

Free Fermi and Bose Fields in TQFT and GBF

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Abstract. We present a rigorous and functorial quantization scheme for linear fermionic and bosonic field theory targeting the topological quantum field theory (TQFT) that is part of the general boundary formulation (GBF). Motivated by geometric quantization, we generalize a previous axiomatic characterization of classical linear bosonic field theory to include the fermionic case. We proceed to describe the quantization scheme, combining a Fock space quantization for state spaces with the Feynman path integral for amplitudes. We show rigorously that the resulting quantum theory satisfies the axioms of the TQFT, in a version generalized to include fermionic theories. In the bosonic case we show the equivalence to a previously developed holomorphic quantization scheme. Remarkably, it turns out that consistency in the fermionic case requires state spaces to be Krein spaces rather than Hilbert spaces. This is also supported by arguments from geometric quantization and by the explicit example of the Dirac field theory. Contrary to intuition from the standard formulation of quantum theory, we show that this is compatible with a consistent probability interpretation in the GBF. Another surprise in the fermionic case is the emergence of an algebraic notion of time, already in the classical theory, but inherited by the quantum theory. As in earlier work we need to impose an integrability condition in the bosonic case for all TQFT axioms to hold, due to the gluing anomaly. In contrast, we are able to renormalize this gluing anomaly in the fermionic case.

Key words: general boundary formulation; topological quantum field theory; fermions; free field theory; functorial quantization; foundations of quantum theory; quantum field theory

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1 Introduction

Free field theories and their quantization are usually the first examples treated in text books of quantum field theory. Apart from being simple toy examples they also serve to illustrate foundations of the formalism, its elementary objects and elucidate the physical interpretation of quantum field theory at its most basic level. Moreover, they serve to exhibit quantization schemes which are then refined for the treatment of more complicated theories. They also play a crucial role as the basis for perturbation theory and S -matrix theory. Finally, they are part of the small class of quantum field theories that have been defined mathematically rigorously.

It is natural to expect an equally important role for free field theories in approaches to quantum theory that aim to go beyond the context of spacetime with a fixed background metric. Here we shall be interested in the general boundary formulation (GBF), an axiomatic approach to quantum theory with precisely that aim [7]. In the GBF, spacetime is modeled through manifolds that need not carry any structure in addition to a topological one. Its basic objects are that

of a specific version of topological quantum field theory (TQFT) [1]. The other fundamental ingredient of the GBF consists of rules that allow to extract predictions of measurement outcomes from these basic objects.

In quantum field theory in Minkowski spacetime a comprehensive and universal description of free field theory can be given, see e.g. [2]. By universal we mean here that it applies not just to specific examples, but to free field theories as a class. For the GBF such a universal description of free, i.e., linear bosonic field theory has been presented in [9]. More specifically, motivated by properties of Lagrangian field theory, an axiomatic definition of classical linear bosonic field theories as a class was given. Then a functorial quantization scheme was exhibited that produces a corresponding quantum theory in the sense of the GBF. In particular, it was proven rigorously that the quantum theory so produced satisfies the axioms of the GBF. It was also shown that the employed quantization is “correct” for certain examples of quantum field theories. Later, even the formal equivalence of the quantization to the Feynman path integral (where applicable) was established [6, 12].

Our main interest in the present work shall be a functorial quantization scheme for linear fermionic field theory in the GBF, paralleling that of the linear bosonic theory in [9]. That is, we start with an axiomatization of classical field theory, exhibit a quantization prescription, and show that the quantum theory obtained satisfies the axioms of the GBF. We supplement this with an explicit example of a realistic quantum field theory, in this case the Dirac field in Minkowski spacetime. This serves the dual purpose of demonstrating on the one hand that standard quantum field theories fit into the framework, while on the other hand exhibiting features of the latter not accessible via the standard formulation. Concretely, we provide here the consistent implementation of states on certain timelike hypersurfaces.

We recall that the quantization scheme in [9] used the holomorphic representation and relied heavily on coherent states. While there exist notions of coherent states also in the fermionic case, these have rather different properties and are not amenable to an analogous treatment. We thus opt for a Fock space quantization. While this seems simple enough, it turns out to complicate considerably the demonstration that the quantized theory satisfies the GBF axioms. On the other hand, it makes possible a treatment of the bosonic theory alongside the fermionic one in a unified fashion and with little additional effort. In this way we also gain new insight into the bosonic theory from a perspective rather different than the one employed in [9].

Even though we work within the specific framework of the GBF, much of our treatment should be applicable to a large class of TQFTs. In particular, it is designed to be universal also in the sense of being independent of, but compatible with various types of additional structure on the manifolds modeling spacetime, such as a metric or conformal structure. We also note that for other specific types of TQFT frameworks there is a well developed understanding of the notion of free field theory. This is notably so in the TQFT approach to conformal field theory due to Segal [17, 18]. It is beyond the scope of the present work, however, to explore the potential connections to this approach.

An early and intriguing result of the present work is the necessity, in the fermionic case, to abandon Hilbert spaces as (generalized) state spaces of the quantum theory in favor of *Krein spaces*. Apart from the justifications for this laid out in the main body of this work, certain fermionic examples with Krein spaces in [18] served as an additional motivation. The appearance of indefinite inner product spaces in quantization prescriptions in quantum field theory is not at all new, a famous example is the Gupta–Bleuler quantization of the electromagnetic field [3, 5]. What is new here, however, is that such spaces not only appear at intermediate stages of the quantization process, but as the final physical state spaces. The appearance of Krein spaces might lead to concerns about the feasibility of a consistent probability interpretation. We remind the reader, however, that the state spaces of the GBF can not in general be identified with state spaces of the standard formulation of quantum theory, where this would indeed be

a problem. What is more, we show in this work explicitly (in Section 11) that a consistent probability interpretation is possible.

Another intriguing aspect of linear fermionic field theories, present already at the classical level, is something that may be called an *emergent notion of time* in the dynamics. We refer the reader to Section 4.3 for details.

Finally, we remark on the integrability condition that was necessary to impose in [9] for a classical linear bosonic field theory to possess a quantization satisfying all the axioms of the GBF. This integrability condition arises due to the presence of the *gluing anomaly factor*. Roughly speaking the integrability condition amounts to requiring that this anomaly factor be finite. In the present work we exhibit the same gluing anomaly factor. However, in the fermionic case we are able to carry out a renormalization procedure so that even if the gluing anomaly factor is “infinite”, a renormalized version of the relevant gluing axiom still holds. Thus the quantization of any linear fermionic field theory satisfying the axioms of the classical theory satisfies the renormalized version of the GBF axioms, without exception.

In Section 2 we provide motivation for the axiomatization of the classical theory and aspects of the quantization scheme in the fermionic case. In Section 3 we give a short review of some necessary mathematical ingredients, notably Krein spaces and the associated Fock spaces. The axiomatization of classical linear field theory is discussed in Section 4. The example of the free Dirac field is treated in Section 5, both with spacelike and timelike hypersurfaces. The axioms of the GBF are reviewed and generalized to include the fermionic case in Section 6. The basics of the quantization scheme are described in Section 7. In Section 8 relations between amplitude and inner product are examined. Section 9 treats the gluing axioms and exhibits their proofs, while concentrating the most difficult part into a lemma. Due to its length, the proof of this lemma is delegated to Appendix A. The gluing anomaly and its renormalization are the subjects of Section 10. The probability interpretation in the context of Krein spaces is elaborated on in Section 11. In Section 12 a more detailed comparison to the holomorphic approach of [9] is given in the bosonic case. Finally, a brief outlook is presented in Section 13.

2 Motivation

In this section we shall provide some motivation for concrete aspects of the way we axiomatize classical field theory, and also for certain aspects of the quantization scheme we describe in the following.

2.1 Schematic view of quantization

In the following we recall in a very abbreviated and schematic fashion the first steps in the quantization of free fields from the point of view of geometric quantization [19].

In both the bosonic and the fermionic case the complex Hilbert space \mathcal{H} of states for a free field is most conveniently constructed starting from another complex Hilbert space L . This space L may be thought of as the space of solutions to the equations of motion of the underlying classical system to be quantized. Most commonly (and also in the present work), \mathcal{H} is then constructed as the Fock space over L . However, there are other, but equivalent ways to construct \mathcal{H} from L which may be more convenient depending on the context. We shall remark on this in Section 12.

Taking the point of view of geometric quantization, the space L is obtained in the bosonic case as follows: At first we take L to be the real vector space of solutions of the equations of motion with suitable regularity properties. This space becomes a symplectic vector space by equipping it with the symplectic form $\omega : L \times L \rightarrow \mathbb{R}$ that is derived from the Lagrangian of the theory in the standard way. The next step is then to introduce a complex structure $J : L \rightarrow L$ that leaves ω invariant and is compatible with the dynamics. The latter usually means something

along the lines that J leads to a useful notion of “positive energy” versus “negative energy” solutions and respects relevant symmetries, such as Lorentz transformations. The details of this are not important here. Combining the complex structure with the symplectic structure leads to a real inner product $g : L \times L \rightarrow \mathbb{R}$, which is usually required to be positive definite (possibly requiring taking a quotient),

$$g(\phi', \phi) := 2\omega(\phi', J\phi).$$

This in turn is combined with the symplectic structure to yield a complex inner product

$$\{\phi', \phi\} := g(\phi', \phi) + 2i\omega(\phi', \phi). \quad (2.1)$$

This makes L (possibly upon completion) into a complex Hilbert space.

In the fermionic case the construction of L is slightly more complicated. The symplectic structure $\tilde{\omega} : L \times L \rightarrow \mathbb{R}$ obtained from the Lagrangian does not in itself serve to define the final complex inner product. Rather, it has to be combined with a complex structure $\tilde{J} : L \rightarrow L$ which is usually a natural complex structure on the spinor bundle that would serve to define the classical system. This yields the real inner product $g : L \times L \rightarrow \mathbb{R}$,

$$g(\phi', \phi) := 2\tilde{\omega}(\phi', \tilde{J}\phi).$$

Another complex structure $J : L \rightarrow L$, related to the dynamics as in the bosonic case, then yields the relevant symplectic structure

$$\omega(\phi', \phi) := -\frac{1}{2}g(\phi', J\phi). \quad (2.2)$$

The real inner product and the symplectic structure are then combined as in the bosonic case via equation (2.1).

2.2 Quantization in the GBF

For the convenience of the reader we have described quantization so far as if we were interested in a single Hilbert space describing our quantum system. This is not the case. Rather, in the GBF we need one such Hilbert space for each (admissible) hypersurface in spacetime. It turns out that the schematic procedure described above is already naturally adapted to this [9]. Given a hypersurface Σ we think of L_Σ as the space of solutions in a neighborhood of Σ (or more precisely as the space of germs of solutions). Consider the bosonic case first. The symplectic structure ω derived from the Lagrangian is actually not necessarily a structure on a global space of solutions. Rather it arises as the integral of a $(d-1)$ -form (where d is the spacetime dimension) over a hypersurface. Moreover, the dependence of the $(d-1)$ -form on the field is completely local so that the symplectic structure depends only on the field and its derivatives on the hypersurface. Taking the hypersurface to be Σ , it is thus naturally a structure on L_Σ and we denote it by ω_Σ . The complex structure by contrast is generally somewhat non-local, but can nevertheless be specified as a complex structure on L_Σ . We denote it thus by J_Σ . We write the resulting complex inner product as $\{\cdot, \cdot\}_\Sigma$.

Looking more closely at the above description of the space L_Σ and associated structures ω_Σ , g_Σ and $\{\cdot, \cdot\}_\Sigma$ it turns out that the latter not only depend on the choice of hypersurface Σ itself, but also on its *orientation*. We shall write $\bar{\Sigma}$ to denote the same hypersurface Σ , but with opposite orientation. Now, recall that in the bosonic case the symplectic structure ω_Σ arises as the integral of a $(d-1)$ -form over the hypersurface Σ . But reversing the orientation changes the sign of this integral. The complex structure J_Σ , encoding in some sense the distinction between “positive energy” and “negative energy” solutions, also changes sign under change of orientation

of Σ . As a consequence, the real inner product is invariant under orientation change while the complex inner product is complex conjugated. In summary,

$$\omega_{\bar{\Sigma}} = -\omega_{\Sigma}, \quad J_{\bar{\Sigma}} = -J_{\Sigma}, \quad g_{\bar{\Sigma}} = g_{\Sigma}, \quad \{\phi', \phi\}_{\bar{\Sigma}} = \overline{\{\phi', \phi\}_{\Sigma}}. \quad (2.3)$$

Precisely these ingredients (the spaces L_{Σ} with the given structures) have been taken as the “kinematical” part of the data of an axiomatic formalization of classical linear bosonic field theory in [9]. Moreover, they have been shown there to integrate nicely into a functorial quantization scheme which moreover reproduces and extends basic examples from quantum field theory. We consider that this confirms the “correctness” of the above schematic description.

In the fermionic case it is the symplectic form $\tilde{\omega}$ that arises as the integral of a field-local $(d-1)$ -form on a hypersurface. The complex structure \tilde{J} is also completely local in its field dependence. So, the real inner product g is the integral of a field-local $(d-1)$ -form on the hypersurface. For a hypersurface Σ we denote it by g_{Σ} . As in the bosonic case the complex structure J_{Σ} introduces a certain non-locality which is thus inherited by the symplectic structure ω_{Σ} . $\{\cdot, \cdot\}_{\Sigma}$ denotes the resulting complex inner product. The symplectic form $\tilde{\omega}_{\Sigma}$ depends on the orientation of Σ as in the bosonic case. \tilde{J}_{Σ} on the other hand does not depend on the orientation while g_{Σ} does, coming from a field-local $(d-1)$ -form. J_{Σ} does depend on the orientation as in the bosonic case and for the same reason. So ω_{Σ} does not depend on the orientation. In summary,

$$\omega_{\bar{\Sigma}} = \omega_{\Sigma}, \quad J_{\bar{\Sigma}} = -J_{\Sigma}, \quad g_{\bar{\Sigma}} = -g_{\Sigma}, \quad \{\phi', \phi\}_{\bar{\Sigma}} = -\overline{\{\phi', \phi\}_{\Sigma}}. \quad (2.4)$$

This is in precise analogy to the bosonic case (2.3) if we think of the difference between bosons and fermions as due to the interchange of symmetric with anti-symmetric structures.

A closer look at the relations (2.4) reveals a profound and surprising implication. In contrast to the bosonic case, it cannot be consistently required that all the spaces L_{Σ} have positive-definite inner product. Indeed, suppose g_{Σ} was positive-definite for some hypersurface Σ . Then $g_{\bar{\Sigma}}$ would necessarily be negative-definite. Moreover, if L_{Σ} carries an indefinite inner product then so does the Fock space \mathcal{H}_{Σ} based on it. We are thus forced to give up the insistence that state spaces should be Hilbert spaces.

While this narrative of the fermionic case appears compelling in view of its tight analogy to the more established bosonic case, the abandonment of Hilbert spaces calls for the presentation of strong evidence in its favor. The purpose of the present work is in part to provide just that, by embedding it into a functorial quantization scheme that successfully reproduces known quantum field theories. We shall also alleviate fears (in Section 11) that this may destroy a consistent probability interpretation. Moreover, we will explain why the usual formulation of quantum theory gets along nicely without “seeing” these indefinite inner product spaces in fermionic quantum field theories (see Section 8.3).

2.3 Dynamical aspects

We have so far exclusively concentrated on the “kinematical” aspects of the classical theory and its quantization. We proceed to discuss aspects of the dynamics. For linear bosonic field theory this was formalized in [9] as follows. Given a spacetime region M , the “dynamics” in M is given by a real vector space L_M of solutions of the equations of motion. This is a subspace of the space $L_{\partial M}$ of (germs of) solutions on the boundary ∂M of M . Indeed, the key is that it is a *Lagrangian* subspace with respect to the symplectic form $\omega_{\partial M}$ on $L_{\partial M}$. This was motivated extensively in [9] from Lagrangian field theory and confirmed in the sense that it fits precisely into the quantization scheme given there. Moreover, it also fits with the examples of known quantum field theories considered there.

In view of the analogy between fermions and bosons we should thus expect the dynamics in a “classical” fermionic field theory to be encoded in a subspace $L_M \subseteq L_{\partial M}$ that is “Lagrangian” with respect to the symmetric bilinear form $g_{\partial M}$. That is the subspace L_M should be a neutral subspace which is maximal in a certain sense. It turns out that the right notion is that of a *hypermaximal neutral subspace*.

The quantum dynamics for a region M is encoded in the amplitude map $\rho_M : \mathcal{H}_{\partial M} \rightarrow \mathbb{C}$. This was given for the bosonic theory in [9] in terms of a certain integral of the holomorphic wave function on the boundary over the space of solutions in the interior, L_M . It was later shown [6, 12] that this is precisely equivalent to the usual Feynman path integral and thus “correct” from a quantum field theory point of view. This same quantization rule, translated to the Fock space setting is the basis for the quantization of the dynamics in the bosonic case also here. (We shall have much more to say about this in Section 12.) The quantization rule in the fermionic case is deduced by analogy. Again, our overall results may be seen to “confirm” it.

3 Ingredients

3.1 Krein space

Let V be an indefinite inner product space with an orthogonal direct sum decomposition $V = V^+ \oplus V^-$ such that V^+ is positive definite and V^- is negative definite. Let $\overline{V^-}$ denote the space V^- with the sign of its inner product inverted. Then $\overline{V^-}$ is positive definite and V is canonically isomorphic as a vector space to the positive inner product space $V^+ \oplus \overline{V^-}$. If $V^+ \oplus \overline{V^-}$ is a Hilbert space then we say that V is a *Krein space* and equip it with the Hilbert space topology.

In general there are many ways to decompose a Krein space into an orthogonal direct sum $V = V^+ \oplus V^-$ with the described properties. We shall be interested, however, exclusively in Krein spaces that come with a canonical such decomposition. In order not to complicate notation, *Krein space* in the following always refers to this strict version of Krein space.

We call V^+ the *positive part* and V^- the *negative part* of V . We also view the decomposition as a \mathbb{Z}_2 -grading with the notation

$$[v] := \begin{cases} 0 & \text{if } v \in V^+, \\ 1 & \text{if } v \in V^-. \end{cases}$$

Let W be a subset of the Krein space V . Then the subspace

$$W^\perp := \{v \in V : \langle v, w \rangle = 0 \ \forall w \in W\}$$

is called the *orthogonal companion* of W . A subspace W of a Krein space V is called *neutral* iff the inner product of each element of W with itself vanishes. This is equivalent to the property that the real part of the inner product vanishes on W . W is called *maximal neutral* iff moreover, W is not a proper subspace of any neutral subspace of V . Since the closure of a neutral subspace is neutral, a maximal neutral subspace is necessarily closed. A subspace W is called *hypermaximal neutral* iff $W = W^\perp$. This implies in particular that W is maximal neutral. In case of a complex Krein space we use in addition the adjective *real* if the subspace in question is a real subspace only and its property refers to the real part of the inner product only.

Let V_1, V_2 be Krein spaces with decompositions $V_1 = V_1^+ \oplus V_1^-$ and $V_2 = V_2^+ \oplus V_2^-$. A subspace $W \subset V_1$ is called *adapted* iff W admits a decomposition as a direct sum $W = W^+ \oplus W^-$ such that $W^+ \subseteq V_1^+$ and $W^- \subseteq V_1^-$. A linear map $a : V_1 \rightarrow V_2$ is called an *isometry* iff a preserves the inner product. It is called an *adapted isometry* if in addition a respects the decompositions

of V_1 and V_2 by mapping V_1^+ to V_2^+ and V_1^- to V_2^- . A linear map $a : V_1 \rightarrow V_2$ is called a *anti-isometry* iff a reverses the sign of the inner product. It is called an *adapted anti-isometry* if in addition a respects the decompositions of V_1 and V_2 by mapping V_1^+ to V_2^- and V_1^- to V_2^+ . Note that an adapted isometry and an adapted anti-isometry are both isometries for the canonical Hilbert space structures of V_1 and V_2 . In particular, they are continuous. In case of a complex Krein space we use in addition the adjective *real* if the map in question is only real linear and is an isometry or anti-isometry only with respect to the real part of the inner product.

Lemma 3.1. *Let $V = V^+ \oplus V^-$ be a real Krein space. Then there is a natural one-to-one correspondence between involutive adapted anti-isometries $u : V \rightarrow V$ and hypermaximal neutral subspaces $W \subseteq V$. Moreover, this correspondence is such that W is the fixed point set of u .*

Proof. Let u be an involutive adapted anti-isometry $u : V \rightarrow V$. Define W to be the fixed point set of u . Then W is obviously a closed subspace of V . Moreover, we must have

$$\langle v, v \rangle = \langle u(v), u(v) \rangle = -\langle v, v \rangle \quad \forall v \in W.$$

That is, W is neutral. Since we are in the real case this implies $W \subseteq W^\perp$. Now let $v \in V \setminus W$. This implies $v - u(v) \neq 0$. By the non-degeneracy of V there is $w \in V$ such that

$$0 \neq \langle w, v - u(v) \rangle = \langle w, v \rangle - \langle w, u(v) \rangle = \langle w, v \rangle + \langle u(w), v \rangle = \langle w + u(w), v \rangle.$$

Since $w + u(w) \in W$ this implies $v \notin W^\perp$. Hence $W^\perp \subseteq W$. Combining this with the above result yields $W = W^\perp$, i.e., W is hypermaximal neutral.

Conversely, suppose that W is a hypermaximal neutral subspace of V . Let $M \subseteq V^+$ be the subset of vectors $v \in V^+$ such that there exists $w \in V^-$ with $v + w \in W$. It is easy to see that M is a subspace of V^+ and that it is closed (due to W being closed). In particular, M has an orthogonal complement N in the Hilbert space V^+ . Let $n \in N$ and $v \in W$. Decompose $v = v^+ + v^-$ with $v^+ \in V^+$ and $v^- \in V^-$. Then $v^+ \in M$, so n is orthogonal both to v^+ and to v^- . In particular, n is orthogonal to v . We see that any element of N is orthogonal to any element of W . Since W is hypermaximal neutral this implies $N \subseteq W$. On the other hand $V^+ \cap W = \{0\}$ so we must have $N = \{0\}$. This implies in turn $M = V^+$.

On the other hand given $v \in V^+$ let $w, w' \in V^-$ be such that $v + w$ and $v + w'$ are elements of W . Thus we have both $w - w' \in W$ and $w - w' \in V^-$. But $W \cap V^- = \{0\}$, so $w = w'$. Suppose now there was an involutive adapted anti-isometry $u : V \rightarrow V$ with fixed point set W . Then by adaptedness we would need to have $u(v) \in V^-$ while also having $v + u(v) \in W$ by involutiveness. Thus, we would necessarily have $u(v) = w$. We take this here as the definition of a map $u : V^+ \rightarrow V^-$ and define analogously $u : V^- \rightarrow V^+$. It is easy to see that the so defined map $u : V \rightarrow V$ is indeed adapted and involutive. To see that it is an anti-isomorphism take $v, w \in V$ with canonical decompositions $v = v^+ + v^-$ and $w = w^+ + w^-$. Then

$$\begin{aligned} \langle u(v^+ + v^-), u(w^+ + w^-) \rangle &= \langle u(v^+), u(w^+) \rangle + \langle u(v^-), u(w^-) \rangle \\ &= \langle v^+ + u(v^+), w^+ + u(w^+) \rangle - \langle v^+, w^+ \rangle + \langle v^- + u(v^-), w^- + u(w^-) \rangle - \langle v^-, w^- \rangle \\ &= -\langle v^+, w^+ \rangle - \langle v^-, w^- \rangle = -\langle v^+ + v^-, w^+ + w^- \rangle. \end{aligned}$$

This completes the proof. ■

Remark 3.2. The lemma shows in particular, that not every real Krein space admits a hypermaximal neutral subspace. Indeed, the existence of such a subspace is equivalent to the existence of an involutive adapted anti-isometry. The latter is equivalent to an anti-isometry between the positive and negative parts of the Krein space. Taking the Hilbert space inner product this is just an ordinary isometry between Hilbert spaces. Such an isometry thus exists precisely when both parts have the same cardinality.

