.g. [34]. We also define the first hitting times $T_c^\theta \equiv \inf\{t : \theta_t = c\}$ and $T_c^{|\theta|} \equiv \inf\{t : |\theta_t| = c\}$.

3.1. General results

Bougerol's identity in law combined with the symmetry principle of André [3, 14] yields the following identity in law (see e.g. [7, 4]): for every fixed $l > 0$,

$$H_{\tau_l} \stackrel{(law)}{=} \tau_{a(l)} \tag{26}$$

where $(\tau_l, l \geq 0)$ stands for a stable (1/2)-subordinator. An example of this kind of identities in law is given for the planar Brownian motion case in the next Subsection. The main point in [7] is that (26) is not extended in the level of processes indexed by $l \geq 0$.

3.2. Bougerol's identity in terms of planar Brownian motion

Vakeroudis [28] investigated Bougerol's identity in terms of planar Brownian motion and obtained some striking identities in law:

Proposition 3.1. *Let $(\beta_u, u \geq 0)$ be a 1-dimensional Brownian motion independent of the planar Brownian motion $(Z_u, u \geq 0)$, starting from 1. Then, for any $b \geq 0$ fixed, the following identities in law hold:*

$$i)\ H_{T_b^\beta} \overset{(law)}{=} T^B_{a(b)} \qquad ii)\ \theta_{T_b^\beta} \overset{(law)}{=} C_{a(b)} \qquad iii)\ \bar{\theta}_{T_b^\beta} \overset{(law)}{=} |C_{a(b)}|,$$

where C_A is a Cauchy variable with parameter A and $\bar{\theta}_u = \sup_{0\leq s\leq u}\theta_s$.

Proof. $i)$ We identify the laws of the first hitting times of a fixed level b by the processes on each side of (7) and we obtain:

$$T^B_{a(b)} \overset{(law)}{=} H_{T_b^\beta},$$

which is $i)$.
$ii)$ It follows from $i)$ since:

$$\theta_u \overset{(law)}{=} \gamma_{H_u},$$

with $(\gamma_s, s \geq 0)$ a Brownian motion independent of $(H_u, u \geq 0)$ and $(C_u, u \geq 0)$ may be represented as $(\gamma_{T_u^B}, u \geq 0)$.
$iii)$ follows from $ii)$, again with the help of the symmetry principle. □

Using now these identities in law, we can apply William's "pinching" method [33, 24] and recover Spitzer's celebrated asymptotic law which states that [26]:

$$\frac{2}{\log t}\,\theta_t \xrightarrow[t\to\infty]{(law)} C_1\ , \tag{27}$$

with C_1 denoting a standard Cauchy variable (for other proofs, see also e.g. [33, 13, 24, 6, 36, 29]). One can also find a characterization of the distribution of T_c^θ and of $T_c^{|\theta|}$ in [28]. First, applying Bougerol's identity (1) in terms of planar Brownian motion, we have:

Proposition 3.2. *For fixed $c > 0$,*

$$\sinh(C_c) \overset{(law)}{=} \beta_{(T_c^\theta)} \overset{(law)}{=} \sqrt{T_c^\theta}\ N\ , \tag{28}$$

where $(C_c, c \geq 0)$ stands for a standard Cauchy process and $N \sim \mathcal{N}(0,1)$.

Furthermore, we can obtain the following Gauss-Laplace transforms which are equivalent to Bougerol's identity exploited for planar Brownian motion:

Proposition 3.3. *For $x \geq 0$ and $m = \frac{\pi}{2c}$,*

$$c\,E\left[\sqrt{\frac{\pi}{2T_c^\theta}}\exp\left(-\frac{x}{2T_c^\theta}\right)\right] = \frac{1}{\sqrt{1+x}}\,\frac{c^2}{(c^2+\log^2(\sqrt{x}+\sqrt{1+x}))}\ ; \tag{29}$$

$$c\,E\left[\sqrt{\frac{2}{\pi T_c^{|\theta|}}}\exp\left(-\frac{x}{2T_c^{|\theta|}}\right)\right] = \frac{1}{\sqrt{1+x}}\,\frac{2}{(\sqrt{1+x}+\sqrt{x})^m+(\sqrt{1+x}-\sqrt{x})^m}\ . \tag{30}$$

Proof. For the proof of (29), it suffices to identify the densities of the two parts of (28) and to recall that the density of a Cauchy variable with parameter c, equals:

$$\frac{c}{\pi(c^2+y^2)} \ .$$

For (30), we apply Bougerol's identity with $u = T_c^{|\gamma|} \equiv \inf\{t : |\gamma_t| = c\}$ and we obtain:

$$\sinh(B_{T_c^{|\gamma|}}) \stackrel{(law)}{=} \beta_{(T_c^{|\theta|})} \stackrel{(law)}{=} \sqrt{T_c^{|\theta|}}\ N \ . \tag{31}$$

Once again, we identify the densities of the two parts. For the left hand side, we use the following Laplace transform: for $\lambda \geq 0$, $E\left[e^{-\frac{\lambda^2}{2}T_b^{|\gamma|}}\right] = \frac{1}{\cosh(\lambda b)}$ (see e.g. Proposition 3.7, p. 71 in Revuz and Yor [25]). Using now the well-known result [21, 8]:

$$\begin{aligned} E\left[\exp(i\lambda B_{T_c^{|\gamma|}})\right] \quad &= \quad \frac{1}{\cosh(\lambda c)} = \frac{1}{\cosh(\pi\lambda\frac{c}{\pi})} = \int_{-\infty}^{\infty} e^{i(\frac{\lambda c}{\pi})y}\frac{1}{2\pi}\frac{1}{\cosh(\frac{y}{2})}\ dy \\ &\stackrel{x=\frac{cy}{\pi}}{=} \quad \int_{-\infty}^{\infty} e^{i\lambda x}\frac{1}{2\pi}\frac{\frac{\pi}{c}}{\cosh(\frac{x\pi}{2c})}\ dx = \int_{-\infty}^{\infty} e^{i\lambda x}\frac{1}{2c}\frac{1}{\cosh(\frac{x\pi}{2c})}\ dx \ , \end{aligned} \tag{32}$$

we obtain the density of $B_{T_c^{|\gamma|}}$ which equals:

$$\left(2c\cosh(\frac{y\pi}{2c})\right)^{-1} = \left(c(e^{\frac{y\pi}{2c}} + e^{-\frac{y\pi}{2c}})\right)^{-1} ,$$

and finishes the proof. □

Vakeroudis and Yor in [29, 30] investigated further the law of these random times.

