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Pseudo-Bosons from Landau Levels*

Fabio BAGARELLO

Dipartimento di Metodi e Modelli Matematici, Facoltà di Ingegneria,
Università di Palermo, I-90128 Palermo, Italy
E-mail: *bagarell@unipa.it*
URL: http://www.unipa.it/~bagarell/

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Abstract. We construct examples of pseudo-bosons in two dimensions arising from the Hamiltonian for the Landau levels. We also prove a no-go result showing that non-linear combinations of bosonic creation and annihilation operators cannot give rise to pseudo-bosons.

Key words: non-hermitian Hamiltonians; pseudo-bosons

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1 Introduction

In a series of recent papers [1, 2, 3, 4, 5, 6], we have investigated some mathematical aspects of the so-called *pseudo-bosons*, originally introduced by Trifonov[1] in [8]. They arise from the canonical commutation relation $[a, a^\dagger] = 1\!\!1$ upon replacing $a^\dagger$ by another (unbounded) operator b not (in general) related to a: $[a, b] = 1\!\!1$. We have shown that, under suitable assumptions, $N = ba$ and $N^\dagger = a^\dagger b^\dagger$ can be both diagonalized, and that their spectra coincide with the set of natural numbers (including 0), $\mathbb{N}_0$. However the sets of related eigenvectors are not orthonormal (o.n.) bases but, nevertheless, they are automatically biorthonormal. In most of the examples considered so far, they are bases of the Hilbert space of the system, $\mathcal{H}$, and, in some cases, they turn out to be *Riesz bases*.

In [9] and [10] some physical examples arising from quantum mechanics have been discussed. In particular, these examples have suggested the introduction of a difference between what we have called *regular pseudo-bosons* and *pseudo-bosons*, to better focus on what we believe are the mathematical or on the physical aspects of these *particles*. Indeed all the examples of regular pseudo-bosons considered so far arise from Riesz bases [4], with a rather mathematical construction, while pseudo-bosons are those which one can find when starting with the Hamiltonian of some realistic quantum system.

In this paper, after a short review of the general framework, we discuss a two-dimensional example arising from the Hamiltonian of the Landau levels. It should be stressed that this example is of a completely different kind than those considered in [10], where a modified version of the Landau levels have been considered.

We close the paper with a no-go result, suggesting that non-linear combinations of ordinary bosonic creation and annihilation operators, even if they produce pseudo-bosonic commutation rules, cannot satisfy the Assumptions of our construction, see Section 2.

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[1]It should be mentioned that pseudo-bosons already appeared in [7] but with a different meaning.

2 The commutation rules

In this section we will review a d-dimensional version of what originally proposed in [1, 6].

Let $\mathcal{H}$ be a given Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and related norm $\|\cdot\|$. We introduce d pairs of operators, a_j and b_j, $j=1,2,\ldots,d$, acting on $\mathcal{H}$ and satisfying the following commutation rules

$$[a_j,b_k]=\delta_{j,k}1\!\!1, \tag{1}$$

$j,k=1,2,\ldots,d$. Of course, these collapse to the CCR's for d independent modes if $b_j=a_j^\dagger$, $j=1,2,\ldots,d$. It is well known that a_j and b_j are unbounded operators, so they cannot be defined on all of $\mathcal{H}$. Following [1], and writing $D^\infty(X):=\cap_{p\geq 0}D(X^p)$ (the common domain of all the powers of the operator X), we consider the following:

Assumption 1. *There exists a non-zero $\varphi_{\mathbf{0}}\in\mathcal{H}$ such that $a_j\varphi_{\mathbf{0}}=0$, $j=1,2,\ldots,d$, and $\varphi_{\mathbf{0}}\in D^\infty(b_1)\cap D^\infty(b_2)\cap\cdots\cap D^\infty(b_d)$.*

Assumption 2. *There exists a non-zero $\Psi_{\mathbf{0}}\in\mathcal{H}$ such that $b_j^\dagger\Psi_{\mathbf{0}}=0$, $j=1,2,\ldots,d$, and $\Psi_{\mathbf{0}}\in D^\infty(a_1^\dagger)\cap D^\infty(a_2^\dagger)\cap\cdots\cap D^\infty(a_d^\dagger)$.*

Under these assumptions we can introduce the following vectors in $\mathcal{H}$:

$$\varphi_{\mathbf{n}}:=\varphi_{n_1,n_2,\ldots,n_d}=\frac{1}{\sqrt{n_1!n_2!\cdots n_d!}}b_1^{n_1}b_2^{n_2}\cdots b_d^{n_d}\varphi_{\mathbf{0}},$$
$$\Psi_{\mathbf{n}}:=\Psi_{n_1,n_2,\ldots,n_d}=\frac{1}{\sqrt{n_1!n_2!\cdots n_d!}}a_1^{\dagger\,n_1}a_2^{\dagger\,n_2}\cdots a_d^{\dagger\,n_d}\,\Psi_{\mathbf{0}},$$