Let V be a complex Krein space. Then V is also a real Krein space by forgetting the imaginary part of the inner product. Suppose V has a real hypermaximal neutral subspace W . Denote the complex structure of V by $J : V \rightarrow V$. We say that J and W are *compatible* iff the induced involutive adapted real anti-isometry $u : V \rightarrow V$ is conjugate linear, i.e., $u \circ J = -J \circ u$.

Lemma 3.3. *Let V be a complex Krein space with a compatible real hypermaximal neutral subspace W . Then V as a real vector space decomposes as a direct sum $V = W \oplus JW$. Moreover, JW is the eigenspace of the induced adapted real anti-isometry $u : V \rightarrow V$ with eigenvalue -1 . Also, JW is a real hypermaximal neutral subspace of V . Furthermore, W and JW are symplectic complements with respect to the symplectic form given by the imaginary part of the inner product of V .*

We leave the straightforward proof to the reader.

Lemma 3.4. *Let V be a complex Krein space and W be a closed Lagrangian subspace with respect to the imaginary part of the inner product. Then JW is also a closed Lagrangian subspace and V as a real Krein space has the orthogonal direct sum decomposition $V = W \oplus JW$. Moreover, the real linear map $u : V \rightarrow V$ defined as the identity on W and minus the identity on JW is a conjugate linear involutive real isometry of V . Also, if W is adapted then so are JW and u .*

This lemma is essentially a straightforward generalization of Lemma 4.1 in [9] to the Krein space case. We leave the proof to the reader. In the following it will be useful to view the structures appearing in Lemmas 3.1 and 3.3 on the one hand and in Lemma 3.4 on the other hand from a common perspective. To this end let $\kappa = -1$ in the former and $\kappa = 1$ in the latter case. The map u is then a conjugate linear involution with the property

$$\langle u(v), u(v') \rangle = \kappa \overline{\langle v, v' \rangle}. \quad (3.1)$$

Moreover the map u has precisely the structure of a *complex conjugation* or *real structure* on the Krein space V .

Let V be a Krein space with decomposition $V = V^+ \oplus V^-$. We say that a subset $B \subseteq V$ is an *orthonormal basis* (short: ON-basis) of V iff B is the union of subsets $B^+ \subseteq V^+$ and $B^- \subseteq V^-$ such that B^+ is an ON-basis of V^+ as a Hilbert space and B^- is an ON-basis of $\overline{V^-}$.

3.2 Fock space

We recall in the following some elementary facts about Fock space and introduce our notation. It will be convenient to treat both the bosonic and the fermionic case at once. To this end we introduce the constant κ , defined as

$$\kappa := 1 \quad \text{in the bosonic case,} \quad \kappa := -1 \quad \text{in the fermionic case.}$$

The fermionic or bosonic Fock space \mathcal{F} is a Krein space which arises as the completion of an \mathbb{N}_0 -graded Krein space,

$$\mathcal{F} = \widehat{\bigoplus_{n=0}^{\infty} \mathcal{F}_n}.$$

To denote the degree we write

$$|\psi| := n \quad \text{if } \psi \in \mathcal{F}_n.$$

We refer to this grading as the *Fock grading*. Another important grading is the *fermionic grading* or *f-grading*. This is a \mathbb{Z}_2 -grading defined as follows. All elements of a bosonic Fock space are

assigned degree zero. Elements of a fermionic Fock space have degree zero if their Fock degree is even and degree one if their Fock degree is odd.

The space \mathcal{F}_0 is the canonical one-dimensional Hilbert space over \mathbb{C} . We write it also as $\mathcal{F}_0 = \mathbf{1}\mathbb{C}$ with $\langle \mathbf{1}, \mathbf{1} \rangle = 1$. If L is the Krein space that *generates* \mathcal{F} , then \mathcal{F}_n is defined to be the space of continuous n -linear maps from the n -fold direct product of L with itself to \mathbb{C} that are either symmetric (bosonic case) or anti-symmetric (fermionic case). Thus

$$\mathcal{F}_n := \{ \psi : L \times \cdots \times L \rightarrow \mathbb{C}, \text{ } n\text{-linear continuous} : \psi \circ \sigma = \kappa^{|\sigma|} \psi, \forall \sigma \in S^n \}.$$

Here S^n denotes the symmetric group acting on $L \times \cdots \times L$, while $|\sigma|$ equals 0 or 1 depending on whether $\sigma \in S^n$ is even or odd.

Let $\xi_1, \dots, \xi_n \in L$. We define a corresponding element $\psi[\xi_1, \dots, \xi_n] \in \mathcal{F}_n$ as follows

$$\psi[\xi_1, \dots, \xi_n](\eta_1, \dots, \eta_n) := \frac{1}{n!} \sum_{\sigma \in S^n} \kappa^{|\sigma|} \prod_{i=1}^n \{ \xi_i, \eta_{\sigma(i)} \}.$$

Here, $\{ \cdot, \cdot \}$ denotes the inner product in L . Note that this expression is conjugate linear in the variables ξ_1, \dots, ξ_n . We define the inner product between these elements in \mathcal{F}_n as

$$\langle \psi[\eta_1, \dots, \eta_n], \psi[\xi_1, \dots, \xi_n] \rangle := 2^n \sum_{\sigma \in S^n} \kappa^{|\sigma|} \prod_{i=1}^n \{ \xi_i, \eta_{\sigma(i)} \}. \quad (3.2)$$

This makes \mathcal{F}_n into a Krein space with the subspace generated by elements of the form $\psi[\xi_1, \dots, \xi_n]$ dense. We shall refer to these special states as *generating states*.

Let $L = L^+ \oplus L^-$ be the canonical decomposition of L as a Krein space. We define subspaces of \mathcal{F}_n ,

$$\begin{aligned} \mathcal{F}_n^+ &:= \{ \psi \in \mathcal{F}_n : \psi(\xi_1, \dots, \xi_n) = 0 \text{ if } [\xi_1] + \cdots + [\xi_n] \text{ odd} \}, \\ \mathcal{F}_n^- &:= \{ \psi \in \mathcal{F}_n : \psi(\xi_1, \dots, \xi_n) = 0 \text{ if } [\xi_1] + \cdots + [\xi_n] \text{ even} \}. \end{aligned}$$

Then \mathcal{F}_n^+ is a complete positive-definite inner product space (a Hilbert space) while \mathcal{F}_n^- is a complete negative-definite inner product space. Moreover, \mathcal{F}_n is the orthogonal direct sum $\mathcal{F}_n = \mathcal{F}_n^+ \oplus \mathcal{F}_n^-$. This yields the Krein space structure of \mathcal{F}_n . Note also that $\mathcal{F}_0^+ = \mathcal{F}_0$ while $\mathcal{F}_0^- = \{0\}$. The induced canonical decomposition of \mathcal{F} as a Krein space we denote by $\mathcal{F}^+ \oplus \mathcal{F}^-$.

Lemma 3.5. *Let L be a Krein space. Suppose that $u : L \rightarrow L$ is a conjugate linear involutive adapted real isometry in the bosonic case or anti-isometry in the fermionic case. Define $U_n : \mathcal{F}_n \rightarrow \mathcal{F}_n$ by*

$$(U_n \psi)(\xi_1, \dots, \xi_n) := \overline{\psi(u(\xi_n), \dots, u(\xi_1))}. \quad (3.3)$$

Then U_n is a conjugate linear involutive adapted real anti-isometry in the fermionic case if n is odd and an isometry otherwise. Combining the maps U_n to a map $U : \mathcal{F} \rightarrow \mathcal{F}$ yields a conjugate linear involutive adapted real f -graded isometry in the sense,

$$\langle U \psi', U \psi \rangle = \kappa^{|\psi|} \overline{\langle \psi', \psi \rangle}.$$

We leave the straightforward proof to the reader. It is also straightforward to check the action of U on generating states,

$$U \psi[\xi_1, \dots, \xi_n] = \kappa^n \psi[u(\xi_n), \dots, u(\xi_1)] \quad \forall \xi_1, \dots, \xi_n \in L. \quad (3.4)$$

Note also that U has the structure of a *complex conjugation* or *real structure* for the Fock space \mathcal{F} .

Let L be a Krein space that arises as the orthogonal direct sum of Krein spaces $L = L_1 \oplus L_2$. Let $m, n \in \mathbb{N}_0$. We write elements in L as pairs (η, ξ) with $\eta \in L_1$ and $\xi \in L_2$. Then we have an isometric morphism of Krein spaces $\mathcal{F}_m(L_1) \hat{\otimes} \mathcal{F}_n(L_2) \rightarrow \mathcal{F}_{m+n}(L)$ given by $\psi_1 \otimes \psi_2 \mapsto \psi$, where

$$\begin{aligned} & \psi((\eta_1, \xi_1), \dots, (\eta_{m+n}, \xi_{m+n})) \\ & := \frac{1}{(m+n)!} \sum_{\sigma \in S^{m+n}} \kappa^{|\sigma|} \psi_1(\eta_{\sigma(1)}, \dots, \eta_{\sigma(m)}) \psi_2(\xi_{\sigma(m+1)}, \dots, \xi_{\sigma(m+n)}). \end{aligned} \quad (3.5)$$

For generating states this takes the form

$$\psi[\eta_1, \dots, \eta_m] \otimes \psi[\xi_1, \dots, \xi_n] \mapsto \psi[(\eta_1, 0), \dots, (\eta_m, 0), (0, \xi_1), \dots, (0, \xi_n)]. \quad (3.6)$$

Extending over all degrees we get an isometric isomorphism of Krein spaces $\mathcal{F}(L_1) \hat{\otimes} \mathcal{F}(L_2) \rightarrow \mathcal{F}(L)$, which is additive in the degree. Moreover, this isometric isomorphism induces the natural isometric isomorphism $\mathcal{F}(L_1) \hat{\otimes} \mathcal{F}(L_2) \rightarrow \mathcal{F}(L_2) \hat{\otimes} \mathcal{F}(L_1)$ given by

$$\psi_1 \otimes \psi_2 \mapsto \kappa^{|\psi_1| \cdot |\psi_2|} \psi_2 \otimes \psi_1. \quad (3.7)$$

Let L be a non-trivial separable Krein space. Let $N = \{1, \dots, \dim L\}$ if L is finite-dimensional and $N = \mathbb{N}$ otherwise. Let $\{\xi_i\}_{i \in N}$ be an ON-basis of L . Then an ON-basis of L is given in the fermionic case by

$$\left\{ \frac{1}{\sqrt{2^m}} \psi[\xi_{a_1}, \dots, \xi_{a_m}] \right\}_{m \in \mathbb{N}_0, a_1 < \dots < a_m \in N}, \quad (3.8)$$

and in the bosonic case by

$$\left\{ \frac{1}{\sqrt{2^m K_{a_1, \dots, a_m}}} \psi[\xi_{a_1}, \dots, \xi_{a_m}] \right\}_{m \in \mathbb{N}_0, a_1 \leq \dots \leq a_m \in N}. \quad (3.9)$$

Here K_{a_1, \dots, a_m} denotes the number of ways the indices a_1, \dots, a_m can be permuted without changing the value of the m -tuple (a_1, \dots, a_m) . In particular, K_{a_1, \dots, a_m} differs from 1 only if there are coincidences between some of the indices a_1, \dots, a_m . We also observe that the bosonic Fock space over L is always countably infinite-dimensional while the fermionic Fock space over L is countably infinite-dimensional only if L is. If L has dimension d , then the fermionic Fock space over L has dimension 2^d .

4 Classical data

In this section we provide an axiomatic description of a linear classical field theory, either fermionic or bosonic. Here the attribute ‘‘classical’’ has to be taken with a grain of salt as we already include certain data that belongs more properly into the quantum realm, notably complex structures.

4.1 Geometric data

We recall briefly the formalization of the notion of spacetime in terms of a *spacetime system* in the GBF. The presentation here is a refined version of previous presentations such as the one in [9].

There is a fixed positive integer $d \in \mathbb{N}$, the *dimension* of spacetime. We are given a collection \mathcal{M}_0^c of connected oriented topological manifolds of dimension d , possibly with boundary,

that we call connected *regular regions*. Furthermore, there is a collection \mathcal{M}_1^c of connected oriented topological manifolds without boundary of dimension $d - 1$ that we call *hypersurfaces*. The manifolds are either abstract manifolds or they are all concrete submanifolds of a given fixed *spacetime manifold*. In the former case we call the spacetime system *local*, in the latter we call it *global*.

There is an operation of union both for regular regions and for hypersurfaces. This leads to the collection \mathcal{M}_0 , of all formal finite unions of elements of \mathcal{M}_0^c , and to the collection \mathcal{M}_1 , of all formal finite unions of elements of \mathcal{M}_1^c . In case the spacetime system is global, only unions with members who are disjoint are allowed in \mathcal{M}_1 and only unions with members whose interiors are disjoint are allowed in \mathcal{M}_0 . For simplicity, and to capture the intuitive meaning of these unions we use the term *disjoint unions* uniformly. Note that in the global case the elements of \mathcal{M}_1 are actual submanifolds of the spacetime manifold. This is not the case for all elements of \mathcal{M}_0 as overlaps on boundaries may occur.

The collection \mathcal{M}_1 has to be closed under orientation reversal. That is, given $\Sigma \in \mathcal{M}_1$, the hypersurface $\bar{\Sigma}$ which is identical to Σ , but with reversed orientation is also in \mathcal{M}_1 . Also, any boundary of a regular region in \mathcal{M}_0 is a hypersurface in \mathcal{M}_1 . That is, taking the boundary defines a map $\partial : \mathcal{M}_0 \rightarrow \mathcal{M}_1$. When we want to emphasize explicitly that a given manifold is in one of those collections we also use the attribute *admissible*.

It is convenient to also introduce the concept of *slice regions*¹. A slice region is topologically simply a hypersurface, but thought of as an infinitesimally thin region. Concretely, the slice region associated to a hypersurface Σ will be denoted by $\hat{\Sigma}$ and its boundary is defined to be the disjoint union $\partial\hat{\Sigma} = \Sigma \cup \bar{\Sigma}$. There is one slice region for each hypersurface (forgetting its orientation). The reason for the terminology is that slice regions can be treated in certain respects in the same way as regular regions. Thus, we refer to regular regions and slice regions collectively as *regions*.

There is also a notion of *gluing* of regions. Suppose we are given a region M with its boundary a disjoint union $\partial M = \Sigma_1 \cup \Sigma \cup \bar{\Sigma}'$, where Σ' is a copy of Σ (Σ_1 may be empty). Then we may obtain a new region M_1 by gluing M to itself along $\Sigma, \bar{\Sigma}'$. That is, we identify the points of Σ with corresponding points of Σ' to obtain M_1 . The resulting region M_1 might be inadmissible, in which case the gluing is not allowed.

Depending on the theory one wants to model, the manifolds may carry additional structure such as for example a differentiable structure or a metric. This has to be taken into account in the gluing and will modify the procedure as well as its admissibility in the first place. Our description above is merely meant as a minimal one. Moreover, there might be important information present in different ways of identifying the boundary hypersurfaces that are glued. Such a case can be incorporated into our present setting by encoding this information explicitly through suitable additional structure on the manifolds.

4.2 Axioms of the classical theory

Given a spacetime system, we axiomatize a linear classical theory on the spacetime system as follows. As previously, we set $\kappa = -1$ in the fermionic case and $\kappa = 1$ in the bosonic case. In the bosonic case these axioms coincide with those previously proposed in [9] up to the small difference that we allow for Krein spaces here rather than only for Hilbert spaces.

- (C1) Associated to each hypersurface Σ is a complex separable Krein space L_Σ with indefinite inner product denoted by $\{\cdot, \cdot\}_\Sigma$. We also define $g_\Sigma(\cdot, \cdot) := \Re\{\cdot, \cdot\}_\Sigma$ and $\omega_\Sigma(\cdot, \cdot) := \frac{1}{2}\Im\{\cdot, \cdot\}_\Sigma$ and denote by $J_\Sigma : L_\Sigma \rightarrow L_\Sigma$ the scalar multiplication with i in L_Σ .

¹In previous papers slice regions were called “empty regions”. We modify our terminology here in order to make it more descriptive and at the same time avoid possible confusion with the empty set.

- (C2) Associated to each hypersurface Σ there is a conjugate linear involution $L_\Sigma \rightarrow L_{\overline{\Sigma}}$, written as an identification, under which the inner product is transformed as follows

$$\{\phi', \phi\}_{\overline{\Sigma}} = \kappa \overline{\{\phi', \phi\}_\Sigma} \quad \forall \phi, \phi' \in L_\Sigma. \quad (4.1)$$

- (C3) Suppose the hypersurface Σ decomposes into a disjoint union of hypersurfaces $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_n$. Then there is an isometric isomorphism of complex Krein spaces $L_{\Sigma_1} \oplus \dots \oplus L_{\Sigma_n} \rightarrow L_\Sigma$. We will not write this map explicitly, but rather think of it as an identification.

- (C4) Associated to each region M is a real vector space L_M .

- (C5) Associated to each region M there is a linear map of real vector spaces $r_M : L_M \rightarrow L_{\partial M}$. In the fermionic case the image $L_{\tilde{M}}$ of r_M is a real hypermaximal neutral subspace of $L_{\partial M}$, compatible with the complex structure $J_{\partial M}$. In the bosonic case the image $L_{\tilde{M}}$ of r_M is a closed adapted Lagrangian subspace of $L_{\partial M}$.

- (C6) Let M_1 and M_2 be regions and $M := M_1 \cup M_2$ be their disjoint union. Then L_M is the orthogonal direct sum $L_M = L_{M_1} \oplus L_{M_2}$. Moreover, $r_M = r_{M_1} + r_{M_2}$.

- (C7) Let M be a region with its boundary decomposing as a disjoint union $\partial M = \Sigma_1 \cup \Sigma \cup \overline{\Sigma'}$, where Σ' is a copy of Σ . Let M_1 denote the gluing of M to itself along $\Sigma, \overline{\Sigma'}$ and suppose that M_1 is a region. Note $\partial M_1 = \Sigma_1$. Then there is an injective linear map $r_{M; \Sigma, \overline{\Sigma'}} : L_{M_1} \hookrightarrow L_M$ such that

$$L_{M_1} \hookrightarrow L_M \rightrightarrows L_\Sigma \quad (4.2)$$

is an exact sequence. Here the arrows on the right hand side are compositions of the map r_M with the orthogonal projections of $L_{\partial M}$ to L_Σ and $L_{\overline{\Sigma'}}$ respectively (the latter identified with L_Σ). Moreover, the following diagram commutes, where the bottom arrow is the orthogonal projection,

$$\begin{array}{ccc} L_{M_1} & \xrightarrow{r_{M; \Sigma, \overline{\Sigma'}}} & L_M \\ r_{M_1} \downarrow & & \downarrow r_M \\ L_{\partial M_1} & \longleftarrow & L_{\partial M} \end{array}$$

We add the following observations: g_Σ is a real symmetric non-degenerate bilinear form making L_Σ into a real Krein space. ω_Σ is a real anti-symmetric non-degenerate bilinear form making L_Σ into a symplectic vector space. Also

$$g_{\overline{\Sigma}} = \kappa g_\Sigma, \quad J_{\overline{\Sigma}} = -J_\Sigma, \quad \omega_{\overline{\Sigma}} = -\kappa \omega_\Sigma.$$

Moreover, for all $\phi, \phi' \in L_\Sigma$,

$$\begin{aligned} g_\Sigma(\phi, \phi') &= 2\omega_\Sigma(\phi, J_\Sigma \phi'), & \{\phi, \phi'\}_\Sigma &= g_\Sigma(\phi, \phi') + 2i\omega_\Sigma(\phi, \phi'), \\ g_\Sigma(\phi, \phi') &= g_\Sigma(J_\Sigma \phi, J_\Sigma \phi'), & \omega_\Sigma(\phi, \phi') &= \omega_\Sigma(J_\Sigma \phi, J_\Sigma \phi'). \end{aligned}$$

The structures coming from Lemma 3.3 in the fermionic case or Lemma 3.4 in the bosonic case for a region M will play an important role in the following. In the fermionic case the real subspace $L_{\tilde{M}} \subseteq L_{\partial M}$ is a compatible real hypermaximal neutral subspace and we have the real direct sum decomposition $L_{\partial M} = L_{\tilde{M}} \oplus JL_{\tilde{M}}$. Moreover, $L_{\tilde{M}}$ and $JL_{\tilde{M}}$ are both hypermaximal neutral subspaces with respect to $g_{\partial M}$ and they are symplectic complements with respect to $\omega_{\partial M}$. In the bosonic case the real subspace $L_{\tilde{M}} \subseteq L_{\partial M}$ is a closed adapted Lagrangian subspace and we have the real orthogonal direct sum decomposition $L_{\partial M} = L_{\tilde{M}} \oplus JL_{\tilde{M}}$. Moreover, $L_{\tilde{M}}$ and $JL_{\tilde{M}}$ are both closed adapted Lagrangian subspaces with respect to $\omega_{\partial M}$. In both cases, we denote the associated complex conjugation by $u_M : L_{\partial M} \rightarrow L_{\partial M}$.

4.3 A notion of evolution

For a region M the physical interpretation of the space L_M is that of the space of solutions of the equations of motions inside M . It is as such naturally a subspace of the space $L_{\partial M}$ of germs of solutions on the boundary of M . It is in this sense that we may think of the spaces L_M as encoding the “dynamics” of a theory and of the spaces $L_{\partial M}$ the “kinematics”. In order to connect this to a more traditional view of “dynamics” versus “kinematics” we make use of the map u_M .

Lemma 4.1. *Suppose that $L_2 \oplus L_1 = L_{\partial M}$ is a decomposition as an orthogonal direct sum of complex Krein spaces such that $u_M(L_1) = L_2$. Then we have the identity*

$$L_{\tilde{M}} = \{(u_M(\phi), \phi) : \phi \in L_1\}.$$

The proof is immediate from the properties of u_M exhibited in Section 3.1. This statement admits the following interpretation: u_M (restricted to L_1) describes the “evolution” of the “initial data” encoded in L_1 to the “final data” encoded in L_2 . Of course, the role of L_1 and L_2 may be interchanged at will, since u_M is involutive. Moreover, there may be many such decompositions of $L_{\partial M}$, leading to different such notions of “evolution”.

Such an algebraic notion of “evolution” of classical data may be seen to have a geometric underpinning when the underlying decomposition of the space $L_{\partial M}$ arises from a corresponding decomposition of the boundary hypersurface ∂M .

Proposition 4.2. *Suppose that ∂M decomposes as a disjoint union $\partial M = \overline{\Sigma_2} \cup \Sigma_1$ such that $u_M(L_{\Sigma_1}) = L_{\overline{\Sigma_2}}$. Then the restriction of u_M to L_{Σ_1} viewed as a map $\tilde{u}_M : L_{\Sigma_1} \rightarrow L_{\Sigma_2}$ is a complex linear isometric isomorphism of Krein spaces.*

Proof. Preservation of the inner product follows from the combination of property (3.1) of u_M with property (4.1) associated to hypersurface orientation change according to axiom (C2). Complex linearity comes from the fact that both u_M and the map implementing orientation change are complex conjugate linear. ■

In this context the “initial data” L_{Σ_1} and the “final data” L_{Σ_2} are really localized on separate hypersurfaces, giving stronger justification to the attributes “initial” and “final”.