3.3. The Ornstein-Uhlenbeck case

Vakeroudis in [28] investigated also the case of Ornstein-Uhlenbeck processes. In particular, we consider now a complex valued Ornstein-Uhlenbeck (OU) process:

$$Z_t = z_0 + \tilde{Z}_t - \lambda\int_0^t Z_s ds, \tag{33}$$

where $\tilde{Z}_t$ is a complex valued Brownian motion, $z_0 \in \mathbb{C}$ (for simplicity and without loss of generality, we suppose again $z_0 = 1$) and $\lambda \geq 0$ and $T_c^{(\lambda)} \equiv T_c^{|\theta^Z|} \equiv \inf\left\{t \geq 0 : \left|\theta_t^Z\right| = c\right\}$ (θ_t^Z is the continuous winding process associated to Z) denoting the first hitting time of the symmetric conic boundary of angle c for Z. Then, we have the following:

Proposition 3.4. *Consider $(Z_t^\lambda, t \geq 0)$ and $(U_t^\lambda, t \geq 0)$ two independent Ornstein-Uhlenbeck processes, the first one complex valued and the second one real valued, both starting from a point different from 0, and define $T_b^{(\lambda)}(U^\lambda) = \inf\left\{t \geq 0 : e^{\lambda t}U_t^\lambda = b\right\}$, for any $b \geq 0$. Then, an Ornstein-Uhlenbeck extension of identity in law ii) in Proposition 3.1 is the following:*

$$\theta^{Z^\lambda}_{T_b^{(\lambda)}(U^\lambda)} \stackrel{(law)}{=} C_{a(b)}, \tag{34}$$

where $a(x) = \arg\sinh(x)$ and C_A is a Cauchy variable with parameter A.

Proof. First, for Ornstein-Uhlenbeck processes, is well known that [25], with $(\mathbb{B}_t, t \geq 0)$ denoting a complex valued Brownian motion starting from 1, using Dambis-Dubins-Schwarz Theorem,

$$\begin{aligned} Z_t &= e^{-\lambda t}\left(1 + \int_0^t e^{\lambda s} d\tilde{Z}_s\right) \\ &= e^{-\lambda t}\left(\mathbb{B}_{\alpha_t}\right), \end{aligned} \tag{35}$$

Let us consider a second Ornstein-Uhlenbeck process $(U_t^\lambda, t \geq 0)$ independent of the first one. Taking now equation (35) for U_t^λ (1-dimensional case), we have:

$$e^{\lambda t}U_t^\lambda = \delta_{\left(\frac{e^{2\lambda t}-1}{2\lambda}\right)}, \tag{36}$$

where $(\delta_t, t \geq 0)$ is a real valued Brownian motion starting from 1.
Second, applying Itô's formula to (35), and dividing by Z_s, we obtain ($\alpha_t = \int_0^t e^{2\lambda s} ds = \frac{e^{2\lambda t}-1}{2\lambda}$):

$$\mathrm{Im}\left(\frac{dZ_s}{Z_s}\right) = \mathrm{Im}\left(\frac{d\mathbb{B}_{\alpha_s}}{\mathbb{B}_{\alpha_s}}\right),$$

hence:

$$\theta_t^Z = \theta^{\mathbb{B}}_{\alpha_t} \quad . \tag{37}$$

By inverting α_t, it follows now that:

$$T_c^{(\lambda)} = \frac{1}{2\lambda}\ln\left(1 + 2\lambda T_c^{|\theta|^{\mathbb{B}}}\right). \tag{38}$$

Similarly, for the 1-dimensional case, we have:

$$T_b^{(\lambda)}(U^\lambda) = \frac{1}{2\lambda}\ln\left(1 + 2\lambda T_b^\delta\right). \tag{39}$$

Equation (37) for $t = \frac{1}{2\lambda}\ln\left(1 + 2\lambda T_b^\delta\right)$, equivalently: $\alpha(t) = T_b^\delta$ becomes:

$$\theta^{Z^\lambda}_{T_b^{(\lambda)}(U^\lambda)} = \theta^{Z^\lambda}_{\frac{1}{2\lambda}\ln\left(1+2\lambda T_b^\delta\right)} = \theta^{\mathbb{B}}_{u=T_b^\delta} \stackrel{(law)}{=} C_{a(b)},$$

where the last equation in law follows precisely from statement $ii)$ in Proposition 3.1. ∎

4. Multidimensional extensions of Bougerol's identity

4.1. The law of the couple $(\sinh(\beta_t), \sinh(L_t))$

A first 2-dimensional extension of Bougerol's identity was obtained by Bertoin, Dufresne and Yor in [4] (for a first draft, see also [12]). With $(L_t, t \geq 0)$ denoting the local time at 0 of B, we have:

Theorem 4.1. *For fixed t, the 3 following 2-dimensional random variables are equal in law:*

$$(\sinh(B_t), \sinh(L_t)) \overset{(law)}{=} (\beta_{A_t}, \exp(-B_t)\ \lambda_{A_t}) \overset{(law)}{=} (\exp(-B_t)\ \beta_{A_t}, \lambda_{A_t}), \quad (40)$$

where $(\lambda_u, u \geq 0)$ is the local time of β at 0.

Remark 4.2. *Theorem 4.1 can be equivalently stated as: for fixed t, the 3 following 2-dimensional random variables are equal in law:*

$$(\sinh(|B_t|), \sinh(L_t)) \overset{(law)}{=} (|\beta|_{A_t}, \exp(-B_t)\ \lambda_{A_t}) \overset{(law)}{=} (\exp(-B_t)\ |\beta|_{A_t}, \lambda_{A_t}). \quad (41)$$

Using now Paul Lévy's celebrated identity in law (see e.g. [25]):

$$\left((\bar{B}_t - B_t, \bar{B}_t), t \geq 0\right) \overset{(law)}{=} ((|B_t|, L_t), t \geq 0)\,, \quad (42)$$

we can reformulate (40) or (41), and we obtain:

$$\begin{aligned}(\sinh(\bar{B}_t - B_t), \sinh(\bar{B}_t)) &\overset{(law)}{=} \left((\bar{\beta} - \beta)_{A_t}, \exp(-B_t)\ \bar{\beta}_{A_t}\right) \\ &\overset{(law)}{=} \left(\exp(-B_t)\ (\bar{\beta} - \beta)_{A_t}, \bar{\beta}_{A_t}\right). \quad (43)\end{aligned}$$

Remark 4.3. *Considering only the second processes of the first and the third part of (40) (or equivalently of (41)), we obtain a "local time" version of Bougerol's identity:*

$$\sinh(L_t) \overset{(law)}{=} \lambda_{A_t}, \quad (44)$$

which (as was shown in [7]), similar to the Brownian motion case, is true only for fixed t and not in the level of processes.

