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Dunkl-Type Operators with Projection Terms Associated to Orthogonal Subsystems in Root System

Fethi BOUZEFFOUR

Department of Mathematics, King Saudi University, College of Sciences, P.O. Box 2455 Riyadh 11451, Saudi Arabia
E-mail: *fbouzaffour@ksu.edu.sa*

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Abstract. In this paper, we introduce a new differential-difference operator T_ξ ($\xi \in \mathbb{R}^N$) by using projections associated to orthogonal subsystems in root systems. Similarly to Dunkl theory, we show that these operators commute and we construct an intertwining operator between T_ξ and the directional derivative ∂_ξ. In the case of one variable, we prove that the Kummer functions are eigenfunctions of this operator.

Key words: special functions; differential-difference operators; integral transforms

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1 Introduction

In a series of papers [3, 4, 5, 6], C.F. Dunkl builds up the framework for a theory of differential-difference operators and special functions related to root systems. Beside them, there are now various further Dunkl-type operators, in particular the trigonometric Dunkl operators of Heckman [7, 8], Opdam [14], Cherednik [2], and the important q-analogues of Macdonald and Cherednik [13], see also [1, 11].

The main objective of this paper is to present a new class of differential-difference operators T_ξ, $\xi \in \mathbb{R}^N$ with the help of orthogonal projections related to orthogonal subsystems in root systems. In other words, our operators follow from Dunkl operator after replacing the usual reflections that exist in the definition of the operator with their corresponding orthogonal projections. Several problems related to the Dunkl theory arise in the setting of our operators, in particular, commutativity of $\{T_\xi,\, \xi \in \mathbb{R}^N\}$ and the existence of the intertwining operators.

The outline of the content of this paper is as follows. In Section 2, we collect some definitions and results related to root systems and Dunkl operators which will be relevant for the sequel. In Section 3, we introduce new differential-difference operators T_ξ and we prove the first main result. In Section 4, we give an explicit formula for the intertwining operator between T_ξ and the directional derivative. In Section 5, we study the one variable case. Finally, in Section 6 we study the cases of orthogonal subsets in root systems of type A_{N-1} and B_N.

2 Dunkl operators

Let us begin to recall some results concerning the root systems and Dunkl operators. A useful reference for this topic is the book by Humphreys [9]. Let $\alpha \in \mathbb{R}^N \backslash \{0\}$, we denote by s_α the reflection onto the hyperplane orthogonal to α; that is,

$$s_\alpha(x) = x - 2\frac{\langle x, \alpha\rangle}{|\alpha|^2}\alpha,$$

where $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product on $\mathbb{R}^N$, and $|x| = \sqrt{\langle x, x\rangle}$.

A root system is a finite set R of nonzero vectors in $\mathbb{R}^N$ such that for any $\alpha \in R$ one has

$$s_\alpha(R) = R, \qquad \text{and} \qquad R \cap \mathbb{R}\alpha = \{\pm\alpha\}.$$

A positive subsystem R_+ is any subset of R satisfying $R = R_+ \cup \{-R_+\}$. The Weyl group $W = W(R)$ (or real finite reflection group) generated by the root system $R \subset \mathbb{R}^N$ is the subgroup of orthogonal group $O(N)$ generated by $\{s_\alpha,\, \alpha \in R\}$. A multiplicity function on R is a complex-valued function $\kappa : R \to \mathbb{C}$ which is invariant under the Weyl group W, i.e.,

$$\kappa(\alpha) = \kappa(g\alpha), \qquad \forall\, \alpha \in R, \qquad \forall\, g \in W.$$

Let $\xi \in \mathbb{R}^N$, the Dunkl operator $\mathcal{D}_\xi$ associated with the Weyl group $W(R)$ and the multiplicity function κ, is the first order differential-difference operator:

$$(\mathcal{D}_\xi f)(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} \kappa(\alpha)\langle \alpha, \xi\rangle \frac{f(x) - f(s_\alpha x)}{\langle x, \alpha\rangle}. \tag{1}$$

Here ∂_ξ is the direction derivative corresponding to ξ and s_α is the orthogonal reflection onto the hyperplane orthogonal to α.

The Dunkl operator $\mathcal{D}_\xi$ is a homogeneous differential-difference operator of degree -1. By the W-invariance of the multiplicity function κ, we have

$$g^{-1} \circ \mathcal{D}_\xi \circ g = \mathcal{D}_{g\xi}, \qquad \forall\, g \in W(R), \qquad \xi \in \mathbb{R}^N.$$

The remarkable property of the Dunkl operators is that the family $\{\mathcal{D}_\xi,\, \xi \in \mathbb{R}^N\}$ generates a commutative algebra of linear operators on the $\mathbb{C}$-algebra of polynomial functions.

3 Operators of Dunkl-type

Let R be a root system. A subset R' of R is called a subsystem of R if it satisfies the following conditions:

i) If $\alpha \in R'$, then $-\alpha \in R'$;

ii) If $\alpha, \beta \in R'$ and $\alpha + \beta \in R$, then $\alpha + \beta \in R'$.