$n_j=0,1,2,\ldots$ for all $j=1,2,\ldots,d$. Let us now define the unbounded operators $N_j:=b_ja_j$ and $\mathfrak{N}_j:=N_j^\dagger=a_j^\dagger b_j^\dagger$, $j=1,2,\ldots,d$. Each $\varphi_{\mathbf{n}}$ belongs to the domain of N_j, $D(N_j)$, and $\Psi_{\mathbf{n}}\in D(\mathfrak{N}_j)$, for all possible $\mathbf{n}$. Moreover,

$$N_j\varphi_{\mathbf{n}}=n_j\varphi_{\mathbf{n}},\qquad \mathfrak{N}_j\Psi_{\mathbf{n}}=n_j\Psi_{\mathbf{n}}.$$

Under the above assumptions, and if we chose the normalization of $\Psi_{\mathbf{0}}$ and $\varphi_{\mathbf{0}}$ in such a way that $\langle\Psi_{\mathbf{0}},\varphi_{\mathbf{0}}\rangle=1$, we find that

$$\langle\Psi_{\mathbf{n}},\varphi_{\mathbf{m}}\rangle=\delta_{\mathbf{n},\mathbf{m}}=\prod_{j=1}^d\delta_{n_j,m_j}.$$

This means that the sets $\mathcal{F}_\Psi=\{\Psi_{\mathbf{n}}\}$ and $\mathcal{F}_\varphi=\{\varphi_{\mathbf{n}}\}$ are *biorthonormal* and, because of this, the vectors of each set are linearly independent. If we now call $\mathcal{D}_\varphi$ and $\mathcal{D}_\Psi$ respectively the linear span of $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$, and $\mathcal{H}_\varphi$ and $\mathcal{H}_\Psi$ their closures, then

$$f=\sum_{\mathbf{n}}\langle\Psi_{\mathbf{n}},f\rangle\,\varphi_{\mathbf{n}},\qquad\forall\,f\in\mathcal{H}_\varphi,\qquad h=\sum_{\mathbf{n}}\langle\varphi_{\mathbf{n}},h\rangle\,\Psi_{\mathbf{n}},\quad\forall\,h\in\mathcal{H}_\Psi.$$

What is not in general ensured is that $\mathcal{H}_\varphi=\mathcal{H}_\Psi=\mathcal{H}$. Indeed, we can only state that $\mathcal{H}_\varphi\subseteq\mathcal{H}$ and $\mathcal{H}_\Psi\subseteq\mathcal{H}$. However, motivated by the examples discussed so far in the literature, we consider

Assumption 3. *The above Hilbert spaces all coincide: $\mathcal{H}_\varphi=\mathcal{H}_\Psi=\mathcal{H}$.*

This means, in particular, that both $\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are bases of $\mathcal{H}$. The resolution of the identity in the bra-ket formalism looks like

$$\sum_{\mathbf{n}}|\varphi_{\mathbf{n}}\rangle\langle\Psi_{\mathbf{n}}|=\sum_{\mathbf{n}}|\Psi_{\mathbf{n}}\rangle\langle\varphi_{\mathbf{n}}|=1\!\!1.$$

Let us now introduce the operators S_φ and S_Ψ via their action respectively on $\mathcal{F}_\Psi$ and $\mathcal{F}_\varphi$:

$$S_\varphi \Psi_{\mathbf{n}} = \varphi_{\mathbf{n}}, \qquad S_\Psi \varphi_{\mathbf{n}} = \Psi_{\mathbf{n}},$$

for all $\mathbf{n}$, which also imply that $\Psi_{\mathbf{n}} = (S_\Psi\, S_\varphi)\Psi_{\mathbf{n}}$ and $\varphi_{\mathbf{n}} = (S_\varphi\, S_\Psi)\varphi_{\mathbf{n}}$, for all $\mathbf{n}$. Hence

$$S_\Psi S_\varphi = S_\varphi S_\Psi = 1\!\!1 \quad \Rightarrow \quad S_\Psi = S_\varphi^{-1}.$$

In other words, both S_Ψ and S_φ are invertible and one is the inverse of the other. Furthermore, we can also check that they are both positive, well defined and symmetric [1]. Moreover, it is possible to write these operators as

$$S_\varphi = \sum_{\mathbf{n}} |\varphi_{\mathbf{n}}\rangle\langle\varphi_{\mathbf{n}}|, \qquad S_\Psi = \sum_{\mathbf{n}} |\Psi_{\mathbf{n}}\rangle\langle\Psi_{\mathbf{n}}|.$$

These expressions are only formal, at this stage, since the series may not converge in the uniform topology and the operators S_φ and S_Ψ could be unbounded. Indeed we know [11], that two biorthonormal bases are related by a bounded operator, with bounded inverse, if and only if they are Riesz bases[2]. This is why in [1] we have also considerered

Assumption 4. *$\mathcal{F}_\varphi$ and $\mathcal{F}_\Psi$ are both Riesz bases.*

Therefore, as already stated, S_φ and S_Ψ are bounded operators and their domains can be taken to be all of $\mathcal{H}$. While Assumptions 1, 2 and 3 are quite often satisfied, [12], it is quite difficult to find *physical* examples satisfying also Assumption 4. On the other hand, it is rather easy to find *mathematical* examples satisfying all the assumptions, see [1, 6]. This is why in [9] we have introduced a difference in the notation: we have called *pseudo-bosons* (PB) those satisfying the first three assumptions, while, if they also satisfy Assumption 4, they are called *regular pseudo-bosons* (RPB).