Proof. (Theorem 4.1)
From Remark 4.2 it suffices to prove (41).
First, we denote S_p, $p \geq 0$, an exponential variable with parameter p independent from B and $g_t = \sup\{u < t : B_u = 0\}$. We know that $\left(B_u, u \leq g_{S_p}\right)$ and $\left(B_{g_{S_p}+u}, u \leq S_p - g_{S_p}\right)$ are independent, hence L_{S_p} and B_{S_p} are also independent. We also know that L_t and $|B_t|$ have the same law. Hence, using the following computation: for every $l \geq 0$, with $(\tau_l, l \geq 0)$ denoting the time L reaches l,

$$P\left(L_{S_p} \geq l\right) = P\left(S_p \geq \tau_l\right) = E\left[\exp(-p\tau_l)\right] = \exp(-l\sqrt{2p}),$$

we deduce that the common density of L_{S_p} and $|B_{S_p}|$ is:

$$\sqrt{2p}\exp(-u\sqrt{2p}), \quad u \geq 0.$$

Equivalently, we have:

$$\sqrt{2\mathbf{e}}(|\beta(1)|, \lambda(1)) \overset{(law)}{=} (\mathbf{e}, \mathbf{e}'),$$

where on the left hand side $\mathbf{e}$ and $\mathbf{e}'$ are two independent copies of S_1 independent from β.
For the second identity in law in Theorem 4.1, it suffices to remark that

$$(\beta_{A_t}, \exp(-B_t)\ \lambda_{A_t}) \overset{(law)}{=} (\sqrt{A_t}\beta_1, \exp(-B_t)\sqrt{A_t}\ \lambda_1),$$

and use a time reversal argument.
For the first identity in law, we use an exponential time S_p and we compute the joint Mellin transforms in both sides in order to show that:

$$\sqrt{2\mathbf{e}}(\sinh(|B|_{S_p}), \sinh(L_{S_p})) \overset{(law)}{=} \sqrt{2\mathbf{e}}(\exp(-B_{S_p})\sqrt{A_{S_p}}\ |\beta_1|, \sqrt{A_{S_p}}\ \lambda_{A_t}).$$

For further details, we address the reader to [4]. □

Using now Tanaka's formula, we can also obtain the following identity in law for 2-dimensional processes:

Corollary 4.4.

$$(\sinh(B_t), L_t)_{t\geq 0} \overset{(law)}{=} \left(\exp(-B_t)\ \beta_{A_t}, \int_0^t \exp(-B_s) d\lambda_{A_s}\right)_{t\geq 0}, \tag{45}$$

where in each part, the second process is the local time at level 0 and time t of the first one.

4.2. Another two-dimensional extension

In this Subsection, we will study the joint distribution of:

$$\left(X_u^{(1)}, X_u^{(2)}\right) = \left(\exp(-B_u)\int_0^u d\xi_v^{(1)} \exp(B_v),\ \exp(-2B_u)\int_0^u d\xi_v^{(2)} \exp(2B_v)\right), \tag{46}$$

where $(\xi_v^1, v \geq 0)$, $(\xi_v^2, v \geq 0)$ and $(B_u, u \geq 0)$ are three independent Brownian motions. Hence, we obtain a new 2-dimensional extension, which states the following:

Proposition 4.5. *For u fixed,*

$$\left(X_u^{(1)}, X_u^{(2)}\right) \overset{(law)}{=} \left(\sinh(B_u^{(1)}), \frac{1}{2}\sinh(2B_u^{(2)})\right) \tag{47}$$

$$\overset{(law)}{=} \left(\beta^{(1)}_{\left(\int_0^u dv\ \exp(2B_v)\right)}, \beta^{(2)}_{\left(\int_0^u dv\ \exp(4B_v)\right)}\right). \tag{48}$$

With $p_u(x,y)$ *now denoting the density function of* $(\sinh(B_u^{(1)}), \frac{1}{2}\sinh(2B_u^{(2)}))$*, we have:*

$$p_u(x,y) = E\left[\frac{1}{2\pi}\frac{\exp\left(-x^2/2\int_0^u dv\ \exp(2B_v)\right)}{\sqrt{\int_0^u dv\ \exp(2B_v)}}\frac{\exp\left(-y^2/2\int_0^u dv\ \exp(4B_v)\right)}{\sqrt{\int_0^u dv\ \exp(4B_v)}}\right]. \quad (49)$$

Proof. Let us define:

$$X_u^{(\alpha)} = \exp(-\alpha B_u)\int_0^u d\xi_v \exp(\alpha B_v), \quad (50)$$

where $\alpha = 1, 2$. By Itô's formula, we have:

$$\begin{aligned} X_u^{(\alpha)} &= \xi_u + \int_0^u \left(\exp(-\alpha B_v)(-\alpha dB_v) + \frac{\alpha^2}{2}\exp(-\alpha B_v)dv\right)\left(\int_0^v d\xi_h \exp(\alpha B_h)\right) \\ &= \xi_u + \int_0^u \left(-\alpha dB_v\ X_v^{(\alpha)} + \frac{\alpha^2}{2}\ X_v^{(\alpha)}\ dv\right). \end{aligned}$$

Hence:

$$\begin{aligned} X_u^{(1)} &= \xi_u^{(1)} - \int_0^u dB_v\ X_v^{(1)} + \frac{1}{2}\int_0^u X_v^{(1)}\ dv \\ &= \int_0^u d\eta_v^{(1)}\sqrt{\left(1+\left(X_v^{(1)}\right)^2\right)} + \frac{1}{2}\int_0^u X_v^{(1)}\ dv\ , \end{aligned} \quad (51)$$

and

$$\begin{aligned} X_u^{(2)} &= \xi_u^{(2)} - 2\int_0^u dB_v\ X_v^{(2)} + 2\int_0^u X_v^{(2)}\ dv \\ &= \int_0^u d\eta_v^{(2)}\sqrt{\left(1+4\left(X_v^{(2)}\right)^2\right)} + 2\int_0^u X_v^{(2)}\ dv, \end{aligned} \quad (52)$$

where $(\eta_v^{(1)}, v \geq 0)$ and $(\eta_v^{(2)}, v \geq 0)$ are two dependent Brownian motions, with quadratic variation:

$$d < \eta^{(1)}, \eta^{(2)} >_v = \frac{2X_v^{(1)}X_v^{(2)}dv}{\sqrt{\left(1+\left(X_v^{(1)}\right)^2\right)}\sqrt{\left(1+4\left(X_v^{(2)}\right)^2\right)}}. \quad (53)$$