A subsystem R' of a root system R in $\mathbb{R}^N$ consisting of pairwise orthogonal roots is called orthogonal subsystem. In this case the related Weyl group $W(R')$ is a subgroup of $\mathbb{Z}_2^N$. For a vector $\alpha \in \mathbb{R}^N \setminus \{0\}$, we write

$$\tau_\alpha(x) = x - \frac{\langle x, \alpha\rangle}{|\alpha|^2}\alpha, \qquad x \in \mathbb{R}^N,$$

for the orthogonal projection onto the hyperplane $(\mathbb{R}\alpha)^\perp = \{x,\, \langle x, \alpha\rangle = 0\}$, so that the reflection s_α with respect to hyperplane orthogonal to α is related to τ_α by

$$\tau_\alpha = \frac{1}{2}(1 + s_\alpha).$$

The hyperplane $(\mathbb{R}\alpha)^{\perp}$ is the invariant set of τ_α. If $\langle \alpha, \beta \rangle = 0$, then the orthogonal projections τ_α and τ_β commute. The conjugate of orthogonal projection onto a hyperplane is again an orthogonal projection onto a hyperplane: suppose $u \in O(N)$ and $\alpha \in \mathbb{R}^N \backslash \{0\}$ then

$$u\tau_\alpha u^{-1} = \tau_{u\alpha}.$$

Let R be a root system and R' a positive orthogonal subsystem of R. For $\xi \in \mathbb{R}^N$, we define the differential-difference operator T_ξ by

$$(T_\xi f)(x) = \partial_\xi f(x) + \sum_{\alpha \in R'} \kappa(\alpha)\langle \alpha, \xi \rangle \frac{f(x) - f(\tau_\alpha x)}{\langle x, \alpha \rangle}. \tag{2}$$

where κ is a multiplicity function on R'. For $j = 1, \ldots, N$ denotes T_{e_j} by T_j. The operator T_ξ can be considered as a deformation of the usual directional derivatives and when $\kappa = 0$, the operator T_ξ reduces to the corresponding directional derivative. Furthermore, there is overlap between the notations (2) and (1). In fact, the operator (2) follows from Dunkl operator after replacing the reflections terms that exist in (1) by orthogonal projection terms.

Example 1. In the rank-one case, the root system is of type A_1 and the corresponding reflection s and orthogonal projection τ are given by

$$s(x) = -x, \qquad \tau(x) = \frac{1}{2}(1+s)(x) = 0.$$

The Dunkl-type operator T_κ associated with the projection τ and the multiplicity parameters κ ($\kappa \in \mathbb{C}$) is given by

$$T_\kappa f(x) = f'(x) + \kappa \frac{f(x) - f(\tau(x))}{x} = f'(x) + \kappa \frac{f(x) - f(0)}{x}.$$

Example 2. Let $R = \{\pm(e_1 \pm e_2), \pm e_1, \pm e_2\}$ be a root system of type B_2 in the 2-plane and $R' = \{e_1 \pm e_2\}$ be a positive orthogonal subsystem in R. The related Dunkl-type operators to R' and to the positive parameters (κ_1, κ_2) are given by

$$\begin{aligned} T_1 &= \partial_x + \kappa_1 \frac{f(x,y) - f((x+y)/2, (x+y)/2)}{x-y} + \kappa_2 \frac{f(x,y) - f((x-y)/2, (x-y)/2)}{x+y}, \\ T_2 &= \partial_y - \kappa_1 \frac{f(x,y) - f((x+y)/2, (x+y)/2)}{x-y} + \kappa_2 \frac{f(x,y) - f((x-y)/2, (x-y)/2)}{x+y}. \end{aligned}$$

We denote by Π^N the space of polynomials and by Π^N_n the subspace of homogenous polynomials of degree n.

Let $R' = \{\alpha_1, \ldots, \alpha_n\}$ be a positive orthogonal subsystem of a root system R. Consider the operator ρ_i defined on Π^N by

$$(\rho_i f)(x) = \frac{f(x) - f(\tau_{\alpha_i} x)}{\langle x, \alpha \rangle}, \qquad i = 1, \ldots, n.$$

It follows from the equality

$$(\rho_j f)(x) = -\frac{1}{|\alpha_j|^2} \int_0^1 \partial_{\alpha_j} f\left(x - t\frac{\langle x, \alpha_j \rangle}{|\alpha_j|^2}\alpha_j\right) \mathrm{d}t$$

that T_ξ is a homogeneous operator of degree -1 on Π^N, that is, $T_\xi f \in \Pi^N_{n-1}$, for $f \in \Pi^N_n$, and leaves $\mathcal{S}(\mathbb{R})$ ($\mathcal{S}(\mathbb{R})$ is the Schwartz space of rapidly decreasing functions on $\mathbb{R}$) invariant.

Proposition 1. *The operators ρ_i $(i = 1, \dots, n)$ have the following properties:*

i) for $i, j = 1, \dots, n$, we have $[\rho_i, \rho_j] = 0$;

ii) if α is an orthogonal vector to α_i, then $[\partial_\alpha, \rho_i] = 0$, where the commutator of two operators A, B is defined by $[A, B] := AB - BA$.

The family $\{\alpha_1, \dots, \alpha_n\}$ is orthogonal, then there exist scalars $\xi_1, \dots, \xi_n$ and a vector $\widehat{\xi} \in \mathbb{R}^N$ orthogonal to the subspace $\mathbb{R}\alpha_1 \oplus \cdots \oplus \mathbb{R}\alpha_n$ such that

$$\xi = \sum_{i=1}^{n} \xi_i \alpha_i + \widehat{\xi}.$$

This allows us to decompose the operator T_ξ (2) associated with R' and the multiplicity parameters $(\kappa_1, \dots, \kappa_n)$ in a unique way in the form

$$T_\xi = \sum_{i=1}^{n} \xi_i T_{\alpha_i} + \partial_{\widehat{\xi}}.$$

We now have all ingredients to state and prove the first main result of the paper.