To make the link with a traditional notion of dynamics even more pertinent consider the following setting. Suppose there is a fixed spacetime manifold with a metric. Moreover, suppose the metric is Lorentzian and the spacetime is globally hyperbolic. Now consider only hypersurfaces that are spacelike (and possibly have some additional nice properties). Consider regions that are submanifolds of spacetime, bounded by pairs of such hypersurfaces. Adding finite disjoint unions yields a spacetime system. Suppose that a classical field theory is given by hyperbolic partial differential equations of motion so that all the admissible connected hypersurfaces are Cauchy. Supposing the theory is Lagrangian, proceeding roughly along the lines sketched in Sections 2.1 and 2.2, we should obtain a model satisfying the axioms of Section 4.2. Crucially, we are assuming here that in addition to the purely classical information, sensible complex structures exist on the hypersurfaces. Suppose moreover that the complex structures for the different hypersurfaces are chosen in a compatible way (e.g., if they come from a global complex structure). We would then have the assumption $u_M(L_{\Sigma_1}) = L_{\overline{\Sigma_2}}$ of Proposition 4.2 satisfied for any connected admissible region M with initial boundary component Σ_1 and final boundary component Σ_2 . (Here we take all admissible connected hypersurfaces to be oriented as past boundaries of an admissible region lying in the future.) Then \tilde{u}_M literally is the time-evolution map from the initial data L_{Σ_1} on Σ_1 to the final data L_{Σ_2} on Σ_2 . For future reference we shall refer to this setting as the *standard globally hyperbolic setting*.

So far we have treated here the bosonic and fermionic case on the same footing. However, the axioms of Section 4.2 imply a remarkable asymmetry between the two cases with respect to the (algebraic) notion of “evolution”. In contrast to the bosonic case the fermionic case exhibits a *preferred* decomposition of $L_{\partial M}$ of the required type, inducing thus a *preferred* (algebraic) notion of “evolution”. This is the decomposition into positive and negative parts $L_{\partial M} = L_{\partial M}^+ \oplus L_{\partial M}^-$, which satisfies naturally $u_M(L_{\partial M}^+) = L_{\partial M}^-$. Moreover, for standard examples of free fermionic field theories in globally hyperbolic spacetimes formalized along the lines sketched in the previous paragraph, this decomposition is precisely the one induced by the geometric decomposition of ∂M into initial and final hypersurface. More formally we have $L_{\partial M}^+ = L_{\Sigma_1}$ and $L_{\partial M}^- = L_{\Sigma_2}$ if $\partial M = \Sigma_1 \cup \Sigma_2$ as above². In Section 5 we shall exhibit this concretely for the example of the Dirac field in Minkowski spacetime. On the other hand, the preferred algebraic notion of “evolution” is present even if the spacetime system does not come equipped with a notion of time. Moreover, it behaves very much like time-evolution in that it is directed and compatible with composition of regions etc. This suggests to speak of an *emergent notion of time* inherent in fermionic theories satisfying the axioms of Section 4.2.

5 The free Dirac field in Minkowski spacetime

In this section we review the free Dirac field in Minkowski spacetime and show explicitly how it fits into the axiomatic system of Section 4.2.

5.1 The real inner product

We consider the free Dirac field in Minkowski spacetime. We use the γ -matrices and other notation of high energy physics such as in [15]. The field X representing the Dirac spinor is a section of a trivial vector bundle over Minkowski spacetime with fiber isomorphic to \mathbb{C}^4 . The Lagrangian is

$$L(X) = -\Im(X^\dagger \gamma^0 \gamma^\mu \partial_\mu X) - m X^\dagger \gamma^0 X.$$

Given a hypersurface Σ the space L_Σ is essentially the space of spinors restricted to Σ . The Lagrangian yields on it the symplectic structure³

$$\tilde{\omega}_\Sigma(X, Y) = \int_\Sigma \Im(X^\dagger \gamma^0 \gamma^\mu Y) n_\mu d^3x.$$

Here d^3x denotes the 3-form induced from the metric in the hypersurface and n denotes the normal vector to the hypersurface. This is to be understood in the sense

$$n_\mu d^3x := \partial_\mu \lrcorner d^4x,$$

where the right-hand side denotes contraction of the partial derivative ∂_μ as a vector field with the volume 4-form d^4x of Minkowski spacetime. To fix signs on the left-hand side we introduce the additional convention that the integral over d^3x is locally always positive so that the orientation information of the hypersurface resides exclusively in the sign of n .

Using the complex structure of the fiber, \tilde{J} is simply the natural multiplication with i of the spinor X understood as a complex vector field. This yields the symmetric real bilinear form on L_Σ ,

$$g_\Sigma(X, Y) = 2\tilde{\omega}_\Sigma(X, iY) = 2 \int_\Sigma \Re(X^\dagger \gamma^0 \gamma^\mu Y) n_\mu d^3x.$$

²We have fixed here an overall choice of sign. The other choice would interchange $L_{\partial M}^+$ and $L_{\partial M}^-$.

³We use the sign conventions for the symplectic structure as in [6].

In order to analyze its properties it will be convenient to rewrite it as follows

$$g_{\Sigma}(X, Y) = 2 \int_{\Sigma} \Re(X^{\dagger} P Y) d^3x, \quad (5.1)$$

where

$$P(x) := \gamma^0 \gamma^{\mu} n_{\mu}(x)$$

is a complex 4×4 -matrix valued function on Σ . We note that $P(x)$ is self-adjoint, since $\gamma^0 \gamma^{\mu}$ is self-adjoint and $n_{\mu}(x)$ is real. In particular, we may decompose $P(x)$ as $P(x) = P^{+}(x) + P^{-}(x)$, where $P^{+}(x)$ is positive and $P^{-}(x)$ is negative. If $P(x)$ is non-degenerate the decomposition is unique with $P^{+}(x)$ strictly positive and $P^{-}(x)$ strictly negative. As we will see this is the case if n is not light-like.

For simplicity we shall only consider the cases that Σ is either completely spacelike or completely timelike. We start with the spacelike case. If Σ is in particular an equal-time hypersurface we have

$$n(x) = (1, 0, 0, 0) \quad \text{or} \quad n(x) = (-1, 0, 0, 0) \quad (5.2)$$

depending on the orientation of Σ . Since $\gamma^0 \gamma^0 = \mathbf{1}$ this implies $P(x) = \mathbf{1}$ in the first case and $P(x) = -\mathbf{1}$ in the second⁴. With the positivity of the integral this means that the real inner product g_{Σ} given by (5.1) is either positive-definite or negative-definite.

Suppose now that Σ is an arbitrary spacelike hypersurface in Minkowski spacetime. Picking a point $x \in \Sigma$ we can always use a Lorentz transformation locally to have $n(x)$ aligned with the time axis (i.e. of the form (5.2)). Since the rank of $P(x)$ must change continuously under Lorentz transformations, but is a discrete quantity, it remains constant. So $P(x)$ is strictly positive or strictly negative. Moreover, $P(x)$ must have the same sign for all $x \in \Sigma$. Thus g_{Σ} is positive-definite or negative definite depending on the orientation of Σ . For spacelike hypersurfaces there is a global notion of orientation since we may talk about the normal vector $n(x)$ being future or past pointing. In particular, if we orient them all in the same way the inner products g_{Σ} will be all positive-definite or all negative-definite. Since the inner product associated to spacelike hypersurfaces with a fixed orientation is the starting point for conventional quantization prescriptions, it is natural there to consider only positive-definite inner products.

We proceed to consider the case that the hypersurface Σ is timelike. Proceeding in a manner similar to that of the spacelike case it will be sufficient for our purposes to consider just one special normal vector. All others are related to this by a Lorentz transformation, due to their spacelike nature. We choose

$$n(x) = (0, 0, 0, 1).$$

With either the standard or the chiral representation of the γ -matrices [15] this yields

$$P(x) = -\gamma^0 \gamma^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We read off immediately that $P(x)$ has one positive eigenvalue, 1, and one negative eigenvalue, -1 , each with an associated two-dimensional eigenspace. By the same argument as in the spacelike case we may conclude that a decomposition of $P(x)$ into a positive and a negative

⁴We use the sign convention $(1, -1, -1, -1)$ for the metric.

part $P(x) = P^+(x) + P^-(x)$, each of rank 2, holds everywhere on the hypersurface. Moreover, it holds for any timelike hypersurface. (Note the symmetry of the statement under orientation reversal.)

The decomposition of P induces a corresponding decomposition of the space L_Σ of sections of the spinor bundle on Σ induced by the eigenspace decomposition of each fiber. That is, we have the orthogonal decomposition $L_\Sigma = L_\Sigma^+ \oplus L_\Sigma^-$, where L_Σ^+ is positive-definite and L_Σ^- is negative-definite.

5.2 Plane waves

As already mentioned, the complex structure is in general not local in the field, making its discussion in the abstract setting we have used so far difficult. It is convenient, and for our motivational purposes here sufficient, to consider a global space of solutions of the Dirac equation in Minkowski spacetime in terms of plane wave spinors. Moreover, we shall consider only two special choices of hypersurface, one spacelike and one timelike. In the treatment of spinors we follow mostly text book style conventions, in particular those of [15].

Recall the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)X = 0.$$

We expand its solutions in Minkowski spacetime in terms of plane waves in the standard way as

$$X(t, x) = \int \frac{d^3k}{(2\pi)^3 2E} \sum_{s=1,2} \left(X_a^s(k) u^s(k) e^{-i(Et-kx)} + \overline{X_b^s(k)} v^s(k) e^{i(Et-kx)} \right). \quad (5.3)$$

Here u^s and v^s with $s \in \{1, 2\}$ are certain spinors in momentum space. That is, they are complex 4-dimensional vector valued functions on momentum space \mathbb{R}^3 , forming a convenient basis of the solutions of the equations

$$(\gamma^\mu k_\mu - m)u^s(k) = 0 \quad \text{and} \quad (\gamma^\mu k_\mu + m)v^s(k) = 0. \quad (5.4)$$

They satisfy moreover the following properties

$$u^{r\dagger}(k) \gamma^0 \gamma^\mu u^s(k) = 2k^\mu \delta^{r,s}, \quad v^{r\dagger}(k) \gamma^0 \gamma^\mu v^s(k) = 2k^\mu \delta^{r,s}, \quad (5.5)$$

$$u^{r\dagger}(k) v^s(-k) = 0, \quad v^{r\dagger}(k) u^s(-k) = 0. \quad (5.6)$$

Let Σ be an equal-time hypersurface located at time t . We shall denote the corresponding quantities with a subscript t rather than a subscript Σ . For the space L_t we simply take the global solution space parametrized by (5.3). The inner product g_t is given by (5.1) with $P(x) = 1$. This yields

$$g_t(X, Y) = 2 \int \frac{d^3k}{(2\pi)^3 2E} \sum_{s=1,2} \Re \left(\overline{X_a^s(k)} Y_a^s(k) + X_b^s(k) \overline{Y_b^s(k)} \right).$$

The example of a timelike hypersurface we shall consider is the hypersurface Σ located at a constant value of the coordinate x^3 , which we shall denote by z for simplicity. It is then convenient to parametrize the plane wave solutions in a slightly different (but strictly equivalent) way compared to (5.3). We base the parametrization on Fourier modes on the hypersurface Σ which we shall denote (E, \tilde{k}) where the ‘‘energy’’ E can be positive or negative while $\tilde{k} = (k_1, k_2)$. We restrict the integral over modes to $|E| \geq m$, which means that we only consider the propagating wave modes contained in (5.3) and exclude evanescent waves,

$$X(t, \tilde{x}, z) = \int_{|E| \geq m} \frac{dE d^2 \tilde{k}}{(2\pi)^3 2k_3} \quad (5.7)$$

$$\times \sum_{s=1,2} \left(\tilde{X}_a^s(E, \tilde{k}) \tilde{u}^s(E, \tilde{k}) e^{-i(Et - \tilde{k}\tilde{x} - k_3 z)} + \overline{\tilde{X}_b^s(E, \tilde{k})} \tilde{v}^s(E, \tilde{k}) e^{i(Et - \tilde{k}\tilde{x} - k_3 z)} \right).$$

Here \tilde{x} is a shorthand for $\tilde{x} = (x^1, x^2)$ and $k_3 = \sqrt{E^2 - \tilde{k}^2 - m^2}$ is chosen positive. The momentum space spinors \tilde{u}, \tilde{v} are defined in terms of the spinors u, v as follows

$$\tilde{u}^s(E, \tilde{k}) := \begin{cases} u^s(\tilde{k}, k_3) & \text{if } E > 0, \\ v^s(-\tilde{k}, -k_3) & \text{if } E < 0, \end{cases} \quad \tilde{v}^s(E, \tilde{k}) := \begin{cases} v^s(\tilde{k}, k_3) & \text{if } E > 0, \\ u^s(-\tilde{k}, -k_3) & \text{if } E < 0. \end{cases}$$

They satisfy the equations

$$(\gamma^0 E - \tilde{\gamma}\tilde{k} - \gamma^3 k_3 - m) \tilde{u}^s(E, \tilde{k}) = 0, \quad (\gamma^0 E - \tilde{\gamma}\tilde{k} - \gamma^3 k_3 + m) \tilde{v}^s(E, \tilde{k}) = 0,$$

analogous to (5.4). They have the following important properties analogous to (5.5), (5.6):

$$\tilde{u}^{r\dagger}(E, \tilde{k}) \gamma^0 \gamma^3 \tilde{u}^s(E, \tilde{k}) = -2 \frac{E}{|E|} k_3 \delta^{r,s}, \quad (5.8)$$

$$\tilde{v}^{r\dagger}(E, \tilde{k}) \gamma^0 \gamma^3 \tilde{v}^s(E, \tilde{k}) = -2 \frac{E}{|E|} k_3 \delta^{r,s}, \quad (5.9)$$

$$\tilde{u}^{r\dagger}(E, \tilde{k}) \gamma^0 \gamma^3 \tilde{v}^s(-E, -\tilde{k}) = 0, \quad (5.10)$$

$$\tilde{v}^{r\dagger}(E, \tilde{k}) \gamma^0 \gamma^3 \tilde{u}^s(-E, -\tilde{k}) = 0. \quad (5.11)$$

We denote quantities related to the hypersurface Σ of constant z with a subscript z . The inner product g_z is given by (5.1) with P as in (5.1). Using the identities (5.8)–(5.11), we obtain

$$g_z(X, Y) = 2 \int_{|E| \geq m} \frac{dE d^2 \tilde{k}}{(2\pi)^3 2k_3 |E|} \sum_{s=1,2} \Re \left(\overline{\tilde{X}_a^s(E, \tilde{k})} \tilde{Y}_a^s(E, \tilde{k}) + \tilde{X}_b^s(E, \tilde{k}) \overline{\tilde{Y}_b^s(E, \tilde{k})} \right).$$

We see that the space L_z of spinors on the hypersurface splits as a direct sum $L_z = L_z^+ \oplus L_z^-$ into a part L_z^+ with $E > 0$ where g_z is positive-definite and a part L_z^- where g_z is negative-definite.

It is also instructive to express this inner product in terms of the original parametrization (5.3) of the global solution space. The relation between the two parametrizations is the following

$$\tilde{X}_a^s(E, \tilde{k}) = \begin{cases} X_a^s(\tilde{k}, k_3) & \text{if } E > 0, \\ X_b^s(-\tilde{k}, -k_3) & \text{if } E < 0, \end{cases} \quad \tilde{X}_b^s(E, \tilde{k}) = \begin{cases} X_b^s(\tilde{k}, k_3) & \text{if } E > 0, \\ X_a^s(-\tilde{k}, -k_3) & \text{if } E < 0. \end{cases}$$

We obtain

$$g_z(X, Y) = 2 \int \frac{d^3 k}{(2\pi)^3 2E |k_3|} \sum_{s=1,2} \Re \left(\overline{X_a^s(k)} Y_a^s(k) + X_b^s(k) \overline{Y_b^s(k)} \right).$$

In this parametrization the subspaces L_z^+ and L_z^- are distinguished by the direction of the momentum component k_3 that is perpendicular to the hypersurface.

5.3 The complex structure

We proceed to consider the complex structure J_Σ which, together with the real inner product, gives rise to the complex inner product on L_Σ . As already mentioned this encodes the distinction between “positive energy” and “negative energy” solutions and is not a field-local object.

In the case where the hypersurface Σ is a hypersurface of constant time t the standard complex structure J_t is given by

$$(J_t X)_a^s(k) = i X_a^s(k), \quad (J_t X)_b^s(k) = i X_b^s(k).$$

Looking at the parametrization (5.3) this means that solutions with a temporal dependence of the form e^{-iEt} are multiplied by i , while solutions with a temporal dependence of the form e^{iEt} are multiplied by $-i$. (Here $E > 0$ by convention.) The symplectic form according to (2.2) results to be

$$\omega_t(X, Y) = \int \frac{d^3k}{(2\pi)^3 2E} \sum_{s=1,2} \Im \left(\overline{X_a^s(k)} Y_a^s(k) + \overline{X_b^s(k)} Y_b^s(k) \right). \quad (5.12)$$

Combining real inner product and symplectic form according to (2.1) yields

$$\{X, Y\}_t = 2 \int \frac{d^3k}{(2\pi)^3 2E} \sum_{s=1,2} \left(\overline{X_a^s(k)} Y_a^s(k) + \overline{X_b^s(k)} Y_b^s(k) \right). \quad (5.13)$$

This is indeed the usual complex inner product for the 1-particle space of the Dirac field.

We proceed to the timelike case with Σ being the hypersurface of constant coordinate value $z = x^3$. The analogue of “positive energy” versus “negative energy” solutions is now given by the distinction between solutions with a dependence in z -direction of the form e^{ik_3} versus e^{-ik_3} in the parametrization (5.7). (Here $k_3 > 0$ by convention.) That is, the complex structure J_z is given by

$$(J_z \tilde{X})_a^s(E, \tilde{k}) = i \tilde{X}_a^s(E, \tilde{k}), \quad (J_z \tilde{X})_b^s(E, \tilde{k}) = i \tilde{X}_b^s(E, \tilde{k}).$$

With (2.2) we obtain the symplectic form

$$\omega_z(X, Y) = \int_{|E| \geq m} \frac{dE d^2\tilde{k}}{(2\pi)^3 2k_3} \frac{E}{|E|} \sum_{s=1,2} \Im \left(\overline{\tilde{X}_a^s(E, \tilde{k})} \tilde{Y}_a^s(E, \tilde{k}) + \overline{\tilde{X}_b^s(E, \tilde{k})} \tilde{Y}_b^s(E, \tilde{k}) \right).$$

Combining real inner product and symplectic form according to (2.1) yields the complex inner product

$$\{X, Y\}_z = 2 \int_{|E| \geq m} \frac{dE d^2\tilde{k}}{(2\pi)^3 2k_3} \frac{E}{|E|} \sum_{s=1,2} \left(\overline{\tilde{X}_a^s(E, \tilde{k})} \tilde{Y}_a^s(E, \tilde{k}) + \overline{\tilde{X}_b^s(E, \tilde{k})} \tilde{Y}_b^s(E, \tilde{k}) \right). \quad (5.14)$$

Remarkably, expressing the symplectic form ω_z in terms of the standard field parametrization (5.3) yields

$$\omega_z(X, Y) = \int \frac{d^3k}{(2\pi)^3 2E} \sum_{s=1,2} \Im \left(\overline{X_a^s(k)} Y_a^s(k) + \overline{X_b^s(k)} Y_b^s(k) \right),$$

identical to (5.12) for ω_t . This suggests that the symplectic structure here is the same not only for all reasonable spacelike hypersurfaces (which is known), but also for a class of timelike hypersurfaces.

5.4 Dynamics and axiomatic form

We proceed to integrate the structures derived so far in terms of the axiomatic system of Section 4.2, including the already implicit dynamics. We start with the case of constant-time hypersurfaces. Thus, admissible hypersurfaces are constant-time hypersurfaces and their finite disjoint unions. Admissible regions are regions bounded by pairs of such hypersurfaces and their finite disjoint unions⁵. For a time $t \in \mathbb{R}$ the space L_t associated to the hypersurface Σ_t of

⁵Recall from Section 4.1, that “disjoint” only refers to the interior. An admissible region may well be the formal union of regions that share boundary components.

constant time t is a copy of the Hilbert space L . This in turn is the \mathcal{L}^2 -space of equivalence classes of square-integrable functions $X_{a,b}^s(k)$ with the inner product (5.13). Here, we take Σ_t to have the standard orientation by which we mean that it is oriented as the past boundary of an admissible region. The admissible regions all have the same orientation inherited from some global standard orientation of Minkowski space.

This yields the data of axiom (C1) for connected admissible hypersurfaces with standard orientation. For a connected admissible hypersurface Σ_t with the opposite orientation we define the associated space $L_{\bar{t}}$ simply as identical to L_t as a real vector space, but with opposite complex structure and inner product modified according to relation (4.1). Note in particular that the space $L_{\bar{t}}$ is thus negative-definite. This yields the data of axioms (C1) and (C2) for all connected admissible hypersurfaces. Next, we define the space associated to a finite disjoint union of connected hypersurfaces to be the direct sum of the spaces associated to the individual hypersurfaces. This yields the complete data for axioms (C1), (C2) and (C3).

Consider now a connected admissible region M . M has the structure of a time-interval $[t_1, t_2]$ times all of space \mathbb{R}^3 . We index associated structures with $[t_1, t_2]$. The space $L_{[t_1, t_2]}$ of solutions associated to this region is a copy of the global space of solutions L , understood merely as a real vector space without additional structure. For admissible regions that are disjoint unions of connected regions the associated real vector space is taken to be the direct sum of the vector spaces associated to the components. This yields the data for axiom (C4).

For axiom (C5) it will be sufficient to consider a connected admissible region. Say the region is determined by the time interval $[t_1, t_2]$. We define the map $r_{[t_1, t_2]} : L_{[t_1, t_2]} \rightarrow L_{t_1} \oplus L_{\bar{t}_2}$ to be given by

$$\phi \mapsto (\phi, \phi),$$

recalling that all the involved spaces are copies of L . It is also easy to verify that the image $L_{\widetilde{[t_1, t_2]}}$ of this map is a real hypermaximal subspace of $L_{t_1} \oplus L_{\bar{t}_2}$ since the real inner product on the latter is given by

$$g_{\partial[t_1, t_2]}((\phi_1, \phi_2), (\eta_1, \eta_2)) = g_t(\phi_1, \eta_1) - g_t(\phi_2, \eta_2).$$

So if $\phi_1 = \phi_2$ and $\eta_1 = \eta_2$ the result vanishes while for $\phi_1 \neq \phi_2$ we can always find some $\eta_1 = \eta_2$ giving a non-vanishing result. The according to Lemma 3.1 induced involutive anti-isometry $u_{[t_1, t_2]}$ takes the form

$$u_{[t_1, t_2]}((\phi, \eta)) = (\eta, \phi).$$

In particular, it anti-commutes with the complex structure since the latter takes opposite signs on the hypersurfaces at t_1 and at t_2 , due to opposite orientation. Thus, the compatibility condition with the complex structure is satisfied and so is axiom (C5) and also (C6).

With respect to the discussion in Section 4.3 note that the map $\tilde{u}_{[t_1, t_2]} : L_{t_1} \rightarrow L_{t_2}$ as defined in Proposition 4.2 and encoding time-evolution is here simply the identity between different copies of the same Hilbert space L . In particular, the preferred algebraic notion of evolution discussed in Section 4.3 coincides with the time-evolution. Moreover, given the fixed standard orientation all the connected admissible hypersurfaces L_t are Hilbert spaces.