As already discussed in our previous papers, these d-dimensional pseudo-bosons give rise to interesting intertwining relations among non self-adjoint operators, see in particular [3] and references therein. For instance, it is easy to check that

$$S_\Psi N_j = \mathfrak{N}_j S_\Psi \qquad \text{and} \qquad N_j S_\varphi = S_\varphi \mathfrak{N}_j,$$

$j = 1, 2, \ldots, d$. This is related to the fact that the spectra of, say, N_1 and $\mathfrak{N}_1$, coincide and that their eigenvectors are related by the operators S_φ and S_Ψ, in agreement with the literature on intertwining operators [13, 14].

3 The example

In this section we will consider an example arising from a quantum mechanical system, i.e. a single electron moving on a two-dimensional plane and subject to a uniform magnetic field along the z-direction. Taking $\hbar = m = \frac{eB}{c} = 1$, the Hamiltonian of the electron is given by the operator

$$H_1 = \frac{1}{2}\left(\underline{p} - \underline{A}(r)\right)^2 = \frac{1}{2}\left(p_x + \frac{y}{2}\right)^2 + \frac{1}{2}\left(p_y - \frac{x}{2}\right)^2, \tag{2}$$

where we have used minimal coupling and the symmetric gauge $\vec{A} = \frac{1}{2}(-y, x, 0)$. The Hilbert space of the system is $\mathcal{H} = \mathcal{L}^2(\mathbb{R}^2)$.

[2]Recall that a set of vectors $\phi_1, \phi_2, \phi_3, \ldots$, is a Riesz basis of a Hilbert space $\mathcal{H}$, if there exists a bounded operator V, with bounded inverse, on $\mathcal{H}$, and an o.n. basis of $\mathcal{H}$, $\varphi_1, \varphi_2, \varphi_3, \ldots$, such that $\phi_j = V\varphi_j$, for all $j = 1, 2, 3, \ldots$

The spectrum of this Hamiltonian is easily obtained by first introducing the new variables

$$Q_1 = p_x + y/2, \qquad P_1 = p_y - x/2. \tag{3}$$

In terms of P_1 and Q_1 the single electron Hamiltonian, H_1, can be rewritten as

$$H_1 = \frac{1}{2}(Q_1^2 + P_1^2).$$

The transformation (3) is part of a canonical map from the variables (x, y, p_x, p_y) to (Q_1, Q_2, P_1, P_2), where

$$Q_2 = p_y + x/2, \qquad P_2 = p_x - y/2,$$

which can be used to construct a second Hamiltonian $H_2 = \frac{1}{2}(Q_2^2+P_2^2)$. Since $[x, p_x] = [y, p_y] = i$, $[x, p_y] = [y, p_x] = [x, y] = [p_x, p_y] = 0$, we deduce that

$$[Q_1, P_1] = [Q_2, P_2] = i, \qquad [Q_1, P_2] = [Q_2, P_1] = [Q_1, Q_2] = [P_1, P_2] = 0,$$

so that $[H_1, H_2] = 0$. The two Hamiltonians correspond to two opposite magnetic fields, respectively along $+\hat{k}$ and $-\hat{k}$. Let us now introduce the operators

$$A_k = \frac{1}{\sqrt{2}}\left(Q_k + iP_k\right),$$

$k = 1, 2$, together with their adjoints. Then $[A_k, A_l^\dagger] = \delta_{k,l}1\!\!1$, the other commutators being zero. In terms of these operators we can write $H_k = A_k^\dagger A_k + \frac{1}{2}1\!\!1$, $k = 1, 2$, whose eigenvectors are $\Phi_n^{(k)} = \frac{1}{\sqrt{n!}}(A_k^\dagger)^n \Phi_0^{(k)}$, where $k = 1, 2$, $n = 0, 1, 2, \ldots$ and $\Phi_0^{(k)}$ is the vacuum of A_k: $A_k\Phi_0^{(k)} = 0$. Furthermore we have $\langle\Phi_n^{(k)}, \Phi_m^{(k)}\rangle = \delta_{n,m}$ and $H_k\Phi_n^{(k)} = \left(n + \frac{1}{2}\right)\Phi_n^{(k)}$, for $k = 1, 2$. It is natural to introduce the sets $\mathcal{F}_k := \left\{\Phi_n^{(k)},\, n \geq 0\right\}$, $k = 1, 2$, and the closures of their linear span, $\mathcal{H}_1$ and $\mathcal{H}_2$. Hence, by construction, $\mathcal{F}_k$ is an o.n. basis of $\mathcal{H}_k$. Moreover, we can also introduce an o.n. basis of $\mathcal{H}$ as the set $\mathcal{F}_\Phi$ whose vectors are defined as follows:

$$\Phi_{n,m} := \frac{1}{\sqrt{n!m!}}(A_1^\dagger)^n (A_2^\dagger)^m \Phi_{0,0},$$

where $\Phi_{0,0} := \Phi_0^{(1)} \otimes \Phi_0^{(2)}$ is such that $A_1\Phi_{0,0} = A_2\Phi_{0,0} = 0$. It is clear that $\Phi_{n,m} = \Phi_n^{(1)} \otimes \Phi_m^{(2)}$ and that $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$.

