Symmetry, Integrability and Geometry: Methods and Applications SIGMA 11 (2015), 059, 33 pages

A Perturbation of the Dunkl Harmonic Oscillator on the Line

Jesús A. ÁLVAREZ LÓPEZ † *and Manuel CALAZA* ‡

† *Departamento de Xeometría e Topoloxía, Facultade de Matemáticas,*
Universidade de Santiago de Compostela, 15782 Santiago de Compostela, Spain
E-mail: *jesus.alvarez@usc.es*

‡ *Laboratorio de Investigación 2 and Rheumatology Unit,*
Hospital Clinico Universitario de Santiago, Santiago de Compostela, Spain
E-mail: *manuel.calaza@usc.es*

Received February 19, 2015, in final form July 20, 2015; Published online July 25, 2015
http://dx.doi.org/10.3842/SIGMA.2015.059

Abstract. Let J_σ be the Dunkl harmonic oscillator on $\mathbb{R}$ ($\sigma > -1/2$). For $0 < u < 1$ and $\xi > 0$, it is proved that, if $\sigma > u - 1/2$, then the operator $U = J_\sigma + \xi|x|^{-2u}$, with appropriate domain, is essentially self-adjoint in $L^2(\mathbb{R}, |x|^{2\sigma}dx)$, the Schwartz space $\mathcal{S}$ is a core of $\overline{U}^{1/2}$, and $\overline{U}$ has a discrete spectrum, which is estimated in terms of the spectrum of $\overline{J_\sigma}$. A generalization $J_{\sigma,\tau}$ of J_σ is also considered by taking different parameters σ and τ on even and odd functions. Then extensions of the above result are proved for $J_{\sigma,\tau}$, where the perturbation has an additional term involving, either the factor x^{-1} on odd functions, or the factor x on even functions. Versions of these results on $\mathbb{R}_+$ are derived.

Key words: Dunkl harmonic oscillator; perturbation theory

2010 Mathematics Subject Classification: 47A55; 47B25; 33C45

1 Introduction

The Dunkl operators on $\mathbb{R}^n$ were introduced by Dunkl [6, 7, 8], and gave rise to what is now called the Dunkl theory [20]. They play an important role in physics and stochastic processes (see, e.g., [10, 19, 22]). In particular, the Dunkl harmonic oscillators on $\mathbb{R}^n$ were studied in [9, 14, 15, 18]. We will consider only this operator on $\mathbb{R}$, where it is uniquely determined by one parameter. In this case, a conjugation of the Dunkl operator was previously introduced by Yang [23] (see also [16]).

Let us fix some notation that is used in the whole paper. Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ be the Schwartz space on $\mathbb{R}$, with its Fréchet topology. It decomposes as direct sum of subspaces of even and odd functions, $\mathcal{S} = \mathcal{S}_{\rm ev} \oplus \mathcal{S}_{\rm odd}$. The even/odd component of a function in $\mathcal{S}$ is denoted with the subindex ev/odd. Since $\mathcal{S}_{\rm odd} = x\mathcal{S}_{\rm ev}$, where x is the standard coordinate of $\mathbb{R}$, $x^{-1}\phi \in \mathcal{S}_{\rm ev}$ is defined for $\phi \in \mathcal{S}_{\rm odd}$. Let $L^2_\sigma = L^2(\mathbb{R}, |x|^{2\sigma}dx)$ ($\sigma \in \mathbb{R}$), whose scalar product and norm are denoted by $\langle\ ,\ \rangle_\sigma$ and $\|\ \|_\sigma$. The above decomposition of $\mathcal{S}$ extends to an orthogonal decomposition, $L^2_\sigma = L^2_{\sigma,\rm ev} \oplus L^2_{\sigma,\rm odd}$, because the function $|x|^{2\sigma}$ is even. $\mathcal{S}$ is a dense subspace of L^2_σ if $\sigma > -1/2$, and $\mathcal{S}_{\rm odd}$ is a dense subspace of $L^2_{\tau,\rm odd}$ if $\tau > -3/2$. Unless otherwise stated, we assume $\sigma > -1/2$ and $\tau > -3/2$. The domain of a (densely defined) operator P in a Hilbert space is denoted by $\mathsf{D}(P)$. If P is closable, its closure is denoted by $\overline{P}$. The domain of a (densely defined) sesquilinear form $\mathfrak{p}$ in a Hilbert space is denoted by $\mathsf{D}(\mathfrak{p})$. The quadratic form of $\mathfrak{p}$ is also denoted by $\mathfrak{p}$. If $\mathfrak{p}$ is closable, its closure is denoted by $\bar{\mathfrak{p}}$. For an operator in L^2_σ preserving the above decomposition, its restrictions to $L^2_{\sigma,\rm ev/odd}$ will be indicated with the subindex ev/odd. The operator of multiplication by a continuous function h in L^2_σ is also denoted by h. The harmonic oscillator is the operator $H = -\frac{d^2}{dx^2} + s^2x^2$ $(s > 0)$ in L^2_0 with $\mathsf{D}(H) = \mathcal{S}$.

The Dunkl operator on $\mathbb{R}$ is the operator T in L^2_σ, with $\mathsf{D}(T) = \mathcal{S}$, determined by $T = \frac{d}{dx}$ on $\mathcal{S}_{\mathrm{ev}}$ and $T = \frac{d}{dx} + 2\sigma x^{-1}$ on $\mathcal{S}_{\mathrm{odd}}$, and the Dunkl harmonic oscillator on $\mathbb{R}$ is the operator $J = -T^2 + s^2x^2$ in L^2_σ with $\mathsf{D}(J) = \mathcal{S}$. Thus J preserves the above decomposition of $\mathcal{S}$, being $J_{\mathrm{ev}} = H - 2\sigma x^{-1}\frac{d}{dx}$ and $J_{\mathrm{odd}} = H - 2\sigma\frac{d}{dx}x^{-1}$. The subindex σ is added to J if needed. This J is essentially self-adjoint, and the spectrum of $\overline{J}$ is well known [18]; in particular, $\overline{J} > 0$. In fact, even for $\tau > -3/2$, the operator $J_{\tau,\mathrm{odd}}$ is defined in $L^2_{\tau,\mathrm{odd}}$ with $\mathsf{D}(J_{\sigma,\mathrm{odd}}) = \mathcal{S}_{\mathrm{odd}}$ because it is a conjugation of $J_{\tau+1,\mathrm{ev}}$ by a unitary operator (Section 2). Some operators of the form $J + \xi x^{-2}$ $(\xi \in \mathbb{R})$ are conjugates of J by powers $|x|^a$ $(a \in \mathbb{R})$, and therefore their study can be reduced to the case of J [3]. Our first theorem analyzes a different perturbation of J.

Theorem 1.1. *Let $0 < u < 1$ and $\xi > 0$. If $\sigma > u - 1/2$, then there is a positive self-adjoint operator $\mathcal{U}$ in L^2_σ satisfying the following:*

(i) *$\mathcal{S}$ is a core of $\mathcal{U}^{1/2}$, and, for all $\phi, \psi \in \mathcal{S}$,*

$$\langle \mathcal{U}^{1/2}\phi, \mathcal{U}^{1/2}\psi\rangle_\sigma = \langle J\phi, \psi\rangle_\sigma + \xi\langle |x|^{-u}\phi, |x|^{-u}\psi\rangle_\sigma. \tag{1.1}$$

(ii) *$\mathcal{U}$ has a discrete spectrum. Let $\lambda_0 \le \lambda_1 \le \cdots$ be its eigenvalues, repeated according to their multiplicity. There is some $D = D(\sigma, u) > 0$, and, for each $\epsilon > 0$, there is some $C = C(\epsilon, \sigma, u) > 0$ so that, for all $k \in \mathbb{N}$,*

$$(2k + 1 + 2\sigma)s + \xi D s^u (k + 1)^{-u} \le \lambda_k \le (2k + 1 + 2\sigma)(s + \xi\epsilon s^u) + \xi C s^u. \tag{1.2}$$

Remark 1.2. In Theorem 1.1, observe the following:

(i) The second term of the right hand side of (1.1) makes sense because $|x|^{-u}\mathcal{S} \subset L^2_\sigma$ since $\sigma > u - 1/2$.

(ii) $\mathcal{U} = \overline{U}$, where $U := J + \xi|x|^{-2u}$ with $\mathsf{D}(U) = \bigcap_{m=0}^{\infty} \mathsf{D}(\mathcal{U}^m)$ (see [11, Chapter VI, § 2.5]). The more explicit notation U_σ will be also used if necessary.

(iii) The restrictions $\mathcal{U}_{\mathrm{ev/odd}}$ are self-adjoint in $L^2_{\sigma,\mathrm{ev/odd}}$ and satisfy (1.1) with $\phi, \psi \in \mathcal{S}_{\mathrm{ev/odd}}$ and (1.2) with k even/odd. In fact, by the comments before the statement, $\mathcal{U}_{\tau,\mathrm{odd}}$ is defined and satisfies these properties if $\tau > u - 3/2$.

To prove Theorem 1.1, we consider the positive definite symmetric sesquilinear form $\mathfrak{u}$ defined by the right hand side of (1.1). Perturbation theory [11] is used to show that $\mathfrak{u}$ is closable and $\bar{\mathfrak{u}}$ induces a self-adjoint operator $\mathcal{U}$, and to relate the spectra of $\mathcal{U}$ and $\overline{J}$. Most of the work is devoted to check the conditions to apply this theory so that (1.2) follows; indeed, (1.2) is stronger than a general eigenvalue estimate given by that theory (Remark 3.21).

The following generalizations of Theorem 1.1 follow with a simple adaptation of the proof. If $\xi < 0$, we only have to reverse the inequalities of (1.2). In (1.1), we may use a finite sum $\sum_i \xi_i\langle |x|^{-u_i}\phi, |x|^{-u_i}\psi\rangle_\sigma$, where $0 < u_i < 1$, $\sigma > u_i - 1/2$ and $\xi_i > 0$; then (1.2) would be modified by using $\max_i u_i$ and $\min_i \xi_i$ in the left hand side, and $\max_i \xi_i$ in the right hand side. In turn, this can be extended by taking $\mathbb{R}^p$-valued functions $(p \in \mathbb{Z}_+)$, and a finite sum $\sum_i\langle |x|^{-u_i}\Xi_i\phi, |x|^{-u_i}\psi\rangle_\sigma$ in (1.1), where each Ξ_i is a positive definite self-adjoint endomorphism of $\mathbb{R}^p$; then the minimum and maximum eigenvalues of all Ξ_i would be used in (1.2).