For axiom (C7) it is sufficient to consider the case of a region M that is the disjoint union of regions $[t_1, t_2]$ and $[t_2, t_3]$ for $t_1 < t_2 < t_3$. Gluing amounts to producing the region $[t_1, t_3]$. It is then straightforward to check the details of the axiom, which we leave to the reader. This completes the construction of our model of the axioms encoding the Dirac field theory in Minkowski space with constant-time hypersurfaces.

We proceed to consider the setting based on hypersurfaces of constant z . The construction is very much analogous to the one just performed, with the role of the time coordinate t now played

by the space coordinate z . We will therefore limit ourselves to highlighting the differences. We equip the global solution space L to this end with the inner product (5.14). More precisely, we take L^+ to be the \mathcal{L}^2 -space of equivalence classes of square-integrable functions $\tilde{X}_{a,b}^s(E, \tilde{k})$ with $E > 0$ and the inner product (5.14). We take $\overline{L^-}$ to be the \mathcal{L}^2 -space of equivalence classes of square-integrable functions $\tilde{X}_{a,b}^s(E, \tilde{k})$ with $E < 0$ and the inner product the negative of (5.14). The two Hilbert spaces L^+ and $\overline{L^-}$ are then combined to the Krein space $L = L^+ \oplus \overline{L^-}$ with positive part L^+ and negative part $\overline{L^-}$. Apart from this difference the implementation of the axioms (C1)–(C7) is analogous to the setting with constant-time hypersurfaces.

Concerning the discussion in Section 4.3 the induced evolution map $\tilde{u}_{[z_1, z_2]} : L_{z_1} \rightarrow L_{z_2}$ from the hypersurface Σ_{z_1} to the hypersurface Σ_{z_2} is again the identity, this time between copies of the same Krein space (which is not a Hilbert space). However, this notion of spatial evolution is distinct from the preferred algebraic notion of evolution since the decomposition of the boundary solution space $L_{\partial[z_1, z_2]}$ into positive part $L_{\partial[z_1, z_2]}^+$ and negative part $L_{\partial[z_1, z_2]}^-$ now does not coincide with its decomposition into the hypersurface solution spaces L_{z_1} and $\overline{L_{z_2}}$. Indeed, the preferred algebraic notion of evolution continues to agree with time-evolution.

6 Axioms of the quantum theory

6.1 Core axioms of the GBF

The core axioms of the GBF are generalized here to include fermionic theories in addition to bosonic ones. Firstly, this is reflected in the appearance of a \mathbb{Z}_2 -grading on state spaces. Secondly, this is manifest in the fact that state spaces may be Krein spaces rather than merely Hilbert spaces. In the special case that all state spaces are purely of degree 0 and are moreover Hilbert spaces we recover the purely bosonic core axioms in the version put forward in [9].

- (T1) Associated to each hypersurface Σ is a complex separable \mathbb{Z}_2 -graded Krein space \mathcal{H}_Σ , called the *state space* of Σ . We denote its indefinite inner product by $\langle \cdot, \cdot \rangle_\Sigma$.
- (T1b) Associated to each hypersurface Σ is a conjugate linear adapted \mathbb{Z}_2 -graded isometry $\iota_\Sigma : \mathcal{H}_\Sigma \rightarrow \mathcal{H}_{\overline{\Sigma}}$. This map is an involution in the sense that $\iota_{\overline{\Sigma}} \circ \iota_\Sigma$ is the identity on \mathcal{H}_Σ .
- (T2) Suppose the hypersurface Σ decomposes into a disjoint union of hypersurfaces $\Sigma = \Sigma_1 \cup \dots \cup \Sigma_n$. Then there is an isometric isomorphism of Krein spaces $\tau_{\Sigma_1, \dots, \Sigma_n; \Sigma} : \mathcal{H}_{\Sigma_1} \hat{\otimes} \dots \hat{\otimes} \mathcal{H}_{\Sigma_n} \rightarrow \mathcal{H}_\Sigma$. The maps τ satisfy obvious associativity conditions. Moreover, in the case $n = 2$ the map $\tau_{\Sigma_2, \Sigma_1; \Sigma}^{-1} \circ \tau_{\Sigma_1, \Sigma_2; \Sigma} : \mathcal{H}_{\Sigma_1} \hat{\otimes} \mathcal{H}_{\Sigma_2} \rightarrow \mathcal{H}_{\Sigma_2} \hat{\otimes} \mathcal{H}_{\Sigma_1}$ is the \mathbb{Z}_2 -graded transposition

$$\psi_1 \otimes \psi_2 \mapsto (-1)^{|\psi_1| \cdot |\psi_2|} \psi_2 \otimes \psi_1.$$

- (T2b) Orientation change and decomposition are compatible in a \mathbb{Z}_2 -graded sense. That is, for a disjoint decomposition of hypersurfaces $\Sigma = \Sigma_1 \cup \Sigma_2$ we have

$$\tau_{\overline{\Sigma_1}, \overline{\Sigma_2}, \overline{\Sigma}}((\iota_{\Sigma_1} \otimes \iota_{\Sigma_2})(\psi_1 \otimes \psi_2)) = (-1)^{|\psi_1| \cdot |\psi_2|} \iota_\Sigma(\tau_{\Sigma_1, \Sigma_2; \Sigma}(\psi_1 \otimes \psi_2)).$$

- (T4) Associated with each region M is a \mathbb{Z}_2 -graded linear map from a dense subspace $\mathcal{H}_{\partial M}^\circ$ of the state space $\mathcal{H}_{\partial M}$ of its boundary ∂M (which carries the induced orientation) to the complex numbers, $\rho_M : \mathcal{H}_{\partial M}^\circ \rightarrow \mathbb{C}$. This is called the *amplitude* map.
- (T3x) Let Σ be a hypersurface. The boundary $\partial \hat{\Sigma}$ of the associated slice region $\hat{\Sigma}$ decomposes into the disjoint union $\partial \hat{\Sigma} = \overline{\Sigma} \cup \Sigma'$, where Σ' denotes a second copy of Σ . Then $\rho_{\hat{\Sigma}}$ is well defined on $\tau_{\overline{\Sigma}, \Sigma'; \partial \hat{\Sigma}}(\mathcal{H}_{\overline{\Sigma}} \hat{\otimes} \mathcal{H}_{\Sigma'}) \subseteq \mathcal{H}_{\partial \hat{\Sigma}}$. Moreover, $\rho_{\hat{\Sigma}} \circ \tau_{\overline{\Sigma}, \Sigma'; \partial \hat{\Sigma}}$ restricts to a bilinear pairing $(\cdot, \cdot)_\Sigma : \mathcal{H}_{\overline{\Sigma}} \times \mathcal{H}_{\Sigma'} \rightarrow \mathbb{C}$ such that $\langle \cdot, \cdot \rangle_\Sigma = (\iota_\Sigma(\cdot), \cdot)_\Sigma$.

(T5a) Let M_1 and M_2 be regions and $M := M_1 \cup M_2$ be their disjoint union. Then $\partial M = \partial M_1 \cup \partial M_2$ is also a disjoint union and $\tau_{\partial M_1, \partial M_2; \partial M}(\mathcal{H}_{\partial M_1}^\circ \otimes \mathcal{H}_{\partial M_2}^\circ) \subseteq \mathcal{H}_{\partial M}^\circ$. Moreover, for all $\psi_1 \in \mathcal{H}_{\partial M_1}^\circ$ and $\psi_2 \in \mathcal{H}_{\partial M_2}^\circ$,

$$\rho_M(\tau_{\partial M_1, \partial M_2; \partial M}(\psi_1 \otimes \psi_2)) = \rho_{M_1}(\psi_1)\rho_{M_2}(\psi_2).$$

(T5b) Let M be a region with its boundary decomposing as a disjoint union $\partial M = \Sigma_1 \cup \Sigma \cup \overline{\Sigma'}$, where Σ' is a copy of Σ . Let M_1 denote the gluing of M with itself along $\Sigma, \overline{\Sigma'}$ and suppose that M_1 is a region. Note $\partial M_1 = \Sigma_1$. Then $\tau_{\Sigma_1, \Sigma, \overline{\Sigma'}; \partial M}(\psi \otimes \xi \otimes \iota_\Sigma(\xi)) \in \mathcal{H}_{\partial M}^\circ$ for all $\psi \in \mathcal{H}_{\partial M_1}^\circ$ and $\xi \in \mathcal{H}_\Sigma$. Moreover, for any ON-basis $\{\zeta_i\}_{i \in I}$ of \mathcal{H}_Σ , we have for all $\psi \in \mathcal{H}_{\partial M_1}^\circ$,

$$\rho_{M_1}(\psi) \cdot c(M; \Sigma, \overline{\Sigma'}) = \sum_{i \in I} (-1)^{|\zeta_i|} \rho_M \left(\tau_{\Sigma_1, \Sigma, \overline{\Sigma'}; \partial M}(\psi \otimes \zeta_i \otimes \iota_\Sigma(\zeta_i)) \right), \quad (6.1)$$

where $c(M; \Sigma, \overline{\Sigma'}) \in \mathbb{C} \setminus \{0\}$ is called the *gluing anomaly factor* and depends only on the geometric data.

6.2 Vacuum axioms

In addition to the core axioms we shall impose the vacuum axioms, which we recall in the following. These formalize the notion of a vacuum state, suitably adapted to the GBF. Their present form is a slight adaption from their form in [14, Section 5.1]⁶.

- (V1) For each hypersurface Σ there is a distinguished state $\psi_{\Sigma,0} \in \mathcal{H}_\Sigma$ of degree 0, called the *vacuum state*.
- (V2) The vacuum state is compatible with the involution. That is, for any hypersurface Σ , $\psi_{\overline{\Sigma},0} = \iota_\Sigma(\psi_{\Sigma,0})$.
- (V3) The vacuum state is compatible with decompositions. Suppose the hypersurface Σ decomposes into components $\Sigma_1 \cup \dots \cup \Sigma_n$. Then $\psi_{\Sigma,0} = \tau_{\Sigma_1, \dots, \Sigma_n; \Sigma}(\psi_{\Sigma_1,0} \otimes \dots \otimes \psi_{\Sigma_n,0})$.
- (V5) The amplitude of the vacuum state is unity. That is, for any region M , $\rho_M(\psi_{\partial M,0}) = 1$.

7 Quantization

We proceed to specify in this section the quantization scheme that yields, starting from classical data satisfying the axioms of Section 4, the ingredients of a corresponding quantum theory. We shall also show here the validity of some of the core axioms of Section 6.1 for these data, namely (T1), (T1b), (T2), (T2b) and (T4). Moreover, we show that all of the vacuum axioms hold.

7.1 State spaces

The state space \mathcal{H}_Σ associated to a hypersurface Σ is the bosonic or fermionic Fock space $\mathcal{F}(L_\Sigma)$ in the sense of Section 3.2. The \mathbb{Z}_2 -grading of \mathcal{H}_Σ is the f-grading of $\mathcal{F}(L_\Sigma)$. This yields (T1). Moreover, we define the vacuum state $\psi_{\Sigma,0}$ to be the element $\mathbf{1}$ of the Fock space. Thus, (V1) is satisfied.

For $n \in \mathbb{N}$ define the map $\iota_n : \mathcal{F}_n(L_\Sigma) \rightarrow \mathcal{F}_n(L_{\overline{\Sigma}})$ as follows

$$(\iota_n(\psi))(\xi_1, \dots, \xi_n) := \overline{\psi(\xi_n, \dots, \xi_1)}.$$

⁶We recall that the strange numbering comes from the fact that in the original proposal [7] there was an additional axiom (V4) which was later made redundant due to a modification of the core axioms.

ι_n is a conjugate-linear adapted real isometry of Krein spaces in the bosonic case and in the fermionic case if n is even and a conjugate-linear real anti-isometry of Krein spaces in the fermionic case if n is odd. Also, ι_n is involutive. Combining these maps for all $n \in \mathbb{N}_0$ (for $n = 0$ take the identity map) and completing yields a conjugate-linear adapted real f -graded isometry $\mathcal{F}(L_\Sigma) \rightarrow \mathcal{F}(L_{\widehat{\Sigma}})$ which we denote as ι_Σ . This satisfies (T1b) and (V2). Given $\xi_1, \dots, \xi_n \in L_\Sigma$ we also note for later use

$$\iota_\Sigma(\psi[\xi_1, \dots, \xi_n]) = \kappa^n \psi[\xi_n, \dots, \xi_1]. \quad (7.1)$$

Let $\Sigma_1, \Sigma_2, \Sigma$ be hypersurfaces such that $\Sigma = \Sigma_1 \cup \Sigma_2$ is a disjoint union. We define the map $\tau_{\Sigma_1, \Sigma_2; \Sigma} : \mathcal{F}(L_{\Sigma_1}) \otimes \mathcal{F}(L_{\Sigma_2}) \rightarrow \mathcal{F}(L_\Sigma)$ as in Section 3.2 by equation (3.5). Concretely

$$\begin{aligned} & (\tau_{\Sigma_1, \Sigma_2; \Sigma}(\psi_1 \otimes \psi_2))((\eta_1, \xi_1), \dots, (\eta_{m+n}, \xi_{m+n})) \\ & := \frac{1}{(m+n)!} \sum_{\sigma \in S^{m+n}} \kappa^{|\sigma|} \psi_1(\eta_{\sigma(1)}, \dots, \eta_{\sigma(m)}) \psi_2(\xi_{\sigma(m+1)}, \dots, \xi_{\sigma(m+n)}). \end{aligned}$$

As seen there this is an isometric isomorphism of Krein spaces. Due to associativity, which is easily verified, this extends to a prescription for decompositions of hypersurfaces with an arbitrary finite number of components. We thus satisfy axioms (T2) and (V3). In particular, the graded transposition property comes from equation (3.7). It is also straightforward to verify axiom (T2b). Given $\eta_1, \dots, \eta_m \in L_{\Sigma_1}$ and $\xi_1, \dots, \xi_n \in L_{\Sigma_2}$ we also note, using (3.6),

$$\tau_{\Sigma_1, \Sigma_2; \Sigma}(\psi[\eta_1, \dots, \eta_m] \otimes \psi[\xi_1, \dots, \xi_n]) = \psi[(\eta_1, 0), \dots, (\eta_m, 0), (0, \xi_1), \dots, (0, \xi_n)]. \quad (7.2)$$

7.2 Amplitudes

Let M be a region. Given $\xi \in L_{\partial M}$ we decompose it as $\xi = \xi^R + J_{\partial M} \xi^I$, where $\xi^R, \xi^I \in L_{\widetilde{M}}$. We define $\widehat{\xi} \in L_{\widetilde{M}}^{\mathbb{C}} \subseteq L_{\partial M}^{\mathbb{C}}$ as

$$\widehat{\xi} := \xi^R - i\xi^I.$$

Here $L_{\widetilde{M}}^{\mathbb{C}}$ and $L_{\partial M}^{\mathbb{C}}$ are complexifications of the real vector spaces $L_{\widetilde{M}}$ and $L_{\partial M}$, where we denote the new complex structure by i . The real bilinear forms $\{\cdot, \cdot\}_{\partial M}, g_{\partial M}, \omega_{\partial M}$ all extend to $L_{\partial M}^{\mathbb{C}}$ as complex bilinear forms. This is understood in the following.

We define the subspace $\mathcal{H}_{\partial M}^{\circ} \subseteq \mathcal{H}_{\partial M}$ on which ρ_M will be defined to be the space of linear combinations of vectors $\psi[\xi_1, \dots, \xi_n]$ for $n \in \mathbb{N}$ and $\xi_1, \dots, \xi_n \in L_{\partial M}$ together with the vector $\mathbf{1}$.

If $n \in \mathbb{N}$ is odd and $\xi_1, \dots, \xi_n \in L_{\partial M}$ are arbitrary we define the amplitude map to vanish,

$$\rho_M(\psi[\xi_1, \dots, \xi_n]) := 0. \quad (7.3)$$

Since we view the target space \mathbb{C} of ρ_M as of even f -degree, this is equivalent to saying that ρ_M is f -graded in the fermionic case. In the bosonic case the fact that ρ_M is f -graded is trivial. For the vacuum we define, in accordance with axiom (V5),

$$\rho_M(\mathbf{1}) := 1. \quad (7.4)$$

Now let $n \in \mathbb{N}$ and $\xi_1, \dots, \xi_{2n} \in L_{\partial M}$. We define then

$$\rho_M(\psi[\xi_1, \dots, \xi_{2n}]) := \frac{1}{n!} \sum_{\sigma \in S^{2n}} \kappa^{|\sigma|} \prod_{j=1}^n \{\widehat{\xi_{\sigma(j)}}, \widehat{\xi_{\sigma(2n+1-j)}}\}_{\partial M}. \quad (7.5)$$

Note that the right-hand side is symmetric in the bosonic and anti-symmetric in the fermionic case under interchange of any two elements ξ_i, ξ_j with $i \neq j$. In particular, in the fermionic case

we note that the real parts of the inner products $\{\cdot, \cdot\}_{\partial M}$ appearing on the right-hand side vanish since the real subspace $L_{\tilde{M}}$ and its complexification $L_{\tilde{M}}^{\mathbb{C}}$ is neutral with respect to $g_{\partial M}$. Thus, we could equally well replace $\{\cdot, \cdot\}_{\partial M}$ with the symplectic form $2i\omega_{\partial M}(\cdot, \cdot)$ in (7.5), making the anti-symmetry manifest. In the bosonic case the imaginary parts of the inner products $\{\cdot, \cdot\}_{\partial M}$ vanish due to the isotropic nature of the subspace $L_{\tilde{M}}^{\mathbb{C}} \subseteq L_{\partial M}^{\mathbb{C}}$. Thus we could replace these inner products with their real parts $g_{\partial M}$, making the symmetry manifest. This makes ρ_M well defined and completes its definition on $\mathcal{H}_{\partial M}^{\circ}$, satisfying (T4).

8 Amplitude and inner product

8.1 The inner product as an amplitude

We turn to axiom (T3x). Let Σ be a hypersurface. Then Σ defines a slice region $\hat{\Sigma}$ with boundary $\partial\hat{\Sigma} = \bar{\Sigma} \cup \Sigma'$. Here, Σ' denotes a second copy of Σ . We have then, $L_{\partial\hat{\Sigma}} = L_{\bar{\Sigma}} \times L_{\Sigma'}$ and $\mathcal{H}_{\partial\hat{\Sigma}} = \mathcal{H}_{\bar{\Sigma}} \hat{\otimes} \mathcal{H}_{\Sigma'}$. Moreover,

$$\begin{aligned} \omega_{\partial\hat{\Sigma}} &= \omega_{\bar{\Sigma}} + \omega_{\Sigma'} = \omega_{\Sigma'} - \kappa\omega_{\Sigma}, & J_{\partial\hat{\Sigma}} &= J_{\bar{\Sigma}} + J_{\Sigma'} = J_{\Sigma'} - J_{\Sigma}, \\ g_{\partial\hat{\Sigma}} &= g_{\bar{\Sigma}} + g_{\Sigma'} = g_{\Sigma'} + \kappa g_{\Sigma}, & \{\cdot, \cdot\}_{\partial\hat{\Sigma}} &= \{\cdot, \cdot\}_{\bar{\Sigma}} + \{\cdot, \cdot\}_{\Sigma'} = \kappa\overline{\{\cdot, \cdot\}_{\Sigma}} + \{\cdot, \cdot\}_{\Sigma'}. \end{aligned}$$

It follows from axioms (C6) and (C7) applied to two copies of $\hat{\Sigma}$ and their gluing that the subspace $L_{\hat{\Sigma}} \subseteq L_{\partial\hat{\Sigma}}$ is precisely the space of pairs (ϕ, ϕ) for $\phi \in L_{\Sigma}$. Correspondingly, the subspace $J_{\partial\hat{\Sigma}}L_{\hat{\Sigma}} \subseteq L_{\partial\hat{\Sigma}}$ is the space of pairs $(\phi, -\phi)$ for $\phi \in L_{\Sigma}$. Now note that given $\eta, \xi \in L_{\Sigma}$ the expressions $(\widehat{\eta, 0}), (\widehat{0, \xi}) \in L_{\hat{\Sigma}}^{\mathbb{C}}$ may be expanded as follows

$$(\widehat{\eta, 0}) = \left(\frac{1}{2}\eta - iJ_{\Sigma}\eta, \frac{1}{2}\eta - iJ_{\Sigma}\eta\right), \quad (\widehat{0, \xi}) = \left(\frac{1}{2}\xi + iJ_{\Sigma}\xi, \frac{1}{2}\xi + iJ_{\Sigma}\xi\right).$$

Thus we have for $\eta, \eta', \xi, \xi' \in L_{\Sigma}$,

$$\{(\widehat{\eta, 0}), (\widehat{\eta', 0})\}_{\partial\hat{\Sigma}} = 0 \quad \{(\widehat{0, \xi}), (\widehat{0, \xi'})\}_{\partial\hat{\Sigma}} = 0 \quad (8.1)$$

$$\{(\widehat{\eta, 0}), (\widehat{0, \xi})\}_{\partial\hat{\Sigma}} = \kappa\{\xi, \eta\}_{\Sigma}. \quad (8.2)$$

The following proposition shows that axiom (T3x) is satisfied.

Proposition 8.1. $\rho_{\hat{\Sigma}}$ is well defined on $\tau_{\bar{\Sigma}, \Sigma'; \partial\hat{\Sigma}}(\mathcal{H}_{\bar{\Sigma}} \otimes \mathcal{H}_{\Sigma'}) \subseteq \mathcal{H}_{\partial\hat{\Sigma}}$. Moreover,

$$\rho_{\hat{\Sigma}}(\tau_{\bar{\Sigma}, \Sigma'; \partial\hat{\Sigma}}(\iota_{\Sigma}(\psi') \otimes \psi)) = \langle \psi', \psi \rangle_{\Sigma} \quad \forall \psi, \psi' \in \mathcal{H}_{\Sigma}. \quad (8.3)$$

Proof. Let $\eta_1, \dots, \eta_m \in L_{\Sigma}$ and $\xi_1, \dots, \xi_n \in L_{\Sigma}$. We have

$$\rho_{\hat{\Sigma}}(\tau_{\bar{\Sigma}, \Sigma'; \partial\hat{\Sigma}}(\iota_{\Sigma}(\psi[\eta_1, \dots, \eta_m]) \otimes \psi[\xi_1, \dots, \xi_n])) \quad (8.4)$$

$$= \kappa^m \rho_{\hat{\Sigma}}(\tau_{\bar{\Sigma}, \Sigma'; \partial\hat{\Sigma}}(\psi[\eta_m, \dots, \eta_1] \otimes \psi[\xi_1, \dots, \xi_n])) \quad (8.5)$$

$$= \kappa^m \rho_{\hat{\Sigma}}(\psi[(\eta_m, 0), \dots, (\eta_1, 0), (0, \xi_1), \dots, (0, \xi_n)]) \quad (8.6)$$

$$= \delta_{n,m} \kappa^n 2^n \sum_{\sigma \in S^n} \kappa^{|\sigma|} \prod_{j=1}^n \{(\widehat{\eta_j, 0}), (\widehat{0, \xi_{\sigma(j)}})\}_{\partial\hat{\Sigma}} \quad (8.7)$$

$$= \delta_{n,m} 2^n \sum_{\sigma \in S^n} \kappa^{|\sigma|} \prod_{j=1}^n \{\xi_j, \eta_{\sigma(j)}\}_{\Sigma} \quad (8.8)$$

$$= \langle \psi[\eta_1, \dots, \eta_m], \psi[\xi_1, \dots, \xi_n] \rangle_{\Sigma}$$

The step from (8.4) to (8.5) consists in the application of (7.1), while the step to (8.6) arises from (7.2). In applying the amplitude map (7.5), many permutations do not contribute due to (8.1) and the remaining ones can be reorganized leading to (8.7). Applying (8.2) and moving the permutations from the ξ -variables to the η -variables yields (8.8). This expression coincides with the inner product on \mathcal{H}_Σ , see (3.2). This shows the equality (8.3) on a dense subspace. Since the right hand side extends continuously to the whole Krein space this defines the left hand side there also. This implies the claimed well definedness of $\rho_{\hat{\Sigma}}$ on $\tau_{\bar{\Sigma}, \Sigma'; \partial \hat{\Sigma}}(\mathcal{H}_{\bar{\Sigma}} \otimes \mathcal{H}_{\Sigma'})$, completing the proof. \blacksquare

8.2 Complex conjugation

Let M be a region. Then the subspace $L_{\tilde{M}} \subseteq L_{\partial M}$ gives rise to a conjugate linear involutive adapted real (anti-)isometry $u_M : L_{\partial M} \rightarrow L_{\partial M}$ due to axiom (C5) as explained in Section 4. Recall also that u_M is a complex conjugation on $L_{\partial M}$. With the conventions of Section 7.2 we have the identity

$$\{\widehat{\xi}, \widehat{\eta}\}_{\partial M} = \{\xi, u_M(\eta)\}_{\partial M} \quad \forall \xi, \eta \in L_{\partial M}.$$

Using this we may rewrite formula (7.5) for the amplitude map as

$$\rho_M(\psi[\xi_1, \dots, \xi_{2n}]) := \frac{1}{n!} \sum_{\sigma \in S^{2n}} \kappa^{|\sigma|} \prod_{j=1}^n \{\xi_{\sigma(j)}, u_M(\xi_{\sigma(2n+1-j)})\}_{\partial M}. \quad (8.9)$$

Given u_M , Lemma 3.5 gives rise to a conjugate linear involutive adapted real f -graded isometry $U_M : \mathcal{H}_{\partial M} \rightarrow \mathcal{H}_{\partial M}$ defined by expression (3.3). Recall from Section 3.2 that the map U_M may also be seen as a complex conjugation. It turns out that the amplitude map commutes with this complex conjugation in the following sense.