3.1 Pseudo-bosons in $\mathcal{H}_1$

Let us now define the following operators: $A_1(\alpha) = A_1$ and $B_1(\alpha) = A_1^\dagger + 2\alpha A_1$, where α is a fixed complex number. It is clear that, for $\alpha \neq 0$, $A_1(\alpha)^\dagger \neq B_1(\alpha)$. Moreover, $[A_1(\alpha), B_1(\alpha)] = 1\!\!1$, $\forall\, \alpha$. Hence, we recover (1) for $d = 1$ in $\mathcal{H}_1$. We want to show that $A_1(\alpha)$ and $B_1(\alpha)$ generate PB in $\mathcal{H}_1$ which are not regular.

To begin with, we define $\varphi_0^{(1)}(\alpha) := \Phi_0^{(1)}$. This non zero vector of $\mathcal{H}_1$ satisfies Assumption 1: $A_1(\alpha)\varphi_0^{(1)}(\alpha) = 0$, clearly, and $\varphi_0^{(1)}(\alpha) \in D^\infty(B_1(\alpha))$. This follows from the fact that, since $B_1(\alpha) = A_1^\dagger + 2\alpha A_1$, $B_1(\alpha)^n\varphi_0^{(1)}(\alpha)$ is a finite linear combination of the vectors $\Phi_0^{(1)}, \Phi_1^{(1)}, \ldots, \Phi_n^{(1)}$, which is clearly a vector of $\mathcal{H}_1$.

Before considering Assumption 2, it is convenient to observe that, introducing the following invertible and densely defined operator $U_1(\alpha) := e^{\alpha A_1^2}$, we can write

$$A_1(\alpha) = U_1(\alpha)A_1U_1(\alpha)^{-1}, \qquad B_1(\alpha) = U_1(\alpha)A_1^\dagger U_1(\alpha)^{-1},$$

$$\varphi_n^{(1)}(\alpha) := \frac{1}{\sqrt{n!}} B_1(\alpha)^n \varphi_0^{(1)}(\alpha) = U_1(\alpha)\Phi_n^{(1)}, \tag{4}$$

for all $n \geq 0$. Of course, $\varphi_n^{(1)}(\alpha)$ is well defined for all $n \geq 0$ since, as we have seen, $B_1(\alpha)^n \varphi_0^{(1)}(\alpha)$ is well defined for all complex α. Now, if we define (at least formally, at this stage)

$$\Psi_0^{(1)}(\alpha) := \left(U_1(\alpha)^\dagger\right)^{-1}\Phi_0^{(1)}, \tag{5}$$

it is possible to show that, if $|\alpha| < \frac{1}{2}$: (i) $\Psi_0^{(1)}(\alpha)$ is well defined in $\mathcal{H}_1$, and is different from zero; (ii) $B_1(\alpha)^\dagger \Psi_0^{(1)}(\alpha) = 0$; (iii) $\Psi_0^{(1)}(\alpha) \in D^\infty(A_1^\dagger)$. It is furthermore possible to check that, for the same values of α,

$$\Psi_n^{(1)}(\alpha) := \frac{1}{\sqrt{n!}} \left(A_1(\alpha)^\dagger\right)^n \Psi_0^{(1)}(\alpha) = \left(U_1(\alpha)^\dagger\right)^{-1}\Phi_n^{(1)}. \tag{6}$$

Let us prove point (iii) above. We have, for all $n \geq 0$,

$$\left(A_1^\dagger\right)^n e^{-\overline{\alpha} {A_1^\dagger}^2} \Phi_0^{(1)} = \sum_{k=0}^\infty \frac{(-\overline{\alpha})^k}{k!} \sqrt{(2k+n)!}\,\Phi_{2k+n}^{(1)},$$

so that

$$\left\|(A_1^\dagger)^n e^{-\overline{\alpha} {A_1^\dagger}^2} \Phi_0^{(1)}\right\|^2 = \sum_{k=0}^\infty \frac{|\alpha|^{2k}}{(k!)^2}(2k+n)!,$$