As an open problem, we may ask for a version of Theorem 1.1 using Dunkl operators on $\mathbb{R}^n$, but we are interested in the following different type of extension. For $\sigma > -1/2$ and $\tau > -3/2$, let $L^2_{\sigma,\tau} = L^2_{\sigma,\mathrm{ev}} \oplus L^2_{\tau,\mathrm{odd}}$, whose scalar product and norm are denoted by $\langle\ ,\ \rangle_{\sigma,\tau}$ and $\|\ \|_{\sigma,\tau}$. Matrix expressions of operators refer to this decomposition. Let $J_{\sigma,\tau} = J_{\sigma,\mathrm{ev}} \oplus J_{\tau,\mathrm{odd}}$ in $L^2_{\sigma,\tau}$, with $\mathsf{D}(J_{\sigma,\tau}) = \mathcal{S}$. The hypotheses of the generalization of Theorem 1.1 are rather involved to cover enough cases of certain application that will be indicated. The following sets are used:

- $\mathfrak{J}_1$ is the set of points $(\sigma,\tau)\in\mathbb{R}^2$ such that:

$$\begin{aligned}
&\tfrac{1}{2}\le\tau<\sigma\Longrightarrow\sigma-1<\tau<\tfrac{\sigma}{2}+\tfrac{1}{4},\\
&\tfrac{1}{2},\sigma\le\tau\Longrightarrow\tau<\tfrac{\sigma}{2}+\tfrac{1}{4},\sigma+1,\\
&\tau<\tfrac{1}{2},\sigma\Longrightarrow\tfrac{\sigma}{3},\sigma-1<\tau<\tfrac{\sigma}{2}+\tfrac{1}{4},\\
&\sigma\le\tau<\tfrac{1}{2}\Longrightarrow-\sigma<\tau<\tfrac{\sigma}{2}+\tfrac{1}{4},\sigma+1.
\end{aligned}$$

- $\mathfrak{J}_2$ is the set of points $(\sigma,\tau)\in\mathbb{R}^2$ such that:

$$\begin{aligned}
&\tfrac{1}{2}\le\tau<\sigma-\tfrac{1}{2}\Longrightarrow\sigma-1<\tau<\tfrac{\sigma}{2}+\tfrac{1}{4},\\
&\tfrac{1}{2},\sigma-\tfrac{1}{2}\le\tau\Longrightarrow\tau<\tfrac{\sigma}{2}+\tfrac{1}{4},\sigma,\\
&0<\tau<\tfrac{1}{2},\sigma-\tfrac{1}{2}\Longrightarrow\begin{cases}-\tfrac{\sigma}{3},\sigma-1<\tau<\tfrac{\sigma}{2}+\tfrac{1}{4}, \text{ or}\\ \sigma-1<\tau<\tfrac{\sigma}{2}-\tfrac{1}{4},\end{cases}\\
&0<\tau<\tfrac{1}{2},\ \sigma-\tfrac{1}{2}\le\tau\Longrightarrow\begin{cases}1-\sigma<\tau<\tfrac{\sigma}{2}+\tfrac{1}{4},\sigma, \text{ or}\\ \tau<\tfrac{\sigma}{2}-\tfrac{1}{4},\sigma,\end{cases}\\
&0=\tau<\sigma-\tfrac{1}{2}\Longrightarrow\tfrac{1}{2}<\sigma<1,\\
&\sigma-\tfrac{1}{2}\le\tau=0\Longrightarrow\tfrac{1}{2}<\sigma,\\
&\tau<0,\sigma-\tfrac{1}{2}\Longrightarrow\tfrac{1}{4}-\tfrac{\sigma}{2},\tfrac{\sigma-1}{3},\sigma-1<\tau,\\
&\sigma-\tfrac{1}{2}\le\tau<0\Longrightarrow\tfrac{1}{4}-\tfrac{\sigma}{2},-\sigma<\tau<\sigma.
\end{aligned}$$

- $\mathfrak{K}_1$ is the set of points $(\sigma,\tau,\theta)\in\mathbb{R}^3$ such that:

$$\begin{aligned}
&\theta\le\sigma-1,\ \theta<\tau+1\Longrightarrow\theta>\tfrac{\sigma}{2}-\tfrac{3}{4},\tfrac{\sigma+\tau}{4},\\
&\tau+1\le\theta\le\sigma-1\Longrightarrow\theta>\tfrac{\sigma}{2}-\tfrac{3}{4},\tfrac{\sigma-\tau}{2}-1,\\
&\sigma-1<\theta<\tau+1\Longrightarrow\theta>\tfrac{\sigma}{2}-\tfrac{3}{4},\tfrac{\tau-\sigma}{2}+1,\tfrac{\sigma+\tau}{4},\\
&\sigma-1<\theta,\ \tau+1\le\theta\Longrightarrow\theta>\tfrac{\sigma}{2}-\tfrac{3}{4},\tfrac{\sigma-\tau}{2}-1,\ \sigma+\tau>0.
\end{aligned}$$

- $\mathfrak{K}'_1$ is the set of points $(\sigma,\tau,\theta)\in\mathbb{R}^3$ such that:

$$\begin{aligned}
&\theta<\sigma,\ \theta\le\tau\Longrightarrow\theta>\tfrac{\tau}{2}-\tfrac{1}{4},\tfrac{\sigma+\tau}{4},\\
&\sigma\le\theta\le\tau\Longrightarrow\theta>\tfrac{\tau}{2}-\tfrac{1}{4},\tfrac{\tau-\sigma}{2},\\
&\tau<\theta<\sigma\Longrightarrow\theta>\tfrac{\tau}{2}-\tfrac{1}{4},\tfrac{\sigma-\tau}{2},\tfrac{\sigma+\tau}{4},\\
&\sigma\le\theta,\ \tau<\theta\Longrightarrow\theta>\tfrac{\tau}{2}-\tfrac{1}{4},\tfrac{\tau-\sigma}{2},\ \sigma+\tau>0.
\end{aligned}$$

- $\mathfrak{K}_2$ is the set of points $(\sigma,\tau,\theta)\in\mathbb{R}^3$ such that:

$$\begin{aligned}
&\theta\le\sigma-1,\ \theta<\tau+\tfrac{1}{2}\Longrightarrow\theta>\tfrac{\sigma}{2}-\tfrac{3}{4},\tfrac{\sigma+\tau}{4},\\
&\tau+\tfrac{1}{2}\le\theta\le\sigma-1\Longrightarrow\theta>\tfrac{\sigma}{2}-\tfrac{3}{4},\tfrac{\sigma-\tau-1}{2},\\
&\sigma-1<\theta<\sigma-\tfrac{1}{2},\tau+\tfrac{1}{2}\Longrightarrow\begin{cases}\theta>\tfrac{\sigma}{2}-\tfrac{3}{4},\tfrac{\tau-\sigma}{2}+1,\tfrac{\sigma+\tau}{4}, \text{ or}\\ \theta>\tfrac{\sigma}{2}-\tfrac{1}{4},\tfrac{\sigma+\tau}{4},\end{cases}\\
&\sigma-1<\theta<\sigma-\tfrac{1}{2},\ \tau+\tfrac{1}{2}\le\theta\Longrightarrow\begin{cases}\theta>\tfrac{\sigma}{2}-\tfrac{3}{4},\tfrac{\sigma-\tau-1}{2},\ \sigma+\tau>1, \text{ or}\\ \theta>\tfrac{\sigma}{2}-\tfrac{1}{4},\tfrac{\sigma-\tau-1}{2},\end{cases}\\
&\sigma-\tfrac{1}{2}=\theta<\tau+\tfrac{1}{2}\Longrightarrow\sigma>\tfrac{1}{2},\tfrac{\tau+2}{3},
\end{aligned}$$

$$\tau+\tfrac12\le\theta=\sigma-\tfrac12\Longrightarrow\sigma>\tfrac12,-\tau,$$
$$\sigma-\tfrac12<\theta<\tau+\tfrac12\Longrightarrow\theta>\tfrac{\sigma}{2}-\tfrac14,\tfrac{\tau-\sigma+1}{2},\tfrac{\sigma+\tau}{4},$$
$$\sigma-\tfrac12<\theta,\ \tau+\tfrac12\le\theta\Longrightarrow\theta>\tfrac{\sigma}{2}-\tfrac14,\tfrac{\sigma-\tau-1}{2},\ \sigma+\tau>0.$$

- $\mathfrak{K}_2'$ be the set of points $(\sigma,\tau,\theta)\in\mathbb{R}^3$ such that:

$$\theta\le\sigma-\tfrac12,\ \theta<\tau\Longrightarrow\theta>\tfrac{\tau}{2}-\tfrac14,\tfrac{\sigma+\tau}{4},$$
$$\sigma-\tfrac12\le\theta\le\tau\Longrightarrow\theta>\tfrac{\tau}{2}-\tfrac14,\tfrac{\tau-\sigma+1}{2},$$
$$\tau<\theta<\sigma-\tfrac12,\tau+\tfrac12\Longrightarrow\begin{cases}\theta>\tfrac{\tau}{2}-\tfrac14,\tfrac{\sigma-\tau}{2},\tfrac{\sigma+\tau}{4},\ \text{or}\\ \theta>\tfrac{\tau}{2}+\tfrac14,\tfrac{\sigma+\tau}{4},\end{cases}$$
$$\sigma-\tfrac12\le\theta,\ \tau<\theta<\tau+\tfrac12\Longrightarrow\begin{cases}\theta>\tfrac{\tau}{2}-\tfrac14,\tfrac{\tau-\sigma+1}{2},\ \sigma+\tau>1,\ \text{or}\\ \theta>\tfrac{\tau}{2}+\tfrac14,\tfrac{\tau-\sigma+1}{2},\end{cases}$$
$$\tau+\tfrac12=\theta<\sigma-\tfrac12\Longrightarrow\tau>-\tfrac12,\tfrac{\sigma-2}{3},$$
$$\sigma-\tfrac12\le\theta=\tau+\tfrac12\Longrightarrow\tau>-\tfrac12,-\sigma,$$
$$\tau+\tfrac12<\theta<\sigma-\tfrac12\Longrightarrow\theta>\tfrac{\tau}{2}+\tfrac14,\tfrac{\sigma-\tau-1}{2},\tfrac{\sigma+\tau}{4},$$
$$\sigma-\tfrac12\le\theta,\ \tau+\tfrac12<\theta\Longrightarrow\theta>\tfrac{\tau}{2}+\tfrac14,\tfrac{\tau-\sigma+1}{2},\ \sigma+\tau>0.$$