Proposition 8.2.

$$\rho_M(U_M(\psi)) = \overline{\rho_M(\psi)} \quad \forall \psi \in \mathcal{H}_{\partial M}^{\circ}.$$

Proof. It is sufficient to carry out the proof for generating states. Moreover, it is trivial for states of odd Fock degree so we consider states of even Fock degree only. Let $\xi_1, \dots, \xi_{2n} \in L_{\partial M}$. Then

$$\begin{aligned} \rho_M(U_M(\psi[\xi_1, \dots, \xi_{2n}])) &= \rho_M(\psi[u_M(\xi_{2n}), \dots, u_M(\xi_1)]) \\ &= \frac{1}{n!} \sum_{\sigma \in S^{2n}} \kappa^{|\sigma|} \prod_{j=1}^n \{u_M(\xi_{\sigma(2n+1-j)}), \xi_{\sigma(j)}\}_{\partial M} \\ &= \frac{1}{n!} \sum_{\sigma \in S^{2n}} \kappa^{|\sigma|} \prod_{j=1}^n \overline{\{\xi_{\sigma(j)}, u_M(\xi_{\sigma(2n+1-j)})\}_{\partial M}} \\ &= \overline{\rho_M(\psi[\xi_1, \dots, \xi_{2n}])}. \end{aligned}$$

Here we have used the definition of U_M given by expression (3.3), the amplitude map in the form (8.9) and the involutiveness of u_M . \blacksquare

8.3 Quantum evolution

A quite different perspective on the roles of the maps u_M and U_M is afforded by comparing formula (8.9) for the amplitude to formula (3.2) for the inner product. As we shall see this suggest to think of (a restricted version of) U_M as an ‘‘evolution’’ operator, providing a quantum analogue of the role of u_M in the classical theory as discussed in Section 4.3.

The following is in a sense the quantum analogue of Lemma 4.1.

Lemma 8.3. *Suppose that $L_2 \oplus L_1 = L_{\partial M}$ is a decomposition as an orthogonal direct sum of complex Krein spaces such that $u_M(L_1) = L_2$. Then $U_M(\mathcal{F}(L_1)) = \mathcal{F}(L_2)$ with $\mathcal{F}(L_1)$ and $\mathcal{F}(L_2)$ understood as subspaces of $\mathcal{H}_{\partial M} = \mathcal{F}(L_{\partial M})$. With the isometric isomorphism of Fock spaces $\mathcal{F}(L_2) \hat{\otimes} \mathcal{F}(L_1) \rightarrow \mathcal{F}(L_{\partial M})$ given by expression (3.5) we have the following identity,*

$$\rho_M(\psi' \otimes \psi) = \kappa^{|\psi|} \langle U_M(\psi), \psi' \rangle_2, \quad \forall \psi \in \mathcal{F}(L_1), \quad \forall \psi' \in \mathcal{F}(L_2), \quad (8.10)$$

where $\langle \cdot, \cdot \rangle_2$ denotes the inner product of $\mathcal{H}_{\partial M} = \mathcal{F}(L_{\partial M})$ restricted to $\mathcal{F}(L_2)$.

Proof. The claimed property of U_M follows from the definition (3.3). For the proof of equation (8.10) it is enough to consider generating states. Thus, let $\xi_1, \dots, \xi_n \in L_1$ and $\eta_1, \dots, \eta_m \in L_2$. Then

$$\begin{aligned} \rho_M(\psi[\eta_1, \dots, \eta_m] \otimes \psi[\xi_1, \dots, \xi_n]) &= \rho_M(\psi[(\eta_1, 0), \dots, (\eta_m, 0), (0, \xi_1), \dots, (0, \xi_n)]) \\ &= \delta_{n,m} 2^n \sum_{\sigma \in S^n} \kappa^{|\sigma|} \prod_{j=1}^n \{(\eta_j, 0), u_M((0, \xi_{\sigma(n+1-j)}))\}_{\partial M} \\ &= \delta_{n,m} 2^n \sum_{\sigma \in S^n} \kappa^{|\sigma|} \prod_{j=1}^n \{\eta_j, u_M(\xi_{\sigma(n+1-j)})\}_2 \\ &= \langle \psi[u_M(\xi_n), \dots, u_M(\xi_1)], \psi[\eta_1, \dots, \eta_m] \rangle_2 \\ &= \kappa^n \langle \tilde{U}_M(\psi[\xi_1, \dots, \xi_n]), \psi[\eta_1, \dots, \eta_m] \rangle_2 \end{aligned}$$

Here we have used equation (3.6), formula (8.9) for the amplitude and the fact that the inner products of the type

$$\{(\eta', 0), u_M((\eta, 0))\}_{\partial M} \quad \text{and} \quad \{(0, \xi'), u_M((0, \xi))\}_{\partial M}$$

vanish. Moreover, we have used the inner product formula (3.2) and the formula (3.4) for U_M on generating states. The notation $\{\cdot, \cdot\}_2$ denotes the restriction of the inner product of $L_{\partial M}$ to the subspace L_2 . \blacksquare

Analogous to the classical case, such an algebraic notion of “evolution” underlying the amplitude map may be seen to arise from a geometric notion of “evolution” when the underlying decomposition of the space $L_{\partial M}$ arises from a corresponding decomposition of the hypersurface ∂M . This yields the following quantum analogue of Proposition 4.2.

Proposition 8.4. *Suppose that ∂M decomposes as a disjoint union $\partial M = \overline{\Sigma_2} \cup \Sigma_1$ such that $u_M(L_{\Sigma_1}) = L_{\overline{\Sigma_2}}$. Define $\tilde{U}_M : \mathcal{H}_{\Sigma_1} \rightarrow \mathcal{H}_{\Sigma_2}$ as the composition of $\iota_{\overline{\Sigma_2}}$ with the restriction of U_M to \mathcal{H}_{Σ_1} . Then \tilde{U}_M is a complex linear isometric isomorphism. Moreover,*

$$\rho_M(\tau_{\overline{\Sigma_2}, \Sigma_1; \partial M}(\iota_{\Sigma_2}(\psi') \otimes \psi)) = \langle \psi', \tilde{U}_M(\psi) \rangle_{\Sigma_2}, \quad \forall \psi \in \mathcal{H}_{\Sigma_1}, \quad \forall \psi' \in \mathcal{H}_{\Sigma_2}.$$

Proof. This follows rather straightforwardly from Lemma 8.3. Setting $L_1 = L_{\Sigma_1}$ and $L_2 = L_{\overline{\Sigma_2}}$ we have for $\psi \in L_{\Sigma_1}$ and $\psi' \in L_{\Sigma_2}$,

$$\begin{aligned} \rho_M(\tau_{\overline{\Sigma_2}, \Sigma_1; \partial M}(\iota_{\Sigma_2}(\psi') \otimes \psi)) &= \kappa^{|\psi|} \langle U_M(\psi), \iota_{\Sigma_2}(\psi') \rangle_{\overline{\Sigma_2}} \\ &= \kappa^{|\psi|} \langle \iota_{\Sigma_2} \circ \tilde{U}_M(\psi), \iota_{\Sigma_2}(\psi') \rangle_{\overline{\Sigma_2}} = \langle \psi', \tilde{U}_M(\psi) \rangle_{\Sigma_2}. \end{aligned} \quad \blacksquare$$

This suggests to view M as a cobordism from Σ_1 to Σ_2 , inducing the unitary “evolution” map \tilde{U}_M between the associated state spaces. This would then be in line with the often preferred categorial view of TQFT, in the sense of assigning to the morphism $M : \Sigma_1 \rightarrow \Sigma_2$ the morphism $\tilde{U}_M : \mathcal{H}_{\Sigma_1} \rightarrow \mathcal{H}_{\Sigma_2}$. In particular, if M as a cobordism takes the role of time-evolution, \tilde{U}_M is the

associated unitary quantum operator. We also note that in the special case of a slice region with $\Sigma_1 \approx \Sigma_2$, we recover Proposition 8.1. The “evolution” operator \tilde{U}_M is then simply the identity.

Recall from Section 4.3 that in the fermionic case (but not the bosonic one) there is a preferred algebraic notion of evolution, coming from the canonical decomposition $L_{\partial M} = L_{\partial M}^+ \oplus L_{\partial M}^-$ which satisfies automatically $u_M(L_{\partial M}^+) = L_{\partial M}^-$. This property is inherited by the quantum theory. Denoting the Fock space $\mathcal{F}(L_{\partial M}^+)$ by $\mathcal{H}_{\partial M}^{(+)}$ and $\mathcal{F}(L_{\partial M}^-)$ by $\mathcal{H}_{\partial M}^{(-)}$ we have the tensor product decomposition $\mathcal{H}_{\partial M} = \mathcal{H}_{\partial M}^{(+)} \hat{\otimes} \mathcal{H}_{\partial M}^{(-)}$. (This is not to be confused with the direct sum decomposition of $\mathcal{H}_{\partial M}$ as a Krein space into positive and negative parts.) The map U_M then restricts to a conjugate linear real f -graded isometry $\mathcal{H}_{\partial M}^{(+)} \rightarrow \mathcal{H}_{\partial M}^{(-)}$. If this decomposition of $L_{\partial M}$ into positive and negative parts is induced by a geometric decomposition of ∂M in the sense of Proposition 8.4, the spaces $\mathcal{H}_{\Sigma_1} = \mathcal{H}_{\partial M}^{(+)}$ and $\mathcal{H}_{\Sigma_2} = \mathcal{H}_{\partial M}^{(-)}$ are Hilbert spaces and $\tilde{U}_M : \mathcal{H}_{\Sigma_1} \rightarrow \mathcal{H}_{\Sigma_2}$ is a unitary operator between them⁷.

Recall now the standard globally hyperbolic setting described in Section 4.3. As already stated in that Section and as exhibited explicitly in the example of the Dirac field theory (Section 5), for standard fermionic field theories in this setting the algebraic notion of evolution coincides with the usual notion of time-evolution. That is, the decomposition of the boundary of a region M into initial and final component induces precisely the decomposition of the Krein space $L_{\partial M}$ into its positive and negative part. Applying Proposition 8.4, we obtain a description of the dynamics of the quantum theory in terms of the unitary maps \tilde{U}_M between Hilbert spaces associated to these hypersurfaces (all oriented in the same way). In particular, no indefinite inner product space appears in this way of describing the dynamics. This clarifies the absence of indefinite inner product spaces in the standard description of conventional fermionic quantum field theories.

9 Gluing

9.1 Disjoint gluing

We proceed to consider the gluing axioms (T5a) and (T5b). We start with the proof of (T5a), describing the disjoint gluing of regions.

Proposition 9.1. *Let M_1 and M_2 be regions and $M = M_1 \cup M_2$ be their disjoint union. Then*

$$\rho_M(\tau_{\partial M_1, \partial M_2; \partial M}(\psi_1 \otimes \psi_2)) = \rho_{M_1}(\psi_1)\rho_{M_2}(\psi_2) \quad \forall \psi_1 \in \mathcal{H}_{\partial M_1}^{\circ}, \psi_2 \in \mathcal{H}_{\partial M_2}^{\circ}. \quad (9.1)$$

Proof. Let $\eta_1, \dots, \eta_m \in L_{\partial M_1}$ and $\xi_1, \dots, \xi_n \in L_{\partial M_2}$. Then with (7.2) we get

$$\begin{aligned} & \rho_M(\tau_{\partial M_1, \partial M_2; \partial M}(\psi[\eta_1, \dots, \eta_m] \otimes \psi[\xi_1, \dots, \xi_n])) \\ &= \rho_M(\psi[(\eta_1, 0), \dots, (\eta_m, 0), (0, \xi_1), \dots, (0, \xi_n)]). \end{aligned}$$

If $n + m$ is odd, then (7.3) applies here (i.e., to the left hand side of (9.1)) and also to one of the two factors on the right hand side of (7.3), confirming the equality. We may thus suppose that $n + m$ is even and (7.5) applies. Since mixed terms of the type

$$\{\widehat{(\eta_i, 0)}, \widehat{(0, \xi_j)}\}_{\partial M} = \{(\widehat{\eta_i}, 0), (0, \widehat{\xi_j})\}_{\partial M} = \{\widehat{\eta_i}, 0\}_{\partial M_1} + \{0, \widehat{\xi_j}\}_{\partial M_2}$$

vanish, only pairings of the type $\{\widehat{(\eta_i, 0)}, \widehat{(\eta_j, 0)}\}_{\partial M}$ or $\{\widehat{(0, \xi_i)}, \widehat{(0, \xi_j)}\}_{\partial M}$ contribute. This restricts the permutations we need to sum over considerably. In particular, if m (and hence also n)

⁷Alternatively we may have $\mathcal{H}_{\Sigma_1} = \mathcal{H}_{\partial M}^{(-)}$ and $\mathcal{H}_{\Sigma_2} = \mathcal{H}_{\partial M}^{(+)}$ in which case we would have anti-Hilbert spaces, i.e., negative definite spaces. For the considerations that follow we may in this case simply invert the overall sign by convention.

is odd we do not have any contributing term and the result vanishes, confirming (9.1) in this case also. It remains to consider the case where both m and n are even.

Thus, let $\eta_1, \dots, \eta_{2m} \in L_{\partial M_1}$ and $\xi_1, \dots, \xi_{2n} \in L_{\partial M_2}$. Then

$$\begin{aligned} & \rho_M(\tau_{\partial M_1, \partial M_2; \partial M}(\psi[\eta_1, \dots, \eta_{2m}] \otimes \psi[\xi_1, \dots, \xi_{2n}])) \\ &= \rho_M(\psi[(\eta_1, 0), \dots, (\eta_{2m}, 0), (0, \xi_1), \dots, (0, \xi_{2n})]) \end{aligned} \quad (9.2)$$

$$\begin{aligned} &= \frac{1}{m!n!} \sum_{\sigma \in S^{2m}, \sigma' \in S^{2n}} (-1)^{|\sigma|+|\sigma'|} \left(\prod_{i=1}^m \{(\widehat{\eta_{\sigma(i)}}, 0), (\widehat{\eta_{\sigma(2m+1-i)}}, 0)\}_{\partial M} \right) \\ &\quad \times \left(\prod_{j=1}^n \{(\widehat{(0, \xi_{\sigma(j)})}, (0, \widehat{\xi_{\sigma(2n+1-j)})}\}_{\partial M} \right) \end{aligned} \quad (9.3)$$

$$\begin{aligned} &= \frac{1}{m!n!} \sum_{\sigma \in S^{2m}, \sigma' \in S^{2n}} (-1)^{|\sigma|+|\sigma'|} \left(\prod_{i=1}^m \{\widehat{\eta_{\sigma(i)}}, \widehat{\eta_{\sigma(2m+1-i)}}\}_{\partial M_1} \right) \\ &\quad \times \left(\prod_{j=1}^n \{\widehat{\xi_{\sigma(j)}}, \widehat{\xi_{\sigma(2n+1-j)}}\}_{\partial M_2} \right). \end{aligned} \quad (9.4)$$

In the application of (7.5) in the step from (9.2) to (9.3) we make use of the already mentioned simplification due to the elimination of permutations from the sum that yield no contribution. Finally (9.4) can be easily recognized as the product of two amplitudes of the form (7.5). This completes the proof of (9.1). \blacksquare

9.2 Gluing along a hypersurface

Axiom (T5b) is considerably more delicate. Consider a region M with boundary decomposing as a disjoint union $\partial M = \Sigma_1 \cup \Sigma \cup \overline{\Sigma'}$ where Σ' is a copy of Σ . We suppose that M can be glued to itself along Σ/Σ' to form the new region M_1 with boundary Σ_1 . In view of the definition (7.4) of the amplitude on the vacuum state the composition identity (6.1) would imply the following identity for the gluing anomaly,

$$c(M; \Sigma, \overline{\Sigma'}) = \sum_{i \in I} (-1)^{[\zeta_i]} \rho_M(\tau_{\Sigma_1, \Sigma, \overline{\Sigma'}; \partial M}(\mathbf{1} \otimes \zeta_i \otimes \iota_{\Sigma}(\zeta_i))), \quad (9.5)$$

given any ON-basis $\{\zeta_i\}_{i \in I}$ of \mathcal{H}_{Σ} . However, if the dimension of \mathcal{H}_{Σ} is infinite, it is not even clear whether the sum converges, given some particular ordered basis. Indeed, it is easy to construct simple models of a spacetime system together with data satisfying the axioms of Section 4 such that for a particular gluing this sum never converges. For the core axioms to be valid we thus need to introduce an additional assumption, not contained in the axioms of Section 4. This is precisely the assumption that the gluing anomaly $c(M; \Sigma, \overline{\Sigma'})$ is well defined for all admissible gluings. In the bosonic case this was discussed in [9] where it was shown that this is equivalent to an integrability condition in the holomorphic representation. Here we need to deal with this from a Fock space point of view.

Given the Fock space structure of \mathcal{H}_{Σ} there is a natural family of ON-basis arising from ON-basis of the underlying space L_{Σ} . It will be convenient to formulate the well-definedness criterion for the gluing anomaly using these special basis only. Let $\{\xi_i\}_{i \in N}$ be an ON-basis of L_{Σ} , where $N = \{1, \dots, \dim L_{\Sigma}\}$ if L_{Σ} is finite-dimensional and $N = \mathbb{N}$ otherwise. Using the ON-basis (3.9) of Fock space in the bosonic case we can then write the sum

$$\sum_{i \in I} (-1)^{[\zeta_i]} \zeta_i \otimes \iota_{\Sigma}(\zeta_i) \quad (9.6)$$

appearing in the identities (6.1) and (9.5) as follows

$$\sum_{m=0}^{\infty} \sum_{a_1 \leq \dots \leq a_m \in N} (-1)^{\sum_{i=1}^m [\xi_{a_i}]} \frac{1}{2^m K_{a_1, \dots, a_m}} \psi[\xi_{a_1}, \dots, \xi_{a_m}] \otimes \iota_{\Sigma} (\psi[\xi_{a_1}, \dots, \xi_{a_m}]).$$

In fact, this very same formula serves also in the fermionic case since the fermionic ON-basis (3.8) corresponds to a subset of the bosonic one and the extraneous terms then simply vanish due to antisymmetry. Moreover, we can simplify the formula by eliminating the combinatorial factors K_{a_1, \dots, a_m} in exchange for an unrestricted sum of the indices over N^m while dividing by a factor $m!$. This yields

$$\sum_{m=0}^{\infty} \frac{1}{2^m m!} \sum_{a_1, \dots, a_m \in N} (-1)^{\sum_{i=1}^m [\xi_{a_i}]} \psi[\xi_{a_1}, \dots, \xi_{a_m}] \otimes \iota_{\Sigma} (\psi[\xi_{a_1}, \dots, \xi_{a_m}]). \quad (9.7)$$

This continues to apply to both the bosonic and the fermionic case.

Suppose now that the Krein space L_{Σ} is finite-dimensional and N thus finite. The inner sum in (9.7) is then a well defined element of the tensor product $\mathcal{H}_{\Sigma} \otimes \mathcal{H}_{\bar{\Sigma}}$. Moreover, it does not depend on the choice of basis of L_{Σ} . This follows from the conjugate linearity of the states $\psi[\xi_{a_1}, \dots, \xi_{a_m}]$ in terms of the variables $\xi_{a_1}, \dots, \xi_{a_m}$ combined with the conjugate linearity of ι_{Σ} . To see this explicitly note that any other ON-basis of L_{Σ} is of the form $\{\Lambda \xi_i\}_{i \in N}$ for a unitary map $\Lambda : L_{\Sigma} \rightarrow L_{\Sigma}$. The sum for the modified basis reduces to that of the original basis as follows

$$\begin{aligned} & \sum_{a_1, \dots, a_m \in N} (-1)^{\sum_{i=1}^m [\Lambda \xi_{a_i}]} \psi[\Lambda \xi_{a_1}, \dots, \Lambda \xi_{a_m}] \otimes \iota_{\Sigma} (\psi[\Lambda \xi_{a_1}, \dots, \Lambda \xi_{a_m}]) \\ &= \sum_{a_1, \dots, a_m \in N} \sum_{b_1, \dots, b_m \in N} \sum_{d_1, \dots, d_m \in N} \left(\prod_{i=1}^m \{ \xi_{d_i}, \Lambda \xi_{a_i} \}_{\Sigma} (-1)^{[\Lambda \xi_{a_i}]} \{ \Lambda \xi_{a_i}, \xi_{b_i} \}_{\Sigma} \right) \\ & \quad \times \psi[\xi_{b_1}, \dots, \xi_{b_m}] \otimes \iota_{\Sigma} (\psi[\xi_{d_1}, \dots, \xi_{d_m}]) \\ &= \sum_{b_1, \dots, b_m \in N} (-1)^{\sum_{i=1}^m [\xi_{b_i}]} \psi[\xi_{b_1}, \dots, \xi_{b_m}] \otimes \iota_{\Sigma} (\psi[\xi_{b_1}, \dots, \xi_{b_m}]). \end{aligned}$$

In the fermionic case the outer sum in (9.7) is then also finite, since terms with $m > \dim L_{\Sigma}$ vanish.

