ON THE MOTION OF A CURVE TOWARDS ELASTICA

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Abstract. We consider a non-linear 4-th order parabolic equation derived from bending energy of wires in the 3-dimensional Euclidean space. We show that a solution exists for all time, and converges to an elastica when t goes to ∞.

Résumé. On considère une équation parabolique du 4^{e} ordre non linéaire provenant de l'énergie de flexion d'un câble dans l'espace euclidien de dimension 3. On montre qu'une solution existe pour tout temps, et converge lorsque t tend vers l'infini vers un "elastica".

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INTRODUCTION

Consider a springy circle wire in the Euclidean space $\mathbb{R}^3$. We characterize such a wire as a closed curve γ with fixed line element and fixed length. We treat curves $\gamma : S^1 = \mathbb{R}/\mathbb{Z} \to \mathbb{R}^3$ with $|\gamma'| \equiv 1$. We denote by x the parameter of the curve, and denote by $'$, $''$ or $^{(n)}$ the derivatives with respect to x.

For such a curve, its elastic energy is given by

$$E(\gamma) := \oint |\gamma''|^2 \, dx \ .$$

Solutions of the corresponding Euler-Lagrange equation are called *elastic curves*. We discuss the corresponding parabolic equation in this paper. We will see that the equation becomes

$$\begin{cases} \partial_t \gamma = -\gamma^{(4)} + ((v - 2|\gamma''|^2)\gamma')' \ , \\ -v'' + |\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \ . \end{cases}$$

Theorem. — *For any* C^∞ *initial data* $\gamma_0(x)$ *with* $|\gamma_0'| = 1$, *the above equation has a unique solution* $\gamma(x,t)$ *for all time, and the solution converges to an elastica when* $t \to \infty$.

We refer to Langer and Singer [13] for the classification of closed elasticae in the Euclidean space. They also discuss Palais-Smale's condition C and the gradient flow in [14]. However, their flow is completely different from ours. Our equation represents the physical motion of springy wire under high viscosity, while their flow has no physical meaning.

This paper is organized as follows. First, we prepare some basic facts. Section 1 : The equation (introduce the above equation), Section 2 : Notations, Section 3 : Basic inequalities, Section 4 : Estimations for ODE ($-v'' + av = b$). After this preparation, the proof of Theorem goes as usual. Section 5 : Linearized equation, Section 6 : Short time existence (by open–closed method), Section 7 : Long time existence, Section 8 : Convergence (using real analyticity of the Euclidean space).

1. THE EQUATION

To derive an equation of motion governed by energy, we perturb the curve $\gamma = \gamma(x)$ with a time parameter t : $\gamma = \gamma(x,t)$. Then, the elastic energy changes as

$$\frac{d}{dt}|_{t=0}E(\gamma) = 2\oint(\gamma'', \partial_t|_{t=0}\gamma'')\,dx = 2\oint(\gamma^{(4)}, \partial_t|_{t=0}\gamma)\,dx\ ,$$

where $\gamma(x,0) = \gamma(x)$. Therefore, $-\gamma^{(4)}$ would be the most efficient direction to minimize the elastic energy. However, this direction does not preserve the condition $|\gamma'| \equiv 1$. To force to preserve the condition we have to add certain terms. Let V be the space of all directions satisfying the condition in the sense of first derivative, i.e., $V = \{\eta \mid (\gamma', \eta') = 0\}$.

We can check that a direction is L^2 orthogonal to V if and only if it has a form $(w\gamma')'$ for some function $w(x)$. Therefore, the "true" direction has a form $-\gamma^{(4)} + (w\gamma')'$, where the function w has to satisfy the condition $\big((-\gamma^{(4)} + (w\gamma')')', \gamma'\big) = 0$. Namely, we consider the equation

$$\begin{cases} \partial_t\gamma = -\gamma^{(4)} + (w\gamma')'\ , \\ (-\gamma^{(5)} + (w\gamma')'', \gamma') = 0\ , \\ |\gamma'| = 1\ . \end{cases} \tag{1.1}$$

Note that both γ and w are unknown functions on $S^1 \times \mathbb{R}_+$.

The second equality of (1.1) is reduced as follows. By the third condition, we see

$$\begin{aligned} (\gamma'', \gamma') &= 0\ , \\ (\gamma^{(3)}, \gamma') &= -|\gamma''|^2\ , \\ (\gamma^{(4)}, \gamma') &= -\frac{3}{2}(|\gamma''|^2)'\ , \\ (\gamma^{(5)}, \gamma') &= -2(|\gamma''|^2)'' + |\gamma^{(3)}|^2\ . \end{aligned}$$

Hence,

$$((w\gamma')'', \gamma') = w'' - |\gamma''|^2 w \ ,$$

and the second equality in (1.1) becomes

$$-w'' + |\gamma''|^2 w = 2(|\gamma''|^2)'' - |\gamma^{(3)}|^2 \ .$$

If we put $v = w + 2|\gamma''|^2$, then we get

$$-v'' + |\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \ .$$

We conclude that equation (1.1) is equivalent to the equation

$$\text{EP} \qquad \begin{cases} \partial_t \gamma = -\gamma^{(4)} + ((v - 2|\gamma''|^2)\gamma')' \ , \\ -v'' + |\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \ . \end{cases}$$

The equation of elastic curves is

$$\text{EE} \qquad \begin{cases} -\gamma^{(4)} + ((v - 2|\gamma''|^2)\gamma')' = 0 \ , \\ -v'' + |\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \ , \\ |\gamma'| = 1 \ . \end{cases}$$

The first equality gives

$$0 = (\gamma', -\gamma^{(4)} + ((v - 2|\gamma''|^2)\gamma')') = \frac{3}{2}(|\gamma''|^2)' + (v - 2|\gamma''|^2)' \ .$$

Hence, the equation of elastic curves reduces to the equation

$$-\gamma^{(4)} - \big((\frac{3}{2}|\gamma''|^2 + c)\gamma'\big)' = 0 \ ,$$

where c is an arbitrary number.

2. NOTATIONS

Throughout this paper, we use variables x on $S^1 = \mathbb{R}/\mathbb{Z}$ and t on $\mathbb{R}_+ = [0, \infty)$. Symbols $*'$ and $*^{(n)}$ denote the derivation with respect to the variable x, even for a function on $S^1 \times \mathbb{R}_+$. The derivative with respect to the variable t is always denoted by ∂_t.

For functions on S^1, we use the C^n norm $\|*\|_{C^n}$, the L^2 norm $\|*\|$, the Sobolev space H^n with norm $\|*\|_n$, and the Hölder space $C_x^{n+4\mu}$ with norm $\|*\|_{x,(n+4\mu)}$, where n denotes a non-negative integer and μ denotes a positive real number with $4\mu < 1$. When these norms are applied to a function on $S^1 \times \mathbb{R}_+$, we get a function on $\mathbb{R}_+$. We also use the L^2 inner product $\langle *, * \rangle$.

For functions on $S^1 \times [0, T)$, we use weighted Hölder space $C^{n+4\mu}$ with norm $\|*\|_{(n+4\mu)}$. This norm is defined as follows.

$$
\begin{aligned}
\|f\|_{(n+4\mu)} &= [f]_{(x,n+4\mu)} + [f]_{(t,n/4+\mu)} + \sum_{0 \le 4r+s \le n} \sup |\partial_t^r f^{(s)}| \ , \\
[f]_{(x,n+4\mu)} &= \sum_{4r+s=n} [\partial_t^r f^{(s)}]_{(x,4\mu)} \ , \\
[f]_{(t,n+\mu)} &= \sum_{n-4<4r+s \le n} [\partial_t^r f^{(s)}]_{(t,(n-4r-s)/4+\mu)} \ , \\
[f]_{(x,4\mu)} &= \sup \frac{|f(x_1,t) - f(x_2,t)|}{|x_1 - x_2|^{4\mu}} \ , \\
[f]_{(t,i/4+\mu)} &= \sup \frac{|f(x,t_1) - f(x,t_2)|}{|t_1 - t_2|^{i/4+\mu}} \ ,
\end{aligned}
$$

where n, r, s and i denote non-negative integers with $i < 4$, and $0 < \mu < 1/4$.

3. BASIC INEQUALITIES

Lemma 3.1. — *For any $\mathbb{R}^N$ valued H^1-function u on S^1, we have*

$$\sup |v|^2 \le 2\|v\|(\|v\| + \|v'\|) \ .$$

Moreover, if $\oint v \, dx = 0$, then

$$\sup |v|^2 \le 2\|v\| \cdot \|v'\| \ .$$

Proof. The proof reduces to the case of $N = 1$. Since

$$\sup|v|^2 \leq \min|v|^2 + \oint |(v^2)'|\,dx \leq \|v\|^2 + 2\|v\| \cdot \|v'\| .$$

If v takes the value 0 at some point, then the term $\|v\|^2$ on the right hand side can be omitted.