Theorem 1.3. *Let $0<u<1$, $\xi>0$, $\eta\in\mathbb{R}$, $\sigma>u-1/2$, $\tau>u-3/2$ and $\theta>-1/2$, and let $v=\sigma+\tau-2\theta$. Suppose that the following conditions hold:*

(a) *If $\sigma=\theta\ne\tau$ and $\tau-\sigma\notin-\mathbb{N}$, then $\sigma-1<\tau<\sigma+1,2\sigma+\frac12$.*

(b) *If $\sigma\ne\theta=\tau$ and $\sigma-\tau\notin-\mathbb{N}$, then $(\sigma,\tau)\in\mathfrak{J}_1\cup\mathfrak{J}_2$.*

(c) *If $\sigma\ne\theta=\tau+1$ and $\sigma-\tau-1\notin-\mathbb{N}$, then $\tau<\frac{3\sigma}{2}-\frac94,\sigma-\frac53$.*

(d) *If $\sigma\ne\theta\ne\tau$ and $\sigma-\theta,\tau-\theta\notin-\mathbb{N}$, then $(\sigma,\tau,\theta)\in(\mathfrak{K}_1\cup\mathfrak{K}_2)\cap(\mathfrak{K}_1'\cup\mathfrak{K}_2')$.*

Then there is a positive self-adjoint operator $\mathcal{V}$ in $L^2_{\sigma,\tau}$ satisfying the following:

(i) *$\mathcal{S}$ is a core of $\mathcal{V}^{1/2}$, and, for all $\phi,\psi\in\mathcal{S}$,*

$$\begin{aligned}\langle\mathcal{V}^{1/2}\phi,\mathcal{V}^{1/2}\psi\rangle_{\sigma,\tau}&=\langle J_{\sigma,\tau}\phi,\psi\rangle_{\sigma,\tau}+\xi\langle|x|^{-u}\phi,|x|^{-u}\psi\rangle_{\sigma,\tau}\\&\quad+\eta\left(\langle x^{-1}\phi_{\mathrm{odd}},\psi_{\mathrm{ev}}\rangle_\theta+\langle\phi_{\mathrm{ev}},x^{-1}\psi_{\mathrm{odd}}\rangle_\theta\right).\end{aligned}\tag{1.3}$$

(ii) *Let $\varsigma_k=\sigma$ if k is even, and $\varsigma_k=\tau$ if k is odd. $\mathcal{V}$ has a discrete spectrum. Its eigenvalues form two groups, $\lambda_0\le\lambda_2\le\cdots$ and $\lambda_1\le\lambda_3\le\cdots$, repeated according to their multiplicity, such that there is some $D=D(\sigma,\tau,u)>0$, and, for each $\epsilon>0$, there are some $C=C(\epsilon,\sigma,\tau,u)>0$ and $E=E(\epsilon,\sigma,\tau,\theta)>0$ so that, for all $k\in\mathbb{N}$,*

$$\begin{aligned}&(2k+1+2\varsigma_k)s+\xi D(k+1)^{-u}\le\lambda_k\\&\quad\le(2k+1+2\varsigma_k)\big(s+\epsilon\big(\xi s^u+2|\eta|s^{(v+1)/2}\big)\big)+\xi Cs^u+2|\eta|Es^{(v+1)/2}.\end{aligned}\tag{1.4}$$

Remark 1.4. Note the following in Theorem 1.3:

(i) In (b), the condition $(\sigma,\tau)\in\mathfrak{J}_1$ holds if

$$-\sigma,\tfrac{\sigma}{3},\sigma-1<\tau<\tfrac{\sigma}{2}+\tfrac14,\sigma+1,$$

which requires $-1/6<\sigma<5/4$. In (d), the condition $(\sigma,\tau,\theta)\in\mathfrak{K}_1\cap\mathfrak{K}_1'$ holds if

$$\theta>\tfrac{\sigma}{2}-\tfrac34,\tfrac{\sigma-\tau}{2},\tfrac{\tau-\sigma}{2}+1,\tfrac{\sigma+\tau}{4},\tfrac{\tau}{2}-\tfrac14,\qquad\sigma+\tau>0.$$

(ii) Like in Remark 1.2(ii), we have $\mathcal{V} = \overline{V}$, where

$$V = \begin{pmatrix} U_{\sigma,\mathrm{ev}} & \eta|x|^{2(\theta-\sigma)}x^{-1} \\ \eta|x|^{2(\theta-\tau)}x^{-1} & U_{\tau,\mathrm{odd}} \end{pmatrix},$$

with $\mathsf{D}(V) = \bigcap_{m=0}^{\infty} \mathsf{D}(\mathcal{V}^m)$.

(iii) Taking $\theta' = \theta - 1 > -3/2$, since

$$\langle x\phi, \psi\rangle_{\theta'} = \langle \phi, x^{-1}\psi\rangle_{\theta}$$

for all $\phi \in \mathcal{S}_{\mathrm{ev}}$ and $\psi \in \mathcal{S}_{\mathrm{odd}}$, we can write (1.3) as

$$\begin{aligned}\langle \mathcal{V}^{1/2}\phi, \mathcal{V}^{1/2}\psi\rangle_{\sigma,\tau} &= \langle J_{\sigma,\tau}\phi, \psi\rangle_{\sigma,\tau} + \xi\langle |x|^{-u}\phi, |x|^{-u}\psi\rangle_{\sigma,\tau} \\ &\quad + \eta\left(\langle \phi_{\mathrm{odd}}, x\psi_{\mathrm{ev}}\rangle_{\theta'} + \langle x\phi_{\mathrm{ev}}, \psi_{\mathrm{odd}}\rangle_{\theta'}\right),\end{aligned}$$

and, correspondingly,

$$V = \begin{pmatrix} U_{\sigma,\mathrm{ev}} & \eta|x|^{2(\theta'-\sigma)}x \\ \eta|x|^{2(\theta'-\tau)}x & U_{\tau,\mathrm{odd}} \end{pmatrix}.$$

(iv) A slight improvement of (d) could be achieved according to Remark 5.7, but it is omitted because it is useless in our application (Section 7).

Versions of these results on $\mathbb{R}_+$ are also derived (Corollaries 6.1, 6.2 and 6.3). In [4], these corollaries are used to study a version of the Witten's perturbation Δ_s of the Laplacian on strata with the general adapted metrics of [5, 12, 13]. This gives rise to an analytic proof of Morse inequalities in strata involving intersection homology of arbitrary perversity, which was our original motivation. The simplest case of adapted metrics, corresponding to the lower middle perversity, was treated in [2] using an operator induced by J on $\mathbb{R}_+$. The perturbations of J studied here show up in the local models of Δ_s when general adapted metrics are considered. Some details of this application are given in Section 7.

2 Preliminaries

The *Dunkl annihilation* and *creation* operators are $B = sx + T$ and $B' = sx - T$ ($s > 0$). Like J, the operators B and B' are considered in L^2_σ with domain $\mathcal{S}$. They are perturbations of the usual annihilation and creation operators. The operators T, B, B' and J are continuous on $\mathcal{S}$. The following properties hold [3, 18]:

- B' is adjoint of B, and J is essentially self-adjoint.
- The spectrum of $\overline{J}$ consists of the eigenvalues[1] $(2k + 1 + 2\sigma)s$ ($k \in \mathbb{N}$), of multiplicity one.
- The corresponding normalized eigenfunctions ϕ_k are inductively defined by

$$\phi_0 = s^{(2\sigma+1)/4}\Gamma(\sigma + 1/2)^{-1/2}e^{-sx^2/2}, \tag{2.1}$$

$$\phi_k = \begin{cases} (2ks)^{-1/2}B'\phi_{k-1} & \text{if } k \text{ is even,} \\ (2(k+2\sigma)s)^{-1/2}B'\phi_{k-1} & \text{if } k \text{ is odd,} \end{cases} \qquad k \geq 1. \tag{2.2}$$

[1] It is assumed that $0 \in \mathbb{N}$.

- The eigenfunctions ϕ_k also satisfy

$$B\phi_0 = 0, \tag{2.3}$$

$$B\phi_k = \begin{cases} (2ks)^{1/2}\phi_{k-1} & \text{if } k \text{ is even,} \\ (2(k+2\sigma)s)^{1/2}\phi_{k-1} & \text{if } k \text{ is odd,} \end{cases} \qquad k \geq 1. \tag{2.4}$$

- $\bigcap_{m=0}^{\infty} \mathsf{D}(\overline{J}^m) = \mathcal{S}$.

By (2.1) and (2.2), we get $\phi_k = p_k e^{-sx^2/2}$, where p_k is the sequence of polynomials inductively given by $p_0 = s^{(2\sigma+1)/4}\Gamma(\sigma+1/2)^{-1/2}$ and

$$p_k = \begin{cases} (2ks)^{-1/2}(2sxp_{k-1} - Tp_{k-1}) & \text{if } k \text{ is even,} \\ (2(k+2\sigma)s)^{-1/2}(2sxp_{k-1} - Tp_{k-1}) & \text{if } k \text{ is odd,} \end{cases} \qquad k \geq 1.$$

Up to normalization, p_k is the sequence of generalized Hermite polynomials [21, p. 380, Problem 25], and ϕ_k is the sequence of generalized Hermite functions. Each p_k is of degree k, even/odd if k is even/odd, and with positive leading coefficient. They satisfy the recursion formula [3, equation (13)]

$$p_k = \begin{cases} k^{-1/2}\big((2s)^{1/2}xp_{k-1} - (k-1+2\sigma)^{1/2}p_{k-2}\big) & \text{if } k \text{ is even,} \\ (k+2\sigma)^{-1/2}\big((2s)^{1/2}xp_{k-1} - (k-1)^{1/2}p_{k-2}\big) & \text{if } k \text{ is odd.} \end{cases} \tag{2.5}$$

When $k = 2m+1$ ($m \in \mathbb{N}$), we have [3, equation (14)]

$$x^{-1}p_k = \sum_{i=0}^{m} (-1)^{m-i} \sqrt{\frac{m!\Gamma(i+\frac{1}{2}+\sigma)s}{i!\Gamma(m+\frac{3}{2}+\sigma)}}\, p_{2i}. \tag{2.6}$$