Theorem 1. *Let $\xi, \eta \in \mathbb{R}^N$, then $[T_\xi, T_\eta] = 0$.*

Proof. A straightforward computation yields

$$[T_\xi, T_\eta] = \sum_{i,j=1}^{n} \xi_i \eta_j [T_{\alpha_i}, T_{\alpha_j}] + [\partial_{\widehat{\xi}}, \partial_{\widehat{\eta}}] + \sum_{i=1}^{n} \xi_i [T_{\alpha_i}, \partial_{\widehat{\eta}}] - \eta_i [T_{\alpha_i}, \partial_{\widehat{\xi}}].$$

On the other hand,

$$\begin{aligned} [T_{\alpha_i}, T_{\alpha_j}] &= [\partial_{\alpha_i} + \kappa_i \|\alpha_i\| \rho_i, \partial_{\alpha_j} + \kappa_j \|\alpha_j\| \rho_j] \\ &= [\partial_{\alpha_i}, \partial_{\alpha_j}] + \kappa_j \|\alpha_j\| [\partial_{\alpha_i}, \rho_j] - \kappa_i \|\alpha_i\| [\partial_{\alpha_j}, \rho_i] + \kappa_i \kappa_j \|\alpha_i\| \|\alpha_j\| [\rho_i, \rho_j], \end{aligned}$$

and

$$[T_{\alpha_i}, \partial_{\widehat{\xi}}] = [\partial_{\alpha_i}, \partial_{\widehat{\xi}}] + \kappa_i \|\alpha_i\| [\rho_i, \partial_\xi].$$

From Proposition 1, we get

$$[T_{\alpha_i}, T_{\alpha_j}] = 0 \qquad \text{and} \qquad [T_{\alpha_i}, \partial_{\widehat{\xi}}] = 0.$$

This proves the result. ∎

One important consequence of the Theorem 1, is that the operators $T_{\alpha_1}, \dots, T_{\alpha_m}$ generate a commutative algebra.

4 Intertwining operator

In this section, we give an intertwining operator between T_ξ and the directional derivative ∂_ξ. Consider a positive orthogonal subsystem $R' = \{\alpha_1, \dots, \alpha_n\}$ composed of n vectors in a root system R, and $\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{C}^n$ and $\xi \in \mathbb{R}^N$. The associated Dunkl-type operator T_ξ with R' and κ takes the form

$$(T_\xi f)(x) = \partial_\xi f(x) + \sum_{j=1}^{n} \kappa_j \langle \alpha_j, \xi \rangle \frac{f(x) - f(\tau_{\alpha_j} x)}{\langle x, \alpha_j \rangle}.$$

Let $h : \mathbb{R}^n \times \mathbb{R}^N \to \mathbb{R}^N$ be the function defined by

$$h(t,x) = x + \sum_{j=1}^{n} (t_j - 1)\frac{\langle x, \alpha_j\rangle}{|\alpha_j|^2}\alpha_j,$$

where $t = (t_1, \dots, t_n) \in \mathbb{R}^n$ and $x \in \mathbb{R}^N$.

We define

$$\chi_\kappa(f)(x) = \frac{1}{\Gamma(\kappa)} \int_{[0,1]^n} f(h(t,x))w(t)\,\mathrm{d}t, \tag{3}$$

where $w(t) = \prod_{j=1}^{n}(1-t_j)^{\kappa_j - 1}$ and $\Gamma(\kappa) = \prod_{j=1}^{n}\Gamma(\kappa_j)$.

Theorem 2. *Let $f \in C^\infty(\mathbb{R}^N)$, then we have*

$$T_\xi \circ \chi_\kappa f(x) = \chi_\kappa \circ \partial_\xi f(x).$$

Proof. For $j = 1, \dots, n$, we denote by θ_j the orthogonal projection in $\mathbb{R}^n$ with respect to the hyperplane $(\mathbb{R}e_j)^\perp$ orthogonal to the vector e_j of the canonical basis $(e_1, \dots, e_n)$ of $\mathbb{R}^n$. The orthogonal projection θ_j acts on $\mathbb{R}^n$ as

$$\theta_j(t) = (t_1, \dots, t_{j-1}, 0, t_{j+1}, \dots, t_n).$$

The system R is orthogonal, then for $j = 1, \dots, n$, we have

$$\begin{aligned} h(t, \tau_{\alpha_j}x) &= \tau_{\alpha_j}x + \sum_{k=1}^{n}(t_k - 1)\frac{\langle \tau_{\alpha_j}x, \alpha_k\rangle}{|\alpha_k|^2}\alpha_k \\ &= x - \frac{\langle x, \alpha_j\rangle}{|\alpha_j|^2}\alpha_j + \sum_{k=1, k\neq j}^{n}(t_k - 1)\frac{\langle x, \alpha_k\rangle}{|\alpha_k|^2}\alpha_k = h(\theta_j t, x). \end{aligned}$$

Let $f \in C^\infty(\mathbb{R}^N)$ and $\xi \in \mathbb{R}^N$. The mapping $x \to h(t,x)$ is linear on $\mathbb{R}^N$, then we can write

$$\partial_\xi(f(h(t,x))) = \partial_{h(t,\xi)}f(h(t,x)) = \partial_\xi f(h(t,x)) + \sum_{j=1}^{n}(t_j - 1)\frac{\langle \xi, \alpha_j\rangle}{|\alpha_j|^2}\partial_{\alpha_j} f(h(t,x)).$$

Hence,

$$\begin{aligned} \partial_\xi \chi_\kappa(f)(x) &= \frac{1}{\Gamma(\kappa)}\int_{[0,1]^n} \partial_\xi(f(h(t,x)))w(t)\,\mathrm{d}t = \frac{1}{\Gamma(\kappa)}\int_{[0,1]^n}\partial_\xi f(h(t,x))w(t)\,\mathrm{d}t \\ &\quad + \frac{1}{\Gamma(\kappa)}\sum_{j=1}^{n}\frac{\langle \xi, \alpha_j\rangle}{|\alpha_j|^2}\int_{[0,1]^n}(t_j - 1)\partial_{\alpha_j} f(h(t,x))w(t)\,\mathrm{d}t. \end{aligned}$$