Thus, we deduce that the infinitesimal generator of $\left(X_u^{(1)}, X_u^{(2)}\right)$ is:

$$\frac{1}{2}\left[\left(1+x_1^2\right)\frac{\partial^2}{\partial x_1^2} + \left(1+4x_2^2\right)\frac{\partial^2}{\partial x_2^2} + 4x_1x_2\frac{\partial^2}{\partial x_1\partial x_2}\right] + \frac{x_1}{2}\frac{\partial}{\partial x_1} + 2x_2\frac{\partial}{\partial x_2}\ . \quad (54)$$

Let us now study the couple:

$$(x_t^{(1)}, x_t^{(2)}) = \left(\sinh(B_t^{(1)}), \frac{1}{2}\sinh(2B_t^{(2)})\right), \quad (55)$$

where $(B_t^{(1)}, t \geq 0)$ and $(B_t^{(2)}, t \geq 0)$ are two dependent Brownian motions. By Itô's formula we have:

$$\begin{aligned} x_t^{(1)} &= \sinh(B_t^{(1)}) \\ &= \int_0^t \cosh(B_v^{(1)})\, dB_v^{(1)} + \frac{1}{2}\int_0^t \sinh(B_v^{(1)})\, dv \\ &= \int_0^t \sqrt{\left(1+(x_v^{(1)})^2\right)}\, dB_v^{(1)} + \frac{1}{2}\int_0^t x_v^{(1)}\, dv, \end{aligned} \tag{56}$$

and:

$$\begin{aligned} x_t^{(2)} &= \frac{1}{2}\sinh(2B_t^{(2)}) \\ &= \int_0^t \cosh(2B_v^{(2)})\, dB_v^{(2)} + \int_0^t \sinh(2B_v^{(2)})\, dv \\ &= \int_0^t \sqrt{\left(1+4(x_v^{(2)})^2\right)}\, dB_v^{(2)} + 2\int_0^t x_v^{(2)}\, dv. \end{aligned} \tag{57}$$

Moreover:

$$\begin{aligned} d < \sinh(B^{(1)}), \frac{1}{2}\sinh(2B^{(2)}) >_v &= \cosh(B_v^{(1)})\ \cosh(2B_v^{(2)})\ d < B^{(1)}, B^{(2)} >_v \\ &= 2\sinh(B_v^{(1)})\ \frac{1}{2}\sinh(2B_v^{(2)})\ dv. \end{aligned} \tag{58}$$

Hence, the dependence of $(B_t^{(1)}, t \geq 0)$ and $(B_t^{(2)}, t \geq 0)$ is characterized by:

$$d < B^{(1)}, B^{(2)} >_v = \tanh(B_v^{(1)})\ \tanh(2B_v^{(2)})\ dv. \tag{59}$$

Finally, we have that $(x_u^{(1)}, x_u^{(2)})$ has the same infinitesimal generator with $\left(X_u^{(1)}, X_u^{(2)}\right)$. Hence:

$$\begin{aligned} &\left(\exp(-B_u)\int_0^u d\xi_v^{(1)} \exp(B_v),\ \exp(-2B_u)\int_0^u d\xi_v^{(2)} \exp(2B_v),\ u \geq 0\right) \\ \overset{(law)}{=}\ &\left(\sinh(B_u^{(1)}), \frac{1}{2}\sinh(2B_u^{(2)}),\ u \geq 0\right). \end{aligned} \tag{60}$$

Thus, if we fix u:

$$\left(\sinh(B_u^{(1)}), \frac{1}{2}\sinh(2B_u^{(2)})\right) \overset{(law)}{=} \left(\beta^{(1)}_{\left(\int_0^u dv\ \exp(2B_v)\right)}, \beta^{(2)}_{\left(\int_0^u dv\ \exp(4B_v)\right)}\right), \tag{61}$$

where $(\beta_v^{(1)}, v \geq 0)$ and $(\beta_v^{(2)}, v \geq 0)$ are two dependent Brownian motions and $(B_v, v \geq 0)$ is another Brownian motion independent from them.
Now, from (61), we obtain (49). $\square$

4.3. A three-dimensional extension

Alili, Dufresne and Yor, in [1], obtained a 3-dimensional extension of Bougerol's identity:

Proposition 4.6. *The two following processes have the same law:*

$$\left\{ e^{B_t} \int_0^t e^{B_u} d\beta_u, B_t, \beta_t; t \geq 0 \right\} \overset{(law)}{=} \left\{ \sinh(B_t), B'_t, G'_t; t \geq 0 \right\}, \tag{62}$$

where:

$$\begin{cases} B'_t = \int_0^t \tanh(B_s) dB_s + \int_0^t \frac{dc_s}{\cosh(B_s)} \ ; \\ G'_t = \int_0^t \frac{dB_s}{\cosh(B_s)} - \int_0^t \tanh(B_s) dG_s \ , \end{cases} \tag{63}$$

with $(G_t, t \geq 0)$ *denoting another Brownian motion, independent from* B.

Remark 4.7. *We remark that with:*

$$\alpha(x) = \begin{pmatrix} \tanh(x) & -\frac{1}{\cosh(x)} \\ \frac{1}{\cosh(x)} & \tanh(x) \end{pmatrix}, \tag{64}$$

we have:

$$\begin{pmatrix} dB'_t \\ dG'_t \end{pmatrix} = \alpha(B_t) \begin{pmatrix} dB_t \\ d\beta_t \end{pmatrix}, \tag{65}$$

and

$$\left\{ \begin{pmatrix} B'_t \\ G'_t \end{pmatrix}, t \geq 0 \right\}$$

is a 2-dimensional Brownian motion.

Proof. (Proposition 4.6)
First proof: Using Itô's formula, we deduce easily that each of these triplets is a Markov process with infinitesimal generator (in $C^2(\mathbb{R}^3)$):

$$\frac{1}{2}(1+x^2)\frac{d^2}{dx^2} + \frac{1}{2}\frac{d^2}{dy^2} + \frac{1}{2}\frac{d^2}{dz^2} + x\frac{d^2}{dxdy} + \frac{d^2}{dxdz} + x\frac{d}{dx}. \tag{66}$$

The proof finishes by the unicity (in law) of the solutions of the corresponding martingale problem.