Replacing in (9.5) expression (9.6) with expression (9.7) yields for the gluing anomaly,

$$\begin{aligned} c(M; \Sigma, \bar{\Sigma}') &= \sum_{m=0}^{\infty} \frac{1}{2^m m!} \sum_{a_1, \dots, a_m \in N} (-1)^{\sum_{i=1}^m [\xi_{a_i}]} \\ & \quad \times \rho_M \left(\tau_{\Sigma_1, \Sigma, \bar{\Sigma}', \partial M} (\mathbf{1} \otimes \psi[\xi_{a_1}, \dots, \xi_{a_m}] \otimes \iota_{\Sigma} (\psi[\xi_{a_1}, \dots, \xi_{a_m}])) \right). \end{aligned} \quad (9.8)$$

If L_{Σ} is finite-dimensional, the gluing anomaly is automatically well defined in the fermionic case, as can also be seen from the fact that \mathcal{H}_{Σ} is then finite-dimensional. In the bosonic case, \mathcal{H}_{Σ} is infinite-dimensional even when L_{Σ} is finite-dimensional. However, only the outer sum in (9.8) may diverge then. We declare $c(M; \Sigma, \bar{\Sigma}')$ well defined whenever this sum converges absolutely.

If L_{Σ} is infinite-dimensional the situation is more complicated. Consider the projective system $\{L_{\Sigma, \alpha}\}_{\alpha \in A}$ of all finite-dimensional complex subspaces of L_{Σ} . (Here A denotes a suitable index set.) The projective limit of $\{L_{\Sigma, \alpha}\}_{\alpha \in A}$ is L_{Σ} . Associated to each subspace $L_{\Sigma, \alpha} \subseteq L_{\Sigma}$ we define

the corresponding restricted anomaly factor $c_\alpha(M; \Sigma, \overline{\Sigma'})$ by the evaluation of the right hand side of (9.8) if it exists. We then say that the anomaly factor is well defined if the projective limit exists,

$$c(M; \Sigma, \overline{\Sigma'}) := \varprojlim_{\alpha} c_\alpha(M; \Sigma, \overline{\Sigma'}).$$

More explicitly, this may be formulated as follows. In the fermionic case, every $c_\alpha(M; \Sigma, \overline{\Sigma'})$ is well defined. We say then that the limit exists and is equal to a quantity $c \in \mathbb{C}$, i.e., $\varprojlim_{\alpha} c_\alpha(M; \Sigma, \overline{\Sigma'}) = c$ if c has the following property: For any $\epsilon > 0$ there exists $\beta \in A$ such that for any $\gamma \in A$ with $\gamma \geq \beta$ we have

$$|c_\gamma(M; \Sigma, \overline{\Sigma'}) - c| < \epsilon.$$

In the bosonic case the definition is the same with the additional requirement that $c_\beta(M; \Sigma, \overline{\Sigma'})$ as well as $c_\gamma(M; \Sigma, \overline{\Sigma'})$ for any $\gamma \geq \beta$ must be well defined.

Our definition of the anomaly factor really amounts to saying that equation (9.5) has to make sense, but only for certain particular choices of ON-basis of \mathcal{H}_Σ and certain particular choices of orderings of the sum. We apply this same definition to the right-hand side of (6.1). Thus, our main theorem shall consist of a proof of equation (6.1) and thus of axiom (T5b), but with the slight modification of restricting the ON-basis $\{\zeta_i\}_{i \in I}$ and their orderings.

Theorem 9.2. *Assume the geometric context of axiom (T5b). Moreover, assume that the anomaly factor $c(M; \Sigma, \overline{\Sigma'})$ is well defined and non-zero. Then the composition identity (6.1) holds in the sense discussed and thus axiom (T5b) holds (for a restricted choice of ON-basis and orderings).*

Proof. By our definition of the subspace $\mathcal{H}_{\partial M_1}^\circ \subseteq \mathcal{H}_{\partial M_1}$ it is sufficient to prove the composition identity (6.1) for states of the form $\psi[\phi_1, \dots, \phi_n]$ for $\phi_1, \dots, \phi_n \in L_{\partial M_1}$. Consider a finite-dimensional complex Krein subspace $L_{\Sigma, \alpha}$ of L_Σ , let $N_\alpha = \{1, \dots, \dim L_{\Sigma, \alpha}\}$ and let $\{\xi_a\}_{a \in N_\alpha}$ be an ON-basis of $L_{\Sigma, \alpha}$ as a complex Krein space. We define

$$R_\alpha := \sum_{m=0}^{\infty} \frac{1}{2^m m!} \sum_{a_1, \dots, a_m \in N_\alpha} (-1)^{\sum_{i=1}^m [\xi_{a_i}]} \\ \times \rho_M(\tau_{\Sigma_1, \Sigma, \overline{\Sigma'}, \partial M}(\psi[\phi_1, \dots, \phi_n] \otimes \psi[\xi_{a_1}, \dots, \xi_{a_m}] \otimes \iota_\Sigma(\psi[\xi_{a_1}, \dots, \xi_{a_m}]))) ,$$

if the outer sum converges absolutely. The version of the identity (6.1) we need to prove amounts to showing the following limit,

$$\rho_{M_1}(\psi[\phi_1, \dots, \phi_n]) \cdot c(M; \Sigma, \overline{\Sigma'}) = \varprojlim_{\alpha} R_\alpha.$$

More explicitly this means that given $\epsilon > 0$ we need to show the existence of $\beta \in A$ such that for all $\gamma \geq \beta$ we have

$$|\rho_{M_1}(\psi[\phi_1, \dots, \phi_n]) \cdot c(M; \Sigma, \overline{\Sigma'}) - R_\gamma| < \epsilon.$$

Instead we are going to show the stronger statement that there exists $\beta \in A$ such that for all $\gamma \geq \beta$ we have

$$\rho_{M_1}(\psi[\phi_1, \dots, \phi_n]) \cdot c_\gamma(M; \Sigma, \overline{\Sigma'}) = R_\gamma. \quad (9.9)$$

To simplify the proof it is convenient to exploit complex conjugate linearity of the states $\psi[\phi_1, \dots, \phi_n]$ in terms of the variables ϕ_1, \dots, ϕ_n in conjunction with the decomposition $L_{\partial M_1} = L_{\tilde{M}_1} \oplus J_{\partial M_1} L_{\tilde{M}_1}$. Decomposing $\phi_i = \phi_i^0 + J_{\Sigma_1} \phi_i^1$ where $\phi_i^0, \phi_i^1 \in L_{\tilde{M}_1}$ we have

$$\psi[\phi_1, \dots, \phi_n] = \sum_{l_1, \dots, l_n \in \{0,1\}} (-i)^{l_1 + \dots + l_n} \psi[\phi_1^{l_1}, \dots, \phi_n^{l_n}].$$

Since the amplitude map is complex linear we can apply this decomposition to both sides of (9.9). It is thus sufficient to restrict to the special case $\phi_1, \dots, \phi_n \in L_{\tilde{M}_1}$, as we shall do in the following.

Given $\phi \in L_{\tilde{M}_1}$ we observe that due to the exact sequence (4.2) of axiom (C7) there is a unique $\tilde{\phi} \in L_\Sigma$ such that $(\phi, \tilde{\phi}, \tilde{\phi}) \in L_{\tilde{M}} \subseteq L_{\Sigma_1} \times L_\Sigma \times L_{\overline{\Sigma'}}$. With this notation let $\beta' \in A$ be determined such that $L_{\Sigma, \beta'}$ is the complex Krein subspace of L_Σ generated by $\{\tilde{\phi}_1, \dots, \tilde{\phi}_n\}$. On the other hand let $\beta'' \in A$ be such that for any $\gamma \geq \beta''$, $c_\gamma(M; \Sigma, \overline{\Sigma'})$ is well defined. Such a β'' must exist since by assumption $c(M; \Sigma, \overline{\Sigma'})$ is well defined. Now let $\beta \in A$ such that $\beta \geq \beta'$ and $\beta \geq \beta''$. We claim that with this choice of β we satisfy (9.9) for all $\gamma \geq \beta$. This is precisely the content of the following Lemma 9.3. ■

Lemma 9.3. *Assume the geometric context of axiom (T5b). Let $\phi_1, \dots, \phi_n \in L_{\tilde{M}_1}$. Let $L_{\Sigma, \alpha}$ be a complex Krein subspace of L_Σ such that $\tilde{\phi}_1, \dots, \tilde{\phi}_n \in L_{\Sigma, \alpha}$ with the notation as above. Suppose that $c_\alpha(M; \Sigma, \overline{\Sigma'})$ is well defined. Define $N_\alpha := \{1, \dots, \dim L_{\Sigma, \alpha}\}$ and let $\{\xi_a\}_{a \in N_\alpha}$ be an ON-basis of $L_{\Sigma, \alpha}$. Then*

$$\begin{aligned} \rho_{M_1}(\psi[\phi_1, \dots, \phi_n]) \cdot c_\alpha(M; \Sigma, \overline{\Sigma'}) &= \sum_{m=0}^{\infty} \frac{1}{2^m m!} \sum_{a_1, \dots, a_m \in N_\alpha} (-1)^{\sum_{i=1}^m [\xi_{a_i}]} \\ &\times \rho_M(\tau_{\Sigma_1, \Sigma, \overline{\Sigma'}, \partial M}(\psi[\phi_1, \dots, \phi_n] \otimes \psi[\xi_{a_1}, \dots, \xi_{a_m}] \otimes \iota_\Sigma(\psi[\xi_{a_1}, \dots, \xi_{a_m}]))). \end{aligned} \quad (9.10)$$

The rather lengthy proof of this lemma is contained in Appendix A.

To summarize: Given a spacetime system and a model satisfying the axioms of Section 4, we need an additional assumption, namely the well definedness of the gluing anomaly for any admissible gluing. If this assumption is satisfied, the quantization described in this section yields a general boundary quantum field theory satisfying the core axioms for a slightly modified version of axiom (T5b).

10 The gluing anomaly

10.1 An important special case

Although it is not obvious in general how to ensure the well definedness of the gluing anomaly and thus the validity of all of the core axioms, there is an important special case where well definedness can be demonstrated easily. This is the case where two regions are glued together, one of which can be seen as a cobordism with an associated notion of evolution in the sense of Sections 4.3 and 8.3.

Proposition 10.1. *Let P be a region with boundary decomposing disjointly as $\partial P = \Sigma_1 \cup \overline{\Sigma_2}$ and Q a region with boundary decomposing disjointly as $\partial Q = \Sigma'_2 \cup \Sigma_3$, with Σ'_2 a copy of Σ_2 . Suppose furthermore that $u_P(L_{\Sigma_1}) = L_{\overline{\Sigma_2}}$. Let M be the disjoint union of P and Q and consider the gluing of M to itself along Σ_2 . Then the associated gluing anomaly is well defined and equal to 1, that is*

$$c(M; \Sigma'_2, \overline{\Sigma_2}) = 1.$$

Proof. We write ρ_M as a map $\mathcal{H}_{\Sigma_1} \hat{\otimes} \mathcal{H}_{\Sigma_3} \hat{\otimes} \mathcal{H}_{\Sigma'_2} \hat{\otimes} \mathcal{H}_{\overline{\Sigma_2}} \rightarrow \mathbb{C}$. Then given an ON-basis $\{\zeta_i\}_{i \in I}$ of \mathcal{H}_{Σ_2} we obtain

$$\begin{aligned} c(M; \Sigma'_2, \overline{\Sigma_2}) &= \sum_{i \in I} (-1)^{[\zeta_i]} \rho_M(\tau_{\Sigma_1, \Sigma'_2, \overline{\Sigma_2}; \partial M}(\mathbf{1} \otimes \mathbf{1} \otimes \zeta_i \otimes \iota_{\Sigma_2}(\zeta_i))) \\ &= \sum_{i \in I} (-1)^{[\zeta_i]} \rho_P(\tau_{\Sigma_1, \overline{\Sigma_2}; \partial P}(\mathbf{1} \otimes \iota_{\Sigma_2}(\zeta_i))) \rho_Q(\tau_{\Sigma_3, \Sigma'_2; \partial Q}(\mathbf{1} \otimes \zeta_i)) \\ &= \sum_{i \in I} (-1)^{[\zeta_i]} \langle \zeta_i, \mathbf{1} \rangle_{\Sigma_2} \rho_Q(\tau_{\Sigma_3, \Sigma'_2; \partial Q}(\mathbf{1} \otimes \zeta_i)) = \rho_Q(\tau_{\Sigma_3, \Sigma'_2; \partial Q}(\mathbf{1} \otimes \mathbf{1})) = 1. \end{aligned}$$

We have used here equation (9.5), Proposition 9.1, Proposition 8.4, and Definition (7.4). \blacksquare

In particular, this applies to the standard globally hyperbolic setting described in Section 4.3. The quantization of this setting thus proceeds without any obstruction from a potential ill-definedness of the gluing anomaly.

10.2 Renormalization

The necessity of an additional integrability condition, here in the form of the existence of the gluing anomaly factor, in order for the quantum theory to be well defined is somewhat unsatisfactory. This is especially so given that we know that allowing for both, non-trivial topologies and infinite-dimensional state spaces, will likely lead to a violation of this condition. Clearly, throwing out axiom (T5b) is not an option as this forms a corner stone of the coherence of the whole edifice of TQFT and consequently of the GBF. It is therefore natural to look for some kind of “renormalization” of the gluing anomaly. That is, one would “regularize” the quantities on both sides of relation (6.1) by introducing some “cut-off”. Then the equation would have to hold in the “limit” that the “cut-off” is taken away.

In the fermionic case, to which we restrict in the following, this informal description of renormalization can be realized in a precise way. Moreover, the hard work for this has already been done in Section 9.2 and Appendix A. The role of the “cut-off” will be played by restricting the space L_Σ associated to the gluing hypersurface Σ to the subspace $L_{\Sigma, \alpha}$. The role of the “limit” is taken by the projective limit in the system $\{L_{\Sigma, \alpha}\}_{\alpha \in A}$.

Define the orthogonal projector P_α as the projector onto the Fock subspace $\mathcal{F}(L_{\Sigma, \alpha})$ of the Fock space $\mathcal{F}(L_\Sigma)$. The way we have treated the right hand side of relation (6.1) in Theorem 9.2 then amounts to saying that it is well defined and coincides with the limit in α of the expression

$$\sum_{i \in I} (-1)^{[\zeta_i]} \rho_M(\tau_{\Sigma_1, \Sigma, \overline{\Sigma'}; \partial M}(\psi \otimes P_\alpha \zeta_i \otimes \iota_\Sigma(P_\alpha \zeta_i))) \quad (10.1)$$

whenever this limit exists. Here $\{\zeta_i\}_{i \in I}$ is an arbitrary ON-basis of $\mathcal{H}_\Sigma = \mathcal{F}(L_\Sigma)$ whose choice is irrelevant, as we saw already in Section 9.2. Expression (10.1) will play the role of the regularized version of the right hand side of relation (6.1). Our definition of the regularized gluing anomaly factor from Section 9.2, essentially equation (9.8), can be rewritten in a similar way as follows

$$c_\alpha(M; \Sigma, \overline{\Sigma'}) := \sum_{i \in I} (-1)^{[\zeta_i]} \rho_M(\tau_{\Sigma_1, \Sigma, \overline{\Sigma'}; \partial M}(\mathbf{1} \otimes P_\alpha \zeta_i \otimes \iota_\Sigma(P_\alpha \zeta_i))). \quad (10.2)$$

Again, the choice of ON-basis is irrelevant. Note that P_α projects onto a finite-dimensional subspace, so expressions (10.1) and (10.2) are always well defined. We are now ready to state the renormalized version of axiom (T5b).

(T5b*) Let M be a region with its boundary decomposing as a disjoint union $\partial M = \Sigma_1 \cup \Sigma \cup \overline{\Sigma'}$, where Σ' is a copy of Σ . Let M_1 denote the gluing of M with itself along $\Sigma, \overline{\Sigma'}$ and

suppose that M_1 is a region. Note $\partial M_1 = \Sigma_1$. Then $\tau_{\Sigma_1, \Sigma, \overline{\Sigma}'; \partial M}(\psi \otimes \xi \otimes \iota_\Sigma(\xi)) \in \mathcal{H}_{\partial M}^7$ for all $\psi \in \mathcal{H}_{\partial M_1}^\circ$ and $\xi \in \mathcal{H}_\Sigma$. Moreover, there is $\{c_\alpha(M; \Sigma, \overline{\Sigma}')\}_{\alpha \in A}$ such that for any ON-basis $\{\zeta_i\}_{i \in I}$ of \mathcal{H}_Σ and any $\psi \in \mathcal{H}_{\partial M_1}^\circ$,

$$\lim_{\alpha} \left(\rho_{M_1}(\psi) \cdot c_\alpha(M; \Sigma, \overline{\Sigma}') - \sum_{i \in I} (-1)^{|\zeta_i|} \rho_M(\tau_{\Sigma_1, \Sigma, \overline{\Sigma}'; \partial M}(\psi \otimes P_\alpha \zeta_i \otimes \iota_\Sigma(P_\alpha \zeta_i))) \right) = 0. \quad (10.3)$$

Theorem 10.2. *Assume the geometric context of axiom (T5b*). Then the renormalized composition identity (10.3) holds and thus axiom (T5b*) holds. Moreover, $c_\alpha(M; \Sigma, \overline{\Sigma}')$ can be taken to be specified by equation (10.2).*

Proof. By our definition of the subspace $\mathcal{H}_{\partial M_1}^\circ \subseteq \mathcal{H}_{\partial M_1}$ it is sufficient to prove the renormalized composition identity (10.3) for states of the form $\psi[\phi_1, \dots, \phi_n]$ for $\phi_1, \dots, \phi_n \in L_{\partial M_1}$. Moreover, as in the proof of Theorem 9.2 it will be sufficient to even suppose $\phi_1, \dots, \phi_n \in L_{\tilde{M}_1}$. Also, proceeding as in that proof there are due to axiom (C2) unique elements $\tilde{\phi}_1, \dots, \tilde{\phi}_n \in L_\Sigma$ such that $(\phi_i, \tilde{\phi}_i, \tilde{\phi}_i) \in L_{\tilde{M}}$ for $i \in \{1, \dots, n\}$. Now let $\beta \in A$ such that $L_{\Sigma, \beta}$ is the Krein subspace of L_Σ generated by $\{\tilde{\phi}_1, \dots, \tilde{\phi}_n\}$. Then Lemma 9.3 implies for any $\gamma \geq \beta$ the equality

$$\rho_{M_1}(\psi[\phi_1, \dots, \phi_n]) \cdot c_\gamma(M; \Sigma, \overline{\Sigma}') - \sum_{i \in I} (-1)^{|\zeta_i|} \rho_M(\tau_{\Sigma_1, \Sigma, \overline{\Sigma}'; \partial M}(\psi[\phi_1, \dots, \phi_n] \otimes P_\gamma \zeta_i \otimes \iota_\Sigma(P_\gamma \zeta_i))) = 0.$$

(This is just a rewrite of equation (9.10).) This implies the validity of the equation in the limit, i.e., (10.3) is valid for $\psi = \psi[\phi_1, \dots, \phi_n]$. This completes the proof. \blacksquare

11 Remarks on the probability interpretation

In the standard formulation of quantum theory an indefinite inner product on a state space would lead to serious problems with the probability interpretation. The situation turns out to be more nuanced in the GBF. A key fact to keep in mind is that the concept of state space is considerably more general in the GBF. Only state spaces associated with certain rather special hypersurfaces may be compared in their role to the state spaces of the standard formulation. In particular, as explained in Section 8.3, in the standard globally hyperbolic setting, restricting ourselves to state spaces that play this role leaves us with only Hilbert spaces, even in fermionic theories.

In the following we shall consider this issue from an intrinsic GBF perspective. Recall to this end the probability interpretation for amplitudes in the GBF in the setting of Hilbert spaces [7]. That is, we have a region M and a boundary Hilbert space $\mathcal{H}_{\partial M}$. To define a measurement we need to specify a closed subspace $\mathcal{S} \subseteq \mathcal{H}_{\partial M}$ that encodes “preparation” and another closed subspace $\mathcal{A} \subseteq \mathcal{S}$ that encodes the “question” we are asking in the measurement. The probability for an affirmative outcome of the measurement (if defined) is then given by the formula

$$P(\mathcal{A}|\mathcal{S}) = \frac{\sum_{i \in J} |\rho_M(\xi_i)|^2}{\sum_{i \in I} |\rho_M(\xi_i)|^2}. \quad (11.1)$$

Here $\{\xi_i\}_{i \in I}$ is an orthonormal basis of \mathcal{S} that restricts to an orthonormal basis $\{\xi_i\}_{i \in J}$ of \mathcal{A} with $J \subseteq I$. The quantity (11.1) has then all the right properties of a probability, reduces to

the usual probabilities in the standard formulation, admits a notion of probability conservation in spacetime etc. [7, 11].

So what changes if we wish to generalize from Hilbert spaces to Krein spaces? Apparently not much as the quantity (11.1) still has all the right properties and all the relevant arguments in [7, 11] still apply. However, there is a crucial difference. For expression (11.1) to be defined in the first place the subspaces \mathcal{A} and \mathcal{S} have to be Krein spaces in the sense of Section 3.1. This amounts to \mathcal{A} and \mathcal{S} being decomposable as direct sums in terms of positive and negative parts. Concretely, given the decomposition $\mathcal{H}_{\partial M} = \mathcal{H}_{\partial M}^+ \oplus \mathcal{H}_{\partial M}^-$ we must have $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ and $\mathcal{S} = \mathcal{S}^+ \oplus \mathcal{S}^-$, where $\mathcal{A}^+ \subseteq \mathcal{S}^+ \subseteq \mathcal{H}_{\partial M}^+$ and $\mathcal{A}^- \subseteq \mathcal{S}^- \subseteq \mathcal{H}_{\partial M}^-$.

The Krein space case is thus more restrictive than the Hilbert space case in the following sense. Recall that $\mathcal{H}_{\partial M}$ is also a Hilbert space by taking the inner product of $\mathcal{H}_{\partial M}^+ \oplus \mathcal{H}_{\partial M}^-$. All the Krein subspaces of $\mathcal{H}_{\partial M}$ are Hilbert subspaces of $\mathcal{H}_{\partial M}^+ \oplus \mathcal{H}_{\partial M}^-$, but there are many more Hilbert subspaces. One potential way to make physical sense of this limitation in the Krein space case would be to think of it as originating from a kind of ‘‘superselection rule’’. This superselection rule would here amount to saying that superpositions between states in the subspace $\mathcal{H}_{\partial M}^+$ and states in the subspace $\mathcal{H}_{\partial M}^-$ do not make physical sense.

An undesirable feature of the setting as described so far is its apparent dependence on conventions. Consider a general boundary quantum field theory given in terms of the axioms of Section 6.1. If we invert the global orientation of spacetime, i.e., the orientation of all regions and hypersurfaces of the spacetime system, we obtain again a theory satisfying the axioms. Indeed, physically the new theory should really be equivalent to the old theory. However, given a region M , the decomposition of its boundary Krein space $\mathcal{H}_{\partial M}$ into a direct sum of positive and negative parts is generally *different* from the corresponding decomposition of $\mathcal{H}_{\overline{\partial M}} = \iota_{\partial M}(\mathcal{H}_{\partial M})$ (the boundary Krein space $\mathcal{H}_{\partial M}$ in the orientation-reversed theory). This would seem to be an indication against a physical significance of this decomposition.