which converges inside the disk $|\alpha| < \frac{1}{2}$. In particular, if $n = 0$, this implies the statement in (i) above. The proof of (ii) is trivial and the last equality in (6) can be deduced using (4) and (5) in the definition $\Psi_n^{(1)}(\alpha) := \frac{1}{\sqrt{n!}}(A_1(\alpha)^\dagger)^n \Psi_0^{(1)}(\alpha)$. This, as we have seen, is well defined if $|\alpha| < \frac{1}{2}$, while, for $|\alpha| > \frac{1}{2}$ all the procedure makes no sense, since the vectors we are using do not belong to the Hilbert space. Now, biorthonormality of the two sets $\mathcal{F}_{\varphi^{(1)}} := \{\varphi_n^{(1)}(\alpha),\, n \geq 0\}$ and $\mathcal{F}_{\Psi^{(1)}} := \{\Psi_n^{(1)}(\alpha),\, n \geq 0\}$ follows directly from their definitions:

$$\langle \varphi_n^{(1)}(\alpha), \Psi_m^{(1)}(\alpha)\rangle = \langle U_1(\alpha)\Phi_n^{(1)}, \left(U_1(\alpha)^\dagger\right)^{-1}\Phi_m^{(1)}\rangle = \langle \Phi_n^{(1)}, \Phi_m^{(1)}\rangle = \delta_{n,m}.$$

The proof of Assumption 3 goes as follows:

First of all, as we have already stated, it is possible to check that for all $n \geq 0$ we have $\varphi_n^{(1)}(\alpha) = \Phi_n^{(1)} + \sum_{k=0}^{n-1} d_k \Phi_k^{(1)}$, for some constants $\{d_k,\, k = 0, 1, \ldots, n-1\}$.

Secondly, using induction on n and this simple remark we can prove that, *if $f \in \mathcal{H}_1$ is such that $\langle f, \varphi_k^{(1)}(\alpha)\rangle = 0$ for $k = 0, 1, \ldots, n$, then $\langle f, \Phi_k^{(1)}\rangle = 0$ for $k = 0, 1, \ldots, n$ as well.* Therefore, if f is orthogonal to all the $\varphi_k^{(1)}(\alpha)$'s, it is also orthogonal to all the $\Phi_k^{(1)}$'s, whose set is complete in $\mathcal{H}_1$. Hence $f = 0$, so that $\mathcal{F}_{\varphi^{(1)}}$ is also complete in $\mathcal{H}_1$.

As a consequence, being the vectors of $\mathcal{F}_{\varphi^{(1)}}$ linearly independent and complete in $\mathcal{H}_1$, they are a basis of $\mathcal{H}_1$. In particular we find that, for all $f \in \mathcal{H}_1$, the following expansion holds true: $f = \sum_{k=0}^\infty \langle \Psi_n^{(1)}(\alpha), f\rangle \varphi_n^{(1)}(\alpha)$. Then, for all $f, g \in \mathcal{H}_1$,

$$\langle g, f\rangle = \left\langle g, \sum_{k=0}^\infty \langle \Psi_n^{(1)}(\alpha), f\rangle \varphi_n^{(1)}(\alpha)\right\rangle = \left\langle \sum_{k=0}^\infty \langle \varphi_n^{(1)}(\alpha), g\rangle \Psi_n^{(1)}(\alpha), f\right\rangle,$$

which, since f could be any vector in $\mathcal{H}_1$, implies that $g = \sum_{k=0}^\infty \langle \varphi_n^{(1)}(\alpha), g\rangle \Psi_n^{(1)}(\alpha)$: $\mathcal{F}_{\Psi^{(1)}}$ is a basis of $\mathcal{H}_1$ as well, and Assumption 3 is satisfied. Finally, Assumption 4 is not satisfied since, for instance, the operator $\left(U_1(\alpha)^\dagger\right)^{-1}$ is unbounded [11].

Remark 1. It might be worth stressing that, while it is quite easy to check that the set $\mathcal{F}_{\varphi^{(1)}}$ is complete in $D(U(\alpha)^\dagger)$, it is not trivial at all to check that it is also complete in $\mathcal{H}_1$. This is the reason why we have used the above procedure.

It is not hard to deduce the expression of two non self-adjoint operators which admit $\varphi_n^{(1)}(\alpha)$ and $\Psi_n^{(1)}(\alpha)$ as eigenstates. For that we define first $h_1(\alpha) := U_1(\alpha)H_1U_1(\alpha)^{-1} = B_1(\alpha)A_1(\alpha) + \frac{1}{2}1\!\!1$, which, in coordinate representation, looks like

$$h_1(\alpha) = \left(\frac{1}{2}+\alpha\right)\left(p_x+\frac{y}{2}\right)^2 + \left(\frac{1}{2}-\alpha\right)\left(p_y-\frac{x}{2}\right)^2 + 2i\alpha\left(p_x+\frac{y}{2}\right)\left(p_y-\frac{x}{2}\right) + \alpha 1\!\!1.$$

We can also introduce $h_1(\alpha)^\dagger$, which is clearly different from $h_1(\alpha)$. Now, as expected from general facts in the theory of intertwining operators [13], we see that

$$h_1(\alpha)\varphi_n^{(1)}(\alpha) = (n+1/2)\varphi_n^{(1)}(\alpha), \qquad h_1(\alpha)^\dagger\Psi_n^{(1)}(\alpha) = (n+1/2)\Psi_n^{(1)}(\alpha),$$

for all $n \geq 0$.