Second proof: First, we admit that the identity in law is true. Then, if we replace on the left hand side (B_s) by (B'_s) and (β_s) by (G'_s), we have necessarily:

$$\sinh(B_t) \overset{(law)}{=} e^{B'_t} \int_0^t e^{B'_u} dG'_t, \tag{67}$$

which is essentially a (partial) inversion formula of the transformation (65). Equation (67) can be proved by using Itô's formula on the right hand side. ☐

Gruet in [1] also remarked that:

Proposition 4.8. *There exist two independent linear Brownian motions V and W and a diffusion J starting from 0 satisfying the following equation*

$$dJ_t = dW_t + \frac{1}{2}\ \tanh(J_t)dt, \tag{68}$$

such that,

$$\begin{pmatrix} d\beta_t \\ dB_t \end{pmatrix} = \alpha(-J_t) \begin{pmatrix} dV_t \\ dW_t \end{pmatrix}. \tag{69}$$

Hence, the two following 3-dimensional processes:

$$\left(\exp\left(B_t + \frac{t}{2}\right) \int_0^t \exp\left(-B_s - \frac{s}{2}\right) d\beta_s, B_t, \beta_t; t \geq 0 \right)$$

and

$$\left(\sinh(J_t), B_t, \beta_t; t \geq 0\right),$$

are equal.

Proof. This result follows from a geometric proof and it is essentially an explanation of the second proof of Proposition 4.6, at least for $\nu = 0$. For this purpose, we can compare the writing of a hyperbolic Brownian motion in the half-plane of Poincaré, decomposed in rectangular coordinates with the equidistant coordinates [31]. For further details, see the Appendix in [1] due to Gruet. □

5. The diffusion version of Bougerol's identity

5.1. Bougerol's diffusion

Bertoin, Dufresne and Yor in a recent work [5] generalized Bougerol's identity in terms of diffusions. First, we remark that from Proposition 2.1, we have that (see also [1]):

$$\left(\sinh(B_t), t \geq 0\right) \overset{(law)}{=} \left(\exp(-B_t)\beta_{A_t^{(0)}}, t \geq 0 \right). \tag{70}$$

In particular, using Lamperti's relation (see e.g. [19] or [25]) we can invoke a Bessel process $R^{(\delta)}$ independent from B in order to replace the right hand side of (70) by:

$$\left(\exp\left(-B_t^{(\nu)}\right) R^{(\delta)}_{A_t^{(\nu)}}, t \geq 0 \right),$$

which turns out to be a diffusion (named *Bougerol's diffusion*) with a certain infinitesimal generator. Hence, we obtain the following:

Theorem 5.1. *With $Z = Z^{(\delta)}$ and $Z' = Z^{(\delta')}$ denoting two independent squared Bessel process of dimension $\delta = 2(1+\mu)$ and $\delta' = 2(1+\nu)$ respectively, starting from z and z', the process:*

$$X_t \equiv X_t^{(\nu,\delta)} \equiv \exp\left(-2B_t^{(\nu)}\right) Z_{A_t^{(\nu)}} = \left.\frac{Z_u}{Z'_u}\right|_{u=A_t^{(\nu)}}, \ t \geq 0, \tag{71}$$

is a diffusion, with infinitesimal generator:

$$2x(1+x)D^2 + (\delta + (4-\delta')x)\, D, \tag{72}$$

where $\delta' = 2(1+\nu)$.

Proof. Applying Itô's formula to the process X, we obtain:

$$X_t = \int_0^t \exp\left(-2B_u^{(\nu)}\right) d\left(Z_{A_u^{(\nu)}}\right) + \int_0^t Z_{A_u^{(\nu)}} d\left(\exp\left(-2B_u^{(\nu)}\right)\right). \tag{73}$$

For the second integral in (73), Itô's formula once more yields:

$$\begin{aligned} d\left(\exp\left(-2B_u^{(\nu)}\right)\right) &= -2\exp\left(-2B_u^{(\nu)}\right)(dB_u + \nu du) + 2\exp\left(-2B_u^{(\nu)}\right) du \\ &= -2\int_0^t X_u dB_u + \underbrace{2(1-\nu)}_{\delta'} \int_0^t X_u du\ . \end{aligned} \tag{74}$$

For the first integral in (73), we recall that, with γ denoting another Brownian motion independent from B (thus independent also from Z):

$$dZ_s = 2\sqrt{Z_s} d\gamma_s + \delta\ ds\ . \tag{75}$$

Hence:

$$\begin{aligned} dZ_{A_u^{(\nu)}} &= 2\sqrt{Z_s} d\gamma_{A_u^{(\nu)}} + \delta \exp\left(2B_u^{(\nu)}\right) du \\ &= 2\sqrt{Z_s}\exp\left(2B_u^{(\nu)}\right) d\hat{\gamma}_u + \delta\exp\left(2B_u^{(\nu)}\right) du\ , \end{aligned} \tag{76}$$

with $\hat{\gamma}$ denoting another Brownian motion, independent from γ and from B. The proof finishes by some elementary computations from (73), using (74) and (76).
Finally, using Lamperti's relation, which states that:

$$\exp\left(2B_t^{(\nu)}\right) = Z'_{A_t^{(\nu)}}\ , \tag{77}$$

we obtain the last identity in (71). □

We may continue a little further in order to obtain the following result relating the diffusion X with its reciprocal (recall that, $A_u^{(\nu)} = \int_0^u ds\ \exp\left(2B_s^{(\nu)}\right)$):

Corollary 5.2. *The following relation holds:*

$$X_t^{(\nu,\mu)} = \frac{1}{X^{(\mu,\nu)}_{\int_0^t \frac{du}{X^{(\nu,\mu)}_{A_u^{(\nu)}}}}} \ , \tag{78}$$

or equivalently:

$$\frac{1}{X_t^{(\nu,\mu)}} = X^{(\mu,\nu)}_{\int_0^t \frac{du}{X^{(\nu,\mu)}_{A_u^{(\nu)}}}} \ . \tag{79}$$

Proof. It follows easily by some relations involving the changes of time:

$$A_t^{(\nu)} = \int_0^t ds \ \exp\left(2B_s^{(\nu)}\right); \quad A_t^{(\mu)} = \int_0^t ds \ \exp\left(2B_s^{(\mu)}\right); \tag{80}$$

$$H_u^{(\nu)} = \int_0^u \frac{ds}{Z'_s}; \quad H_u^{(\mu)} = \int_0^u \frac{ds}{Z_s} \ . \tag{81}$$

Moreover, we remark that $(H_t^{(\nu)})$ is the inverse of $(A_t^{(\nu)})$ and $(H_t^{(\mu)})$ is the inverse of $(A_t^{(\mu)})$.
We also need to use:

$$H_u^{(\nu,\mu)} = \int_0^u \frac{ds}{X_s^{(\nu,\mu)}}; \quad H_u^{(\mu,\nu)} = \int_0^u \frac{ds}{X_s^{(\mu,\nu)}} \ . \tag{82}$$