Let $\mathfrak{j}$ be the positive definite symmetric sesquilinear form in L^2_σ, with $\mathsf{D}(\mathfrak{j}) = \mathcal{S}$, given by $\mathfrak{j}(\phi,\psi) = \langle J\phi,\psi\rangle_\sigma$. Like in the case of J, the subindex σ will be added to the notation T, B, B' and ϕ_k and $\mathfrak{j}$ if necessary. Observe that

$$B_\sigma = \begin{cases} B_\tau & \text{on } \mathcal{S}_{\mathrm{ev}}, \\ B_\tau + 2(\sigma-\tau)x^{-1} & \text{on } \mathcal{S}_{\mathrm{odd}}, \end{cases} \tag{2.7}$$

$$B'_\sigma = \begin{cases} B'_\tau & \text{on } \mathcal{S}_{\mathrm{ev}}, \\ B'_\tau + 2(\tau-\sigma)x^{-1} & \text{on } \mathcal{S}_{\mathrm{odd}}. \end{cases} \tag{2.8}$$

The operator $x\colon \mathcal{S}_{\mathrm{ev}} \to \mathcal{S}_{\mathrm{odd}}$ is a homeomorphism [3], which extends to a unitary operator $x\colon L^2_{\sigma,\mathrm{ev}} \to L^2_{\sigma-1,\mathrm{odd}}$. We get $xJ_{\sigma,\mathrm{ev}}x^{-1} = J_{\sigma-1,\mathrm{odd}}$ because $x[\frac{d^2}{dx^2}, x^{-1}] = -2\frac{d}{dx}x^{-1}$. Thus, even for any $\tau > -3/2$, the operator $J_{\tau,\mathrm{odd}}$ is densely defined in $L^2_{\tau,\mathrm{odd}}$, with $\mathsf{D}(J_{\tau,\mathrm{odd}}) = \mathcal{S}_{\mathrm{odd}}$, and has the same spectral properties as $J_{\tau+1,\mathrm{ev}}$; in particular, the eigenvalues of $\overline{J_{\tau,\mathrm{odd}}}$ are $(4k+1+2\tau)s$ ($k \in 2\mathbb{N}+1$), and $\phi_{\tau,k} = x\phi_{\tau+1,k-1}$.

To prove the results of the paper, alternative arguments could be given by using the expression of the generalized Hermite polynomials in terms of the Laguerre ones (see, e.g., [19, p. 525] or [20, p. 23]).

3 The sesquilinear form $\mathfrak{t}$

Let $0 < u < 1$ such that $\sigma > u - 1/2$. Then $|x|^{-u}\mathcal{S} \subset L^2_\sigma$, and therefore a positive definite symmetric sesquilinear form $\mathfrak{t}$ in L^2_σ, with $\mathsf{D}(\mathfrak{t}) = \mathcal{S}$, is defined by

$$\mathfrak{t}(\phi,\psi) = \langle |x|^{-u}\phi, |x|^{-u}\psi\rangle_\sigma = \langle \phi,\psi\rangle_{\sigma-u}.$$

The notation $\mathfrak{t}_\sigma$ may be also used. The goal of this section is to study $\mathfrak{t}$ and apply it to prove Theorem 1.1. Precisely, an estimation of the values $\mathfrak{t}(\phi_k, \phi_\ell)$ is needed.

Lemma 3.1. *For all $\phi \in \mathcal{S}_{\mathrm{odd}}$ and $\psi \in \mathcal{S}_{\mathrm{ev}}$,*

$$\mathfrak{t}(B'\phi, \psi) - \mathfrak{t}(\phi, B\psi) = \mathfrak{t}(\phi, B'\psi) - \mathfrak{t}(B\phi, \psi) = -2u\mathfrak{t}\big(x^{-1}\phi, \psi\big).$$

Proof. By (2.7) and (2.8), for all $\phi \in \mathcal{S}_{\mathrm{odd}}$ and $\psi \in \mathcal{S}_{\mathrm{ev}}$,

$$\begin{aligned} \mathfrak{t}(B'_\sigma\phi, \psi) - \mathfrak{t}(\phi, B_\sigma\psi) &= \langle B'_{\sigma-u}\phi, \psi\rangle_{\sigma-u} - 2u\langle x^{-1}\phi, \psi\rangle_{\sigma-u} - \langle \phi, B_{\sigma-u}\psi\rangle_{\sigma-u} \\ &= -2u\mathfrak{t}(x^{-1}\phi, \psi), \\ \mathfrak{t}(\phi, B'_\sigma\psi) - \mathfrak{t}(B_\sigma\phi, \psi) &= \langle \phi, B'_{\sigma-u}\psi\rangle_{\sigma-u} - \langle B_{\sigma-u}\phi, \psi\rangle_{\sigma-u} - 2u\langle x^{-1}\phi, \psi\rangle_{\sigma-u} \\ &= -2u\mathfrak{t}(x^{-1}\phi, \psi). \end{aligned}$$

■

In the whole of this section, k, ℓ, m, n, i, j, p and q will be natural numbers. Let $c_{k,\ell} = \mathfrak{t}(\phi_k, \phi_\ell)$ and $d_{k,\ell} = c_{k,\ell}/c_{0,0}$. Thus $d_{k,\ell} = d_{\ell,k}$, and $d_{k,\ell} = 0$ when $k + \ell$ is odd. Since

$$\int_{-\infty}^{\infty} e^{-sx^2}|x|^{2\kappa}dx = s^{-(2\kappa+1)/2}\Gamma(\kappa + 1/2) \tag{3.1}$$

for $\kappa > -1/2$, we get

$$c_{0,0} = \Gamma(\sigma - u + 1/2)\Gamma(\sigma + 1/2)^{-1}s^u. \tag{3.2}$$

Lemma 3.2. *If $k = 2m > 0$, then*

$$d_{k,0} = \frac{u}{\sqrt{m}}\sum_{j=0}^{m-1}(-1)^{m-j}\sqrt{\frac{(m-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(m+\frac{1}{2}+\sigma)}}d_{2j,0}.$$

Proof. By (2.2), (2.3), (2.6) and Lemma 3.1,

$$\begin{aligned} c_{k,0} &= \frac{1}{\sqrt{2sk}}\mathfrak{t}(B'\phi_{k-1}, \phi_0) \\ &= \frac{1}{\sqrt{2sk}}\mathfrak{t}(\phi_{k-1}, B\phi_0) - \frac{2u}{\sqrt{2sk}}\mathfrak{t}(x^{-1}\phi_{k-1}, \phi_0) = -\frac{2u}{\sqrt{2sk}}\mathfrak{t}(x^{-1}\phi_{k-1}, \phi_0) \\ &= \frac{u}{\sqrt{m}}\sum_{j=0}^{m-1}(-1)^{m-j}\sqrt{\frac{(m-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(m+\frac{1}{2}+\sigma)}}c_{2j,0}. \end{aligned}$$

■

Lemma 3.3. *If $k = 2m > 0$ and $\ell = 2n > 0$, then*

$$d_{k,\ell} = \sqrt{\frac{m}{n}}d_{k-1,\ell-1} + \frac{u}{\sqrt{n}}\sum_{j=0}^{n-1}(-1)^{n-j}\sqrt{\frac{(n-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(n+\frac{1}{2}+\sigma)}}d_{k,2j}.$$

Proof. By (2.2), (2.4), (2.6) and Lemma 3.1,

$$\begin{aligned} c_{k,\ell} &= \frac{1}{\sqrt{2s\ell}}\mathfrak{t}(\phi_k, B'\phi_{\ell-1}) = \frac{1}{\sqrt{2\ell s}}\mathfrak{t}(B\phi_k, \phi_{\ell-1}) - \frac{2u}{\sqrt{2\ell s}}\mathfrak{t}(\phi_k, x^{-1}\phi_{\ell-1}) \\ &= \sqrt{\frac{m}{n}}c_{k-1,\ell-1} + \frac{u}{\sqrt{n}}\sum_{j=0}^{n-1}(-1)^{n-j}\sqrt{\frac{(n-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(n+\frac{1}{2}+\sigma)}}c_{k,2j}. \end{aligned}$$

■

Lemma 3.4. *If $k = 2m + 1$ and $\ell = 2n + 1$, then*

$$d_{k,\ell} = \sqrt{\frac{n + \frac{1}{2} + \sigma}{m + \frac{1}{2} + \sigma}} d_{k-1,\ell-1} - \frac{u}{\sqrt{m + \frac{1}{2} + \sigma}} \sum_{j=0}^{n} (-1)^{n-j} \sqrt{\frac{n!\Gamma(j + \frac{1}{2} + \sigma)}{j!\Gamma(n + \frac{3}{2} + \sigma)}} d_{k-1,2j}.$$

Proof. By (2.2), (2.4), (2.6) and Lemma 3.1,

$$\begin{aligned} c_{k,\ell} &= \frac{1}{\sqrt{2(k+2\sigma)s}} \mathfrak{t}(B'\phi_{k-1}, \phi_\ell) \\ &= \frac{1}{\sqrt{2(k+2\sigma)s}} \mathfrak{t}(\phi_{k-1}, B\phi_\ell) - \frac{2u}{\sqrt{2(k+2\sigma)s}} \mathfrak{t}(\phi_{k-1}, x^{-1}\phi_\ell) \\ &= \sqrt{\frac{n + \frac{1}{2} + \sigma}{m + \frac{1}{2} + \sigma}} c_{k-1,\ell-1} - \frac{u}{\sqrt{m + \frac{1}{2} + \sigma}} \sum_{j=0}^{n} (-1)^{n-j} \sqrt{\frac{n!\Gamma(j + \frac{1}{2} + \sigma)}{j!\Gamma(n + \frac{3}{2} + \sigma)}} c_{k-1,2j}. \end{aligned}$$ ■