□

Lemma 3.2. — *For integers $0 \leq p \leq q \leq r$, we have*

$$\|v^{(q)}\| \leq \|v^{(p)}\|^{(r-q)/(r-p)} \cdot \|v^{(r)}\|^{(q-p)/(r-p)} .$$

For integers $0 \leq p \leq q < r$ and $q > 0$, we have

$$\sup|v^{(q)}| \leq 2\|v^{(p)}\|^{(2(r-q)-1)/(2(r-p))} \cdot \|v^{(r)}\|^{(2(q-p)+1)/(2(r-p))} .$$

Proof. Since

$$\|v^{(n)}\|^2 = -\langle v^{(n-1)}, v^{(n+1)}\rangle \leq \|v^{(n-1)}\| \cdot \|v^{(n+1)}\| ,$$

we see that the function $\log\|v^{(n)}\|$ is concave with respect to $n \geq 0$. Therefore, the first inequality holds. From Lemma 3.1, we get

$$\sup|v^{(q)}| \leq \sqrt{2}\|v^{(q)}\|^{1/2} \cdot \|v^{(q+1)}\|^{1/2} .$$

Combining it with the first inequality, we get the second inequality. □

4. ESTIMATIONS FOR ODE

Lemma 4.1. — *The equation*

$$-v'' + av = b$$

for a, $b \in L^1$, $a \geq 0$ *and* $\|a\|_{L^1} > 0$ *has a unique solution* v, *and* v *is bounded in* C^1 *as*

$$\max|v| \leq 2(1 + \|a\|_{L^1}^{-1})\|b\|_{L^1} \ ,$$
$$\max|v'| \leq 2(1 + \|a\|_{L^1})\|b\|_{L^1} \ .$$

Proof. Set $B = \|b\|_{L^1}$. Since $v'' = av - b$,

$$v'(q) - v'(p) = \int_p^q av\,dx - \int_p^q b\,dx \ .$$

Therefore, if $v'(p) = 0$,

$$\int_p^q av\,dx - B \leq v'(q) \leq \int_p^q av\,dx + B \ . \tag{4.1}$$

Assume that $\max v \geq 0$ and the maximum is attained at $x = p$. If $v \geq 0$ on $[p, q]$, then $-B \leq v'(q)$. It implies that for $x \in [p, q]$, $-B \leq v'(x)$ and $v(q) \geq v(p) - B$. Therefore, if $\min v \geq 0$, then $\min v \geq \max v - B$, and if $\min v \leq 0$, then $\max v \leq B$. Combining it with similar estimations for $-v$, we get

$$\max v - \min v \leq 2B \ .$$

Thus, from the equality

$$0 = \oint av\,dx - \oint b\,dx \ ,$$

we see

$$B \geq \oint av\,dx \geq \min v \cdot \|a\|_{L^1} \geq (\max v - 2B)\|a\|_{L^1} \ .$$

This leads to

$$\max v \leq \|a\|_{L^1}^{-1} B + 2B \ .$$

Again from (4.1), we get

$$v'(q) \leq \max v \cdot \|a\|_{L^1} + B \leq (B + 2B\|a\|_{L^1}) + B \ .$$

Thus, $\max v' \leq 2(1 + \|a\|_{L^1})B$. □

Lemma 4.2. — *Let*

$$-v'' + av = b \ ,$$

where $a \geq 0$ *and* $\|a\|_{L^1} \geq 1$. *Then we have*

$$\|v\|_{C^{n+2}} \leq C \cdot (1 + \|a\|_{C^n}^N) \cdot \|b\|_{C^n} \ ,$$
$$\|v\|_{n+2} \leq C \cdot (1 + \|a\|_n^N) \cdot \|b\|_n \ .$$

The positive integer N *and the positive number* C *depend only on* n.

Proof. We check the first inequality. By Lemma 4.1, we see that

$$\begin{aligned}\sup|v''| &\leq \sup|av| + \sup|b| \leq \sup|a| \cdot \sup|v| + \sup|b| \\ &\leq \sup|a| \cdot C_1\|b\| + \sup|b| \ .\end{aligned}$$

Thus, the inequality holds for $n = 0$. Suppose that the inequality holds up to n. Then,

$$\begin{aligned}\sup|v^{(n+3)}| &\leq \sup|(av)^{(n+1)}| + \sup|b^{(n+1)}| \\ &\leq C_2\|a\|_{C^{n+1}} \cdot \|v\|_{C^{n+1}} + \|b\|_{C^{n+1}} \\ &\leq C_2\|a\|_{C^{n+1}} \cdot C_3 \cdot (1 + \|a\|_{C^n}^{N_1})\|b\|_{C^n} + \|b\|_{C^{n+1}} \ .\end{aligned}$$

Therefore, the inequality holds for $n + 1$.

Next, we check the second inequality. By Lemma 4.1,

$$\begin{aligned}\|v''\| &\leq \|av\| + \|b\| \leq \|a\| \cdot \sup|v| + \|b\| \\ &\leq \|a\| \cdot C_4\|b\|_{L^1} + \|b\| \ .\end{aligned}$$

Thus, the inequality holds for $n = 0$. Suppose that the inequality holds up to n. Then,

$$\begin{aligned}\|v^{(n+3)}\| &\leq \|(av)^{(n+1)}\| + \|b^{(n+1)}\| \\ &\leq C_5\|a\|_{C^n} \cdot \|v\|_{n+1} + \|a^{(n+1)}\| \cdot \sup|v| + \|b\|_{n+1} \\ &\leq C_6\|\cdot\|_{n+1}a(1 + \|a\|_n^{N_2})\|b\|_n + C_7\|a\|_{n+1} \cdot \|b\| + \|b\|_{n+1} \ .\end{aligned}$$

Therefore, the inequality holds for $n + 1$. □

Lemma 4.3. — *Suppose that functions* $a = a(x,t)$, $b = b(x,t)$ *and* $v = v(x,t)$ *satisfies*

$$-v'' + av = b \ ,$$

where $a \geq 0$ and $\|a\|_{L^1} \geq 1$. If $a, b \in C^{n+4\mu}$, then v, v' and $v'' \in C^{n+4\mu}$, and we have

$$\|v\|_{(n+4\mu)}, \|v'\|_{(n+4\mu)}, \|v''\|_{(n+4\mu)} \leq C \cdot (1 + \|a\|_{(n+4\mu)}^N)\|b\|_{(n+4\mu)} \ .$$

The positive integer N and the positive number C depend only on n.

Proof. Note that if we have bound only for $\|v\|_{(n+4\mu)}$, then the assumption leads the bounds for others. By definition,

$$\|v\|_{(4\mu)} = \sup |v| + [v]_{(x,4\mu)} + [v]_{(t,\mu)} \ .$$

Here, $\sup |v|$, $\sup |v'|$ and $[v]_{(x,4\mu)}$ are bounded by

$$C_1 \cdot (1 + \sup\|a\|_{L^1}) \sup\|b\|_{L^1} \leq C_1 \cdot (1 + \|a\|_{(4\mu)})\|b\|_{(4\mu)} \ .$$

To check $[v]_{(t,\mu)}$, let $t_+ = t + \delta t$ and put $f_+(t) = f(t_+)$, $\delta f = f_+ - f$ for a function f. Then,

$$-\delta v'' + a\delta v = \delta b - v_+\delta a \ .$$

Therefore, by Lemma 4.1,

$$\begin{aligned} \sup |\frac{\delta v}{\delta t^\mu}| &\leq C_2 \cdot \big(\sup |\frac{\delta b}{\delta t^\mu}| + \sup |v| \cdot \sup |\frac{\delta a}{\delta t^\mu}|\big) \\ &\leq C_3 \cdot \big(\|b\|_{(4\mu)} + \sup\|b\|_{L^1} \cdot \|a\|_{(4\mu)}\big) \ . \end{aligned}$$

Thus, also $[v]_{(t,\mu)}$ is bounded.

Suppose that the claim holds up to n (< 3). Then,

$$\|v\|_{(n+1+4\mu)} = \|v'\|_{(n+4\mu)} + [v]_{(t,(n+1)/4+\mu)} + \sup |v| \ .$$

By similar estimation with the case $n = 0$, the term $[v]_{(t,(n+1)/4+\mu)}$ is estimated as desired. Thus, the claim holds up to $n = 3$.

Now suppose that the claim holds up to $n = 4m + 3$ $(m \geq 0)$. Then, for $n = 4(m+1) + i$ $(0 \leq i \leq 3)$, the claim holds if both $\partial_t v$ and $v^{(4)}$ can be estimated in $C^{4m+i+4\mu}$. Since

$$\begin{aligned}
-\partial_t v'' + a\partial_t v &= \partial_t b - \partial_t a \cdot v \ , \\
-(v^{(4)})'' + av^{(4)} &= b^{(4)} - a^{(4)}v - 4a^{(3)}v' - 6a''v'' - 4a'v^{(3)} \\
&= b^{(4)} - a^{(4)}v - 4a^{(3)}v' - 6a''v'' - 4a' \cdot (av - b)' \ ,
\end{aligned}$$

we have

$$\begin{aligned}
\|\partial_t v\|_{(4m+i+4\mu)} &\le C_4 \cdot (1 + \|a\|^{N_1}_{(4m+i+4\mu)}) \\
&\qquad \times (\|\partial_t b\|_{(4m+i+4\mu)} + \|\partial_t a\|_{(4m+i+4\mu)} \cdot \|v\|_{(4m+i+4\mu)}) \\
&\le C_5 \cdot (1 + \|a\|^{N_2}_{(4m+i+4+4\mu)}) \|b\|_{(4m+i+4+4\mu)} \ , \\
\|v^{(4)}\|_{(4m+i+4\mu)} &\le C_6 \cdot (1 + \|a\|^{N_3}_{(4m+i+4+4\mu)}) \|b\|_{(4m+i+4+4\mu)} \ .
\end{aligned}$$

□

5. LINEARIZED EQUATION

In this section, we use the following basic facts concerning a parabolic equation with constant coefficients. We omit the proof of the first three lemmas. They are direct modifications of corresponding facts on a heat equation. See [1].

Proposition 5.1. (cf. [1] p. 237 (2.2), p. 262 (1.6)]) — *The equation*

$$\partial_t u + u^{(4)} = 0, \qquad u(x,0) = \phi(x) \ ,$$

with $\phi \in C_x^{4\mu}$ has a unique solution $u \in C^{4\mu}$. Moreover, we have

$$\|u\|_{(4\mu)} \le C\|\phi\|_{x,(4\mu)} \ ,$$

where C is a universal positive constant.

Proposition 5.2. (cf. [1] p. 298 Theorem 4.2]) — *The equation*

$$\partial_t u + u^{(4)} = f, \qquad u(x,0) = \phi(x) \ ,$$

with $\phi \in C_x^{4+4\mu}$ and $f \in C^{4\mu}$ has a unique solution $u \in C^{4+4\mu}$. Moreover, we have

$$\|u\|_{(4+4\mu)} \leq C \cdot (\|f\|_{(4\mu)} + \|\phi\|_{x,(4+4\mu)}) ,$$

where C is a universal positive constant.

Proposition 5.3. (cf. [1] p. 298 Theorem 4.3]) — *For any $\phi_0 \in C_x^{4+4\mu}$ and $\phi_1 \in C_x^{4\mu}$, there exists $u \in C^{4+4\mu}$ such that*

$$u(x,0) = \phi_0(x), \qquad \partial_t u(x,0) = \phi_1(x) ,$$

and

$$\|u\|_{(4+4\mu)} \leq C \cdot (\|\phi_0\|_{x,(4+4\mu)} + \|\phi_1\|_{x,(4\mu)}) .$$

Here, C is a universal positive constant.