However, there is another superselection rule at play here: The amplitude of a state with odd fermionic degree must vanish. This is stipulated in our formulation in axiom (T4), via the \mathbb{Z}_2 -gradedness of the amplitude map. It is of course a basic and long established fact of quantum field theory, interacting or not⁸. Let us denote by $\mathcal{H}_{\partial M,0}$ the part of $\mathcal{H}_{\partial M}$ with \mathbb{Z}_2 -degree 0 (equivalent to even fermion number). By axiom (T1b) the restriction of $\iota_{\partial M}$ to $\mathcal{H}_{\partial M,0}$ is an isometry. In particular, the decomposition into positive and negative part of $\mathcal{H}_{\partial M,0}$ is the same as that of $\iota_{\partial M}(\mathcal{H}_{\partial M,0})$. So, if we also enforce this superselection rule in the choice of the subspaces \mathcal{S} and \mathcal{A} , the new rule behaves well. Concretely, this enforcement consists of restricting the subspaces \mathcal{S} and \mathcal{A} to be subspaces of $\mathcal{H}_{\partial M,0}$.⁹

Let us emphasize that the new superselection rule does not show in predictions that can also be made within the standard formalism for conventional bosonic or fermionic field theories. Indeed, for bosonic theories there does not seem to be any realistic example theory that would require genuine Krein spaces instead of Hilbert spaces. For fermionic theories, Krein spaces do necessarily appear in a GBF setting as we have seen. However, the predictions of measurement outcomes that can be made in the standard formalism are tied to transition amplitudes. In these, \mathcal{S} encodes a condition on the initial hypersurface and \mathcal{A} an additional condition on the final hypersurface. However, due to the coincidence of preferred algebraic and geometric notion of time in the standard globally hyperbolic setting (recall Sections 4.3 and 8.3), this implies that \mathcal{S} and \mathcal{A} then automatically satisfy the new superselection rule, supposing that we explicitly enforce the usual fermionic \mathbb{Z}_2 -superselection rule.

⁸The conventional way of expressing this would be to say that the transition amplitude between a state with even and a state with odd fermion number must vanish.

⁹We could more generally let them be direct sums with another part of \mathbb{Z}_2 -degree 1. However, there is little point in considering these extra parts as they do not contribute to the amplitude and hence to the probability formula (11.1).

12 A brief comparison to holomorphic quantization

The previous treatment of the free bose field and its quantization in the GBF in [9] differs from the present one principally in the quantization method used. While we employ a Fock space construction here, a holomorphic quantization was performed in [9]. Nevertheless, the results are equivalent. We shall provide a basic dictionary between the two approaches here, restricted to the Hilbert space case.

We first recall the quantization of state spaces in the holomorphic approach. Let Σ be a hypersurface. Denote by \hat{L}_Σ the algebraic dual of the topological dual of L_Σ , equipped with the weak* topology. Note that there is a natural inclusion $L_\Sigma \subseteq \hat{L}_\Sigma$. The inner product of L_Σ induces a Gaussian Borel measure ν_Σ on \hat{L}_Σ . The square integrable holomorphic functions on \hat{L}_Σ form a separable Hilbert space $H^2(\hat{L}_\Sigma, \nu_\Sigma)$ with inner product

$$\langle \psi', \psi \rangle = \int_{\hat{L}_\Sigma} \overline{\psi'(\xi)} \psi(\xi) d\nu_\Sigma(\xi).$$

It turns out that elements of $H^2(\hat{L}_\Sigma, \nu_\Sigma)$ are uniquely determined by their values on L_Σ . That is, we may think of $H^2(\hat{L}_\Sigma, \nu_\Sigma)$ as a space of holomorphic functions on L_Σ rather than on \hat{L}_Σ . This defines the state space \mathcal{H}_Σ in the holomorphic approach. The Fock space $\mathcal{F}(L_\Sigma)$, which is the model for \mathcal{H}_Σ in the present work, is related to it by an isometric isomorphism $T : \mathcal{F}(L_\Sigma) \rightarrow H^2(\hat{L}_\Sigma, \nu_\Sigma)$ with the following characteristics. T is the sum of isometries $T_n : \mathcal{F}_n(L_\Sigma) \rightarrow H^2(\hat{L}_\Sigma, \nu_\Sigma)$ given by

$$(T_n(\psi))(\xi) = \psi(\xi, \dots, \xi) \quad \forall \psi \in \mathcal{F}_n(L_\Sigma), \quad \forall \xi \in L_\Sigma.$$

In particular, a generating state $\psi[\xi_1, \dots, \xi_n]$ with $\xi_1, \dots, \xi_n \in L_\Sigma$ is mapped to the wave function

$$(T_n(\psi[\xi_1, \dots, \xi_n]))(\eta) = \prod_{i=1}^n \{\xi_i, \eta\}_\Sigma \quad \forall \eta \in L_\Sigma. \quad (12.1)$$

Of particular importance in the holomorphic setting are the coherent states K_ξ which are associated to elements $\xi \in L_\Sigma$. As elements of $H^2(\hat{L}_\Sigma, \nu_\Sigma)$ these take the form

$$K_\xi(\eta) = \exp\left(\frac{1}{2}\{\xi, \eta\}_\Sigma\right) \quad \forall \eta \in L_\Sigma.$$

Expanding the exponential and comparing to formula (12.1), we see that the coherent state K_ξ may be written in terms of generating states as the sum

$$K_\xi = \sum_{n=0}^{\infty} \frac{1}{n!2^n} T_n(\psi[\xi, \dots, \xi]).$$

Conversely, we can recover a generating state $\psi[\xi_1, \dots, \xi_n]$ for $\xi_1, \dots, \xi_n \in L_\Sigma$ from a coherent state by using partial derivatives,

$$T_n(\psi[\xi_1, \dots, \xi_n]) = 2^n \frac{\partial}{\partial \lambda_1} \cdots \frac{\partial}{\partial \lambda_n} K_{\lambda_1 \xi_1 + \cdots + \lambda_n \xi_n} \Big|_{\lambda_1=0, \dots, \lambda_n=0}. \quad (12.2)$$

We proceed to the comparison of the amplitude maps $\rho_M : \mathcal{H}_{\partial M} \rightarrow \mathbb{C}$. This was given in the holomorphic approach through an integral

$$\rho_M(\psi) = \int_{\hat{L}_{\bar{M}}} \psi(\xi) d\nu_{\bar{M}}(\xi). \quad (12.3)$$

The integral is over the subspace $\hat{L}_{\tilde{M}} \subseteq \hat{L}_{\partial M}$ that carries an induced Gaussian measure $\nu_{\tilde{M}}$. This, however, was shown to lead for coherent states to the remarkably simple expression

$$\rho_M(K_\xi) = \exp\left(\frac{1}{4}\{\widehat{\xi}, \widehat{\xi}\}_{\partial M}\right),$$

where we are using the conventions of Section 7.2. It is then straightforward, using formula (12.2) to extract the amplitude for a generating state. The result is precisely given by the definitions (7.4), (7.3) and (7.5) of Section 7.2. Thus, not only the state spaces, but also the amplitudes are equivalent in both approaches. In fact, this is actually the way in which we obtained the bosonic versions of formulas (7.4), (7.3) and (7.5).

It was shown in [6, 12] that the quantization rule (12.3) is precisely equivalent to the usual Feynman path integral. So this applies also to the quantization scheme put forward in the present work.

13 Outlook

An obvious question concerns the generalization to interacting field theories. Neither the core axioms (Section 6.1) nor the probability interpretation (Section 11) are specific to the linear setting. What needs to be generalized is the encoding of the classical theory and the quantization prescription. In the bosonic case a perturbative approach to interactions was briefly discussed in [6]. This basically amounts to the standard techniques of quantum field theory motivated through the Feynman path integral, involving generating functions and leading to Feynman diagrams. It should not be difficult to adapt this to the fermionic case. In the bosonic case the holomorphic quantization approach followed in [9] is suggestive of non-perturbative generalizations in view of its motivation from geometric quantization. This was exploited in [6] in a mild way by generalizing from linear to affine theories. Due to the large body of work on geometric quantization of non-linear systems (although mostly limited to systems with finitely many degrees of freedom) [19] one should expect that this line of investigation could be pushed much further. In the purely fermionic case this looks much less promising, but one would expect interesting fermionic systems to contain also bosonic degrees of freedom.

Apart from state spaces and amplitudes, the GBF also naturally accommodates a notion of observable [10]. For linear bosonic field theory this was comprehensively developed in [12]. An essentially parallel treatment should be possible for linear fermionic field theory. A likely limitation from the axiomatic point of view would be the restriction for observables to have fermionic degree 0. In the light of the remarks in Section 11 this would likely also ensure compatibility with the probability interpretation and thus a consistent notion of expectation value.

As shown in Section 4.3, classical linear fermionic field theory, axiomatized according to Section 4.2, turns out to exhibit an “emergent” (algebraic) notion of time. As explained in Section 8.3, the quantized theory inherits this notion. Moreover, for theories with Lorentzian metric backgrounds (such as the Dirac theory considered in Section 5) this notion seems to agree precisely with the usual (geometric) notion of time evolution. This suggests to examine linear fermionic theories that do not come with a (Lorentzian) metric background to analyze the physical meaning of this emergent notion of time. Going beyond the linear case, the core axioms of the quantum theory (Section 6.1) alone do not seem to induce any such notion. This begs the question whether this phenomenon is strictly limited to the linear theory or has a more general counterpart.

The generalization of state spaces from Hilbert spaces to Krein spaces is a serious step in view of the justified reservations against giving up Hilbert spaces in quantum theory. However, as we have shown in Section 11, taking this step in the GBF is not at all the same thing as taking it in the standard formulation of quantum theory. Indeed, as we have seen there, a consistent

probability interpretation with Krein spaces in the GBF is perfectly possible. Even though the necessity for Krein spaces only arises in the fermionic case, this suggests to permit them also in bosonic theories. Indeed, throughout this work, we have allowed for this theoretical possibility. On the other hand, the known realistic bosonic field theories work fine with Hilbert spaces. This might change though when we consider them on non-standard hypersurfaces. (However, in the specific examples considered in [4, 8, 13] there seems to be no indication to this effect.) Also, Krein spaces might become useful when abandoning Lorentzian background metrics.

A subject we have touched upon neither in [9] nor here is that of *corners*, i.e., the admission of hypersurfaces with boundaries. The problem is not that it would be difficult to write down axioms for this and have them satisfied. Indeed, we could easily include corners as follows. First we modify the axioms for a spacetime system to include hypersurfaces with boundaries. The notion of a disjoint decomposition of a hypersurface would be replaced by a more general notion that allows the components to have boundaries. (Precisely this was done in [14].) We could then proceed to modify both the classical axioms of Section 4.2 and the core axioms of Section 6.1 by simply replacing the disjoint notion of decomposition of hypersurfaces with the more general one. This would not affect at all the “algebraic” side of things. Everything would still work, including the quantization scheme and the proofs. The problem with this is that we would no longer be able to capture standard examples of quantum field theories. Generically, there would not be a suitable Hilbert or Krein space to be associated with a hypersurface with boundary, even in the classical linear theory. This is related to the non-locality of the complex structure J_Σ associated with the hypersurface Σ , compare Sections 2.2 and 5.3. More generally, this problem can be seen in the light of the Reeh–Schlieder theorem [16]. This essentially implies that state spaces cannot be localized in parts of hypersurfaces. We remark that, on the other hand, in the case of 2-dimensional Yang–Mills theory a certain modification of the core axioms was shown to correctly implement corners [14].

A Proof of the main lemma

To prepare for the proof of Lemma 9.3 we note some useful identities. We assume the geometric context of axiom (T5b) or equivalently that of (T5b*). Consider $\phi \in L_{M_1}$. Due to the exact sequence (4.2) of axiom (C7) there is then $\check{\phi} \in L_\Sigma$ such that $(\phi, \check{\phi}, \check{\phi}) \in L_M \subseteq L_{\Sigma_1} \times L_\Sigma \times L_{\overline{\Sigma}}$.¹⁰ With this notation we have the following identities. We omit the straightforward proofs.

Lemma A.1. *Let $\xi \in L_\Sigma$ and $\phi \in L_{M_1}$. Then*

$$\{(\widehat{0, \xi, 0}), (\widehat{\phi, 0, 0})\}_{\partial M} = \{\xi, \check{\phi}\}_\Sigma - \{(\widehat{0, \xi, 0}), (\widehat{0, \check{\phi}, \check{\phi}})\}_{\partial M}, \quad (\text{A.1})$$

$$\{(\widehat{0, 0, \xi}), (\widehat{\phi, 0, 0})\}_{\partial M} = \{\xi, \check{\phi}\}_{\overline{\Sigma}} - \{(\widehat{0, 0, \xi}), (\widehat{0, \check{\phi}, \check{\phi}})\}_{\partial M}. \quad (\text{A.2})$$

Lemma A.2. *Let $\phi_1, \phi_2 \in L_{M_1}$. Then*

$$\begin{aligned} \{\phi_1, \phi_2\}_{\Sigma_1} &= \{(\widehat{\phi_1, 0, 0}), (\widehat{\phi_2, 0, 0})\}_{\partial M} - \{(\widehat{0, \check{\phi}_1, \check{\phi}_1}), (\widehat{0, \check{\phi}_2, \check{\phi}_2})\}_{\partial M} \\ &\quad + \{\check{\phi}_1, \check{\phi}_2\}_\Sigma + \{\check{\phi}_1, \check{\phi}_2\}_{\overline{\Sigma}}. \end{aligned} \quad (\text{A.3})$$

Proof of Lemma 9.3. We first note that the left hand side and the right hand side of expression (9.10) are both necessarily zero if n is odd, due to definition (7.3). We may thus assume that n is even. To simplify notation we replace n in the following by $2n$, where the new variable n is not necessarily even. We start by evaluating the right hand side of expression (9.10), which

¹⁰As elsewhere, our simplified notation does not explicitly distinguish between L_M and $L_{\overline{M}}$ etc.

we shall denote by R (in order to avoid sub-indices we shall abbreviate ξ_{a_i} by ξ_i throughout this proof)

$$R = \sum_{m=0}^{\infty} \frac{1}{2^m m!} \sum_{a_1, \dots, a_m \in N_\alpha} (-1)^{\sum_{i=1}^m [\xi_i]} \times \rho_M \left(\tau_{\Sigma_1, \Sigma, \overline{\Sigma}'; \partial M} (\psi[\phi_1, \dots, \phi_{2n}] \otimes \psi[\xi_1, \dots, \xi_m] \otimes \iota_\Sigma(\psi[\xi_1, \dots, \xi_m])) \right) \quad (\text{A.4})$$

$$= \sum_{m=0}^{\infty} \frac{\kappa^m}{2^m m!} \sum_{a_1, \dots, a_m \in N_\alpha} (-1)^{\sum_{i=1}^m [\xi_i]} \rho_M (\psi[(\phi_1, 0, 0), \dots, (\phi_{2n}, 0, 0), (0, \xi_1, 0), \dots, (0, \xi_m, 0), (0, 0, \xi_m), \dots, (0, 0, \xi_1)]) \quad (\text{A.5})$$

$$= \sum_{m=0}^{\infty} \sum_{a_1, \dots, a_m \in N_\alpha} (-1)^{\sum_{i=1}^m [\xi_i]} \sum_{\substack{\{l, k, a, b, c, d: 2n=2l+a+b, \\ m=2c+b+k=2d+a+k\}} \frac{2^{n-l-c-d}}{l!k!a!b!c!d!} \sum_{\substack{\sigma \in S^{2n} \\ \nu \in S^m}} \kappa^{|\sigma|+|\nu|} \times \kappa^{d+a+k + \frac{l(l-1)+b(b-1)+c(c-1)+d(d-1)}{2}} \left(\prod_{i=1}^l \{(\widehat{\phi_{\sigma(i)}}, 0, 0), (\widehat{\phi_{\sigma(l+i)}}, 0, 0)\}_{\partial M} \right) \times \left(\prod_{i=1}^a \{(\widehat{\phi_{\sigma(2l+i)}}, 0, 0), (\widehat{0, 0, \xi_{\nu(i)}})\}_{\partial M} \right) \left(\prod_{i=1}^b \{(\widehat{\phi_{\sigma(2l+a+i)}}, 0, 0), (\widehat{0, \xi_i, 0})\}_{\partial M} \right) \times \left(\prod_{i=1}^k \{(\widehat{0, \xi_{b+i}}, 0), (\widehat{0, 0, \xi_{\nu(a+i)}})\}_{\partial M} \right) \left(\prod_{i=1}^c \{(\widehat{0, \xi_{b+k+i}}, 0), (\widehat{0, \xi_{b+k+c+i}}, 0)\}_{\partial M} \right) \times \left(\prod_{i=1}^d \{(\widehat{0, 0, \xi_{\nu(a+k+i)}}), (\widehat{0, 0, \xi_{\nu(a+k+d+i)}})\}_{\partial M} \right). \quad (\text{A.6})$$

The step from expression (A.4) to (A.5) consists of evaluating the maps ι_Σ and $\tau_{\Sigma_1, \Sigma, \overline{\Sigma}'; \partial M}$. The next step consists of applying the amplitude formula (7.5). This generates all possible pairings of 1-particle states. In order to organize these comprehensively, we introduce the variables l, k, a, b, c, d . These are non-negative integers that obey certain relations as indicated. They count the different types of pairing that appear. This reorganization and the associated particular summation over certain permutations leads to combinatorial factors as indicated in the first line of expression (A.6). At the same time, reorderings that occur lead in the fermionic case to a sign factor, which is written as a power of κ at the end of the first line and in the second line of expression (A.6). Note in particular, that implicit use was made of the fact that the summation over the indices a_1, \dots, a_m is symmetrical. This allowed to remove a summation over permutations of the factors ξ_1, \dots, ξ_m and thus a cancellation of the factor $1/(m!)$ that appears in (A.5). We proceed to perform the substitution of the identities (A.1) and (A.2) of Lemma A.1, yielding

$$R = \sum_{m=0}^{\infty} \sum_{a_1, \dots, a_m \in N_\alpha} (-1)^{\sum_{i=1}^m [\xi_i]} \sum_{\substack{\{l, k, a, b, c, d: 2n=2l+a+b, \\ m=2c+b+k=2d+a+k\}} \frac{2^{n-l-c-d}}{l!k!a!b!c!d!} \times \sum_{\substack{\sigma \in S^{2n} \\ \nu \in S^m}} \kappa^{|\sigma|+|\nu|} \kappa^{d+a+k + \frac{l(l-1)+b(b-1)+c(c-1)+d(d-1)}{2}} \left(\prod_{i=1}^l \{(\widehat{\phi_{\sigma(i)}}, 0, 0), (\widehat{\phi_{\sigma(l+i)}}, 0, 0)\}_{\partial M} \right) \times \left(\prod_{i=1}^a \left(\kappa \{ \widehat{\xi_{\nu(i)}}, \widehat{\phi_{\sigma(2l+i)}} \}_{\overline{\Sigma}'} - \{(\widehat{0, \phi_{\sigma(2l+i)}}, 0), (\widehat{0, 0, \xi_{\nu(i)}})\}_{\partial M} \right) \right)$$

$$\begin{aligned}
& - \left\{ (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+i)}}, (0, 0, \widehat{\xi_{\nu(i)}}) \}_{\partial M} \right\} \\
& \times \left(\prod_{i=1}^b \left(\kappa \{ \xi_i, \widehat{\tilde{\phi}_{\sigma(2l+a+i)}} \}_{\Sigma} - \{ (0, \widehat{\tilde{\phi}_{\sigma(2l+a+i)}}, 0), (0, \widehat{\xi_i}, 0) \}_{\partial M} \right. \right. \\
& \left. \left. - \{ (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+a+i)}}, (0, \widehat{\xi_i}, 0) \}_{\partial M} \right) \right) \\
& \times \left(\prod_{i=1}^k \{ (0, \widehat{\xi_{b+i}}, 0), (0, 0, \widehat{\xi_{\nu(a+i)}}) \}_{\partial M} \right) \left(\prod_{i=1}^c \{ (0, \widehat{\xi_{b+k+i}}, 0), (0, \widehat{\xi_{b+k+c+i}}, 0) \}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^d \{ (0, 0, \widehat{\xi_{\nu(a+k+i)}}, (0, 0, \widehat{\xi_{\nu(a+k+d+i)}}) \}_{\partial M} \right).
\end{aligned}$$

We proceed to expand the products of sums that have arisen from the substitutions. In doing so we introduce new variables e, f, g, h, j, o counting the occurrences of the summands, splitting $a = e + f + g$ and $b = h + j + o$. At the same time the variables a, b are eliminated. This yields

$$\begin{aligned}
R &= \sum_{m=0}^{\infty} \sum_{a_1, \dots, a_m \in N_{\alpha}} (-1)^{\sum_{i=1}^m [\xi_i]} \sum_{\substack{\{l, k, c, d, e, f, g, h, j, o\}: \\ 2n = 2l + e + f + g + h + j + o, \\ m = 2c + h + j + o + k \\ = 2d + e + f + g + k}} \frac{2^{n-l-c-d}}{l!c!d!e!f!g!h!j!o!} \\
& \times \sum_{\substack{\sigma \in S^{2n} \\ \nu \in S^m}} \kappa^{|\sigma| + |\nu| + d + h + f + g + k + hj + ho + jo} (-1)^{j+o+e+f} \\
& \times \kappa^{\frac{l(l-1) + h(h-1) + j(j-1) + o(o-1) + c(c-1) + d(d-1)}{2}} \left(\prod_{i=1}^l \{ (\widehat{\phi_{\sigma(i)}}, 0, 0), (\widehat{\phi_{\sigma(l+i)}}, 0, 0) \}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^e \{ \widehat{\xi_{\nu(i)}}, \widehat{\tilde{\phi}_{\sigma(2l+i)}} \}_{\overline{\Sigma}} \right) \left(\prod_{i=1}^f \{ (0, \widehat{\tilde{\phi}_{\sigma(2l+e+i)}}), (0, 0, \widehat{\xi_{\nu(e+i)}}) \}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^g \{ (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+e+f+i)}}), (0, 0, \widehat{\xi_{\nu(e+f+i)}}) \}_{\partial M} \right) \left(\prod_{i=1}^h \{ \xi_i, \widehat{\tilde{\phi}_{\sigma(2l+e+f+g+i)}} \}_{\Sigma} \right) \\
& \times \left(\prod_{i=1}^j \{ (0, \widehat{\tilde{\phi}_{\sigma(2l+e+f+g+h+i)}}), (0, \widehat{\xi_{h+i}}, 0) \}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^o \{ (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+e+f+g+h+j+i)}}), (0, \widehat{\xi_{h+j+i}}, 0) \}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^k \{ (0, \widehat{\xi_{h+j+o+i}}, 0), (0, 0, \widehat{\xi_{\nu(e+f+g+i)}}) \}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^c \{ (0, \widehat{\xi_{h+j+o+k+i}}, 0), (0, \widehat{\xi_{h+j+o+k+c+i}}, 0) \}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^d \{ (0, 0, \widehat{\xi_{\nu(e+f+g+k+i)}}), (0, 0, \widehat{\xi_{\nu(e+f+g+k+d+i)}}) \}_{\partial M} \right).
\end{aligned}$$

Next, we contract the factors of the form

$$\{ \xi_{\nu(i)}, \widehat{\tilde{\phi}_{\sigma(2l+i)}} \}_{\overline{\Sigma}}$$

with corresponding pairs, by viewing the summation in the participating ξ variables as sums over complete basis. To this end we use the identity

$$\sum_{a_i \in N_\alpha} (-1)^{[\xi_i]} \xi_i \{\xi_i, \eta\}_{\overline{\Sigma'}} = \kappa \eta$$

for $\eta \in L_{\overline{\Sigma'}}$. Note that this leads to a factor κ^e . The contraction involves a distribution of factors over the different types of matching pairs. This makes necessary the introduction of additional counting variables $p, q, r, s, t, u, v, w, x, y, z$. At the same time the variables e, h, j, o, k, c are eliminated. The relation between old and new variables is the following

$$\begin{aligned} e &= p + q + r + s + t + 2u, & h &= p + v, & j &= q + w, \\ o &= r + x, & k &= s + y, & c &= z + t + u. \end{aligned}$$