Simple calculations now yield:

$$H_t^{(\nu,\mu)} = H^{(\mu)}_{A_t^{(\nu)}} \ , \tag{83}$$

Finally, using (71), we have:

$$H^{(\mu)}_{A_t^{(\nu)}} = \int_0^t \frac{ds}{X^{(\nu,\mu)}_{A_s^{(\nu)}}} \ , \tag{84}$$

and we obtain easily the result. □

5.2. Relations involving Jacobi processes

In this Subsection, we present a particular study of Theorem 5.1 in terms of the Jacobi processes $Y^{(\delta,\delta')}$ as introduced in Warren and Yor [32] (see also the references therein for Jacobi processes), due to Bertoin, Dufresne and Yor [5]. First, we recall some results involving Jacobi processes:

Proposition 5.3. (Warren and Yor [32], Proposition 8)
With $T \equiv \inf\{u : Z_u + Z'_u = 0\}$*, there exists a diffusion process* $\left(Y_u \equiv Y_u^{\delta,\delta'}, u \geq 0\right)$ *on* $[0,1]$*, independent from* $Z + Z'$ *such that:*

$$\frac{Z_u}{Z_u + Z'_u} = Y_{\int_0^u \frac{ds}{Z_s + Z'_s}}, \; u < T. \tag{85}$$

We remark that $Y' = 1 - Y$ *is the Jacobi process with dimensions* (δ', δ)*, and* Y *has infinitesimal generator:*

$$2y(1-y)D^2 + (\delta - (\delta + \delta')y)\, D \; . \tag{86}$$

Now, X defined in (71) and Y can be related as following:

Proposition 5.4. *The following relation holds:*

$$\frac{Y_h}{1 - Y_h} = X_{\int_0^h \frac{dv}{Y'_v}} = X_{\int_0^h \frac{dv}{1-Y_v}} \;, \tag{87}$$

or equivalently:

$$X_k = \left.\frac{Y_h}{1 - Y_h}\right|_{h=\int_0^k \frac{dv}{1+X_v}} \; . \tag{88}$$

Proof. (Proposition 5.4)
First, from (71), we have:

$$X_t = \left.\frac{Z_u}{Z'_u}\right|_{u=A_t^{(\nu)}} \; .$$

Conversely,

$$\frac{Z_u}{Z'_u} = X_{H_u^{(\nu)}} \;, \tag{89}$$

where $H_u^{(\nu)} = \int_0^u \frac{ds}{Z'_s}$ is the inverse of $A^{(\nu)}$. However, using the Jacobi process Y,

$$\frac{Z_u}{Z'_u} = \frac{Y_h}{1 - Y_h}\Big|_{h=\mathcal{H}_u=\int_0^u \frac{ds}{Z_s + Z'_s}} \;, \tag{90}$$

and moreover:

$$H_u^{(\nu)} = \int_0^u \frac{ds}{Z'_s} = \int_0^u \frac{ds}{(Z_s + Z'_s)} \frac{1}{(1 - Y_{\mathcal{H}_s})} = \int_0^{\mathcal{H}_u} \frac{dv}{Y'_v} \; . \tag{91}$$

Plugging now (91) to (89) and comparing to (90), we obtain (87). For (88), it suffices to remark that $k \to \int_0^k \frac{dv}{1+X_v}$ is the inverse of the increasing process $h \to \int_0^h \frac{dv}{1-Y_v}$. □

6. Bougerol's identity and peacocks

Hirsch, Profeta, Roynette and Yor in [15], studied the processes which are increasing in the convex order, named *peacocks* (coming from the French term: Processus Croissant pour l'Ordre Convexe, which yields the acronym PCOC). Let us first introduce a notation: for W and V, two real-valued random variables, W is said to be dominated by V for the convex order if, for every convex function $\psi : \mathbb{R} \to \mathbb{R}$ such that $E[|\psi(W)|] < \infty$ and $E[|\psi(V)|] < \infty$, we have:

$$E[\psi(W)] \leq E[\psi(V)], \tag{92}$$

and we write: $W \overset{(c)}{\leq} V$.

A process $(G_t, t \geq 0)$ is a peacock if, for every $s \leq t$, $G_s \overset{(c)}{\leq} G_t$. Kelleler's Theorem now (see e.g. [18, 15, 16]) states that, to every peacock, we can associate a martingale (defined possibly on another probability space than G). In other words, there exists a martingale $(M_t, t \geq 0)$ such that, for every fixed $t \geq 0$,

$$G_t \overset{(law)}{=} M_t \ . \tag{93}$$

The main subject of [15] is to give several examples of peacocks and the associated martingales.
We return now to Bougerol's identity and we remark that (see also [15], paragraph 7.5.4, p. 322), for every $\lambda \geq 0$, $(\sinh(\lambda B_t), t \geq 0)$ is a peacock with associated martingale $\left(\lambda \int_0^t e^{\lambda \beta_s} d\gamma_s, t \geq 0\right)$ (see e.g. (3)).
Moreover, for every λ real, $\left(e^{-\frac{\lambda^2}{2}t} \sinh(\lambda B_t), \geq 0\right)$ is obviously a peacock, as it is a martingale. This is generalized in the following:

Proposition 6.1. ([15], Proposition 7.2)
The process $(e^{\mu t} \sinh(\lambda B_t), \geq 0)$ *is a peacock if and only if* $\mu \geq -\frac{\lambda^2}{2}$.

Proof. $i)$ First, we suppose $\mu \geq -\frac{\lambda^2}{2}$. Then, for $s < t$:

$$\begin{aligned} e^{\mu t} \sinh(\lambda B_t) &= e^{(\mu + \frac{\lambda^2}{2})t} \left(\sinh(\lambda B_t) e^{-\frac{\lambda^2}{2}t}\right) \overset{(c)}{\geq} e^{(\mu + \frac{\lambda^2}{2})s} \left(\sinh(\lambda B_t) e^{-\frac{\lambda^2}{2}t}\right) \\ &\overset{(c)}{\geq} e^{(\mu + \frac{\lambda^2}{2})s} \left(\sinh(\lambda B_s) e^{-\frac{\lambda^2}{2}s}\right). \end{aligned}$$

$ii)$ Conversely, Itô-Tanaka's formula yields:

$$E\left[|\sinh(\lambda B_t)|\right] = E\left[\sinh(\lambda |B_t|)\right] = e^{\frac{\lambda^2}{2}t} \lambda \int_0^t \frac{ds}{\sqrt{2\pi s}}\, e^{-\frac{\lambda^2}{2}s},$$

hence:

$$E\left[|e^{\mu t} \sinh(\mu B_t)|\right] \overset{t \to +\infty}{\sim} \lambda e^{(\frac{\lambda^2}{2} + \mu)t} \lambda \int_0^{+\infty} \frac{ds}{\sqrt{2\pi s}}\, e^{-\frac{\lambda^2}{2}s},$$

which means that if $\mu < -\frac{\lambda^2}{2}$,

$$E\left[|e^{\mu t}\sinh(\mu B_t)|\right] \overset{t\to+\infty}{\longrightarrow} 0 \ .$$

However, $x \to |x|$ is convex and if $(e^{\mu t}\sinh(\mu B_t), t \geq 0)$ was a peacock, then $E\left[|e^{\mu t}\sinh(\mu B_t)|\right]$ would increase on t, which is a contradiction. □

7. Further extensions and open questions

In this Section, we propose some possible directions to continue studying and possibly extending Bougerol's celebrated identity in law (for fixed time or as a process).