Proposition 5.4. (cf. [1] p. 302 Lemma 4.1]) — *There is a universal positive constant C such that for any $u \in C^{4+4\mu}$ with $u(x,0) = \partial_t u(x,0) = 0$, we have,*

$$\|u\|_{(3+4\mu)} \leq CT^{1/4}\|u\|_{(4+4\mu)} ,$$

where both norms are taken on $S^1 \times [0,T)$.

Proof. We give the proof for completeness. By definition,

$$\begin{aligned}\|u\|_{(4+4\mu)} &= [\partial_t u]_{(x,4\mu)} + [u^{(4)}]_{(x,4\mu)} + [\partial_t u]_{(t,\mu)} \\ &\qquad + \sum_{1\leq i\leq 4} [u^{(i)}]_{(t,(4-i)/4+\mu)} + \sum_{0\leq i\leq 4} \sup|u^{(i)}| + \sup|\partial_t u| , \\ \|u\|_{(3+4\mu)} &= [u^{(3)}]_{(x,4\mu)} + \sum_{0\leq i\leq 3} [u^{(i)}]_{(t,(3-i)/4+\mu)} + \sum_{0\leq i\leq 3} \sup|u^{(i)}| .\end{aligned}$$

We see that

$$\sup|\partial_t u| \leq \sup\big(|t-0|^\mu \cdot \frac{|\partial_t u(x,t) - \partial_t u(x,0)|}{|t-0|^\mu}\big) \leq T^\mu \cdot [\partial_t u]_{(t,\mu)} .$$

A similar computation leads to

$$\begin{aligned}&\sup|u| \leq T \cdot \sup|\partial_t u| \leq T^{1/4+\mu} \cdot [\partial_t u]_{(t,\mu)} , \\ &[u]_{(t,3/4+\mu)} \leq T^{1/4-\mu} \cdot \sup|\partial_t u| \leq T^{1/4}[\partial_t u]_{(t,\mu)} , \\ &\sup|u^{(i)}| \leq T^{(4-i)/4+\mu} \cdot [u^{(i)}]_{(t,1/4+\mu)} \quad \text{for } 1 \leq i \leq 4 , \\ &[u^{(i)}]_{(t,(3-i)/4+\mu)} \leq T^{1/4} \cdot [u^{(i)}]_{(t,(4-i)/4+\mu)} \quad \text{for } 1 \leq i \leq 3 .\end{aligned}$$

Finally,

$$
\begin{aligned}
[u^{(3)}]_{(x,4\mu)} &= \sup \frac{|u^{(3)}(x_1,t)-u^{(3)}(x_2,t)|}{|x_1-x_2|^{4\mu}} \\
&\le \max\Big\{ \sup_{|x_1-x_2|\le T^{1/4}} |x_1-x_2|^{1-4\mu}\cdot|u^{(4)}|, \sup_{|x_1-x_2|>T^{1/4}} T^{-\mu}\cdot|u^{(3)}| \Big\} \\
&\le \max\Big\{ T^{1/4}\cdot[u^{(4)}]_{(t,\mu)}, T^{1/4}\cdot[u^{(3)}]_{(t,1/4+\mu)} \Big\} \ .
\end{aligned}
$$

□

Lemma 5.5. — *The equation for u and v*

$$
\begin{cases}
\partial_t u + u^{(4)} + \sum_{i=0}^{3} c_i u^{(i)} + \sum_{i=0}^{1} d_i v^{(i)} = f \ , \\
-v'' + a^2 v = \sum_{i=0}^{3} b_i u^{(i)} \ , \\
u(x,0) = \partial_t u(x,0) = 0 \ ,
\end{cases}
$$

with f, a, b_i, c_i, $d_i \in C^{4\mu}$ and $f(x,0)=0$, $\|a\| \ge 1$ has a solution on some time interval $[0,T)$. The norm $\|u\|_{(4+4\mu)}$ and the positive time T are bounded by a constant depending on the $C^{4\mu}$ norms of f, a, b_i, c_i and d_i.

Proof. We follow the proof of [1] p. 322, Theorem 5.4. We define spaces $C_0^{4\mu}$ and $C_0^{4+4\mu}$ by setting

$$C_0^{4\mu} = \{f \in C^{4\mu} \mid f(x,0)=0\} \ ,$$

and

$$C_0^{4+4\mu} = \{u \in C^{4+4\mu} \mid u(x,0) = \partial_t u(x,0) = 0\} \ .$$

For $u \in C_0^{4+4\mu}$, take v so that

$$-v'' + a^2 v = \sum b_i u^{(i)} \ ,$$

and put

$$
\begin{aligned}
Pu &:= \partial_t u + u^{(4)} + \sum c_i u^{(i)} + \sum d_i v^{(i)} \ , \\
P_0 u &:= \partial_t u + u^{(4)} \ , \\
P_1 u &:= \sum c_i u^{(i)} + \sum d_i v^{(i)} \ .
\end{aligned}
$$

We regard P, P_0 and P_1 as operators from $C_0^{4+4\mu}$ to $C_0^{4\mu}$. Using Proposition 5.1, we also define an operator $R : C_0^{4\mu} \to C_0^{4+4\mu}$ by setting

$$\partial_t(Rf) + (Rf)^{(4)} = f, \qquad (Rf)(x,0) = 0 \ .$$

Note that $P = P_0 + P_1$, $P_0 R = \mathrm{id}$, $RP_0 = \mathrm{id}$ and

$$PR = (P_0 + P_1)R = \mathrm{id} + P_1 R \ ,$$
$$RP = R(P_0 + P_1) = \mathrm{id} + RP_1 \ .$$

Put $S = P_1 R$. If the norm of S is sufficiently small, then

$$P(R(\mathrm{id} + S)^{-1}) = (\mathrm{id} + S)(\mathrm{id} + S)^{-1} = \mathrm{id} \ .$$

Since operators R and $(\mathrm{id} + S)^{-1}$ are isomorphisms, P has $R(\mathrm{id} + S)^{-1}$ as inverse.

Therefore, it is sufficient to prove that, if the time T is sufficiently small, then the operator $P_1 R$ is sufficiently small.

Let $f \in C^{4\mu}$ and put $u = Rf$. By Proposition 5.2, we know that

$$\|u\|_{(4+4\mu)} \le C_1 \|f\|_{(4\mu)} \ .$$

By Proposition 5.4, for any positive ε, there is a time T so that

$$\|u\|_{(3+4\mu)} \le \varepsilon \|u\|_{(4+4\mu)}$$

holds. Moreover, by Lemma 4.3, we know that

$$\|v\|_{(4\mu)}, \|v'\|_{(4\mu)} \le C_2 \|u\|_{(3+4\mu)} \ .$$

Combining these, we see that

$$\|P_1 u\|_{(4\mu)} \le \varepsilon \|u\|_{(4+4\mu)} \ .$$

□

Proposition 5.6. — *The equation for u and v*

$$\begin{cases} \partial_t u + u^{(4)} + \sum_{i=0}^{3} c_i u^{(i)} + \sum_{i=0}^{1} d_i v^{(i)} = f \ , \\ -v'' + a^2 v = \sum_{i=0}^{3} b_i u^{(i)} \ , \\ u(x,0) = \phi(x) \ , \end{cases}$$

with f, a, b_i, c_i, $d_i \in C^{4\mu}$, $\phi \in C_x^{4+4\mu}$ and $\|a\| \geq 1$ has a solution on the whole time interval $[0,\infty)$. The norm $\|u\|_{(4+4\mu)}$ is bounded by a constant depending on the $C^{4\mu}$ norms of f, a, b_i, c_i, d_i and the $C_x^{4+4\mu}$ norm of ϕ.

Proof. We follow the proof of [1] p. 320, Theorem 5.1. We construct a function $\bar{u} \in C^{4+4\mu}$ such that

$$\begin{aligned} \bar{u}(x,0) &= \phi(x) \ , \\ \partial_t \bar{u}(x,0) &= f(x,0) - \phi^{(4)}(x) - \sum c_i(x,0)\phi^{(i)}(x) - \sum d_i(x,0)\psi^{(i)}(x) \ , \\ -\psi''(x) + a(x,0)^2\psi(x) &= \sum c_i(x,0)\phi^{(i)}(x) \ , \end{aligned}$$

by Proposition 5.3. Let $\bar{v}$ satisfy the equation $-\bar{v}'' + a^2\bar{v} = \sum c_i \bar{u}^{(i)}$, and put

$$\begin{aligned} \tilde{u} &= u - \bar{u} \ , \\ \tilde{v} &= v - \bar{v} \ , \\ \tilde{f} &= f - (\partial_t \bar{u} + \bar{u}^{(4)} + \sum c_i \bar{u}^{(i)} + \sum d_i \bar{v}^{(i)}) \ . \end{aligned}$$

Then, the equation for $\tilde{u}$ and $\tilde{v}$ becomes

$$\begin{aligned} &\partial_t \tilde{u} + \tilde{u}^{(4)} + \sum c_i \tilde{u}^{(i)} + \sum d_i \tilde{v}^{(i)} = \tilde{f} \ , \\ &-\tilde{v}'' + a^2 \tilde{v} = \sum b_i \tilde{u}^{(i)} \ , \\ &\tilde{u}(x,0) = \partial_t \tilde{u}(x,0) = 0 \ . \end{aligned}$$

Here, we know by Lemma 4.3 that $\|\bar{v}\|_{(4\mu)}$, $\|\bar{v}'\|_{(4\mu)} \leq C_1 \|\bar{u}\|_{(4+4\mu)}$. Therefore,

$$\begin{aligned} \|\tilde{f}\|_{(4\mu)} &\leq C_2 \|\bar{u}\|_{(4+4\mu)} + \|f\|_{(4\mu)} \\ &\leq C_3 \cdot (\|\phi\|_{x,(4+4\mu)} + \|f(x,0)\|_{x,(4\mu)}) + \|f\|_{(4\mu)} \ . \end{aligned}$$

Thus, by Lemma 5.5, we have solutions $\tilde{u}$ and $\tilde{v}$ for some short time $[0,T)$. Hence, we can construct u and v. But, we know how to estimate the time T and u. Thus, we can repeat this procedure and get a solution on the whole line $[0,\infty)$. □

6. SHORT TIME SECTION

In this section, we consider a modified equation

$$\mathrm{EP}_* \qquad \begin{cases} \partial_t\gamma = -\gamma^{(4)} + \lambda((v-2|\gamma''|^2)\gamma')' \ , \\ -v'' + \|\gamma'\|^{-2}|\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \ , \end{cases}$$

where λ is a constant in $[0,1]$. We will give a C^∞ initial data γ_0.