3.2 Pseudo-bosons in $\mathcal{H}_2$

In this subsection we will consider an analogous construction in $\mathcal{H}_2$, i.e. in the Hilbert space related to the uniform magnetic field along $-\hat{k}$. To make the situation more interesting, and to avoid repeating essentially the same procedure considered above, instead of introducing an operator like $e^{\beta A_2^2}$ we consider

$$U_2(\beta) := e^{\beta {A_2^\dagger}^2},$$

with $\beta \in \mathbb{C}$. Then we define

$$\begin{aligned} A_2(\beta) &:= U_2(\beta)A_2U_2(\beta)^{-1} = A_2 - 2\beta A_2^\dagger, \\ B_2(\beta) &:= U_2(\beta)A_2^\dagger U_2(\beta)^{-1} = A_2^\dagger. \end{aligned} \tag{7}$$

These are pseudo-bosonic operators in $\mathcal{H}_2$: $[A_2(\beta), B_2(\beta)] = 1\!\!1$, and $A_2(\beta)^\dagger \neq B_2(\beta)$, for $\beta \neq 0$. Then, once again, it may be interesting to consider Assumptions 1–4.

If $|\beta| < \frac{1}{2}$ Assumption 1 is satisfied: let us define (formally, for the moment) $\varphi_0^{(2)}(\beta) = U_2(\beta)\Phi_0^{(2)}$. Then $A_2(\beta)\varphi_0^{(2)}(\beta) = 0$. Moreover, since $[B_2(\beta), U_2(\beta)] = 0$, $B_2(\beta)^n\varphi_0^{(2)}(\beta) = U_2(\beta){A_2^\dagger}^n\,\Phi_0^{(2)}$, which implies in particular that

$$\varphi_n^{(2)}(\beta) := \frac{1}{\sqrt{n!}}B_2(\beta)^n\varphi_0^{(2)}(\beta) = U_2(\beta)\Phi_n^{(2)}.$$

Of course we have now to check that $\varphi_n^{(2)}(\beta)$ is a well defined vector of $\mathcal{H}_2$ for all $n \geq 0$. This would make the above formal definition rigorous. The computation of $\|U_2(\beta)\Phi_n^{(2)}\|$ follows the same steps as that for $\|{U_1(\alpha)^\dagger}^{-1}\Phi_n^{(1)}\|$ of the previous section, and we get the same conclusion: the power series obtained for $\|U_2(\beta)\Phi_n^{(2)}\|^2$ converges if $|\beta| < \frac{1}{2}$, so that $\Phi_n^{(2)} \in D(U_2(\beta))$ for all $n \geq 0$, inside this disk.

As for Assumption 2, this is also satisfied: to prove this it is enough to take $\Psi_0^{(2)}(\beta) = \Phi_0^{(2)}$. Then $B_2(\beta)^\dagger\Psi_0^{(2)}(\beta) = A_2\Phi_0^{(2)} = 0$. Also, since $U_2(\beta)^\dagger\Phi_0^{(2)} = \Phi_0^{(2)}$, formula (7) implies that $\frac{1}{\sqrt{n!}}(A_2^\dagger(\beta))^n\Psi_0^{(2)}(\beta) = e^{-\overline{\beta}\,A_2^2}\Phi_n^{(2)}$, which is clearly a vector in $\mathcal{H}_2$ since it is a finite linear combination of $\Phi_0^{(2)}, \Phi_1^{(2)}, \ldots, \Phi_n^{(2)}$. This means that the vectors

$$\Psi_n^{(2)}(\beta) := \frac{1}{\sqrt{n!}}{A_2(\beta)^\dagger}^n\Psi_0^{(2)}(\beta) = \left(U_2(\beta)^\dagger\right)^{-1}\Phi_n^{(2)}$$

are well defined in $\mathcal{H}_2$ for all n, independently of β. Once again we deduce that the vectors constructed here are biorthonormal,

$$\langle\varphi_n^{(2)}(\beta), \Psi_m^{(2)}(\beta)\rangle = \delta_{n,m},$$

and that they are eigenstates of two operators which are the adjoint one of the other, and which are related to H_2 by a similarity transformation:

$$h_2(\beta) := U_2(\beta)H_2U_2(\beta)^{-1} = B_2(\beta)A_2(\beta) + \frac{1}{2}1\!\!1,$$

which in coordinate representation looks like

$$h_2(\beta) = \left(\frac{1}{2} - \beta\right)\left(p_y + \frac{x}{2}\right)^2 + \left(\frac{1}{2} + \beta\right)\left(p_x - \frac{y}{2}\right)^2 + 2i\beta\left(p_y + \frac{x}{2}\right)\left(p_x - \frac{y}{2}\right) + \beta 1\!\!1.$$

In particular we find that

$$h_2(\beta)\varphi_n^{(2)}(\beta) = (n + 1/2)\varphi_n^{(2)}(\beta), \qquad h_2(\beta)^\dagger\Psi_n^{(2)}(\beta) = (n + 1/2)\Psi_n^{(2)}(\beta),$$

for all $n \geq 0$. The same arguments used previously prove that $\mathcal{F}_{\varphi^{(2)}} := \{\varphi_n^{(2)}(\beta),\, n \geq 0\}$ and $\mathcal{F}_{\Psi^{(2)}} := \{\Psi_n^{(2)}(\beta),\, n \geq 0\}$ are both complete in $\mathcal{H}_2$. More than this: they are biorthonormal bases but not Riesz bases.