The following definitions are given for $k \geq \ell$ with $k + \ell$ even. Let

$$\Pi_{k,\ell} = \sqrt{\frac{m!\Gamma(n + \frac{1}{2} + \sigma)}{n!\Gamma(m + \frac{1}{2} + \sigma)}} \tag{3.3}$$

if $k = 2m \geq \ell = 2n$, and

$$\Pi_{k,\ell} = \sqrt{\frac{m!\Gamma(n + \frac{3}{2} + \sigma)}{n!\Gamma(m + \frac{3}{2} + \sigma)}} \tag{3.4}$$

if $k = 2m + 1 \geq \ell = 2n + 1$. Let $\Sigma_{k,\ell}$ be inductively defined as follows[2]:

$$\Sigma_{k,0} = \prod_{i=1}^{m} \left(1 - \frac{1-u}{i}\right) \tag{3.5}$$

if $k = 2m$;

$$\Sigma_{k,\ell} = \Sigma_{k-1,\ell-1} + u \sum_{j=0}^{n-1} \frac{(n-1)!\Gamma(j + \frac{1}{2} + \sigma)}{j!\Gamma(n + \frac{1}{2} + \sigma)} \Sigma_{k,2j} \tag{3.6}$$

if $k = 2m \geq \ell = 2n > 0$; and

$$\Sigma_{k,\ell} = \Sigma_{k-1,\ell-1} - u \sum_{j=0}^{n} \frac{n!\Gamma(j + \frac{1}{2} + \sigma)}{j!\Gamma(n + \frac{3}{2} + \sigma)} \Sigma_{k-1,2j} \tag{3.7}$$

$$= \left(1 - \frac{u}{n + \frac{1}{2} + \sigma}\right) \Sigma_{k-1,\ell-1} - \frac{nu}{n + \frac{1}{2} + \sigma} \sum_{j=0}^{n-1} \frac{(n-1)!\Gamma(j + \frac{1}{2} + \sigma)}{j!\Gamma(n + \frac{1}{2} + \sigma)} \Sigma_{k-1,2j} \tag{3.8}$$

if $k = 2m + 1 \geq \ell = 2n + 1$. Thus $\Sigma_{0,0} = 1$, $\Sigma_{2,0} = u$, $\Sigma_{4,0} = \frac{1}{2}u(1+u)$, and

$$\Sigma_{k,1} = \left(1 - \frac{u}{\frac{1}{2} + \sigma}\right) \Sigma_{k-1,0} \tag{3.9}$$

[2] We use the convention that a product of an empty set of factors is 1. Such empty products are possible in (3.5) (when $m = 0$), in Lemma 3.10 and its proof, and in the proofs of Lemma 3.11 and Proposition 3.18. Consistently, the sum of an empty set of terms is 0. Such empty sums are possible in Lemma 4.4 and its proof, and in the proof of Proposition 4.7.

if k is odd. From (3.5) and using induction on m, it easily follows that

$$\Sigma_{k,0} = \frac{u}{m} \sum_{j=0}^{m-1} \Sigma_{2j,0} \tag{3.10}$$

for $k = 2m > 0$. Combining (3.6) with (3.7), and (3.8) with (3.6), we get

$$\Sigma_{k,\ell} = \Sigma_{k-2,\ell-2} - u \sum_{j=0}^{n-1} \frac{(n-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(n-\frac{1}{2}+\sigma)} (\Sigma_{k-2,2j} - \Sigma_{k,2j}) \tag{3.11}$$

if $k = 2m \geq \ell = 2n > 0$; and

$$\begin{aligned}\Sigma_{k,\ell} &= \left(1 - \frac{u}{n+\frac{1}{2}+\sigma}\right) \Sigma_{k-2,\ell-2} \\ &\quad + \left(1 - \frac{u+n}{n+\frac{1}{2}+\sigma}\right) u \sum_{j=0}^{n-1} \frac{(n-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(n-\frac{1}{2}+\sigma)} \Sigma_{k-1,2j}\end{aligned} \tag{3.12}$$

if $k = 2m+1 \geq \ell = 2n+1 > 1$.

Proposition 3.5. *$d_{k,\ell} = (-1)^{m+n}\Pi_{k,\ell}\Sigma_{k,\ell}$ if $k = 2m \geq \ell = 2n$, or if $k = 2m+1 \geq \ell = 2n+1$.*

Proof. We proceed by induction on k and l. The statement is obvious for $k = \ell = 0$ because $d_{0,0} = \Pi_{0,0} = \Sigma_{0,0} = 1$.

Let $k = 2m > 0$, and assume that the result is true for all $d_{2j,0}$ with $j < m$. Then, by Lemma 3.2, (3.3) and (3.10),

$$\begin{aligned}d_{k,0} &= \frac{u}{\sqrt{m}} \sum_{j=0}^{m-1} (-1)^{m-j} \sqrt{\frac{(m-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(m+\frac{1}{2}+\sigma)}} (-1)^j \Pi_{2j,0}\Sigma_{2j,0} \\ &= (-1)^m \frac{u}{\sqrt{m}} \sum_{j=0}^{m-1} \sqrt{\frac{(m-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(m+\frac{1}{2}+\sigma)}} \sqrt{\frac{j!\Gamma(\frac{1}{2}+\sigma)}{\Gamma(j+\frac{1}{2}+\sigma)}} \Sigma_{2j,0} \\ &= (-1)^m \Pi_{k,0} \frac{u}{m} \sum_{j=0}^{m-1} \Sigma_{2j,0} = (-1)^m \Pi_{k,0}\Sigma_{k,0}.\end{aligned}$$

Now, take $k = 2m \geq \ell = 2n > 0$ so that the equality of the statement holds for $d_{k-1,\ell-1}$ and all $d_{k,2j}$ with $j < n$. Then, by Lemma 3.3,

$$\begin{aligned}d_{k,\ell} &= \sqrt{\frac{m}{n}} (-1)^{m+n} \Pi_{k-1,\ell-1}\Sigma_{k-1,\ell-1} \\ &\quad + \frac{u}{\sqrt{n}} \sum_{j=0}^{n-1} (-1)^{n-j} \sqrt{\frac{(n-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(n+\frac{1}{2}+\sigma)}} (-1)^{m+j} \Pi_{k,2j}\Sigma_{k,2j}.\end{aligned}$$

Here, by (3.3) and (3.4), $\sqrt{m/n}\Pi_{k-1,\ell-1} = \Pi_{k,\ell}$, and

$$\begin{aligned}\frac{1}{\sqrt{n}} \sqrt{\frac{(n-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(n+\frac{1}{2}+\sigma)}} \Pi_{k,2j} &= \frac{1}{\sqrt{n}} \sqrt{\frac{m!\Gamma(n+\frac{1}{2}+\sigma)}{(n-1)!\Gamma(m+\frac{1}{2}+\sigma)}} \frac{(n-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(n+\frac{1}{2}+\sigma)} \\ &= \Pi_{k,\ell} \frac{(n-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(n+\frac{1}{2}+\sigma)}.\end{aligned}$$

Thus, by (3.6), $d_{k,\ell} = (-1)^{m+n}\Pi_{k,\ell}\Sigma_{k,\ell}$.

Finally, take $k = 2m + 1 \geq \ell = 2n + 1$ such that the equality of the statement holds for all $d_{k-1,2j}$ with $j \leq n$. Then, by Lemma 3.4,

$$\begin{aligned} d_{k,\ell} &= \sqrt{\frac{n + \frac{1}{2} + \sigma}{m + \frac{1}{2} + \sigma}} (-1)^{m+n} \Pi_{k-1,\ell-1} \Sigma_{k-1,\ell-1} \\ &\quad - \frac{u}{\sqrt{m + \frac{1}{2} + \sigma}} \sum_{j=0}^{n} (-1)^{n-j} \sqrt{\frac{n!\Gamma(j + \frac{1}{2} + \sigma)}{j!\Gamma(n + \frac{3}{2} + \sigma)}} (-1)^{m+j} \Pi_{k-1,2j} \Sigma_{k-1,2j}. \end{aligned}$$

Here, by (3.3) and (3.4),

$$\sqrt{\frac{n + \frac{1}{2} + \sigma}{m + \frac{1}{2} + \sigma}} \Pi_{k-1,\ell-1} = \Pi_{k,\ell},$$

and

$$\begin{aligned} &\frac{1}{\sqrt{m + \frac{1}{2} + \sigma}} \sqrt{\frac{n!\Gamma(j + \frac{1}{2} + \sigma)}{j!\Gamma(n + \frac{3}{2} + \sigma)}} \Pi_{k-1,2j} \\ &\qquad = \frac{1}{\sqrt{m + \frac{1}{2} + \sigma}} \sqrt{\frac{m!\Gamma(n + \frac{3}{2} + \sigma)}{n!\Gamma(m + \frac{1}{2} + \sigma)}} \frac{n!\Gamma(j + \frac{1}{2} + \sigma)}{j!\Gamma(n + \frac{3}{2} + \sigma)} = \Pi_{k,\ell} \frac{n!\Gamma(j + \frac{1}{2} + \sigma)}{j!\Gamma(n + \frac{3}{2} + \sigma)}. \end{aligned}$$

Thus, by (3.7), $d_{k,\ell} = (-1)^{m+n} \Pi_{k,\ell} \Sigma_{k,\ell}$. ■

Lemma 3.6. $\Sigma_{k,\ell} > 0$ *for all* k *and* ℓ.

Proof. We proceed by induction on ℓ. For $\ell \in \{0, 1\}$, this is true by (3.5) and (3.9) because $\sigma > u - 1/2$. If $\ell > 1$ and the results holds for $\Sigma_{k',\ell'}$ with $\ell' < \ell$, then $\Sigma_{k,\ell} > 0$ by (3.6) and (3.12) since $\sigma > u - 1/2$. ■

Lemma 3.7. *If* $k = 2m > \ell = 2n$ *or* $k = 2m + 1 > \ell = 2n + 1$, *then*

$$\Sigma_{k,\ell} \leq \left(1 - \frac{1 - u}{m}\right) \Sigma_{k-2,\ell}.$$

Proof. We proceed by induction on ℓ. This is true for $\ell \in \{0, 1\}$ by (3.5) and (3.9).