Remark. — We put the parameter λ in the first equality to use the so-called open-closed method. Unfortunately, it destroys the equality $|\gamma'|=1$ in (EP), and disturb us from applying estimates from section 4. This is the reason why we put the factor $\|\gamma'\|^{-2}$ in the second equality. However, when the space is $\mathbb{R}^2$ and the initial data has a non-zero rotation number, we may omit this factor. See [6].

Proposition 6.1. — *Let γ be a C^∞ solution of* (EP_*) *on a finite time interval* $[0,T)$. *Suppose that $\|\gamma\|_3 \le C_1$ and $\|\gamma'\| \ge C_2 > 0$. Then, we have $\|\gamma\|_n \le C_3$, where the constant C_3 depends only on the initial data γ_0, C_1, C_2, T and n, but not on λ.*

Proof. We start from the following inequality.

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\gamma^{(n+1)}\|^2 &= \langle \gamma^{(n+1)}, \partial_t\gamma^{(n+1)}\rangle = \langle \gamma^{(n+3)}, \partial_t\gamma^{(n-1)}\rangle \\ &= -\|\gamma^{(n+3)}\|^2 + \lambda\langle \gamma^{(n+3)}, ((v-2|\gamma''|^2)\gamma')^{(n)}\rangle \\ &\le -\|\gamma^{(n+3)}\|^2 + \frac{1}{2}\|\gamma^{(n+3)}\|^2 + \frac{1}{2}\|((v-2|\gamma''|^2\gamma')^{(n)}\|^2 \\ &\le -\frac{1}{2}\|\gamma^{(n+3)}\|^2 + \frac{1}{2}\|((v-2|\gamma''|^2)\gamma')^{(n)}\|^2 \ . \end{aligned}$$

Suppose that $\|\gamma\|_n$ is bounded, where $n \ge 3$. Then, the right hand side and coefficients of the second equation of (EP_*) are bounded in H^{n-3} when $n>3$ and in

L^1 when $n = 3$. In both cases we see that $\|v\|_{C^{n-2}}$ is bounded by Lemma 4.1 and 4.2. Therefore,

$$\begin{aligned}\|(v\gamma')^{(n)}\| &\le C_1 \cdot (1 + \|v^{(n)}\gamma'\| + \|v^{(n-1)}\gamma''\| + \|v\gamma^{(n+1)}\| + \|v'\gamma^{(n)}\|)\\ &\le C_2 \cdot (1 + \|v\|_n + \|\gamma^{(n+1)}\|) \ .\end{aligned}$$

To estimate $\|v\|_n$ and $\|\gamma^{(n+1)}\|$, we need the following inequalities from Lemma 3.1 and 3.2.

$$\begin{aligned}&\|\gamma^{(n+i)}\| \le C_3\|\gamma^{(n)}\|^{(3-i)/3} \cdot \|\gamma^{(n+3)}\|^{i/3} \le C_4\|\gamma^{(n+3)}\|^{i/3} \ ,\\ &\sup|\gamma^{(2+i)}| \le C_5\|\gamma^{(3+i)}\| \le C_6\|\gamma^{(n+i)}\| \le C_7\|\gamma^{(n+3)}\|^{i/3} \ ,\end{aligned}$$

where $i = 1, 2$. We again apply Lemma 4.2 to the second equation of (EP$_*$). Note that

$$\big\|\|\gamma'\|^{-2}|\gamma''|^2\big\|_{n-2} = \|\gamma'\|^{-2}\big\||\gamma''|^2\big\|_{n-2} \le C_8 \ .$$

Therefore,

$$\begin{aligned}\|v\|_n &\le C_9\big\|2|\gamma''|^4 - |\gamma^{(3)}|^2\big\|_{n-2} \le C_{10} \cdot (1 + \big\||\gamma^{(3)}|^2\big\|_{n-2})\\ &\le C_{11} \cdot (1 + \|(|\gamma^{(3)}|^2)^{(n-2)}\|)\\ &\le C_{12} \cdot (1 + \sup|\gamma^{(3)}| \cdot \|\gamma^{(n+1)}\| + \sup|\gamma^{(4)}| \cdot \|\gamma^{(n)}\|)\\ &\le C_{13} \cdot (1 + \|\gamma^{(n+3)}\|^{1/3+1/3} + \|\gamma^{(n+3)}\|^{2/3})\\ &\le C_{14} \cdot (1 + \|\gamma^{(n+3)}\|^{2/3}) \ .\end{aligned}$$

Thus, we have

$$\|(v\gamma')^{(n)}\| \le C_{15} \cdot (1 + \|\gamma^{(n+3)}\|^{2/3} + \|\gamma^{(n+3)}\|^{1/3}) \le C_{16} \cdot (1 + \|\gamma^{(n+3)}\|^{2/3}) \ .$$

To estimate $\|(|\gamma''|^2\gamma')^{(n)}\|$ is done as above. It suffices to consider $\|(|\gamma''|^2)^{(n)}\|$. We see that

$$\begin{aligned}\|(|\gamma''|^2)^{(n)}\| &\le C_{17} \cdot (1 + \|\gamma^{(n+2)}\| + \sup|\gamma^{(3)}| \cdot \|\gamma^{(n+1)}\| + \sup|\gamma^{(4)}| \cdot \|\gamma^{(n)}\|)\\ &\le C_{18} \cdot (1 + \|\gamma^{(n+3)}\|^{2/3}) \ .\end{aligned}$$

Combining these, we have

$$|d|dt\left\|\gamma^{(n+1)}\right\|^2 \leq -\left\|\gamma^{(n+3)}\right\|^2 + C_{19}\cdot(1+\left\|\gamma^{(n+3)}\right\|^{2/3})^2 \leq C_{20} \ .$$

□

Proposition 6.2. — *Let γ be a C^∞ solution of the equation* (EP_*) *with non-constant initial data γ_0. Then, there are positive constants T and C so that, on the time interval $[0,T)$, γ is bounded in C^∞ topology and $\left\|\gamma'\right\| \geq C > 0$. The constants T, C and the C^∞ bound of γ depend only on the C^∞ norm of the initial data γ_0, but not on λ.*

Proof. Note that we do not assume that $\left\|\gamma'\right\|$ is bounded away from 0. First, we estimate $\left\|\gamma^{(3)}\right\|$. We have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\gamma^{(3)}\right\|^2 &= \langle\gamma^{(5)}, \partial_t\gamma'\rangle \\ &= -\left\|\gamma^{(5)}\right\|^2 + \lambda\langle\gamma^{(5)}, ((v-2|\gamma''|^2)\gamma')''\rangle \\ &\leq -\frac{1}{2}\left\|\gamma^{(5)}\right\|^2 + \frac{1}{2}\left\|((v-2|\gamma''|^2\gamma')''\right\|^2 \ .\end{aligned}$$

Therefore, if we have estimates of the form

$$\left\|((v-2|\gamma''|^2\gamma')''\right\| \leq C_1\cdot(1+\left\|\gamma^{(5)}\right\|^p)\cdot(1+\left\|\gamma^{(3)}\right\|^q)$$

for some constant $p<1$, then we will get

$$\frac{d}{dt}\left\|\gamma^{(3)}\right\|^2 \leq C_2\cdot(1+\left\|\gamma^{(3)}\right\|^r) \ .$$

This will imply the existence of a time T such that $\left\|\gamma^{(3)}\right\|$ is bounded from above on $[0,T)$.

We take a term from the expansion of $(|\gamma''|^2\gamma')''$. If it contains $\gamma^{(4)}$, then it is bounded by

$$\left\|(\gamma^{(4)},\gamma'')\gamma'\right\| \leq \left\|\gamma^{(3)}\right\|^q\cdot\left\|\gamma^{(4)}\right\| \ .$$

If it contains $\gamma^{(3)}$, then it is bounded by

$$C_3\left\|\gamma^{(3)}\right\|^q\cdot\sup|\gamma^{(3)}| \leq C_4\left\|\gamma^{(3)}\right\|^{q+1/2}\cdot\left\|\gamma^{(4)}\right\|^{1/2} \ .$$

In both cases, we get the desired estimation, since $\|\gamma^{(4)}\| \leq \|\gamma^{(3)}\|^{1/2} \cdot \|\gamma^{(5)}\|^{1/2}$.