3.3 Pseudo-bosons in $\mathcal{H}$

We begin this section with the following remark: none of the above sets of functions is complete in $\mathcal{H}$. Hence we could try to find a different set of vectors, also labeled by a single quantum number, which is complete in $\mathcal{H}$. It is not hard to check that this is not possible, in general. Let us introduce, for instance, the following pseudo-bosonic operators: $X_{\alpha,\beta} := \frac{1}{\sqrt{2}}\left(A_1(\alpha) + A_2(\beta)\right)$ and $Y_{\alpha,\beta} := \frac{1}{\sqrt{2}}\left(B_1(\alpha) + B_2(\beta)\right)$. Then $[X_{\alpha,\beta}, Y_{\alpha,\beta}] = 1\!\!1$, $X_{\alpha,\beta} \neq Y_{\alpha,\beta}^\dagger$ in general and the vectors $\varphi_{0,0}(\alpha,\beta) := \varphi_0^{(1)}(\alpha) \otimes \varphi_0^{(2)}(\beta)$ and $\Psi_{0,0}(\alpha,\beta) := \Psi_0^{(1)}(\alpha) \otimes \Psi_0^{(2)}(\beta)$ satisfy Assumptions 1 and 2 of Section 2. However, it is not hard to check that the vectors $\eta_n(\alpha,\beta) := \frac{1}{\sqrt{n!}}Y_{\alpha,\beta}^n\varphi_{0,0}(\alpha,\beta)$, $n \geq 0$, are not complete in $\mathcal{H}$: for that it is enough to consider the non zero vector $f = \Psi_1^{(1)}(\alpha) \otimes \Psi_0^{(2)}(\beta) - \Psi_0^{(1)}(\alpha) \otimes \Psi_1^{(2)}(\beta)$, which is non zero and orthogonal to all the $\eta_n(\alpha,\beta)$'s.

This is not surprising: in Section 2, in fact, we have proposed a different way to produce two biorthonormal bases of $\mathcal{H}$, in dimension larger than 1. For instance, in $d = 2$ we expect that the vectors of these bases depend on two quantum numbers rather than just one. So we may proceed as follows: let $T(\alpha,\beta)$ be the following unbounded operator:

$$T(\alpha,\beta) := U_1(\alpha)U_2(\beta) = e^{\alpha A_1^2 + \beta A_2^{\dagger\,2}}.$$

Then the vectors $\varphi_{0,0}(\alpha,\beta)$ and $\Psi_{0,0}(\alpha,\beta)$ introduced above can be defined as $\varphi_{0,0}(\alpha,\beta) = T(\alpha,\beta)\Phi_{0,0}$ and $\Psi_{0,0}(\alpha,\beta) = \left(T(\alpha,\beta)^\dagger\right)^{-1}\Phi_{0,0}$. For what we have seen in the previous sections, these two vectors satisfy Assumptions 1 and 2: $A_1(\alpha)\varphi_{0,0}(\alpha,\beta) = A_2(\beta)\varphi_{0,0}(\alpha,\beta) = 0$, $B_1(\alpha)^\dagger\Psi_{0,0}(\alpha,\beta) = B_2(\beta)^\dagger\Psi_{0,0}(\alpha,\beta) = 0$, and $\varphi_{0,0}(\alpha,\beta) \in D^\infty(B_1(\alpha)) \cap D^\infty(B_2(\beta))$, $\Psi_{0,0}(\alpha,\beta) \in D^\infty(A_1(\alpha)^\dagger) \cap D^\infty(A_2(\beta)^\dagger)$.

Furthermore, since $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, the sets $\mathcal{F}_\varphi := \{\varphi_{n,m}(\alpha,\beta) := \frac{1}{\sqrt{n!m!}}B_1(\alpha)^n B_2(\beta)^m\varphi_{0,0}(\alpha,\beta)\}$ and $\mathcal{F}_\Psi := \{\Psi_{n,m}(\alpha,\beta) := \frac{1}{\sqrt{n!m!}}A_1(\alpha)^{\dagger\,n}A_2(\beta)^{\dagger\,m}\Psi_{0,0}(\alpha,\beta)\}$ are complete in $\mathcal{H}$, so that Assumption 3 is also satisfied. Finally, Assumption 4 is not verified, so that we have found PB which are not regular. This is because $T(\alpha,\beta)$ is unbounded and since we can write $\varphi_{n,m}(\alpha,\beta) = T(\alpha,\beta)\Phi_{n,m}$ and $\Psi_{n,m}(\alpha,\beta) = \left(T(\alpha,\beta)^\dagger\right)^{-1}\Phi_{n,m}$, for all n and m, [11].