First, the natural question posed is wether this identity can be extended to higher dimensions. This very challenging question has already been attempted to be dealt with, and in this paper we've presented several extensions, at least for the 2-dimensional (and partly for the 3-dimensional) case.

Another natural question is wether we can generalize Bougerol's identity to other processes. For this purpose, we may think in terms of a diffusion, as introduced in Section 5. It seems more intelligent to start from the right hand side of (1) and try to see, e.g. in (71), for every particular ratio of processes, which is the corresponding process on the left hand side (this process could be named "*Bougerol's process*").

In particular, it seems interesting to investigate a possible extension in the case of Lévy or stable processes. A possible approach would be to replace the ratio of the two squared independent Bessel processes in (71) by e.g. the ratio of two exponentials of Lévy processes, and investigate the process obtained after the time-change. However, this perspective is not in the aims of the present work.

Finally, another aspect which could be further studied is the applications that one may obtain by the subordination method, as presented in Section 3. Following the lines of this Section, one may retrieve further results and applications, others than for the planar Brownian motion case (see also [7, 4]).

Appendix A: Tables of Bougerol's Identity and other equivalent expressions

Using now the notations introduced in the whole text, we can summarize all the results in the following tables ($u > 0$, wherever used is considered as fixed).

A.1. Table: Bougerol's Identity in law and equivalent expressions ($u > 0$ fixed)

With $a(x) \equiv \arg\sinh(x)$, and B, β denoting two independent real Brownian motions, for $u > 0$ fixed, we have:

1) $\sinh(B_u) \overset{(law)}{=} \beta_{(A_u(B)\equiv\int_0^u ds \exp(2B_s))}$ (Bougerol's Identity)
2) $\sinh(\lvert B_u\rvert) \overset{(law)}{=} \lvert\beta\rvert_{(A_u(B))}$
3) $\sinh(\bar{B}_u) \overset{(law)}{=} \bar{\beta}_{(A_u(B))}, \quad \bar{B}_u = \sup_{s\le u}\beta_s$
4) $E\left[\frac{1}{\sqrt{2\pi A_u(B)}}\exp\left(-\frac{x}{2A_u(B)}\right)\right] = \frac{1}{\sqrt{2\pi u}}\,\frac{1}{\sqrt{1+x}}\,\exp\left(-\frac{(a(\sqrt{x}))^2}{2u}\right), \ x\ge 0$

A.2. Table: Bougerol's Identity for other 1-dimensional processes ($u > 0$ fixed)

We use μ, ν reals and we define:

$$B_t^{(\mu)} = B_t + \mu t, \quad \beta_s^{(\nu)} = \beta_t + \nu t, \quad A_t^{(\nu)} = \int_0^t ds\ \exp(2B_s^{(\nu)}),$$

ε: a Bernoulli variable in $\{-1,1\}$, $(R_t, t\ge 0)$ a 2-dimensional Bessel process started at 0, Ξ an arcsine variable, and $(Y_t^{(\mu,\nu)}, t\ge 0)$ a diffusion with infinitesimal generator:

$$\frac{1}{2}\,\frac{d^2}{dy^2} + \left(\mu\tanh(y) + \frac{\nu}{\cosh(y)}\right)\frac{d}{dy}\ ,$$

starting from $y = \arg\sinh(x)$. $B^{(\mu)}$, $\beta^{(\nu)}$, ε, Ξ and R are independent. Then, for $u > 0$ fixed:

5) $\left(\sinh(Y_t^{(\mu,\nu)}), t\ge 0\right) \overset{(law)}{=} \left(\exp(B_t^{(\mu)})\left(x + \int_0^t \exp(-B_s^{(\mu)})d\beta_s^{(\nu)}\right), t\ge 0\right)$, x: fixed
6) $\sinh(Y_u^{(\mu,\nu)}) \overset{(law)}{=} \int_0^u \exp(B_s^{(\mu)})d\beta_s^{(\nu)}$,
7) $\sinh(B_u + \varepsilon\ t) \overset{(law)}{=} \int_0^u \exp(B_s + s)d\beta_s$, ε: Bernoulli variable in $\{-1,1\}$
8) $\beta_{A_u^{(\nu)}} \overset{(law)}{=} (2\Xi - 1)\phi\left(B_u^{(\nu)}, \sqrt{R_u^2 + (B_u^{(\nu)})^2}\right)$, $\phi(x,z) = \sqrt{2e^x\cosh(z) - e^{2x} - 1}$, $z \ge \lvert x\rvert$

A.3. Table: Bougerol's Identity in terms of planar Brownian motion ($u > 0$ fixed)

We define $(Z_t, t \geq 0)$ a planar Brownian motion starting from 1. Then $\theta_t = \mathrm{Im}(\int_0^t \frac{dZ_s}{Z_s}), t \geq 0$ is well defined. We further define the Bessel clock $H_t = \int_0^t \frac{ds}{|Z_s|^2} = A_u^{-1}(B)$ and the first hitting times: $T_c^{\theta} \equiv \inf\{t : \theta_t = c\}$ and $T_c^{|\theta|} \equiv \inf\{t : |\theta_t| = c\}$. Then, with $(C_c, c \geq 0)$ a standard Cauchy process, C_y a Cauchy variable with parameter y, $N \sim \mathcal{N}(0,1)$, $(Z_t^{\lambda}, t \geq 0)$ and $(U_t^{\lambda}, t \geq 0)$ two independent Ornstein-Uhlenbeck processes, the first one complex valued and the second one real valued, both starting from a point different from 0 and $T_b^{(\lambda)}(U^{\lambda}) = \inf\left\{t \geq 0 : e^{\lambda t} U_t^{\lambda} = b\right\}$, for $b, c > 0$ fixed:

9) $\sinh(C_c) \overset{(law)}{=} \beta_{(T_c^{\theta})} \overset{(law)}{=} \sqrt{T_c^{\theta}} N$, $c > 0$ fixed
10) $H_{T_b^{\beta}} \overset{(law)}{=} T_{a(b)}^{B}, \quad T_y^B = \inf\{t : B_t = y\}$, $a(x) = \arg\sinh(x)$, $\quad b > 0$ fixed
11) $\theta_{T_b^{\beta}} \overset{(law)}{=} C_{a(b)}$, $b > 0$ fixed
12) $\bar{\theta}_{T_b^{\beta}} \overset{(law)}{=} \lvert C_{a(b)} \rvert, \quad \bar{\theta}_u = \sup_{s \leq u} \theta_s$
13) $E\left[\frac{1}{\sqrt{2\pi T_c^{\theta}}} \exp\left(-\frac{x}{2T_c^{\theta}}\right)\right] = \frac{1}{\sqrt{1+x}} \frac{c}{\pi(c^2 + \log^2(\sqrt{x} + \sqrt{1+x}))}, \quad b > 0$ fixed, $x \geq 0$
14) $E\left[\frac{1}{\sqrt{2\pi T_c^{\lvert\theta\rvert}}} \exp(-\frac{x}{2T_c^{\lvert\theta\rvert}})\right] = \left(\frac{1}{c}\right)\left(\frac{1}{\sqrt{1+x}}\right) \frac{1}{(\sqrt{1+x}+\sqrt{x})^{\zeta} + (\sqrt{1+x}-\sqrt{x})^{\zeta}}$, $x \geq 0$, $\zeta = \frac{\pi}{2c}$
15) $\theta^{Z^{\lambda}}_{T_b^{(\lambda)}(U^{\lambda})} \overset{(law)}{=} C_{a(b)} \quad$ (OU version)

A.4. Table: Multi-dimensional extensions of Bougerol's Identity

In the following table, $(L_t, t \geq 0)$ and $(\lambda_t, t \geq 0)$ denote the local times at 0 of B, β respectively and:

$$\left(X_u^{(1)}, X_u^{(2)}\right) = \left(\exp(-B_u) \int_0^u d\xi_v^{(1)} \exp(B_v),\ \exp(-2B_u) \int_0^u d\xi_v^{(2)} \exp(2B_v)\right),$$

where $(\xi_v^1, v \geq 0)$, $(\xi_v^2, v \geq 0)$ and $(B_u, u \geq 0)$ are three independent Brownian motions. Moreover, we denote by $(B^{(1)}, B^{(2)})$ and $(\beta^{(1)}, \beta^{(2)})$ two couples of dependent Brownian motions. Finally,

$$\begin{cases} B'_t = \int_0^t \tanh(B_s) dB_s + \int_0^t \frac{dc_s}{\cosh(B_s)} \ ; \\ G'_t = \int_0^t \frac{dB_s}{\cosh(B_s)} - \int_0^t \tanh(B_s) dG_s \ , \end{cases}$$

with $(G_t, t \geq 0)$ denoting another Brownian motion, independent from B and J a diffusion starting from 0 satisfying: $dJ_t = dW_t + \frac{1}{2}\ \tanh(J_t)dt$, where W stands for an independent Brownian motion. Hence, for $u > 0$ fixed:

16) $\sinh(L_u) \overset{(law)}{=} \lambda_{A_u}$
17) $(\sinh(B_u), \sinh(L_u)) \overset{(law)}{=} (\beta_{A_u}, \exp(-B_u)\ \lambda_{A_u}) \overset{(law)}{=} (\exp(-B_u)\ \beta_{A_u}, \lambda_{A_u})$
18) $(\sinh(\lvert B_u\rvert)\ \sinh(L_u)) \overset{(law)}{=} (\lvert\beta\rvert_{A_u}, \exp(-B_u)\ \lambda_{A_u}) \overset{(law)}{=} (\exp(-B_u)\ \lvert\beta\rvert_{A_u}, \lambda_{A_u})$
19) $(\sinh(\bar{B}_u - B_u), \sinh(\bar{B}_u)) \overset{(law)}{=} \left((\bar{\beta} - \beta)_{A_u}, \exp(-B_u)\ \bar{\beta}_{A_u}\right) \overset{(law)}{=} (\exp(-B_u)\ (\bar{\beta} - \beta)_{A_u}, \bar{\beta}_{A_u})$
20) $(\sinh(B_t), L_t,\ t \geq 0) \overset{(law)}{=} \left(\exp(-B_t)\ \beta_{A_t}, \int_0^t \exp(-B_s) d\lambda_{A_s},\ t \geq 0\right)$
21) $\left(X_u^{(1)}, X_u^{(2)}\right) \overset{(law)}{=} \left(\sinh(B_u^{(1)}), \frac{1}{2}\sinh(2B_u^{(2)})\right) \overset{(law)}{=} \left(\beta^{(1)}_{\left(\int_0^u dv\ \exp(2B_v)\right)}, \beta^{(2)}_{\left(\int_0^u dv\ \exp(4B_v)\right)}\right)$
22) $\left(e^{B_t} \int_0^t e^{B_u} d\beta_u, B_t, \beta_t; t \geq 0\right) \overset{(law)}{=} (\sinh(B_t), B'_t, G'_t; t \geq 0)$
23) $\left(\exp\left(B_t + \frac{t}{2}\right) \int_0^t \exp\left(-B_s - \frac{s}{2}\right) d\beta_s, B_t, \beta_t; t \geq 0\right) \overset{(law)}{=} (\sinh(J_t), B_t, \beta_t; t \geq 0)$

A.5. Table: Diffusion version of Bougerol's Identity (relations involving the Jacobi process)

Let $Z \equiv Z^{(\delta)}$ and $Z' \equiv Z^{(\delta')}$ be two independent squared Bessel process of dimension $\delta = 2(1+\mu)$ and $\delta' = 2(1+\nu)$ respectively, starting from z and z', and $X_t \equiv X_t^{(\nu,\delta)}$ a diffusion (named "Bougerol's diffusion"), with infinitesimal generator:

$$2x(1+x)D^2 + (\delta + (4-\delta')x)\, D,$$

and $Y \equiv Y^{\delta,\delta'}$ the Jacobi process. Then, for $t, h, k > 0$:

24) $X_t^{(\nu,\delta)} \equiv \exp\left(-2B_t^{(\nu)}\right) Z_{A_t^{(\nu)}} = \frac{Z_u}{Z'_u}\Big\vert_{u=A_t^{(\nu)}}$
25) $\frac{Y_h}{1-Y_h} = X_{\int_0^h \frac{dv}{Y'_v}} = X_{\int_0^h \frac{dv}{1-Y_v}}$
26) $X_k = \frac{Y_h}{1-Y_h}\Big\vert_{h=\int_0^k \frac{dv}{1+X_v}}$

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