From the expansion of $(v\gamma')''$, we get $\|v\gamma^{(3)}\|$, $\|v'\gamma''\|$ and $\|v''\gamma'\|$. From Lemma 4.1, the first one $\|v\gamma^{(3)}\|$ is bounded by

$$\sup|v| \cdot \|\gamma^{(3)}\| \leq C_5 \cdot (\||\gamma''|^2\|^2 + \|\gamma^{(3)}\|^2)\|\gamma^{(3)}\| \leq C_6 \cdot (1 + \|\gamma^{(3)}\|^5) \ .$$

Again from Lemma 4.1, the second one $\|v'\gamma''\|$ is bounded by

$$\sup|v'| \cdot \|\gamma''\| \leq C_7 \cdot (1 + \|\gamma'\|^{-2} \cdot \|\gamma''\|^2) \cdot (\||\gamma''|^2\|^2 + \|\gamma^{(3)}\|^2)\|\gamma''\| \ ,$$

and

$$\begin{aligned}
&\|\gamma'\|^{-2}\|\gamma''\|^2 \leq \|\gamma'\|^{-2}\|\gamma'\| \cdot \|\gamma^{(3)}\| = \|\gamma'\|^{-1}\|\gamma^{(3)}\| \ ,\\
&\||\gamma''|^2\|^2 \leq \sup|\gamma''|^2 \cdot \|\gamma''\|^2 \leq \sup|\gamma''|^2 \cdot \|\gamma'\| \cdot \|\gamma^{(3)}\| \ ,\\
&\|\gamma^{(3)}\|^2 \leq \|\gamma''\| \cdot \|\gamma^{(4)}\| \leq \|\gamma'\|^{1/2}\|\gamma^{(3)}\|^{1/2}\|\gamma^{(4)}\| \ ,\\
&\|\gamma''\| \leq \|\gamma'\|^{1/2}\|\gamma^{(3)}\|^{1/2} \ .
\end{aligned}$$

Hence, the negative power of $\|\gamma'\|$ cancels, and we see

$$\sup|v'| \cdot \|\gamma''\| \leq C_8 \cdot (1 + \|\gamma^{(3)}\|^q) \cdot (1 + \|\gamma^{(4)}\|) \ .$$

The last one $\|v''\gamma'\|$ is bounded by

$$(\|\gamma'\|^{-2} \cdot \||\gamma''|^2 v\| + \||\gamma''|^4\| + \||\gamma^{(3)}|^2\|) \cdot \sup|\gamma'| \ .$$

Here, the last two terms are bounded by

$$(\|\gamma^{(3)}\|^4 + \sup|\gamma^{(3)}| \cdot \|\gamma^{(3)}\|) \cdot \|\gamma^{(3)}\| \leq (1 + \|\gamma^{(3)}\|^5) \cdot (1 + \|\gamma^{(4)}\|) \ .$$

For the first term, we see

$$\|\gamma'\|^{-2} \cdot \||\gamma''|^2 v\| \cdot \sup|\gamma'| \leq 2\|\gamma'\|^{-3/2}\|\gamma''\|^{1/2}\||\gamma''|^2 v\| \ ,$$

and

$$\begin{aligned}
\||\gamma''|^2 v\| &\leq \sup|\gamma''| \cdot \sup|v| \cdot \|\gamma''\|\\
&\leq C_9\|\gamma^{(3)}\| \cdot \|\gamma''\|(\||\gamma''|^2\|^2 + \|\gamma^{(3)}\|^2)\\
&\leq C_9\|\gamma^{(3)}\| \cdot \|\gamma''\|(\sup|\gamma''|^2 \cdot \|\gamma''\|^2 + \|\gamma^{(3)}\|^2)\\
&\leq C_{10}\|\gamma^{(3)}\| \cdot \|\gamma''\|(\|\gamma^{(3)}\|^2 \cdot \|\gamma''\|^2 + \|\gamma^{(3)}\|^2)\\
&= C_{10}\|\gamma^{(3)}\|^3 \cdot \|\gamma''\|(1 + \|\gamma''\|^2) \ .
\end{aligned}$$

And,

$$\begin{aligned}\left\|\gamma'\right\|^{-3/2}\cdot\left\|\gamma^{(3)}\right\|^{3}\cdot\left\|\gamma''\right\|^{3/2} &\leq \left\|\gamma'\right\|^{-3/2}\cdot\left\|\gamma'\right\|^{9/8}\cdot\left\|\gamma^{(5)}\right\|^{3/8}\cdot\left\|\gamma^{(3)}\right\|^{3}\\ &\leq \left\|\gamma'\right\|^{-3/8}\cdot\left\|\gamma^{(5)}\right\|^{3/8}\cdot\left\|\gamma'\right\|^{3/8}\cdot\left\|\gamma^{(5)}\right\|^{3/8}\cdot\left\|\gamma^{(3)}\right\|^{9/4}\\ &= \left\|\gamma^{(3)}\right\|^{9/4}\cdot\left\|\gamma^{(5)}\right\|^{3/4}\ .\end{aligned}$$

Thus,

$$\begin{aligned}\left\|v''\gamma'\right\| &\leq C_{11}\cdot(1+\left\|\gamma^{(4)}\right\|+\left\|\gamma^{(5)}\right\|^{3/4})\cdot(1+\left\|\gamma^{(3)}\right\|^{q})\\ &\leq C_{12}\cdot(1+\left\|\gamma^{(5)}\right\|^{3/4})\cdot(1+\left\|\gamma^{(3)}\right\|^{q})\ .\end{aligned}$$

From this estimate, we get positive constants C_{13} and T depending only on $\left\|\gamma_0^{(3)}\right\|$ such that $\left\|\gamma^{(3)}\right\|\leq C_{13}$ on $[0,T)$. In particular, $\left\|\gamma\right\|_{C^2}$ and $\sup|v|$ are bounded from above. Then, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|\gamma'\right\|^{2} = -\langle\gamma'',\partial_t\gamma\rangle &= \langle\gamma^{(3)},-\gamma^{(3)}+\lambda(v-2|\gamma''|^2)\gamma'\rangle\\ &\geq -\left\|\gamma^{(3)}\right\|^{2}-\left\|\gamma^{(3)}\right\|\cdot\left\|(v-2|\gamma''|^2)\gamma\right\|\\ &\geq -C_{14}\ .\end{aligned}$$

Thus, we have a positive time T_1 such that $\left\|\gamma^{(3)}\right\|\leq C_{15}$ and $\left\|\gamma'\right\|\geq C_{16}>0$ on $[0,T_1)$. This completes our proof by Proposition 6.1. □

Proposition 6.3. — *The* C^∞ *solution* γ *in Proposition 6.2 is unique on the time interval* $[0,T)$.

Proof. Let $\{\bar\gamma,\bar v\}$ be another C^∞ solution of the equation (EP_*). Then we have

$$\partial_t(\bar\gamma-\gamma) = -(\bar\gamma-\gamma)^{(4)}+\sum_{i=0}^{3}P_i\cdot(\bar\gamma-\gamma)^{(i)}+\sum_{i=0}^{1}Q_i\cdot(\bar v-v)^{(i)}\ ,$$

$$-(\bar v-v)''+\left\|\gamma'\right\|^{-2}|\gamma''|^2(\bar v-v) = \sum_{i=0}^{3}R_i\cdot(\bar\gamma-\gamma)^{(i)}\ ,$$

where P_i, Q_i and R_i are expressed by γ, $\bar\gamma$, v and $\bar v$, which are bounded from above.

Therefore,

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\|(\bar\gamma-\gamma)'\right\|^2 \\
&\quad= -\left\|(\bar\gamma-\gamma)^{(3)}\right\|^2 + \sum_{i=0}^{3}\left\langle P_i\cdot(\bar\gamma-\gamma)', (\bar\gamma-\gamma)^{(i+1)}\right\rangle \\
&\qquad + \sum_{i=0}^{1}\left\langle Q_i\cdot(\bar\gamma-\gamma)', (\bar v-v)^{(i+1)}\right\rangle \\
&\quad= -\left\|(\bar\gamma-\gamma)^{(3)}\right\|^2 + \sum_{i=0}^{2}\left\langle P_i\cdot(\bar\gamma-\gamma)', (\bar\gamma-\gamma)^{(i+1)}\right\rangle \\
&\qquad - \left\langle (P_3\cdot(\bar\gamma-\gamma)')', (\bar\gamma-\gamma)^{(3)}\right\rangle + \sum_{i=0}^{1}\left\langle Q_i\cdot(\bar\gamma-\gamma)', (\bar v-v)^{(i+1)}\right\rangle \\
&\quad\le -\left\|(\bar\gamma-\gamma)^{(3)}\right\|^2 + C_1\left\|\bar\gamma-\gamma\right\|_2\cdot\left\|\bar\gamma-\gamma\right\|_3 + C_2\left\|\bar\gamma-\gamma\right\|_2\cdot\left\|\bar v-v\right\|_1 .
\end{aligned}
$$

Here, by Lemma 4.2,

$$\left\|\bar v-v\right\|_1 \le C_3\left\|\bar\gamma-\gamma\right\|_3 .$$

Thus,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left\|(\bar\gamma-\gamma)'\right\|^2 &\le -\left\|(\bar\gamma-\gamma)^{(3)}\right\|^2 + C_4\left\|\bar\gamma-\gamma\right\|_2\cdot\left\|\bar\gamma-\gamma\right\|_3 \\
&\le -\frac{1}{2}\left\|(\bar\gamma-\gamma)^{(3)}\right\|^2 + C_5\left\|(\bar\gamma-\gamma)'\right\|^2 \le C_6\left\|(\bar\gamma-\gamma)'\right\|^2 .
\end{aligned}
$$

Since $(\bar\gamma-\gamma)'=0$ at $t=0$, it remains so for all $t<T$. □

Proposition 6.4. — *Let γ be a $C^{4+4\mu}$ solution of the equation* (EP$_*$) *with $C_x^{4+4\mu}$ non-constant initial data γ_0. Then γ is C^∞ on $t>0$. If γ_0 is C^∞, then γ is C^∞ for $t\ge 0$.*

Proof. Let $\hat\gamma$ be a solution of the equation

$$
\begin{cases}
\partial_t\hat\gamma + \hat\gamma^{(4)} = 0 , \\
\hat\gamma(x,0) = \gamma_0(x) .
\end{cases}
$$

Put $\tilde\gamma = \gamma - \hat\gamma$. Then $\tilde\gamma$ satisfies the following equation.

$$
\begin{cases}
\partial_t\tilde\gamma + \tilde\gamma^{(4)} = c := -\gamma^{(4)} + \lambda((v-2|\gamma''|^2)\gamma')' , \\
\tilde\gamma(x,0) = 0 .
\end{cases}
$$

Suppose that $\gamma \in C^{n+4+4\mu}$ on $t > 0$, where n is a non-negative integer. Then by Lemma 4.3, we have v and $v' \in C^{n+1+4\mu}$ on $t > 0$, and $c \in C^{n+1+4\mu}$. Therefore, Proposition 5.2 implies that $\tilde{\gamma}, \gamma \in C^{n+5+4\mu}$.