Since we can write

$$\partial_{t_j} f(h(t,x)) = \frac{\langle x, \alpha_k\rangle}{|\alpha_k|^2}\partial_{\alpha_j} f(h(t,x))$$

and

$$\int_0^1 (1-t_j)^{\kappa_j}\partial_{t_j} f(h(t,x))\,\mathrm{d}t = -f(h(\theta_j(t), x)) + \kappa_j \int_0^1 (1-t_j)^{\kappa_j - 1} f(h(t,x))\,\mathrm{d}t,$$

we are lead to

$$\begin{aligned}\int_{[0,1]^n} \partial_{\alpha_j} f(h(t,x))(t_j-1)w(t)\,\mathrm{d}t &= \frac{|\alpha_j|^2}{\langle x,\alpha_j\rangle}\int_{[0,1]^n} \partial_{t_j} f(h(t,x))(t_j-1)w(t)\,\mathrm{d}t \\ &= \kappa_j \frac{|\alpha_j|^2}{\langle x,\alpha_j\rangle}\int_{[0,1]^n} (f(h(\theta_j(t),x)) - f(h(t,x)))w(t)\,\mathrm{d}t \\ &= -\kappa_j\Gamma(\kappa)\frac{|\alpha_j|^2}{\langle x,\alpha_j\rangle}\big(\chi_\kappa(f)(x) - \chi_\kappa(f)(\tau_{\alpha_j}x)\big).\end{aligned}$$

This, combined with the last expression of $\partial_\xi(\chi_\kappa f)(x)$, yields

$$\partial_\xi \chi_\kappa(f)(x) = \chi_\kappa(\partial_\xi f)(x) - \sum_{j=1}^n \kappa_j \langle \xi,\alpha_j\rangle \frac{\chi_\kappa(f)(x) - \chi_\kappa(f)(\tau_j x)}{\langle x,\alpha_j\rangle}.$$

Therefore,

$$T_\xi(\chi_\kappa f)(x) = \chi_\kappa(\partial_\xi f)(x). \qquad \blacksquare$$

5 The one variable case

The specialization of this theory to the one variable case has its own interest, because everything can be done there in a much more explicit way and new results for special functions in one variable can be obtained. In this setting there is only one Dunkl-type operator T_κ associated up to scaling and it equals to

$$T_\kappa f(x) = f'(x) + \kappa \frac{f(x) - f(0)}{x}. \tag{4}$$

This operator leaves the space of polynomials invariant and acts on the monomials as

$$T_\kappa 1 = 0, \qquad T_\kappa x^n = (n+\kappa)x^{n-1}, \qquad n = 1,2,\dots.$$

Its square is given by

$$T_\kappa^2 f(x) = f''(x) + \frac{2\kappa}{x} f'(x) + \kappa(\kappa-1)\frac{f(x)-f(0)}{x^2} - \frac{\kappa(\kappa+1)}{x} f'(0).$$

Consider the confluent hypergeometric function (see [15, § 7.1])

$$M(a,b;z) = \sum_{n=0}^\infty \frac{(a)_n}{(b)_n}\frac{z^n}{n!},$$

where $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

This is a solution of the confluent hypergeometric differential equation

$$zy''(z) + (b-z)y'(z) = ay(z).$$

This function possesses the following Poisson integral representation (see [15, § 7.1])

$$M(a,b;z) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 t^{a-1}(1-t)^{b-a-1}e^{zt}\,\mathrm{d}t, \qquad \Re(b) > \Re(a) > 0. \tag{5}$$

Theorem 3. *For $\lambda \in \mathbb{C}$ and $\kappa > -1$, the problem*

$$T_\kappa f(x) = i\lambda f(x), \qquad f(0) = 1, \tag{6}$$

has a unique analytic solution $M_\kappa(i\lambda x)$ given by

$$M_\kappa(i\lambda x) = M(1, \kappa + 1; i\lambda x). \tag{7}$$

Proof. Searching a solution of (6) in the form $f(z) = \sum_{n=0}^{\infty} a_n x^n$. Replacing in (6), we obtain

$$\sum_{n=0}^{\infty}(n + 1 + \kappa)a_{n+1}x^n = i\lambda \sum_{n=0}^{\infty} a_n x^n.$$

Thus,

$$a_{n+1} = \frac{i\lambda}{n + 1 + \kappa}a_n \qquad \text{and} \qquad a_n = \frac{(i\lambda)^n}{(\kappa + 1)_n}. \qquad \blacksquare$$

Remark 1. Multiply the equation (6) by x and differentiating both sides, we see that a function u of class C^2 on $\mathbb{R}$, is a solution of the equation (6), if and only if, it is a solution of the generalized eigenvalue problem

$$xu'' + (\kappa + 1)u' = i\lambda(xu' + u).$$

Proposition 2. *The function $\mathbf{M}_\kappa(z)$ defined by*

$$\mathbf{M}_\kappa(z) = \frac{M_\kappa(z)}{\Gamma(\kappa + 1)} = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\kappa + 1 + n)} \tag{8}$$

satisfies the following properties:

(*i*) *$\mathbf{M}_\kappa(z)$ is analytic in κ and z;*

(*ii*) *$\mathbf{M}_0(z) = e^z$;*

(*iii*) *for $\Re(\kappa) > 0$, the function $\mathbf{M}_\kappa(z)$, possesses the integral representation*

$$\mathbf{M}_\kappa(z) = \frac{1}{\Gamma(\kappa)} \int_0^1 (1 - t)^{\kappa - 1} e^{zt}\, \mathrm{d}t;$$