Now, suppose that the result is satisfied by $\Sigma_{k',\ell'}$ with $\ell' < \ell$. If $k = 2m > \ell = 2n > 0$, then, by (3.6) and Lemma 3.6,

$$\begin{aligned} \Sigma_{k,\ell} &\leq \left(1 - \frac{1 - u}{m - 1}\right) \Sigma_{k-3,\ell-1} + u \sum_{j=0}^{n-1} \frac{(n - 1)!\Gamma(j + \frac{1}{2} + \sigma)}{j!\Gamma(n + \frac{1}{2} + \sigma)} \left(1 - \frac{1 - u}{m}\right) \Sigma_{k-2,2j} \\ &\leq \left(1 - \frac{1 - u}{m}\right) \Sigma_{k-2,\ell}. \end{aligned}$$

If $k = 2m + 1 > \ell = 2n + 1 > 1$, then, by (3.12) and Lemma 3.6, and since $\sigma > u - 1/2$,

$$\begin{aligned} \Sigma_{k,\ell} &= \left(1 - \frac{u}{n + \frac{1}{2} + \sigma}\right) \left(1 - \frac{1 - u}{m - 1}\right) \Sigma_{k-3,\ell-2} \\ &\quad + \left(1 - \frac{u + n}{n + \frac{1}{2} + \sigma}\right) u \sum_{j=0}^{n-1} \frac{(n - 1)!\Gamma(j + \frac{1}{2} + \sigma)}{j!\Gamma(n - \frac{1}{2} + \sigma)} \left(1 - \frac{1 - u}{m}\right) \Sigma_{k-1,2j} \\ &< \left(1 - \frac{1 - u}{m}\right) \Sigma_{k-2,\ell}. \end{aligned}$$

■

Corollary 3.8. *If* $k = 2m \geq \ell = 2n > 0$*, then*

$$\Sigma_{k-1,\ell-1} < \Sigma_{k,\ell} \leq \left(1 - \frac{u(1-u)}{m}\right) \Sigma_{k-2,\ell-2}.$$

Proof. The first inequality is a direct consequence of (3.6), and Lemmas 3.6 and 3.7. On the other hand, by (3.11), and Lemmas 3.6 and 3.7,

$$\begin{aligned}\Sigma_{k,\ell} &\leq \Sigma_{k-2,\ell-2} - \frac{u(1-u)}{m} \sum_{j=0}^{n-1} \frac{(n-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(n-\frac{1}{2}+\sigma)} \Sigma_{k-2,2j} \\ &= \left(1 - \frac{u(1-u)}{m}\right) \Sigma_{k-2,\ell-2} - \frac{u(1-u)}{m} \sum_{j=0}^{n-2} \frac{(n-1)!\Gamma(j+\frac{1}{2}+\sigma)}{j!\Gamma(n-\frac{1}{2}+\sigma)} \Sigma_{k-2,2j} \\ &\leq \left(1 - \frac{u(1-u)}{m}\right) \Sigma_{k-2,\ell-2}.\end{aligned}$$

■

Corollary 3.9. *If* $k = 2m+1 \geq \ell = 2n+1$*, then*

$$\left(1 - \frac{u}{n+\frac{1}{2}+\sigma}\right) \Sigma_{k-2,\ell-2} < \Sigma_{k,\ell} < \left(1 - \frac{u}{n+\frac{1}{2}+\sigma}\right) \Sigma_{k-1,\ell-1}.$$

Proof. This follows from (3.8), (3.12) and Lemma 3.6 because $\sigma > u - 1/2$. ■

Lemma 3.10. *For* $0 < t < 1$*, there is some* $C_0 = C_0(t) \geq 1$ *such that, for all* p*,*

$$C_0^{-1}(p+1)^{-t} \leq \prod_{i=1}^{p} \left(1 - \frac{t}{i}\right) \leq C_0(p+1)^{-t}.$$

Proof. For each $t > 0$, by the Weierstrass definition of the gamma function,

$$\Gamma(t) = \frac{e^{-\gamma t}}{t} \prod_{i=1}^{\infty} \left(1 + \frac{t}{i}\right)^{-1} e^{t/i},$$

where $\gamma = \lim_{j\to\infty} \left(\sum_{i=1}^{j} \frac{1}{i} - \ln j\right)$ (the Euler–Mascheroni constant), there is some $K_0 \geq 1$ such that, for all $p \in \mathbb{Z}_+$,

$$K_0^{-1} \prod_{i=1}^{p} e^{-t/i} \leq \prod_{i=1}^{p} \left(1 + \frac{t}{i}\right)^{-1} \leq K_0 \prod_{i=1}^{p} e^{-t/i}. \tag{3.13}$$

Now, assume that $0 < t < 1$, and observe that

$$\prod_{i=1}^{p} \left(1 - \frac{t}{i}\right) = \prod_{i=1}^{p} \left(1 + \frac{t}{i-t}\right)^{-1}.$$

By the second inequality of (3.13), for $p \geq 1$,

$$\begin{aligned}\prod_{i=1}^{p} \left(1 + \frac{t}{i-t}\right)^{-1} &\leq \prod_{i=1}^{p} \left(1 + \frac{t}{i}\right)^{-1} \leq K_0 \prod_{i=1}^{p} e^{-t/i} = K_0 \exp\left(-t \sum_{i=1}^{p} \frac{1}{i}\right) \\ &\leq K_0 \exp\left(-t\left(1 + \int_1^p \frac{dx}{x}\right)\right) = K_0 e^{-t} p^{-t} \leq K_0 e^{-t} 2^t (p+1)^{-t}.\end{aligned}$$

On the other hand, by the first inequality of (3.13), for $p \geq 2$,

$$\begin{aligned}\prod_{i=1}^{p}\left(1+\frac{t}{i-t}\right)^{-1} &\geq (1-t)\prod_{i=1}^{p-1}\left(1+\frac{t}{i}\right)^{-1} \geq (1-t)K_0^{-1}\prod_{i=1}^{p-1}e^{-t/i}\\ &= (1-t)K_0^{-1}\exp\left(-t\sum_{i=1}^{p-1}\frac{1}{i}\right) \geq (1-t)K_0^{-1}\exp\left(-t\left(1+\int_1^{p-1}\frac{dx}{x}\right)\right)\\ &= (1-t)K_0^{-1}e^{-t}(p-1)^{-t} \geq (1-t)K_0^{-1}e^{-t}3^t(p+1)^{-t}.\end{aligned}$$

■

Lemma 3.11. *There is some $C' = C'(u) > 0$ such that*

$$\Sigma_{k,\ell} \leq C'(m+1)^{-u(1-u)}(m-n+1)^{-(1-u)^2}$$

for $k = 2m \geq \ell = 2n$ or $k = 2m+1 \geq \ell = 2n+1$.

Proof. Suppose first that $k = 2m \geq \ell = 2n$. By Lemma 3.7 and (3.10), we get

$$\begin{aligned}\Sigma_{k,\ell} &\leq \prod_{i=m-n+1}^{m}\left(1-\frac{u(1-u)}{i}\right)\prod_{i=1}^{m-n}\left(1-\frac{1-u}{i}\right)\\ &= \prod_{i=m-n+1}^{m}\left(1+\frac{u(1-u)}{i-u(1-u)}\right)^{-1}\prod_{i=1}^{m-n}\left(1+\frac{1-u}{i-1+u}\right)^{-1}\\ &\leq \prod_{i=m-n+1}^{m}\left(1+\frac{u(1-u)}{i}\right)^{-1}\prod_{i=1}^{m-n}\left(1+\frac{1-u}{i}\right)^{-1}\\ &= \prod_{i=1}^{m-n}\left(1+\frac{u(1-u)}{i}\right)\prod_{i=1}^{m}\left(1+\frac{u(1-u)}{i}\right)^{-1}\prod_{i=1}^{m-n}\left(1+\frac{1-u}{i}\right)^{-1}.\end{aligned}$$

Then the result follows in this case from Lemma 3.10.

When $k = 2m+1 \geq \ell = 2n+1$, the result follows from the above case and Corollary 3.9. ■

Lemma 3.12. *For each $t > 0$, there is some $C_1 = C_1(t) \geq 1$ such that, for all p,*

$$C_1^{-1}(p+1)^{1-t} \leq \frac{\Gamma(p+1)}{\Gamma(p+t)} \leq C_1(p+1)^{1-t}.$$

Proof. We can assume that $p \geq 1$. Write $t = q + r$, where $q = \lfloor t \rfloor$. If $q = 0$, then $0 < r < 1$ and the result follows from the Gautschi's inequality, stating that

$$x^{1-r} \leq \frac{\Gamma(x+1)}{\Gamma(x+r)} \leq (x+1)^{1-r} \tag{3.14}$$

for $0 < r < 1$ and $x > 0$, because $x^{1-r} \geq 2^{r-1}(x+1)^{1-r}$ for $x \geq 1$.

If $q \geq 1$ and $r = 0$, then

$$\begin{aligned}\frac{\Gamma(p+1)}{\Gamma(p+t)} &= \frac{p!}{(p+q-1)!} \leq \frac{1}{(p+1)^{q-1}} = (p+1)^{1-t},\\ \frac{\Gamma(p+1)}{\Gamma(p+t)} &= \frac{p!}{(p+q-1)!} \geq \frac{1}{(p+q-1)^{q-1}} \geq \frac{1}{(qp)^{q-1}} \geq t^{1-t}(p+1)^{1-t}.\end{aligned}$$

If $q \geq 1$ and $r > 0$, then, by (3.14),

$$\frac{\Gamma(p+1)}{\Gamma(p+t)} \leq \frac{\Gamma(p+1)}{(p+1)^{q-1}(p+r)\Gamma(p+r)} \leq \frac{(p+1)^{-q-r}}{p+r} \leq \frac{2(p+1)^{1-t}}{1+r},$$

$$\begin{aligned}\frac{\Gamma(p+1)}{\Gamma(p+t)} &\geq \frac{\Gamma(p+1)}{(p+t-1)^q\Gamma(p+r)} \geq \frac{p^{1-r}}{(p+t-1)^q}\\ &\geq \frac{(p+1)^{1-r}}{2^{1-r}(p+t-1)^q} \geq \min\{1,2/t\}2^{r-1}(p+1)^{1-t}.\end{aligned}$$ ■

Corollary 3.13. *There is some $C'' = C''(\sigma) > 0$ such that*

$$\Pi_{k,\ell} \leq \begin{cases} C''\left(\frac{n+1}{m+1}\right)^{\sigma/2-1/4} & \text{if } k = 2m \geq \ell = 2n,\\ C''\left(\frac{n+1}{m+1}\right)^{\sigma/2+1/4} & \text{if } k = 2m+1 \geq \ell = 2n+1.\end{cases}$$

Proof. This follows from (3.3), (3.4) and Lemma 3.12. ■

For the sake of simplicity, let us use the following notation. For real valued functions f and g of (m,n), for (m,n) in some subset of $\mathbb{N}\times\mathbb{N}$, write $f \preccurlyeq g$ if there is some $C > 0$ such that $f(m,n) \leq Cg(m,n)$ for all (m,n). The same notation is used for functions depending also on other variables, $s,\sigma,u,\dots$, taking C independent of m, n and s, but possibly depending on the rest of variables.