Thus, by induction, we know that γ is C^∞. If γ_0 is already C^∞, then the above estimation can be done on $t \geq 0$. □

Theorem 6.5. — *For any* C^∞ *non-constant initial data* γ_0*, there is a positive time* T *such that the equation*

$$EP_{\lambda=1} \qquad \begin{cases} \partial_t \gamma = -\gamma^{(4)} + ((v - 2|\gamma''|^2)\gamma')' \ , \\ -v'' + \|\gamma'\|^{-2}|\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \end{cases}$$

has a unique C^∞ *solution* $\gamma = \gamma(x,t)$ *on* $S^1 \times [0,T)$.

Proof. Let T be as in Proposition 6.2. Suppose that the equation (EP_*) has a $C^{4+4\mu}$ solution on $[0,T)$ for $\lambda = \lambda_0$. Note that γ is C^∞ by Proposition 6.4. Put

$$\Phi(\gamma,\lambda) := \partial_t \gamma + \gamma^{(4)} - \lambda((v - 2|\gamma''|^2)\gamma')' \ ,$$

where v is defined by

$$-v'' + \|\gamma'\|^{-2}|\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \ .$$

The map $\Phi\, C^{4+4\mu} \times \mathbb{R} \to C^{4\mu}$ is a C^∞ map, and its derivative $(\delta_\gamma \Phi)$ in the γ direction is given by

$$(\delta_\gamma \Phi)(\eta) = \partial_t \eta + \eta^{(4)} - \lambda((w - 4(\gamma'',\eta''))\gamma' + (v - 2|\gamma''|^2)\eta')' \ ,$$

where w is given by

$$\begin{aligned} -w'' + \|\gamma'\|^{-2}|\gamma''|^2 w = 8|\gamma''|^2(\gamma'',\eta'') - 2(\gamma^{(3)},\eta^{(3)}) \\ - (-\|\gamma'\|^{-4} 2\langle \gamma',\eta'\rangle |\gamma''|^2 + 2\|\gamma'\|^{-2}(\gamma'',\eta''))v \ . \end{aligned}$$

Since $\big\|\|\gamma'\|^{-1}|\gamma''|\big\| \geq 1$, Proposition 5.6 implies that the map $\eta \to (\delta_\gamma \Phi)(\eta)$ is an isomorphism from $C_0^{4+4\mu}$ to $C_0^{4\mu}$ (for definition, see Lemma 5.5). Therefore, we can apply the implicit function theorem to the map Φ, and conclude that there exists a solution γ for any λ sufficiently close to λ_0. Thus, the set of all λ for which we have a solution is an open set of the interval $[0,1]$.

Note that we have a solution for $\lambda = 0$. Let λ_0 be the supremum of λ's such that we have solutions γ. Then, Proposition 6.2 implies that the solutions are bounded in the C^∞ topology. Therefore, it has a convergent subsequence for $\lambda \to \lambda_0$. The limit γ becomes a solution for $\lambda = \lambda_0$. Thus, we conclude that $\lambda_0 = 1$. This solution is unique by Proposition 6.3. □

Proposition 6.6. — *In Theorem 6.5, if the initial data satisfies $|\gamma_0'|^2 \equiv 1$, then the solution γ satisfies $|\gamma'|^2 \equiv 1$ for all defined t.*

Proof. We can check that

$$\partial_t |\gamma'|^2 = -(|\gamma'|^2)^{(4)} + \sum_{i=0}^{2} P_i \cdot (|\gamma'|^2 - 1)^{(i)} + Q \cdot (\|\gamma'\| - 1) ,$$

where P_i and Q are expressed by γ and v, which are bounded from above. Therefore,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\big\||\gamma'|^2 - 1\big\|^2 &= \big\langle |\gamma'|^2 - 1, \partial_t |\gamma'|^2 \big\rangle \\
&= -\big\|(|\gamma'|^2 - 1)''\big\|^2 + \sum_{i=0}^{2} \big\langle P_i \cdot (|\gamma'|^2 - 1), (|\gamma'|^2 - 1)^{(i)} \big\rangle \\
&\qquad + (\|\gamma'\| - 1) \cdot \big\langle Q, |\gamma'|^2 - 1 \big\rangle \\
&\le -\big\|(|\gamma'|^2 - 1)''\big\|^2 + C_1 \sum_{i=0}^{2} \big\||\gamma'|^2 - 1\big\| \cdot \big\|(|\gamma'|^2 - 1)^{(i)}\big\| \\
&\qquad + C_2 \big\||\gamma'|^2 - 1\big\| \cdot \big|\|\gamma'\| - 1\big| .
\end{aligned}$$

Here,

$$\begin{aligned}
\big|\|\gamma'\|^2 - 1\big| &= \big\|\|\gamma'\|^2 - 1\big\| \\
&\le \big\|\|\gamma'\|^2 - |\gamma'|^2\big\| + \big\||\gamma'|^2 - 1\big\| \\
&\le 2\big\||\gamma'|^2 - 1\big\| .
\end{aligned}$$

Hence,

$$\big|\|\gamma'\| - 1\big| \le 2\big|\|\gamma'\| + 1\big|^{-1} \cdot \big\||\gamma'|^2 - 1\big\| .$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\left\||\gamma'|^2-1\right\|^2 \le -\left\|(|\gamma'|^2-1)''\right\|^2 + C_3\left\||\gamma'|^2-1\right\|\cdot\left\|(|\gamma'|^2-1)''\right\|$$
$$\le -\frac{1}{2}\left\|(|\gamma'|^2-1)''\right\|^2 + \frac{1}{2}(C_3\left\||\gamma'|^2-1\right\|)^2 \ .$$

□

7. LONG TIME EXISTENCE

In this section, we consider the original equation

$$\text{EP}\qquad \begin{cases} \partial_t\gamma = -\gamma^{(4)} + ((v-2|\gamma''|^2)\gamma')' \ , \\ -v'' + |\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \ . \end{cases}$$

We give a C^∞ initial data γ_0 satisfying $|\gamma_0'| = 1$. By Proposition 6.6, the solution of ($\text{EP}_{\lambda=1}$) with initial data γ_0 satisfies $|\gamma'| = 1$, and is the solution of (EP).

Let γ be a solution of (EP) on $[0, T)$ for some positive time T. We will show that γ is uniformly bounded in the C^∞ topology. The next Lemma is obvious.

Lemma 7.1. — *The center of gravity of the curve γ is preserved.*

Therefore, we may assume that $\oint \gamma(x)\,dx = 0$ for all time. Because our problem comes from a variational problem, we observe

Lemma 7.2. — *The quantity $\|\gamma''\|$ is non-increasing. In particular, it is bounded from above and away from 0.*

Proof. We have

$$\frac{1}{2}\frac{d}{dt}\left\|\gamma''\right\|^2 = \langle\gamma^{(4)}, \partial_t\gamma\rangle$$
$$= \langle\gamma^{(4)}, -\gamma^{(4)} + ((v-2|\gamma''|^2)\gamma')'\rangle$$
$$= -\left\|\gamma^{(4)} - ((v-2|\gamma''|^2)\gamma')'\right\|^2 \le 0 \ .$$

Boundedness from below comes from the fact that $\|\gamma''\| \geq \|\gamma'\| = 1$. □

Combining it with Lemma 4.1, we get

Lemma 7.3. — *We have*

$$\sup|v|,\ \sup|v'| \leq C \cdot (\|\gamma^{(3)}\|^2 + 1) \ ,$$

where the positive constant C is independent of T.

Lemma 7.4. — *There are positive constants C_1 and C_2 independent of T, such that the following holds*

$$\frac{d}{dt}\|\gamma''\|^2 \leq -C_1\|\gamma^{(4)}\|^2 + C_2 \ .$$

Proof. Put $\beta = \gamma^{(3)} - (v - 2|\gamma''|^2)\gamma'$ and $\alpha = \beta'$. Note that

$$\frac{1}{2}\frac{d}{dt}\|\gamma''\|^2 = -\|\alpha\|^2 \ .$$

We calculate the tangential factor and normal component of β and α to γ'. Because

$$(\gamma^{(3)}, \gamma') = (\gamma'', \gamma')' - |\gamma''|^2 = -|\gamma''|^2 \ ,$$

$$(\beta, \gamma') = (\gamma^{(3)}, \gamma') - v + 2|\gamma''|^2 = -v + |\gamma''|^2 \ .$$

Hence,

$$\beta^N = \beta - (\beta, \gamma')\gamma' = \gamma^{(3)} - (v - 2|\gamma''|^2)\gamma' + v\gamma' - |\gamma''|^2\gamma' = \gamma^{(3)} + |\gamma''|^2\gamma' \ .$$

Because

$$(\gamma^{(4)}, \gamma') = -\frac{3}{2}(|\gamma''|^2)' \ ,$$

$$(\alpha, \gamma') = (\gamma^{(4)}, \gamma') - (v - 2|\gamma''|^2)' = -v' + \frac{1}{2}(|\gamma''|^2)' \ .$$

Hence,

$$\begin{aligned}
\alpha^N &= \alpha - (\alpha, \gamma')\gamma' \\
&= \gamma^{(4)} - (v - 2|\gamma''|^2)'\gamma' - (v - 2|\gamma''|^2)\gamma'' + v'\gamma' - \frac{1}{2}(|\gamma''|^2)'\gamma' \\
&= \gamma^{(4)} - (v - 2|\gamma''|^2)\gamma'' + \frac{3}{2}(|\gamma''|^2)'\gamma' \ .
\end{aligned}$$

Now, we have

$$\begin{aligned}
|\oint \beta\, dx| &= |\oint (v - 2|\gamma''|^2)\gamma'\, dx| \\
&= |\oint (v - 2|\gamma''|^2)'\gamma\, dx| \\
&\le |\oint (v - \frac{1}{2}|\gamma''|^2)'\gamma\, dx| + \frac{3}{2}|\oint (|\gamma''|^2)'\gamma\, dx| \\
&= |\oint (\alpha, \gamma')\gamma\, dx| + \frac{3}{2}|\oint (|\gamma''|^2)'\gamma\, dx| \\
&\le \sup|\gamma| \cdot \|\alpha\| + \frac{3}{2}\|\gamma''\|^2 \\
&\le C_3 \cdot (\|\alpha\| + 1) \qquad \text{(using Lemma 7.1)} .
\end{aligned}$$