(*iv*) *for $\Re(\kappa) > 0$, we have*

$$\left|\mathbf{M}_\kappa^{(n)}(z)\right| \le |z|^n e^{\Re(z)}, \qquad n \in \mathbb{N}, \qquad z \in \mathbb{C},$$

in particular,

$$|\mathbf{M}_\kappa(i\lambda x)| \le 1, \qquad \lambda, x \in \mathbb{R};$$

(*v*) *for $\Re(\kappa) > 0$, and all $x \in \mathbb{R}^*$,*

$$\lim_{\lambda \to +\infty} \mathbf{M}_\kappa(i\lambda x) = 0.$$

Proof. (i) and (ii) are immediate. (iii) follows from (5). For $n \in \mathbb{N}$, we have

$$\mathbf{M}_\kappa^{(n)}(z) = \frac{z^n}{\Gamma(\kappa)} \int_0^1 (1-t)^\kappa t^n e^{zt}\,\mathrm{d}t.$$

So we find

$$|\mathbf{M}_\kappa^{(n)}(z)| \le \frac{|z|^n}{\Gamma(\kappa)} \int_0^1 (1-t)^\kappa e^{\Re(z)t}\,\mathrm{d}t \le |z|^n e^{\Re(z)}.$$

This proves (iv). (v) follows from (iii) and the Riemann–Lebesgue lemma. ■

Definition 1. We define the Kummer transform on $L^1(\mathbb{R})$ by

$$\forall\, \lambda \in \mathbb{R}, \qquad \mathcal{F}_\kappa(f)(\lambda) = \int_{\mathbb{R}} f(x)\mathbf{M}_\kappa(i\lambda x)(x)\,\mathrm{d}x.$$

When $\kappa = 0$, the transformation $\mathcal{F}_0$ reduces to the usual Fourier transform $\mathcal{F}$ that is given by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)e^{i\lambda x}\,\mathrm{d}x.$$

Theorem 4. *Let f be a function in $L^1(\mathbb{R})$ then $\mathcal{F}_\kappa(f)$ belongs to $C_0(\mathbb{R})$, where $C_0(\mathbb{R})$ is the space of continuous functions having zero as limit at the infinity. Furthermore,*

$$\|\mathcal{F}_\kappa(f)\|_\infty \le \|f\|_1.$$

Proof. It's clear that $\mathcal{F}_\kappa(f)$ is a continuous function on $\mathbb{R}$. From Proposition 2, we get for all $x \in \mathbb{R}^*$,

$$\lim_{\lambda\to\infty} f(x)\mathbf{M}_\kappa(i\lambda x) = 0 \qquad \text{and} \qquad |f(x)\mathbf{M}_\kappa(i\lambda x)| \le |f(x)|.$$

Since f is in $L^1(\mathbb{R})$, we conclude by using the dominated convergence theorem that $\mathcal{F}_\kappa(f)$ belongs to $C_0(\mathbb{R})$ and

$$\|\mathcal{F}_\kappa(f)\|_\infty \le \|f\|_1.$$

We now turn to exhibit a relationship between the Kummer transform and the Fourier transform. The crucial idea is to use the intertwining operator χ_κ. We denote by $C^\infty(\mathbb{R})$ the space of infinitely differentiable functions f on $\mathbb{R}$, provided with the topology defined by the semi norms

$$\|f\|_{n,a} = \sup_{\substack{0\le k\le n\\ x\in[-a,a]}} \big|f^{(k)}(x)\big|, \qquad a > 0, \qquad n \in \mathbb{N}.$$

In the rank-one case the intertwining operator (3) becomes

$$(\chi_\kappa f)(x) = \frac{1}{\Gamma(\kappa)} \int_0^1 (1-t)^{\kappa-1} f(tx)\,\mathrm{d}t. \tag{9}$$

This operator is a particular case of the so called Erdélyi–Kober fractional integral $I^{\gamma,\delta}$, which is given by (see [10])

$$(I^{\gamma,\delta} f)(x) = \frac{1}{\Gamma(\delta)} \int_0^1 (1-t)^{\delta-1} t^\gamma f(tx)\,\mathrm{d}t, \qquad \delta > 0, \qquad \gamma \in \mathbb{R}.$$

It was shown in [12, § 3], that the Erdélyi–Kober fractional integral has a left-inverse

$$D^{\gamma,\delta} I^{\gamma,\delta} f = f, \qquad f \in C^\infty(\mathbb{R}), \tag{10}$$

where

$$D^{\gamma,\delta} = \prod_{k=1}^{n} \left(\gamma + k + x \frac{\mathrm{d}}{\mathrm{d}x} \right) I^{\gamma+\delta, n-\delta},$$

and $n = \lceil \delta \rceil$ ($\lceil \delta \rceil$ denotes the ceiling function the smallest integer $\geq \delta$).

As a consequence of Theorem 2, we deduce that the operator χ_κ (9) has the fundamental intertwining property

$$T_\kappa \circ \chi_\kappa = \chi_\kappa \circ \frac{\mathrm{d}}{\mathrm{d}x}.$$

We regard it as a second main result since it allows us to move from the complicated operator T_κ defined in (4) to the simple derivative operator $\frac{\mathrm{d}}{\mathrm{d}x}$. ■

Theorem 5. *Let $\kappa > 0$, the operator χ_κ is a topological isomorphism from $C^\infty(\mathbb{R})$ onto itself and its inverse χ_κ^{-1} is given for all $f \in C^\infty(\mathbb{R})$ by*

$$\chi_\kappa^{-1} f(x) = D^{0,\kappa} f(x) = \prod_{j=1}^{n} \left(j + x \frac{\mathrm{d}}{\mathrm{d}x} \right) (I^{\kappa+1, n-\kappa} f)(x),$$

where $n = \lceil \kappa \rceil$.