Since some of the summations over basis of L_Σ are removed in the process, the variable m is also replaced by a new variable m' with the relation $m = m' + e$. However, for simplicity of notation we rename the new variable m' again m . After some opportune reorderings we obtain the following expression

$$\begin{aligned} R &= \sum_{m=0}^{\infty} \sum_{a_1, \dots, a_m \in N_\alpha} (-1)^{\sum_{i=1}^m [\xi_i]} \sum_{\substack{\{l, d, f, g, p, q, r, s, t, u, v, w, x, y, z: \\ 2n=2l+2p+2q+2r+2u+f+g+s+t+v+w+x, \\ m=2z+t+v+w+x+y=2d+f+g+s+y\}}} \frac{2^{n-l-d-u-z}}{l!d!f!g!p!q!r!s!t!u!v!w!x!y!z!} \\ &\times \sum_{\substack{\sigma \in S^{2n} \\ \nu \in S^m}} \kappa^{|\sigma|+|\nu|} (-1)^{f+g+q+r+w+x} \kappa^{d+f+g+s+v+y+(t+w)(f+g+s+v+x)+(v+x)(f+g+s)+vx} \\ &\times \kappa^{\frac{l(l-1)+d(d-1)+p(p-1)+q(q-1)+r(r-1)+u(u-1)+v(v-1)+x(x-1)+z(z-1)+(t+w)(t+w-1)}{2}} \\ &\times \left(\prod_{i=1}^l \{(\widehat{\phi_{\sigma(i)}}, 0, 0), (\widehat{\phi_{\sigma(l+i)}}, 0, 0)\}_{\partial M} \right) \left(\prod_{i=1}^p \{\tilde{\phi}_{\sigma(2l+i)}, \tilde{\phi}_{\sigma(2l+p+i)}\}_{\Sigma} \right) \\ &\times \left(\prod_{i=1}^q \{(0, \widehat{\tilde{\phi}_{\sigma(2l+2p+i)}}), (0, \widehat{\tilde{\phi}_{\sigma(2l+2p+q+i)}})\}_{\partial M} \right) \\ &\times \left(\prod_{i=1}^u \{(0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2q+i)}}), (0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2q+u+i)}})\}_{\partial M} \right) \\ &\times \left(\prod_{i=1}^r \{(0, 0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2q+2u+i)}}), (0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2q+2u+r+i)}}), 0\}_{\partial M} \right) \\ &\times \left(\prod_{i=1}^v \{\xi_i, \tilde{\phi}_{\sigma(2l+2p+2q+2u+2r+i)}\}_{\Sigma} \right) \\ &\times \left(\prod_{i=1}^w \{(0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2q+2u+2r+v+i)}}), (0, \widehat{\xi_{v+i}}), 0\}_{\partial M} \right) \\ &\times \left(\prod_{i=1}^t \{(0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2q+2u+2r+v+w+i)}}), (0, \widehat{\xi_{v+w+i}}), 0\}_{\partial M} \right) \\ &\times \left(\prod_{i=1}^x \{(0, 0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2q+2u+2r+v+w+t+i)}}), (0, \widehat{\xi_{v+w+t+i}}), 0\}_{\partial M} \right) \\ &\times \left(\prod_{i=1}^s \{(0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2q+2u+2r+v+w+t+x+i)}}), (0, 0, \widehat{\xi_{\nu(i)}})\}_{\partial M} \right) \end{aligned}$$

$$\begin{aligned}
& \times \left(\prod_{i=1}^f \{ (0, \widehat{\tilde{\phi}}_{\sigma(2l+2p+2q+2u+2r+v+w+t+x+s+i)}, 0), (0, 0, \widehat{\xi}_{\nu(s+i)}) \}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^g \{ (0, 0, \widehat{\tilde{\phi}}_{\sigma(2l+2p+2q+2u+2r+v+w+t+x+s+f+i)}), (0, 0, \widehat{\xi}_{\nu(s+f+i)}) \}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^y \{ (0, \widehat{\xi}_{\nu(s+w+t+x+i)}), (0, 0, \widehat{\xi}_{\nu(s+f+g+i)}) \}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^z \{ (0, \widehat{\xi}_{\nu(s+w+t+x+y+i)}), (0, \widehat{\xi}_{\nu(s+w+t+x+y+z+i)}), 0 \}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^d \{ (0, 0, \widehat{\xi}_{\nu(s+f+g+y+i)}), (0, 0, \widehat{\xi}_{\nu(s+f+g+y+d+i)}) \}_{\partial M} \right). \tag{A.7}
\end{aligned}$$

It turns out that certain sums in the above expression can be identified as binomial sums and simplified. In particular, fix all variables except for q and u . Leaving out all expressions that do not depend on these two variables (but may involve their sum), the remaining sum S is the following

$$\begin{aligned}
S &= \sum_{\{q,u:2n=2l+2p+2q+2r+2u+f+g+s+t+v+w+x\}} \frac{2^{-u}(-1)^q}{q!u!} \kappa^{\frac{q(q-1)+u(u-1)}{2}} \\
& \times \left(\prod_{i=1}^q \{ (0, \widehat{\tilde{\phi}}_{\sigma(2l+2p+i)}), 0 \}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^u \{ (0, \widehat{\tilde{\phi}}_{\sigma(2l+2p+2q+i)}), 0 \}_{\partial M} \right).
\end{aligned}$$

A reordering yields the following expression

$$\begin{aligned}
S &= \sum_{\{q,u:2n=2l+2p+2(q+u)+2r+f+g+s+t+v+w+x\}} \frac{2^{-u}(-1)^q}{q!u!} \kappa^{\frac{(q+u)(q+u-1)}{2}} \\
& \times \left(\prod_{i=1}^q \{ (0, \widehat{\tilde{\phi}}_{\sigma(2l+2p+i)}), 0 \}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^u \{ (0, \widehat{\tilde{\phi}}_{\sigma(2l+2p+q+i)}), 0 \}_{\partial M} \right).
\end{aligned}$$

Introducing the new variable $a = q + u$ we recognize the above as a binomial sum, leading to the following simplification

$$\begin{aligned}
S &= \sum_{\{a:2n=2l+2p+2a+2r+f+g+s+t+v+w+x\}} \frac{1}{a!} \left(\sum_{q=0}^a \binom{a}{q} \left(\frac{1}{2}\right)^{a-q} (-1)^q \right) \kappa^{\frac{a(a-1)}{2}} \\
& \times \left(\prod_{i=1}^a \{ (0, \widehat{\tilde{\phi}}_{\sigma(2l+2p+i)}), 0 \}_{\partial M} \right) \\
&= \sum_{\{a:2n=2l+2p+2a+2r+f+g+s+t+v+w+x\}} \frac{2^{-a}(-1)^a}{a!} \kappa^{\frac{a(a-1)}{2}} \\
& \times \left(\prod_{i=1}^a \{ (0, \widehat{\tilde{\phi}}_{\sigma(2l+2p+i)}), 0 \}_{\partial M} \right).
\end{aligned}$$

We may proceed similarly with the variables t and w . Introducing $b = t + w$ we have a residual sum

$$\begin{aligned}
S &= \sum_{\substack{\{w,t:2n=2l+2p+2a+2r+f+g+s+t+v+w+x, \\ m=2z+t+v+w+x+y\}}} \frac{(-1)^w}{t!w!} \kappa^{(t+w)(f+g+s+v+x) + \frac{(t+w)(t+w-1)}{2}} \\
&\times \left(\prod_{i=1}^w \{ (0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2q+2u+2r+v+i)}}, 0), (0, \widehat{\xi_{v+i}}, 0) \}_{\partial M} \right) \\
&\times \left(\prod_{i=1}^t \{ (0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2q+2u+2r+v+w+i)}}, 0), (0, \widehat{\xi_{v+w+i}}, 0) \}_{\partial M} \right) \\
&= \sum_{\substack{\{b:2n=2l+2p+2a+2r+f+g+s+b+v+x, \\ m=2z+b+v+x+y\}}} \frac{1}{b!} \left(\sum_{w=0}^b \binom{b}{w} (-1)^w \right) \kappa^{b(f+g+s+v+x) + \frac{b(b-1)}{2}} \\
&\times \left(\prod_{i=1}^b \{ (0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2q+2u+2r+v+i)}}, 0), (0, \widehat{\xi_{v+i}}, 0) \}_{\partial M} \right).
\end{aligned}$$

In this case, however, the binomial expression vanishes whenever $b \neq 0$. That is, we may eliminate the variables t and w from expression (A.7) by simply setting $t = 0$ and $w = 0$. The same applies to the pair of variables f and s , as the reader is invited to verify. All in all we obtain the following simplified expression for R ,

$$\begin{aligned}
R &= \sum_{m=0}^{\infty} \sum_{a_1, \dots, a_m \in N_{\alpha}} (-1)^{\sum_{i=1}^m [\xi_i]} \sum_{\substack{\{l,a,d,g,p,r,v,x,y,z: \\ 2n=2l+2a+2p+2r+g+v+x, \\ m=2z+v+x+y=2d+g+y\}}} \frac{2^{n-l-a-d-z}}{l!a!d!g!p!r!v!x!y!z!} \\
&\times \sum_{\substack{\sigma \in S^{2n} \\ \nu \in S^m}} \kappa^{|\sigma|+|\nu|} (-1)^{a+g+r+x} \kappa^{d+g+v+y+(v+x)g+vx} \\
&\times \kappa^{\frac{l(l-1)+a(a-1)+d(d-1)+p(p-1)+r(r-1)+v(v-1)+x(x-1)+z(z-1)}{2}} \\
&\times \left(\prod_{i=1}^l \{ (\widehat{\phi_{\sigma(i)}}), 0, 0), (\widehat{\phi_{\sigma(l+i)}}, 0, 0) \}_{\partial M} \right) \left(\prod_{i=1}^p \{ \tilde{\phi}_{\sigma(2l+i)}, \tilde{\phi}_{\sigma(2l+p+i)} \}_{\Sigma} \right) \\
&\times \left(\prod_{i=1}^a \{ (0, \widehat{\tilde{\phi}_{\sigma(2l+2p+i)}}, 0), (0, \widehat{\tilde{\phi}_{\sigma(2l+2p+a+i)}}, 0) \}_{\partial M} \right) \\
&\times \left(\prod_{i=1}^r \{ (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2a+i)}}, (0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2a+r+i)}}, 0) \}_{\partial M} \right) \\
&\times \left(\prod_{i=1}^v \{ \xi_i, \tilde{\phi}_{\sigma(2l+2p+2a+2r+i)} \}_{\Sigma} \right) \\
&\times \left(\prod_{i=1}^x \{ (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2a+2r+v+i)}}, (0, \widehat{\xi_{v+i}}, 0) \}_{\partial M} \right) \\
&\times \left(\prod_{i=1}^g \{ (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2a+2r+v+x+i)}}, (0, 0, \widehat{\xi_{\nu(i)}}) \}_{\partial M} \right) \\
&\times \left(\prod_{i=1}^y \{ (0, \widehat{\xi_{v+x+i}}, 0), (0, 0, \widehat{\xi_{\nu(g+i)}}) \}_{\partial M} \right)
\end{aligned}$$

$$\begin{aligned} & \times \left(\prod_{i=1}^z \{ (0, \widehat{\xi_{v+x+y+i}}, 0), (0, \widehat{\xi_{v+x+y+z+i}}, 0) \}_{\partial M} \right) \\ & \times \left(\prod_{i=1}^d \{ (0, 0, \widehat{\xi_{\nu(g+y+i)}}), (0, 0, \widehat{\xi_{\nu(g+y+d+i)}}) \}_{\partial M} \right). \end{aligned}$$

We proceed to contract the factors of the form

$$\{ \xi_i, \tilde{\phi}_{\sigma(2l+2p+2a+2r+i)} \}_{\Sigma}$$

with corresponding pairs, by viewing the summation in the participating ξ variables as sums over complete basis. To this end we use the identity

$$\sum_{a_i \in N_\alpha} (-1)^{[\xi_i]} \xi_i \{ \xi_i, \eta \}_{\Sigma} = \eta$$

for $\eta \in L_\Sigma$. The contraction involves a distribution of factors over the different types of matching pairs. This makes necessary the introduction of additional counting variables b, c, e, f, h, j, k . At the same time the variables v, g, y, d are eliminated. The relation between old and new variables is the following

$$v = b + c + e + 2f, \quad g = b + h, \quad y = c + j, \quad d = k + e + f.$$

Since some of the summations over basis of L_Σ are removed in the process, the variable m is also replaced by a new variable m' with the relation $m = m' + v$. However, for simplicity of notation we rename the new variable m' again m . After some reorderings we obtain the following expression

$$\begin{aligned} R &= \sum_{m=0}^{\infty} \sum_{a_1, \dots, a_m \in N_\alpha} (-1)^{\sum_{i=1}^m [\xi_i]} \sum_{\substack{l, a, b, c, e, f, h, j, k, p, r, x, z: \\ 2n = 2l + 2a + 2b + 2f + 2p + 2r + c + e + h + x, \\ m = 2z + c + j + x = 2k + e + h + j}} \frac{2^{n-l-a-f-k-z}}{l!a!b!c!e!f!h!j!k!p!r!x!z!} \\ & \times \sum_{\substack{\sigma \in S^{2n} \\ \nu \in S^m}} \kappa^{|\sigma| + |\nu|} (-1)^{a+b+h+r+x} \kappa^{e+h+j+k+(e+h)(c+x)} \\ & \times \kappa^{\frac{l(l-1)+a(a-1)+b(b-1)+f(f-1)+k(k-1)+p(p-1)+r(r-1)+z(z-1)+(c+x)(c+x-1)}{2}} \\ & \times \left(\prod_{i=1}^l \{ (\widehat{\phi_{\sigma(i)}}, 0, 0), (\widehat{\phi_{\sigma(l+i)}}, 0, 0) \}_{\partial M} \right) \left(\prod_{i=1}^p \{ \tilde{\phi}_{\sigma(2l+i)}, \tilde{\phi}_{\sigma(2l+p+i)} \}_{\Sigma} \right) \\ & \times \left(\prod_{i=1}^a \{ (0, \widehat{\tilde{\phi}_{\sigma(2l+2p+i)}}), (0, \widehat{\tilde{\phi}_{\sigma(2l+2p+a+i)}}) \}_{\partial M} \right) \\ & \times \left(\prod_{i=1}^r \{ (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2a+i)}}), (0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2a+r+i)}}) \}_{\partial M} \right) \\ & \times \left(\prod_{i=1}^b \{ (0, 0, \widehat{\phi_{\sigma(2l+2p+2a+2r+i)}}), (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2a+2r+b+i)}}) \}_{\partial M} \right) \\ & \times \left(\prod_{i=1}^f \{ (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2a+2r+2b+i)}}), (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2a+2r+2b+f+i)}}) \}_{\partial M} \right) \\ & \times \left(\prod_{i=1}^c \{ (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2a+2r+2b+2f+i)}}), (\widehat{\xi_i}, 0) \}_{\partial M} \right) \end{aligned}$$

$$\begin{aligned}
& \times \left(\prod_{i=1}^x \{(\widehat{0, 0, \tilde{\phi}_{\sigma(2l+2p+2a+2r+2b+2f+c+i)}), (\widehat{0, \xi_{c+i}, 0})\}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^h \{(\widehat{0, 0, \tilde{\phi}_{\sigma(2l+2p+2a+2r+2b+2f+c+x+i)}), (\widehat{0, 0, \xi_{\nu(i)}})\}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^e \{(\widehat{0, 0, \tilde{\phi}_{\sigma(2l+2p+2a+2r+2b+2f+c+x+h+i)}), (\widehat{0, 0, \xi_{\nu(h+i)}})\}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^j \{(\widehat{0, \xi_{c+x+i}, 0}), (\widehat{0, 0, \xi_{\nu(h+e+i)}})\}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^z \{(\widehat{0, \xi_{c+x+j+i}, 0}), (\widehat{0, \xi_{c+x+j+z+i}, 0})\}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^k \{(\widehat{0, 0, \xi_{\nu(h+e+j+i)}}), (\widehat{0, 0, \xi_{\nu(h+e+j+k+i)}})\}_{\partial M} \right).
\end{aligned}$$

Again, we identify binomial sums in this expression that can be evaluated. Leaving the details to the reader we note that the variables c and x form a pair, with a vanishing binomial if we sum over c and x , holding their sum fixed. This leads to the substitution $c = 0$ and $x = 0$. Similarly, h and e form a pair yielding a vanishing binomial and we may set $h = 0$ and $e = 0$. Also, b and f form a pair. In this case the binomial does not vanish and we introduce for the sum the new variable $d = b + f$, discarding b and f in the process. This yields

$$\begin{aligned}
R &= \sum_{m=0}^{\infty} \sum_{a_1, \dots, a_m \in N_{\alpha}} (-1)^{\sum_{i=1}^m [\xi_i]} \sum_{\substack{\{l, a, d, j, k, p, r, z: \\ n=l+a+d+p+r, \\ m=2z+j=2k+j\}} \frac{2^{n-l-a-d-k-z}}{l!a!d!j!k!p!r!z!} \\
& \times \sum_{\substack{\sigma \in S^{2n} \\ \nu \in S^m}} \kappa^{|\sigma|+|\nu|} (-1)^{a+d+r} \kappa^{j+k} \kappa^{\frac{l(l-1)+a(a-1)+d(d-1)+k(k-1)+p(p-1)+r(r-1)+z(z-1)}{2}} \\
& \times \left(\prod_{i=1}^l \{(\widehat{\phi_{\sigma(i)}, 0, 0}), (\widehat{\phi_{\sigma(l+i)}, 0, 0})\}_{\partial M} \right) \left(\prod_{i=1}^p \{\tilde{\phi}_{\sigma(2l+i)}, \tilde{\phi}_{\sigma(2l+p+i)}\}_{\Sigma} \right) \\
& \times \left(\prod_{i=1}^a \{(\widehat{0, \tilde{\phi}_{\sigma(2l+2p+i)}), 0), (\widehat{0, \tilde{\phi}_{\sigma(2l+2p+a+i)}}, 0)\}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^r \{(\widehat{0, 0, \tilde{\phi}_{\sigma(2l+2p+2a+i)}), (\widehat{0, \tilde{\phi}_{\sigma(2l+2p+2a+r+i)}}, 0)\}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^d \{(\widehat{0, 0, \phi_{\sigma(2l+2p+2a+2r+i)}), (\widehat{0, 0, \tilde{\phi}_{\sigma(2l+2p+2a+2r+d+i)}})\}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^j \{(\widehat{0, \xi_i}, 0), (\widehat{0, 0, \xi_{\nu(i)}})\}_{\partial M} \right) \left(\prod_{i=1}^z \{(\widehat{0, \xi_{j+i}}, 0), (\widehat{0, \xi_{j+z+i}}, 0)\}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^k \{(\widehat{0, 0, \xi_{\nu(j+i)}}), (\widehat{0, 0, \xi_{\nu(j+k+i)}})\}_{\partial M} \right).
\end{aligned}$$

This expression exhibits the desired factorization into a term depending on the variables ϕ_1, \dots, ϕ_{2n} and a term involving sums over basis of L_{Σ} . Denoting the former term by R_1 and the latter

by R_2 we have $R = R_1 R_2$, with R_1 given as follows

$$\begin{aligned}
R_1 = & \sum_{\substack{\{l,a,d,p,r\} \\ n=l+a+d+p+r}} \frac{2^{n-l-a-d}}{l!a!d!p!r!} \sum_{\sigma \in S^{2n}} \kappa^{|\sigma|} (-1)^{a+d+r} \kappa^{\frac{l(l-1)+a(a-1)+d(d-1)+p(p-1)+r(r-1)}{2}} \\
& \times \left(\prod_{i=1}^l \{(\widehat{\phi_{\sigma(i)}}, 0, 0), (\widehat{\phi_{\sigma(l+i)}}, 0, 0)\}_{\partial M} \right) \left(\prod_{i=1}^p \{\tilde{\phi}_{\sigma(2l+i)}, \tilde{\phi}_{\sigma(2l+p+i)}\}_{\Sigma} \right) \\
& \times \left(\prod_{i=1}^a \{(0, \widehat{\tilde{\phi}_{\sigma(2l+2p+i)}}, 0), (0, \widehat{\tilde{\phi}_{\sigma(2l+2p+a+i)}}, 0)\}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^r \{(0, 0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2a+i)}}, 0), (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2a+r+i)}}, 0)\}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^d \{(0, 0, \widehat{\phi_{\sigma(2l+2p+2a+2r+i)}}, 0), (0, 0, \widehat{\tilde{\phi}_{\sigma(2l+2p+2a+2r+d+i)}}, 0)\}_{\partial M} \right). \tag{A.8}
\end{aligned}$$

Taking into account that the summation constraint implies $z = k$, R_2 may be written as follows

$$\begin{aligned}
R_2 = & \sum_{m=0}^{\infty} \sum_{a_1, \dots, a_m \in N_{\alpha}} (-1)^{\sum_{i=1}^m [\xi_i]} \sum_{\substack{\{j,k\} \\ m=2k+j}} \frac{2^{-2k}}{j!(k!)^2} \sum_{\nu \in S^m} \kappa^{|\nu|} \kappa^{j+k} \\
& \times \left(\prod_{i=1}^j \{(0, \widehat{\xi_i}, 0), (0, 0, \widehat{\xi_{\nu(i)}})\}_{\partial M} \right) \left(\prod_{i=1}^k \{(0, \widehat{\xi_{j+i}}, 0), (0, \widehat{\xi_{j+k+i}}, 0)\}_{\partial M} \right) \\
& \times \left(\prod_{i=1}^k \{(0, 0, \widehat{\xi_{\nu(j+i)}}, 0), (0, 0, \widehat{\xi_{\nu(j+k+i)}}, 0)\}_{\partial M} \right). \tag{A.9}
\end{aligned}$$

Observation reveals that R_2 coincides precisely with expression (A.6) for R in the special case $n = 0$. Therefore, R_2 coincides with expression (9.8) for the regularized gluing anomaly factor associated to α . That is

$$R_2 = c_{\alpha}(M; \Sigma, \overline{\Sigma'}).$$

Returning to expression (A.8) for R_1 , we observe that it may be simplified by identifying the various factors as arising from expanding the left hand side of identity (A.3) of Lemma A.2 in terms of the right hand side. This eliminates the variables l, a, d, p, r , leading to the much simpler expression

$$R_1 = \frac{1}{n!} \sum_{\sigma \in S^{2n}} \kappa^{|\sigma|} \left(\prod_{i=1}^n \{\phi_{\sigma(i)}, \phi_{\sigma(2n+1-i)}\}_{\Sigma_1} \right).$$

But this coincides precisely with the definition (7.5) for an amplitude in M_1 ,

$$R_1 = \rho_{M_1}(\psi[\phi_1, \dots, \phi_{2n}]).$$

So, summarizing, we obtain

$$\rho_{M_1}(\psi[\phi_1, \dots, \phi_{2n}]) \cdot c_{\alpha}(M; \Sigma, \overline{\Sigma'}) = R.$$

This is expression (9.10) of Lemma 9.3 (with our redefinition of the variable n as $2n$). This completes the proof. As a final remark, observe that absolute convergence of expression (A.9) is ensured by the assumption of well definedness of $c_{\alpha}(M; \Sigma, \overline{\Sigma'})$. This then implies absolute convergence of the sums appearing in the preceding expressions for R . This is the case in spite of the fact that some reorderings are taking place (recall the substitutions of the variable m), since the reorderings are bounded in range by the fixed constant $2n$. \blacksquare

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