Therefore,

$$\sup|\beta| \le |\oint \beta\, dx| + \|\beta'\| \le C_4 \cdot (\|\alpha\| + 1) ,$$

and,

$$\sup|v - |\gamma''|^2|,\ \sup|\gamma^{(3)} + |\gamma''|^2\gamma'| \le C_4 \cdot (\|\alpha\| + 1) .$$

It implies that

$$\begin{aligned}
\|\alpha\| &= \|\gamma^{(4)} - (v - 2|\gamma''|^2)\gamma'' + \frac{3}{2}(|\gamma''|^2)'\gamma'\| \\
&\ge \|\gamma^{(4)}\| - \|(v - 2|\gamma''|^2)\gamma''\| - \frac{3}{2}\|(|\gamma''|^2)'\| .
\end{aligned}$$

Here,

$$\begin{aligned}
\|(v - 2|\gamma''|^2)\gamma''\| &\le C_5 \sup|v - 2|\gamma''|^2| \\
&\le C_5 \sup|v - |\gamma''|^2| + C_5 \sup|\gamma''|^2 \\
&\le C_6 \cdot (\|\alpha\| + 1) + C_7 \cdot (1 + \|\gamma^{(3)}\|) \\
&\le C_8 \cdot (\|\alpha\| + \|\gamma^{(4)}\|^{1/2} + 1) ,
\end{aligned}$$

and

$$\begin{aligned}
\|(|\gamma''|^2)'\| &= 2\|(\gamma'', \gamma^{(3)})\| \le C_9 \sup|\gamma^{(3)}| \\
&\le C_{10}\|\gamma^{(3)}\|^{1/2}\|\gamma^{(4)}\|^{1/2} \le C_{11}\|\gamma^{(4)}\|^{3/4} .
\end{aligned}$$

Therefore,

$$(1 + C_8) \cdot \|\alpha\| \ge \|\gamma^{(4)}\| - C_{12}\|\gamma^{(4)}\|^{3/4} - C_{13} \ge C_{14}\|\gamma^{(4)}\| - C_{15} .$$

□

Theorem 7.5. — *The equation* (EP) *has a unique solution on the whole time interval* $[0, \infty)$.

Proof. Using Proposition 6.2, it suffices to prove that $\|\gamma^{(3)}\|$ is bounded for any finite time. To show it, we use

$$\begin{aligned}\frac{d}{dt}\|\gamma^{(3)}\|^2 &= 2\langle \gamma^{(5)}, -\gamma^{(5)} + ((v - 2|\gamma''|^2)\gamma')''\rangle \\ &\leq -\|\gamma^{(5)}\|^2 + \|((v-2|\gamma''|^2)\gamma')''\|^2 .\end{aligned}$$

We will estimate the last line. Note that

$$\begin{aligned}&\sup|\gamma''| \leq C_1\|\gamma^{(3)}\|^{1/2} \leq C_2\|\gamma''\|^{1/3}\|\gamma^{(5)}\|^{1/6} \leq C_3\|\gamma^{(5)}\|^{1/6}, \\ &\|\gamma^{(3)}\| \leq C_4\|\gamma^{(5)}\|^{1/3}, \\ &\|\gamma^{(4)}\| \leq C_5\|\gamma^{(5)}\|^{2/3}, \\ &\sup|\gamma^{(3)}| \leq C_6\|\gamma^{(3)}\|^{1/2}\|\gamma^{(4)}\|^{1/2} \leq C_7\|\gamma^{(5)}\|^{1/2} .\end{aligned}$$

Therefore,

$$\begin{aligned}\|(|\gamma''|^2)''\gamma'\| &\leq C_8\|(|\gamma''|^2)''\| = C_8\|(\gamma^{(4)}, \gamma'') + |\gamma^{(3)}|^2\| \\ &\leq C_9 \cdot (\sup|\gamma''| \cdot \|\gamma^{(4)}\| + \sup|\gamma^{(3)}| \cdot \|\gamma^{(3)}\|) \\ &\leq C_{10} \cdot (\|\gamma^{(5)}\|^{1/6+2/3} + \|\gamma^{(5)}\|^{1/2+1/3}) \\ &\leq C_{11}\|\gamma^{(5)}\|^{5/6},\end{aligned}$$

$$\begin{aligned}\|(|\gamma''|^2)'\gamma''\| &\leq C_{12}\|(\gamma'', \gamma^{(3)})\gamma''\| \leq C_{13}\sup|\gamma''|^2 \cdot \|\gamma^{(3)}\| \\ &\leq C_{14}\|\gamma^{(5)}\|^{1/3+1/3} = C_{15}\|\gamma^{(5)}\|^{2/3},\end{aligned}$$

$$\||\gamma''|^2\gamma^{(3)}\| \leq C_{16}\sup|\gamma''|^2 \cdot \|\gamma^{(3)}\| \leq C_{17}\|\gamma^{(5)}\|^{2/3},$$

$$\begin{aligned}\|v''\gamma'\| &= \||\gamma''|^2 v\gamma' + |\gamma^{(3)}|^2\gamma' - 2|\gamma''|^4\gamma'\| \\ &\leq C_{18} \cdot (\sup|v| \cdot \sup|\gamma''| + \sup|\gamma^{(3)}| \cdot \|\gamma^{(3)}\| + \sup|\gamma''|^3) \\ &\leq C_{19} \cdot (\|\gamma^{(5)}\|^{2/3+1/6} + \|\gamma^{(5)}\|^{1/2+1/3} + \|\gamma^{(5)}\|^{1/2}) \\ &\qquad\qquad \text{(by Lemma 7.3)} \\ &\leq C_{20} \cdot (1 + \|\gamma^{(5)}\|^{5/6}),\end{aligned}$$

$$\|v'\gamma''\| \le C_{21} \sup |v'| \le C_{22} \cdot (1 + \|\gamma^{(5)}\|^{2/3}) \quad \text{(by Lemma 7.3)}$$

$$\begin{aligned} \|v\gamma^{(3)}\| &\le \sup |v| \cdot \|\gamma^{(3)}\| \\ &\le C_{23} \cdot (1 + \|\gamma^{(3)}\|^2)\|\gamma^{(3)}\| \le C_{24}\|\gamma^{(3)}\|(1 + \|\gamma^{(4)}\|). \end{aligned}$$

Combining these, we have

$$\begin{aligned} \frac{d}{dt}\|\gamma^{(3)}\|^2 &\le -\|\gamma^{(5)}\|^2 + C_{25} \cdot (1 + \|\gamma^{(5)}\|^{11/6} + \|\gamma^{(4)}\|^2\|\gamma^{(3)}\|^2) \\ &\le C_{26} \cdot (1 + \|\gamma^{(4)}\|^2\|\gamma^{(3)}\|^2) \ . \end{aligned}$$

It implies that

$$\frac{d}{dt} \log\|\gamma^{(3)}\|^2 \le C_{27} \cdot (1 + \|\gamma^{(4)}\|^2) \ .$$

Combining it with Lemma 7.4

$$\frac{d}{dt}\|\gamma''\|^2 \le -C_{28}\|\gamma^{(4)}\|^2 + C_{29} \ ,$$

we get

$$\frac{d}{dt}\left(\|\gamma''\|^2 + C_{30} \log\|\gamma^{(3)}\|^2\right) \le C_{31} \ .$$

That is, $\|\gamma^{(3)}\|$ is bounded on any finite time interval. □

8. CONVERGENCE

In this section, we continue to consider the original equation

$$\text{EP} \qquad \begin{cases} \partial_t\gamma = -\gamma^{(4)} + ((v - 2|\gamma''|^2)\gamma')' \ , \\ -v'' + |\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \ , \end{cases}$$

with C^∞ initial data γ_0. By Theorem 7.5, we have a unique solution γ for all time. To prove the convergence of the solution γ, we need some preliminaries. As in the proof of Lemma 7.4, put

$$\alpha = \gamma^{(4)} - ((v - 2|\gamma''|^2)\gamma')' .$$

Also, put

$$w = v - 2|\gamma''|^2 .$$

Lemma 8.1. — *There are positive constants C and N independent of the time t such that*

$$\|w\|_2 \le C \cdot (1 + \|\alpha\|^N) .$$

Proof. In the proof of Lemma 7.4, we have shown that $\|\gamma\|_4 \le C_1 \cdot (1 + \|\alpha\|)$. Hence, by Lemma 4.2, $\|v\|_3 \le C_2 \cdot (1 + \|\alpha\|^N)$. Thus, we get the result. □

Lemma 8.2. — *For each non-negative integer n, there are positive constants C and N independent of the time t such that*

$$\|\gamma\|_{n+4},\ \|w\|_{n+2} \le C \cdot (1 + \|\alpha\|_n^N) .$$

Proof. By Lemma 8.1, the result holds for $n = 0$.