Proof. Let $a > 0$ and $f \in C^\infty(\mathbb{R})$. For $x \in [0, a]$, $t \in [0, 1]$ and $l \in \mathbb{N}$, we have the following estimate

$$|t^l (1-t)^{\kappa-1} f^{(l)}(xt)| \leq \|f\|_{l,a} (1-t)^{\kappa-1} \qquad \text{and} \qquad \int_0^1 (1-t)^{\kappa-1}\, \mathrm{d}t = \frac{1}{\kappa}.$$

By the theorem of derivation under the integral sign, we can prove that

$$\chi_\kappa f \in C^\infty(\mathbb{R}) \qquad \text{and} \qquad \|\chi_\kappa(f)\|_{l,a} \leq \frac{1}{\Gamma(\kappa+1)} \|f\|_{l,a}.$$

Then χ_κ is a linear continuous mapping from $C^\infty(\mathbb{R})$ onto its self. From formula (10) the operator

$$D^{0,\kappa} = \prod_{j=1}^{n} \left(j + x \frac{\mathrm{d}}{\mathrm{d}x} \right) \circ I^{\kappa+1, n-\kappa}$$

is a left-inverse of χ_κ. This shows that χ_κ is injective and $D^{0,\kappa}$ is surjective. So it suffices to prove that $D^{0,\kappa}$ is injective.

Let f be a function in $C^\infty(\mathbb{R})$ such that $D^{0,\kappa} f = 0$. Then the function $g = I^{\kappa+1, n-\kappa} f \in C^\infty(\mathbb{R})$ is a solution of the linear differential equation

$$\prod_{j=1}^{n} \left(1 + j + x \frac{\mathrm{d}}{\mathrm{d}x} \right) y(x) = 0.$$

Since, the last differential equation has a unique C^∞-solution, which is equal to $y(x) = 0$, it follows that $g = 0$.

From (10) the operator $I^{\kappa+1,\kappa}$ has a left-inverse, then $f = 0$. This shows that χ_κ is a bijective operator. ■

Let $\kappa > 0$, we define the dual intertwining operator ${}^t\chi_\kappa$ on $\mathcal{D}(\mathbb{R})$ ($\mathcal{D}(\mathbb{R})$ is the space of C^∞-functions on $\mathbb{R}$ with compact support) by

$$\big({}^t\chi_\kappa f\big)(x) = \frac{1}{\Gamma(\kappa)}\int_{|x|}^{+\infty}(t-|x|)^{\kappa-1}t^{-\kappa}f(\operatorname{sgn}(x)t)\,\mathrm{d}t, \qquad x \in \mathbb{R}\setminus\{0\}.$$

Proposition 3. *The operator ${}^t\chi_\kappa$ is a topological automorphism of $\mathcal{D}(\mathbb{R})$, and satisfies the transmutation relation:*

$$\int_{\mathbb{R}}(\chi_\kappa f)(x)g(x)\,\mathrm{d}x = \int_{\mathbb{R}} f(x)\big({}^t\chi_\kappa g\big)(x)\,\mathrm{d}x, \qquad f \in C^\infty(\mathbb{R}).$$

Proof. Let $f \in C^\infty(\mathbb{R})$ and $g \in \mathcal{D}(\mathbb{R})$, we have

$$\begin{aligned}\int_{\mathbb{R}}(\chi_\kappa f)(x)g(x)\,\mathrm{d}x &= \frac{1}{\Gamma(\kappa)}\int_0^{+\infty}\int_0^x (x-t)^{\kappa-1}f(t)\,\mathrm{d}t g(x)x^{-\kappa}\,\mathrm{d}x\\ &\quad - \frac{1}{\Gamma(\kappa)}\int_0^{\infty}\int_0^x (x-t)^{\kappa-1}f(-t)\,\mathrm{d}t g(-x)x^{-\kappa}\,\mathrm{d}x.\end{aligned}$$

Using Fubini's theorem and a change of variable, we get

$$\begin{aligned}\int_{\mathbb{R}}(\chi_\kappa f)(x)g(x)\,\mathrm{d}x &= \frac{1}{\Gamma(\kappa)}\int_0^{+\infty}\int_t^{\infty} x^{-\kappa}(x-t)^{\kappa-1}g(x)\,\mathrm{d}x f(t)\,\mathrm{d}t\\ &\quad + \frac{1}{\Gamma(\kappa)}\int_{-\infty}^{0}\int_{-t}^{\infty} x^{-\kappa}(x+t)^{\kappa-1}g(-x)\,\mathrm{d}x f(t)\,\mathrm{d}t.\end{aligned}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}}(\chi_\kappa f)(x)g(x)\,\mathrm{d}x &= \frac{1}{\Gamma(\kappa)}\int_{\mathbb{R}}\int_{|t|}^{\infty} x^{-\kappa}(x-|t|)^{\kappa-1}g(sign(t)x)\,\mathrm{d}x f(t)\,\mathrm{d}t\\ &= \int_{\mathbb{R}} f(t)\big({}^t\chi_\kappa g\big)(t)\,\mathrm{d}t.\end{aligned}$$

■

Proposition 4. *Let $\kappa > 0$, the Kummer transform $\mathcal{F}_\kappa$ satisfies the decomposition*

$$\mathcal{F}_\kappa(f) = \mathcal{F}\circ {}^t\chi_\kappa(f), \qquad f \in \mathcal{D}(\mathbb{R}).$$

Proof. The result follows from Proposition 3. ■

6 Multivariable case

6.1 Direct product setting

In this subsection, we consider the direct product of the one-dimensional models, which means that the Weyl group of the corresponding subsystem of root system is a subgroup of $\mathbb{Z}_2^N$.