Remark 2. The procedure outlined in this section clearly applies to any pair of uncoupled harmonic oscillators $h_1 = a_1^\dagger a_1$ and $h_2 = a_2^\dagger a_2$, $[a_i, a_j^\dagger] = \delta_{i,j}\, 1\!\!1$, $i, j = 1, 2$, changing properly the definitions of the operators involved.

Remark 3. Bi-coherent states like those in [1] can be easily constructed from the ones for A_1 and A_2 using the operators $U_1(\alpha)$ and $U_2(\beta)$.

4 A no-go result

We devote this short section to prove the following general no-go result: suppose a and $a^\dagger$ are two operators acting on $\mathcal{H}$ and satisfying $[a, a^\dagger] = 1\!\!1$. Then, for all $\alpha \neq 0$, the operators $A := a - \alpha {a^\dagger}^2$ and $B := a^\dagger$ are such that $[A, B] = 1\!\!1$, $A^\dagger \neq B$, but they do not satisfy Assumption 1.

In fact, if such a non zero vector $\varphi_0 \in \mathcal{H}$ exists, then it could be expanded in terms of the eigenvectors $\Phi_n := \frac{{a^\dagger}^n}{\sqrt{n!}}\Phi_0$, $a\Phi_0 = 0$, of the number operator $N = a^\dagger a$: $\varphi_0 = \sum_{n=0}^\infty c_n \Phi_n$, for some sequence $\{c_n, n \geq 0\}$ such that $\sum_{n=0}^\infty |c_n|^2 < \infty$. Condition $A\varphi_0 = 0$ can be rewritten as $a\varphi_0 = \alpha {a^\dagger}^2 \varphi_0$. Now, inserting in both sides of this equality the expansion for φ_0, and recalling that $a^\dagger \Phi_n = \sqrt{n+1}\Phi_{n+1}$ and $a\Phi_n = \sqrt{n}\Phi_{n-1}$, $n \geq 0$, we deduce the following relations between the coefficients c_n: $c_1 = c_2 = 0$ and $c_{n+1}\sqrt{n+1} = \alpha c_{n-2}\sqrt{(n-1)n}$, for all $n \geq 2$. The solution of this recurrence relation is the following:

$$c_3 = \alpha c_0 \frac{\sqrt{3!}}{3}, \qquad c_6 = \alpha^2 c_0 \frac{\sqrt{6!}}{3 \cdot 6}, \qquad c_9 = \alpha^3 c_0 \frac{\sqrt{9!}}{3 \cdot 6 \cdot 9}, \qquad c_{12} = \alpha^4 c_0 \frac{\sqrt{12!}}{3 \cdot 6 \cdot 9 \cdot 12},$$

and so on. Then

$$\varphi_0 = c_0 \left(\Phi_0 + \sum_{k=1}^\infty \alpha^k \frac{\sqrt{(3k)!}}{1 \cdot 3 \cdots 3k}\, \Phi_{3k} \right).$$

However, computing $\|\varphi_0\|$ we deduce that this series only converge if $\alpha = 0$, i.e. if A coincides with a and B with $a^\dagger$.

A similar results can be obtained considering the operators $A := a - \alpha {a^\dagger}^n$ and $B := a^\dagger - \beta 1\!\!1$, $n \geq 2$, $\alpha, \beta \in \mathbb{C}$. Again we find $[A, B] = 1\!\!1$, $A^\dagger \neq B$, and again, with similar techniques, we deduce that they do not satisfy Assumption 1. In the same way, if we define $A := a - \alpha 1\!\!1$ and $B := a^\dagger - \beta a^m$, $m \geq 2$, $\alpha, \beta \in \mathbb{C}$, we find that, in general, $[A, B] = 1\!\!1$, $A^\dagger \neq B$, but they do not satisfy Assumption 2. This suggests that if we try to define, starting from a and $a^\dagger$, new operators $A = a + f(a, a^\dagger)$ and $B = a^\dagger + g(a, a^\dagger)$, only very special choices of f and g are compatible with the pseudo-bosonic structure.

5 Conclusions

We have seen how a non trivial example of two-dimensional PB arises from the Hamiltonian of the Landau levels. We want to stress once again that this is deeply different from what we have done in [10], where the starting point was a generalized Hamiltonian obtained with a *smart* extension of that in (2) with the introduction of two related *superpotentials*. Among the other differences, while the procedure outlined here works in any Hilbert space, the one in [10] works only in $\mathcal{L}^2(\mathbb{R}^2)$. The fact that both here and in [10] we get pseudo-bosons which are not regular is still another indication of the mathematical nature of RPB.

It is not difficult to modify or to generalize the results in Section 3, for instance changing the role of the operators U_1 and U_2, or modifying a bit their definitions. Maybe more interesting is to try to extend the no-go result of Section 4 to other possible combinations of a and $a^\dagger$: this is part of our work in progress.

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