Lemma 3.14. *For $\alpha,\beta,\gamma\in\mathbb{R}$, if $\alpha+\beta,\alpha+\gamma,\alpha+\beta+\gamma<0$, then there is some $\omega>0$ such that, for all naturals $m\geq n$,*

$$(m+1)^\alpha(n+1)^\beta(m-n+1)^\gamma \preccurlyeq (m+1)^{-\omega}(n+1)^{-\omega}.$$

Proof. We consider the following cases:

1. If $\alpha<0$, $\beta<0$ and $\gamma<0$, then
$$(m+1)^\alpha(n+1)^\beta(m-n+1)^\gamma \leq (m+1)^\alpha(n+1)^\beta.$$
2. If $\beta\geq 0$ and $\gamma<0$, then
$$(m+1)^\alpha(n+1)^\beta(m-n+1)^\gamma \leq (m+1)^{\alpha+\beta} \leq (m+1)^{(\alpha+\beta)/2}(n+1)^{(\alpha+\beta)/2}.$$
3. If $\alpha\geq 0$, $\gamma<0$ and $m+1\leq 2(n+1)$, then $\beta<0$ and
$$(m+1)^\alpha(n+1)^\beta(m-n+1)^\gamma \leq 2^{-\beta}(m+1)^{\alpha+\beta} \leq 2^{-\beta}(m+1)^{(\alpha+\beta)/2}(n+1)^{(\alpha+\beta)/2}.$$
4. If $\alpha\geq 0$, $\gamma<0$ and $m+1>2(n+1)$, then $\beta<0$ and $m-n+1>(m+1)/2$, and therefore
$$(m+1)^\alpha(n+1)^\beta(m-n+1)^\gamma \leq 2^{-\gamma}(m+1)^{\alpha+\gamma}(n+1)^\beta.$$
5. If $\beta<0$ and $\gamma\geq 0$, then
$$(m+1)^\alpha(n+1)^\beta(m-n+1)^\gamma \leq (m+1)^{\alpha+\gamma}(n+1)^\beta.$$
6. If $\beta\geq 0$ and $\gamma\geq 0$, then
$$(m+1)^\alpha(n+1)^\beta(m-n+1)^\gamma \leq (m+1)^{\alpha+\beta+\gamma} \leq (m+1)^{(\alpha+\beta+\gamma)/2}(n+1)^{(\alpha+\beta+\gamma)/2}.$$ ■

Proposition 3.15. *There is some $\omega=\omega(\sigma,u)>0$ such that*

$$|d_{k,\ell}| \preccurlyeq (m+1)^{-\omega}(n+1)^{-\omega}$$

for $k=2m$ and $\ell=2n$, or for $k=2m+1$ and $\ell=2n+1$.

Proof. We can assume $k \geq \ell$ because $d_{k,\ell} = d_{\ell,k}$.

If $k = 2m+1 \geq \ell = 2n+1$, then, according to Proposition 3.5, Lemma 3.11 and Corollary 3.13,

$$|d_{k,\ell}| \preccurlyeq (m+1)^{-\sigma/2-1/4-u(1-u)}(n+1)^{\sigma/2+1/4}(m-n+1)^{-(1-u)^2}.$$

Thus the result follows by Lemma 3.14 since

$$-\sigma/2 - 1/4 - u(1-u) - (1-u)^2 = -\sigma/2 + u - 5/4 < u/2 - 1 < 0.$$

If $k = 2m \geq \ell = 2n$, then, according to Proposition 3.5, Lemma 3.11 and Corollary 3.13,

$$|d_{k,\ell}| \preccurlyeq (m+1)^{-\sigma/2+1/4-u(1-u)}(n+1)^{\sigma/2-1/4}(m-n+1)^{-(1-u)^2}.$$

Thus the result follows by Lemma 3.14 since

$$-\sigma/2 + 1/4 - u(1-u) - (1-u)^2 = -\sigma/2 + u - 3/4 < u/2 - 1/2 < 0. \qquad \blacksquare$$

Corollary 3.16. *There is some $\omega = \omega(\sigma, u) > 0$ such that, for $k = 2m$ and $\ell = 2n$, or for $k = 2m+1$ and $\ell = 2n+1$,*

$$|c_{k,\ell}| \preccurlyeq s^u (m+1)^{-\omega}(n+1)^{-\omega}.$$

Proof. This follows from Proposition 3.15 and (3.2). ■

Proposition 3.17. *For any $\epsilon > 0$, there is some $C = C(\epsilon, \sigma, u) > 0$ such that, for all $\phi \in \mathcal{S}$,*

$$\mathfrak{t}(\phi) \leq \epsilon s^{u-1}\mathfrak{j}(\phi) + C s^u \|\phi\|_\sigma^2.$$

Proof. For each k, let $\nu_k = 2k + 1 + \sigma$. By Proposition 3.15, there are $K_0 = K_0(\sigma, u) > 0$ and $\omega = \omega(\sigma, u) > 0$ such that

$$|c_{k,\ell}| \leq K_0 s^u \nu_k^{-\omega} \nu_\ell^{-\omega} \tag{3.15}$$

for all k and ℓ. Since $S = S(\sigma, u) := \sum_k \nu_k^{-1-2\omega} < \infty$, given $\epsilon > 0$, there is some $k_0 = k_0(\epsilon, \sigma, u)$ so that

$$S_0 = S_0(\epsilon, \sigma, u) := \sum_{k \geq k_0} \nu_k^{-1-2\omega} < \frac{\epsilon^2}{4K_0^2 S}.$$

Let $S_1 = S_1(\epsilon, \sigma, u) = \sum_{k \leq k_0} \nu_k^{-\omega}$. For $\phi = \sum_k t_k \phi_k \in \mathcal{S}$, by (3.15) and the Schwartz inequality, we have

$$\begin{aligned}
\mathfrak{t}(\phi) &= \sum_{k,\ell} t_k \overline{t_\ell} c_{k,\ell} \leq \sum_{k,\ell} |t_k||t_\ell||c_{k,\ell}| \\
&\leq K_0 s^{u-1/2} \sum_{k \leq k_0} \frac{|t_k|}{\nu_k^\omega} \sum_\ell \frac{|t_\ell|(\nu_\ell s)^{1/2}}{\nu_\ell^{1/2+\omega}} + K_0 s^{u-1} \sum_{k \geq k_0} \frac{|t_k|(\nu_k s)^{1/2}}{\nu_k^{1/2+\omega}} \sum_\ell \frac{|t_\ell|(\nu_\ell s)^{1/2}}{\nu_\ell^{1/2+\omega}} \\
&\leq K_0 S_1 S^{1/2} s^{u-1/2} \|\phi\|_\sigma \mathfrak{j}(\phi)^{1/2} + K_0 S_0^{1/2} S^{1/2} s^{u-1} \mathfrak{j}(\phi) \\
&\leq K_0 S_1 S^{1/2} s^{u-1/2} \|\phi\|_\sigma \mathfrak{j}(\phi)^{1/2} + \frac{\epsilon s^{u-1}}{2} \mathfrak{j}(\phi) \leq \frac{2K_0^2 S_1^2 S s^u}{\epsilon} \|\phi\|_\sigma^2 + \epsilon s^{u-1} \mathfrak{j}(\phi). \qquad \blacksquare
\end{aligned}$$

Proposition 3.18. *There is some $D = D(\sigma, u) > 0$ such that, for all k,*

$$\mathfrak{t}(\phi_k) \geq D s^u (k+1)^{-u}.$$

Proof. By Proposition 3.5 and (3.2), and since $\Pi_{k,k} = 1$, it is enough to prove that there is some $D_0 = D_0(\sigma, u) > 0$ so that $\Sigma_{k,k} \le D_0(k+1)^{-u}$. Moreover we can assume that $k = 2m+1$ by Corollary 3.8.

We have $p_0 := \lfloor 1/2+\sigma \rfloor \ge 0$ because $1/2+\sigma > u$. According to Corollary 3.9 and Lemma 3.10, there is some $C_0 = C_0(u) \ge 1$ such that

$$\begin{aligned}
\Sigma_{k,k} &\ge \prod_{i=0}^{m} \left(1 - \frac{u}{i + \frac{1}{2} + \sigma}\right) \ge \left(1 - \frac{u}{\frac{1}{2} + \sigma}\right) \prod_{p=1+p_0}^{m+p_0} \left(1 - \frac{u}{p}\right) \\
&= \left(1 - \frac{u}{\frac{1}{2} + \sigma}\right) \prod_{p=1}^{m+p_0} \left(1 - \frac{u}{p}\right) \prod_{p=1}^{p_0} \left(1 - \frac{u}{p}\right)^{-1} \\
&\ge \left(1 - \frac{u}{\frac{1}{2} + \sigma}\right) C_0^{-2}(m+p_0+1)^{-u}(p_0+1)^u \ge \left(1 - \frac{u}{\frac{1}{2} + \sigma}\right) C_0^{-2}(k+1)^{-u}. \qquad \blacksquare
\end{aligned}$$

Remark 3.19. If $0 < u < 1/2$, then $\lim_m \mathfrak{t}(\phi_{2m+1}) = 0$. To check it, we use that there is some $K > 0$ so that $|x|^{2\sigma}\phi_k^2(x) \le K k^{-1/6}$ for all $x \in \mathbb{R}$ and all odd $k \in \mathbb{N}$ [1, Theorem 1.1(ii)]. For any $\epsilon > 0$, take some $x_0 > 0$ and $k_0 \in \mathbb{N}$ such that $x_0^{-2u} < \epsilon/2$ and $K k_0^{-1/6} x_0^{1-2u} < \epsilon(1-2u)/4$. Then, for all odd natural $k \ge k_0$,

$$\begin{aligned}
\mathfrak{t}(\phi_k) &= 2\int_0^{x_0} \phi_k^2(x) x^{2(\sigma-u)}dx + 2\int_{x_0}^{\infty} \phi_k^2(x) x^{2(\sigma-u)}dx \\
&\le 2K k^{-1/6} \int_0^{x_0} x^{-2u}dx + 2x_0^{-2u}\int_{x_0}^{\infty} \phi_k^2(x) x^{2\sigma}dx \le 2K k^{-1/6}\frac{x_0^{1-2u}}{1-2u} + x_0^{-2u} < \epsilon,
\end{aligned}$$

because $1 - 2u > 0$ and $\|\phi_k\|_\sigma = 1$. In the case where $\sigma \ge 0$, this argument is also valid when k is even. We do not know if $\inf_k \mathfrak{t}(\phi_k) > 0$ when $1/2 \le u < 1$.