We denote by τ_k (for each k from 1 to N) the orthogonal projection with respect to the hyperplane orthogonal to e_k, that is to say for every $x = (x_1, \dots, x_N) \in \mathbb{R}^N$

$$\tau_k(x) = x - \frac{\langle x, e_k\rangle}{|e_k|^2}e_k = (x_1, \dots, x_{k-1}, 0, x_{k+1}, \dots, x_N).$$

Let $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_N) \in \mathbb{C}^N$. The associated Dunkl type operators T_j for $j = 1, \dots, N$, are given for $x \in \mathbb{R}^N$ by

$$T_j f(x) = \partial_j f(x) + \sum_{l=1}^N \kappa_l \frac{f(x) - f(\tau_l(x))}{\langle x, e_l\rangle}\,\langle e_k, e_l\rangle$$

$$= \partial_j f(x) + \kappa_j \frac{f(x) - f(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_N)}{x_j}.$$

These operators form a commuting system. The generalized Laplacian associated with T_j is defined in a natural way as

$$\Delta_\kappa = \sum_{j=1}^{N} T_j^2.$$

A straightforward computation yields

$$\Delta_\kappa = \Delta + 2\sum_{j=1}^{N} \kappa_j x_j^{-1}\partial_j f(x) - \sum_{j=1}^{N}(\kappa_j^2 + \kappa_j)x_j^{-1}\partial_j f(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_N)$$
$$+ \sum_{j=1}^{N}(\kappa_j^2 - \kappa_j)x_j^{-2}\big(f(x) - f(x_1, \dots, x_{j-1}, 0, x_{j+1}, \dots, x_N)\big).$$

This operator will play in our context a similar role to that of the Euclidean Laplacian in the classical harmonic analysis. Obviously, the trivial choice of the multiplicity function $\kappa = 0$, reduces our situation to the analysis related to the classical Laplacian Δ.

Let $\kappa = (\kappa_1, \dots, \kappa_N) \in (0, \infty)^N$. For $x, \lambda \in \mathbb{R}^N$, we consider the function $M_\kappa(\lambda, x)$ which is given as the tensor products

$$M_\kappa(\lambda, x) = \prod_{j=1}^{N} M_{\kappa_j}(i\lambda_j x_j).$$

Theorem 6. *For $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$, the function $M_\kappa(\lambda, x)$ is the unique analytic solution of the system*

$$T_\xi u(x) = i\langle \lambda, \xi\rangle u(x), \qquad u(0) = 1, \qquad \forall\, \xi \in \mathbb{C}^N.$$

6.2 Dunkl-type operators associated to an orthogonal subsystem in a root system of type A_{N-1}

Let R be a root system of type A_{N-1}

$$R = \{\pm(e_i - e_j),\ 1 \le i < j \le N\}.$$

Define a positive orthogonal subsystem $R' = \{\alpha_1, \dots, \alpha_{[N/2]}\}$ of R by setting:

$$\alpha_i = e_{2i-1} - e_{2i}, \qquad i = 1, \dots, [N/2].$$

We denote by τ_j (for each j from 1 to $[N/2]$) the orthogonal projection onto the hyperplane perpendicular to α_j, that is to say for every $x = (x_1, \dots, x_N) \in \mathbb{R}^N$

$$\tau_i x = (x_1, \dots, \overline{x}_{2i-1}, \overline{x}_{2i}, \dots, x_N),$$

where $\overline{x}_{2i-1} = \overline{x}_{2i} = \frac{1}{2}(x_{2i-1} + x_{2i})$, $i = 1, \dots, [N/2]$. The vector $\xi \in \mathbb{R}^N$ can be decomposed uniquely in the form

$$\xi = \sum_{i=1}^{[N/2]} \xi_i(e_{2i-1} - e_{2i}) + \widehat{\xi},$$

where $\widehat{\xi}$ is an orthogonal vector to the linear space generated by $R' = \{\alpha_1, \dots, \alpha_{[N/2]}\}$.

A straightforward computation shows that the operator T_ξ ($\xi \in \mathbb{R}^N$) associated with R' and the multiplicity parameters $(\kappa_1, \dots, \kappa_{[N/2]})$ has the following decomposition

$$T_\xi = \sum_{i=1}^{[N/2]} \xi_i T_{\alpha_i} + \partial_{\widehat{\xi}} = \sum_{i=1}^{2[N/2]} (-1)^{i+1} \xi_{[\frac{i+1}{2}]} T_i + \partial_{\widehat{\xi}},$$

where

$$T_i = \partial_i - (-1)^i \kappa_{[\frac{i+1}{2}]} \rho_{[\frac{i+1}{2}]}, \qquad i = 1, \dots, 2[N/2],$$

and

$$(\rho_i f)(x) = \frac{f(x) - f(\tau_i x)}{x_{2i-1} - x_{2i}}.$$

The intertwining operator (3) becomes

$$\chi_\kappa(f)(x) = \frac{1}{\Gamma(\kappa)} \int_{[0,1]^n} f(h(t,x)) w(t) \, \mathrm{d}t,$$

where

$$h(t,x) = x + \sum_{i=1}^{[N/2]} \frac{t_i - 1}{2} (x_{2i-1} - x_{2i})(e_{2i-1} - e_{2i}).$$

Proposition 5. *Let $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N$, and $\kappa = (\kappa_1, \dots, \kappa_{[N/2]}) \in (0, \infty)^{[N/2]}$. The following eigenvalue problem*

$$T_\xi f = i \langle \lambda, \xi \rangle f, \qquad f(0) = 1, \qquad \forall\, \xi \in \mathbb{C}^N, \tag{11}$$

has a unique analytic solution $M_\kappa(\lambda, x)$ given by

$$M_\kappa(\lambda, x) = e^{i\langle \lambda, h(0,x)\rangle} \prod_{j=1}^{[N/2]} M_{\kappa_j} \left(\frac{i}{2} (\lambda_{2j-1} - \lambda_{2j})(x_{2j-1} - x_{2j}) \right).$$

Proof. According to Theorem 2, χ_κ is an intertwining operator between T_ξ and ∂_ξ. So, the function $\chi_\kappa(e^{i\langle \lambda, \cdot \rangle})$ is the unique C^∞-solution of problem (11).