Suppose that the result holds for up to n. Then,

$$\begin{aligned} \|\gamma^{(n+5)}\| &\le \|\gamma^{(n+5)} - (w\gamma')^{(n+2)}\| + \|(w\gamma')^{(n+2)}\| \\ &\le \|\alpha^{(n+1)}\| + C_1 \|\gamma\|_{C^{n+3}} \cdot \|w\|_{n+2} \\ &\le \|\alpha^{(n+1)}\| + C_2 \|\gamma\|_{n+4} \cdot \|w\|_{n+2} \\ &\le \|\alpha^{(n+1)}\| + C_3 \cdot (1 + \|\alpha^{(n)}\|^{N_1}) \\ &\le C_4 \cdot (1 + \|\alpha\|_{n+1}^{N_1}) . \end{aligned}$$

Moreover, by Lemma 4.2,

$$\|v\|_{n+3} \le C_5 \cdot (1 + \|\gamma\|_{n+3}^{N_2}) \cdot (1 + \|\gamma\|_{n+4}^4) .$$

Hence,

$$\|w\|_{n+3} \leq C_6 \cdot (1 + \|\alpha\|_{n+1}^{N_3}) \ .$$

□

Lemma 8.3. — *For each non-negative integer n, there are positive constants C and N independent of the time t such that*

$$\|\partial_t w\|_{n+1} \leq C \cdot (1 + \|\alpha\|_n^N) \cdot \|\alpha\|_{n+3} \ .$$

Proof. Applying Lemma 4.2 to the equality

$$-\partial_t v'' + |\gamma''|^2 \partial_t v = 2(\gamma'', \alpha'')v - 8|\gamma''|^2(\gamma'', \alpha'') - 2(\gamma^{(3)}, \alpha^{(3)}) \ ,$$

we have

$$\begin{aligned}\|\partial_t v\|_{n+2} &\leq C_1 \cdot (1 + \|\gamma\|_{C^{n+3}}^{N_1}) \cdot \|\alpha\|_{n+3} \\ &\leq C_2 \cdot (1 + \|\gamma\|_{n+4}^{N_1}) \cdot \|\alpha\|_{n+3} \ .\end{aligned}$$

Therefore, from Lemma 8.2, we see

$$\begin{aligned}\|\partial_t w\|_{n+1} &= \|\partial_t v + 4(\gamma'', \alpha'')\|_{n+1} \\ &\leq C_3 \cdot (1 + \|\alpha\|_n^{N_2}) \cdot \|\alpha\|_{n+3} + C_4 \cdot (1 + \|\alpha\|_n^{N_3}) \cdot \|\alpha\|_{n+3} \ .\end{aligned}$$

□

Proposition 8.4. — *For each integer $n \geq 0$, we have*

$$\int_0^\infty \|\alpha^{(n)}\|^2 dt < \infty, \qquad \text{and} \qquad \|\alpha^{(n)}\| \to 0 \quad \text{when} \quad t \to \infty \ .$$

Proof. We know that

$$\int_0^\infty \|\alpha\|^2 dt = \frac{1}{2}[\|\gamma''\|^2]_\infty^0 < \infty \ .$$

Moreover,

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\alpha\|^2 &= \langle \alpha, \partial_t \alpha\rangle \\
&= \langle \alpha, \partial_t(\gamma^{(4)} - (w\gamma')')\rangle \\
&= -\langle \alpha', \partial_t\gamma^{(3)} - \partial_t w \cdot \gamma' - w\partial_t\gamma'\rangle \\
&= -\langle \alpha', -\alpha^{(3)} + w\alpha'\rangle \quad \text{since} \quad (\alpha', \gamma') = 0 \\
&= -\|\alpha''\|^2 - \langle w, |\alpha'|^2\rangle \\
&\le -\|\alpha''\|^2 + C_1\cdot(1+\|\alpha\|^N)\|\alpha'\|^2 \quad \text{from Lemma 8.1} \\
&\le -\|\alpha''\|^2 + C_1\cdot(1+\|\alpha\|^N)\|\alpha\|\cdot\|\alpha''\| \\
&\le -\frac{1}{2}\|\alpha''\|^2 + \frac{1}{2}\big(C_1(1+\|\alpha\|^N)\|\alpha\|\big)^2 .
\end{aligned}
$$

Therefore, we see that $\|\alpha\| \to 0$. In particular, $\|\alpha\|$ is bounded. Hence,

$$\frac{1}{2}\frac{d}{dt}\|\alpha\|^2 \le -\frac{1}{2}\|\alpha''\|^2 + C_2\|\alpha\|^2 .$$

Integrating it, we see

$$\int_0^\infty \|\alpha''\|^2 < \infty .$$

Suppose that

$$\int_0^\infty \|\alpha^{(k)}\|^2 < \infty, \qquad \text{and} \qquad \|\alpha^{(k-2)}\| \to 0 \quad \text{when} \quad t\to\infty$$

for $k \le 2m$, where $m \ge 1$. Then,

$$
\begin{aligned}
\frac{d}{dt}\|\alpha^{(2m)}\|^2 &= 2\langle \alpha^{(2m)}, \partial_t\alpha^{(2m)}\rangle = 2\langle \alpha^{(2m)}, \partial_t\gamma^{(2m+4)} - \partial_t(w\gamma')^{(2m+1)}\rangle \\
&= -2\|\alpha^{(2m+2)}\|^2 - 2\langle \alpha^{(2m)}, \partial_t(w\gamma')^{(2m+1)}\rangle \\
&\le -\|\alpha^{(2m+2)}\|^2 + \|\partial_t(w\gamma')^{(2m-1)}\|^2 .
\end{aligned}
$$

Here,

$$\begin{aligned}
\big\|\partial_t(w\gamma')^{(2m-1)}\big\|^2 &= \big\|(\partial_t w\cdot\gamma')^{(2m-1)} + (w\cdot\partial_t\gamma')^{(2m-1)}\big\|^2 \\
&\le 2\big\|(\partial_t w\cdot\gamma')^{(2m-1)}\big\|^2 + 2\big\|(w\cdot\alpha')^{(2m-1)}\big\|^2 \\
&\le C_4\big\|\gamma\big\|^2_{C^{2m}}\cdot\big\|\partial_t w\big\|^2_{2m-1} + C_5\big\|w\big\|^2_{C^{2m-1}}\cdot\big\|\alpha\big\|^2_{2m} \\
&\le C_6\big\|\partial_t w\big\|^2_{2m-1} + C_7\big\|\alpha\big\|^2_{2m} \quad \text{(by Lemma 8.2)} \\
&\le C_8\big\|\alpha\big\|^2_{2m+1} + C_9\big\|\alpha\big\|^2_{2m} \quad \text{(by Lemma 8.3)} \\
&\le \frac{1}{2}\big\|\alpha^{(2m+2)}\big\|^2 + C_{10}\big\|\alpha\big\|^2_{2m} \\
&\le \frac{1}{2}\big\|\alpha^{(2m+2)}\big\|^2 + C_{10}\big\|\alpha^{(2m)}\big\|^2 + C_{11}\ .
\end{aligned}$$

Therefore,

$$\frac{d}{dt}\big\|\alpha^{(2m)}\big\|^2 \le -\frac{1}{2}\big\|\alpha^{(2m+2)}\big\|^2 + C_{10}\big\|\alpha^{(2m)}\big\|^2 + C_{11}\ ,$$

and we see that $\big\|\alpha^{(2m)}\big\| \to 0$ and $\int_0^\infty \big\|\alpha^{(2m+2)}\big\|^2 dt < \infty$. □

We consider the limiting equation of (EP)

$$\text{EE} \qquad \begin{cases} -\hat\gamma^{(4)} + ((\hat v - 2|\hat\gamma''|^2)\hat\gamma')' = 0\ , \\ -\hat v'' + |\hat\gamma''|^2\hat v = 2|\hat\gamma''|^4 - |\hat\gamma^{(3)}|^2, \quad |\hat\gamma'| = 1\ . \end{cases}$$

Proposition 8.5. — *Let $\hat\gamma$ be a solution of* (EE). *Then, there are a constant $\theta \in (0,1/2)$ and a $C_x^{4+4\mu}$ neighbourhood W of $\hat\gamma$ such that*

$$\big\|\alpha\big\| \ge |E(\gamma) - E(\hat\gamma)|^{1-\theta}$$

for any $\gamma \in W$.

This is a direct modification of [2, Theorem 3]. The proof essentially uses real analyticity of the space, the Euclidean space in our case. We omit the proof.

Theorem 8.6. — *The solution of the equation* (EP) *converges to a closed elastic curve in the* C^∞ *topology.*

Proof. From Proposition 8.4 and Lemma 8.2, we know that the solution is bounded in the C^∞ topology. Therefore, there exists a convergent subsequence. For the limit curve $\hat{\gamma}$, we apply Proposition 8.5. Suppose that α is sufficiently small in the C^∞ norm and that γ is sufficiently close to $\hat{\gamma}$ in the L^2 norm. Since γ is bounded in the C^∞ norm, closeness of γ to $\hat{\gamma}$ implies closeness of $\gamma^{(n)}(0)$ to $\hat{\gamma}^{(n)}(0)$ for each n. Moreover, γ satisfies the ODE

$$\begin{cases} -\gamma^{(4)} + ((v - 2|\gamma''|^2)\gamma')' = \alpha \ , \\ -v'' + |\gamma''|^2 v = 2|\gamma''|^4 - |\gamma^{(3)}|^2 \ . \end{cases}$$

Therefore, we see that γ is close to $\hat{\gamma}$ in the C^∞ topology.

In other words, we can restate Proposition 8.5 as follows. There are a positive time T and a positive constant r such that the inequality in Proposition 8.5 holds for any $\gamma = \gamma(x, t_0)$ which satisfies $\|\gamma - \hat{\gamma}\| \leq r$ and $t_0 \geq T$.

Take two L^2 neighbourhoods of $\hat{\gamma}$, W_r with radius r and $W_{r/2}$ with radius $r/2$. Take two positive times $T \leq t_1 < t_2$ so that $\gamma(x, t_1) \in W_{r/2}$ and $\gamma(x, t_0) \in W_r$ for all $t_0 \in [t_1, t_2)$. Then, from the proof of Lemma 7.2,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}E(\gamma) &= -\|\alpha\|^2 = -\|\partial_t\gamma\| \cdot \|\alpha\| \\ &\leq -\|\partial_t\gamma\| \cdot |E(\gamma) - E(\hat{\gamma})|^{1-\theta} \ . \end{aligned}$$

It implies that

$$\frac{1}{2\theta}\frac{d}{dt}(E(\gamma) - E(\hat{\gamma}))^\theta \leq -\|\partial_t\gamma\| \ ,$$

and,

$$\int_{t_1}^{t_2} \|\partial_t\gamma\| dt \leq [(E(\gamma) - E(\hat{\gamma}))^\theta]_{t_2}^{t_1} \ .$$

If $\gamma_{t_2} \notin W_r$, we know that $\int_{t_1}^{t_2} \|\partial_t\gamma\| dt \geq r/2$. But, $[(E(\gamma) - E(\hat{\gamma}))^\theta]_\infty^0$ is bounded. Hence, there is a positive time T_1 such that $\gamma(x, t_0) \in W_r$ for all $t_0 \geq T_1$. Since we can take r arbitrary small, γ converges into $\hat{\gamma}$ in the L^2 norm, and hence in the C^∞ norm.

□

Remark. — We proved the convergence using the real analyticity. In fact, we cannot hope to extend our result to general C^∞ Riemannian manifolds. See [7].

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