Since we can write

$$\langle \lambda, h(t,x) \rangle = \langle \lambda, h(0,x) \rangle + \sum_{j=1}^{[N/2]} \frac{t_j}{2} (\lambda_{2j-1} - \lambda_{2j})(x_{2j-1} - x_{2j}),$$

we are lead to

$$\begin{aligned} M_\kappa(\lambda, x) &= \frac{e^{i\langle \lambda, h(0,x)\rangle}}{\Gamma(\kappa)} \int_{[0,1]^{[N/2]}} e^{\frac{i}{2} \sum\limits_{j=1}^{[N/2]} t_j (\lambda_{2j-1} - \lambda_{2j})(x_{2j-1} - x_{2j})} w(t) \, \mathrm{d}t \\ &= e^{i\langle \lambda, h(0,x)\rangle} \prod_{j=1}^{[N/2]} \frac{1}{\Gamma(\kappa_j)} \int_{[0,1]} e^{\frac{i}{2} (\lambda_{2j-1} - \lambda_{2j})(x_{2j-1} - x_{2j})} (1 - t_j)^{\kappa_j - 1} \, \mathrm{d}t_j. \end{aligned}$$

If we now use (7) and (8) we get

$$M_\kappa(\lambda, x) = e^{i\langle \lambda, h(0,x)\rangle} \prod_{j=1}^{[N/2]} M_{\kappa_j} \left(\frac{i}{2} (\lambda_{2j-1} - \lambda_{2j})(x_{2j-1} - x_{2j}) \right). \qquad \blacksquare$$

6.3 Dunkl-type operators associated to orthogonal subsystem in root system of type B_N

Throughout this subsection R is a root system of type B_N which is given by

$$R = \{\pm e_i \pm e_j,\ 1 \leq i < j \leq N;\ \pm e_i\ 1 \leq i \leq N\},$$

and R' is a positive orthogonal subsystem R' in the root system R given by

$$R' = \{\alpha_i^{\pm} = e_{2i-1} \pm e_{2i},\ 1 \leq i \leq [N/2]\}.$$

Denote by $\tau_i^{\pm}$ (for each i from 1 to $[N/2]$) the orthogonal projection onto the hyperplane perpendicular to $\alpha_i^{\pm}$, that is to say for every $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$

$$\tau_i^{\pm} x = \big(x_1, \ldots, \overline{x}_{2i-1}^{\pm}, \overline{x}_{2i}^{\pm}, \ldots, x_N\big),$$

where $\overline{x}_{2i-1}^{\pm} = \overline{x}_{2i}^{\pm} = \frac{1}{2}(x_{2i-1} \pm x_{2i})$. In this case, the Dunkl type operator T_ξ associated with R' and the multiplicity parameters $\big(\kappa_1^{\pm}, \ldots, \kappa_{[N/2]}^{\pm}\big)$ takes the form

$$(T_\xi f)(x) = \partial_\xi f(x) + \sum_{j=1}^{[N/2]} \kappa_j^- \langle \alpha_j^-, \xi\rangle \frac{f(x) - f(\tau_j^- x)}{\langle x, \alpha_j^-\rangle} + \kappa_j^+ \langle \alpha_j^+, \xi\rangle \frac{f(x) - f(\tau_j^+ x)}{\langle x, \alpha_j^+\rangle}.$$

In particular, for $i = 1, \ldots, 2[N/2]$ we have

$$T_i = \partial_i - (-1)^i \kappa^-_{[\frac{i+1}{2}]} \rho^-_{[\frac{i+1}{2}]} + \kappa^+_{[\frac{i+1}{2}]} \rho^+_{[\frac{i+1}{2}]}.$$

where

$$(\rho_i^{\pm} f)(x) = \frac{f(x) - f(\tau_i^{\pm} x)}{x_{2i-1} \pm x_{2i}}.$$

The operator T_ξ has also the following decomposition

$$T_\xi = \sum_{i=1}^{2[N/2]} \Big(\xi^+_{[\frac{i+1}{2}]} + (-1)^{i+1} \xi^-_{[\frac{i+1}{2}]}\Big) T_i + \varepsilon \xi_N \partial_N,$$

where

$$\xi = \sum_{i=1}^{[N/2]} \xi_i^+ \alpha_i^+ + \xi_i^- \alpha_i^- + \varepsilon \xi_N e_N, \qquad \text{and} \qquad \varepsilon = \begin{cases} 1 & \text{if } N \text{ is odd}, \\ 0 & \text{if } N \text{ is even}. \end{cases}$$

Proposition 6. *Let* $\lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{C}^N$ *and*

$$\kappa = (\kappa^+, \ldots, \kappa^+_{[N/2]}, \kappa^-, \ldots, \kappa^-_{[N/2]}) \in (0, \infty)^{2[N/2]}.$$

The following eigenvalue problem

$$T_\xi f = i\langle \lambda, \xi\rangle f, \qquad f(0) = 1, \qquad \forall\, \xi \in \mathbb{C}^N,$$

has a unique analytic $M_\kappa(\lambda, x)$ *given by*

$$M_\kappa(\lambda, x) = e^{i\langle \lambda, h(0,x)\rangle} \prod_{j=1}^{[N/2]} M_{\kappa_j^-}\left(\frac{i}{2}(\lambda_{2j-1} - \lambda_{2j})(x_{2j-1} - x_{2j})\right)$$
$$\times M_{\kappa_j^+}\left(\frac{i}{2}(\lambda_{2j-1} + \lambda_{2j})(x_{2j-1} + x_{2j})\right).$$

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