Proof of Theorem 1.1. The positive definite sesquilinear form $\mathfrak{j}$ of Section 2 is closable by [11, Chapter VI, Theorems 2.1 and 2.7]. Then, taking $\epsilon > 0$ so that $\xi\epsilon s^{u-1} < 1$, it follows from [11, Chapter VI, Theorem 1.33] and Proposition 3.17 that the positive definite sesquilinear form $\mathfrak{u} := \mathfrak{j} + \xi\mathfrak{t}$ is also closable, and $\mathsf{D}(\bar{\mathfrak{u}}) = \mathsf{D}(\bar{\mathfrak{j}})$. By [11, Chapter VI, Theorems 2.1, 2.6 and 2.7], there is a unique positive definite self-adjoint operator $\mathcal{U}$ such that $\mathsf{D}(\mathcal{U})$ is a core of $\mathsf{D}(\bar{\mathfrak{u}})$, which consists of the elements $\phi \in \mathsf{D}(\bar{\mathfrak{u}})$ so that, for some $\chi \in L^2_\sigma$, we have $\bar{\mathfrak{u}}(\phi, \psi) = \langle \chi, \psi \rangle_\sigma$ for all ψ in some core of $\bar{\mathfrak{u}}$ (in this case, $\mathcal{U}(\phi) = \chi$). By [11, Chapter VI, Theorem 2.23], we have $\mathsf{D}(\mathcal{U}^{1/2}) = \mathsf{D}(\bar{\mathfrak{u}})$, $\mathcal{S}$ is a core of $\mathcal{U}^{1/2}$ (since it is a core of $\mathfrak{u}$), and (1.1) is satisfied. By Proposition 3.18,

$$\mathfrak{u}(\phi_k) \ge (2k+1+2\sigma)s + \xi D s^u (k+1)^{-u}$$

for all k. Therefore $\mathcal{U}$ has a discrete spectrum satisfying the first inequality of (1.2) by the form version of the min-max principle [17, Theorem XIII.2]. The second inequality of (1.2) holds because

$$\bar{\mathfrak{u}}(\phi) \le \left(1 + \xi\epsilon s^{u-1}\right)\bar{\mathfrak{j}}(\phi) + \xi C s^u \|\phi\|_\sigma^2$$

for all $\phi \in \mathsf{D}(\bar{\mathfrak{u}})$ by Proposition 3.17 and [11, Chapter VI, Theorem 1.18], since $\mathcal{S}$ is a core of $\bar{\mathfrak{u}}$ and $\bar{\mathfrak{j}}$. $\blacksquare$

Remark 3.20. In the above proof, note that $\bar{\mathfrak{u}} = \bar{\mathfrak{j}} + \xi\bar{\mathfrak{t}}$ and $\mathsf{D}(\bar{\mathfrak{j}}) = \mathsf{D}(\overline{J}^{1/2})$. Thus (1.1) can be extended to $\phi, \psi \in \mathsf{D}(\mathcal{U}^{1/2})$ using $\langle \overline{J}^{1/2}\phi, \overline{J}^{1/2}\psi \rangle_\sigma$ instead of $\langle J\phi, \psi \rangle_\sigma$.

Remark 3.21. Extend the definition of the above forms and operators to the case of $\xi \in \mathbb{C}$. Then $|\bar{\mathfrak{t}}(\phi)| \le \epsilon s^{u-1} \Re \bar{\mathfrak{j}}(\phi) + C s^u \|\phi\|_\sigma^2$ for all $\phi \in \mathsf{D}(\bar{\mathfrak{j}})$, like in the proof of Theorem 1.1. Thus the family $\bar{\mathfrak{u}} = \bar{\mathfrak{u}}(\xi)$ becomes holomorphic of type (a) by Remark 3.20 and [11, Chapter VII, Theorem 4.8], and therefore $\mathcal{U} = \mathcal{U}(\xi)$ is a self-adjoint holomorphic family of type (B). So the functions $\lambda_k = \lambda_k(\xi)$ ($\xi \in \mathbb{R}$) are continuous and piecewise holomorphic [11, Chapter VII, Remark 4.22, Theorem 3.9, and § 3.4], with $\lambda_k(0) = (2k+1+2\sigma)s$. Moreover [11, Chapter VI, Theorem 4.21] gives an exponential estimate of $|\lambda_k(\xi) - \lambda_k(0)|$ in terms of ξ. But (1.2) is a better estimate.

4 Scalar products of mixed generalized Hermite functions

Let $\sigma, \tau, \theta > -1/2$, and write $v = \sigma + \tau - 2\theta$. This section is devoted to describe the scalar products

$$\hat{c}_{k,\ell} = \hat{c}_{\sigma,\tau,\theta,k,\ell} = \langle \phi_{\sigma,k}, \phi_{\tau,\ell} \rangle_\theta,$$

which will be needed to prove Theorem 1.3. Note that $\hat{c}_{k,\ell} = 0$ if $k + \ell$ is odd, and

$$\hat{c}_{\sigma,\tau,\theta,k,\ell} = \hat{c}_{\tau,\sigma,\theta,\ell,k} \tag{4.1}$$

for all k and ℓ. Of course, $\hat{c}_{k,\ell} = \delta_{k,\ell}$ if $\sigma = \tau = \theta$.

According to Section 2, if k and ℓ are odd, then $\hat{c}_{\sigma,\tau,\theta,k,\ell}$ is also defined when $\sigma, \tau, \theta > -3/2$, and we have

$$\hat{c}_{\sigma,\tau,\theta,k,\ell} = \langle x\phi_{\sigma+1,k-1}, x\phi_{\tau+1,\ell-1} \rangle_\theta = \hat{c}_{\sigma+1,\tau+1,\theta+1,k-1,\ell-1}. \tag{4.2}$$

4.1 Case where $\sigma = \theta \neq \tau$ and $\tau - \sigma \notin -\mathbb{N}$

In this case, we have $v = \tau - \sigma$. By (2.1) and (3.1),

$$\hat{c}_{0,0} = s^{v/2} \Gamma(\sigma + 1/2)^{1/2} \Gamma(\tau + 1/2)^{-1/2}. \tag{4.3}$$

Lemma 4.1. *If $k > 0$ is even, then $\hat{c}_{k,0} = 0$.*

Proof. By (2.2), (2.3) and (2.7),

$$\hat{c}_{k,0} = \frac{1}{\sqrt{2ks}} \langle B'_\sigma \phi_{\sigma,k-1}, \phi_{\tau,0} \rangle_\sigma = \frac{1}{\sqrt{2ks}} \langle \phi_{\sigma,k-1}, B_\tau \phi_{\tau,0} \rangle_\sigma = 0. \qquad \blacksquare$$

Lemma 4.2. *If $\ell = 2n > 0$, then*

$$\hat{c}_{0,\ell} = \frac{v}{\sqrt{n}} \sum_{j=0}^{n-1} (-1)^{n-j} \sqrt{\frac{(n-1)!\Gamma(j + \frac{1}{2} + \tau)}{j!\Gamma(n + \frac{1}{2} + \tau)}} \hat{c}_{0,2j}.$$

Proof. By (2.2), (2.3), (2.6) and (2.8),

$$\begin{aligned}
\hat{c}_{0,\ell} &= \frac{1}{\sqrt{2\ell s}} \langle \phi_{\sigma,0}, B'_\tau \phi_{\tau,\ell-1} \rangle_\sigma = \frac{1}{\sqrt{2\ell s}} \langle \phi_{\sigma,0}, (B'_\sigma - 2vx^{-1}) \phi_{\tau,\ell-1} \rangle_\sigma \\
&= \frac{1}{\sqrt{2\ell s}} \langle B_\sigma \phi_{\sigma,0}, \phi_{\tau,\ell-1} \rangle_\sigma - \frac{2v}{\sqrt{2\ell}} \sum_{j=0}^{n-1} (-1)^{n-1-j} \sqrt{\frac{(n-1)!\Gamma(j + \frac{1}{2} + \tau)}{j!\Gamma(n + \frac{1}{2} + \tau)}} \hat{c}_{0,2j} \\
&= \frac{v}{\sqrt{n}} \sum_{j=0}^{n-1} (-1)^{n-j} \sqrt{\frac{(n-1)!\Gamma(j + \frac{1}{2} + \tau)}{j!\Gamma(n + \frac{1}{2} + \tau)}} \hat{c}_{0,2j}.
\end{aligned} \qquad \blacksquare$$

Lemma 4.3. *If $k = 2m > 0$ and $\ell = 2n > 0$, then $\hat{c}_{k,\ell} = \sqrt{n/m}\,\hat{c}_{k-1,\ell-1}$.*

Proof. By (2.2), (2.4) and (2.7),

$$\hat{c}_{k,\ell} = \frac{1}{\sqrt{2ks}}\langle B'_\sigma \phi_{\sigma,k-1}, \phi_{\tau,\ell}\rangle_\sigma = \frac{1}{\sqrt{2ks}}\langle \phi_{\sigma,k-1}, B_\tau \phi_{\tau,\ell}\rangle_\sigma = \sqrt{\frac{n}{m}}\hat{c}_{k-1,\ell-1}. \qquad \blacksquare$$

Lemma 4.4. *If $k = 2m+1$ and $\ell = 2n+1$, then*

$$\hat{c}_{k,\ell} = \frac{n+\frac{1}{2}+\sigma}{\sqrt{(m+\frac{1}{2}+\sigma)(n+\frac{1}{2}+\tau)}}\hat{c}_{k-1,\ell-1} - \frac{v}{\sqrt{m+\frac{1}{2}+\sigma}}\sum_{j=0}^{n-1}(-1)^{n-j}\sqrt{\frac{n!\Gamma(j+\frac{1}{2}+\tau)}{j!\Gamma(n+\frac{3}{2}+\tau)}}\hat{c}_{k-1,2j}.$$

Proof. By (2.2), (2.4), (2.6) and (2.7),

$$\begin{aligned}\hat{c}_{k,\ell} &= \frac{1}{\sqrt{2(k+\sigma)s}}\langle B'_\sigma \phi_{\sigma,k-1}, \phi_{\tau,\ell}\rangle_\sigma = \frac{1}{\sqrt{2(k+2\sigma)s}}\langle \phi_{\sigma,k-1}, \big(B_\tau - 2vx^{-1}\big)\phi_{\tau,\ell}\rangle_\sigma \\ &= \sqrt{\frac{n+\frac{1}{2}+\tau}{m+\frac{1}{2}+\sigma}}\hat{c}_{k-1,\ell-1} - \frac{v}{\sqrt{m+\frac{1}{2}+\sigma}}\sum_{j=0}^{n}(-1)^{n-j}\sqrt{\frac{n!\Gamma(j+\frac{1}{2}+\tau)}{j!\Gamma(n+\frac{3}{2}+\tau)}}\hat{c}_{k-1,2j} \\ &= \frac{n+\frac{1}{2}+\sigma}{\sqrt{(m+\frac{1}{2}+\sigma)(n+\frac{1}{2}+\tau)}}\hat{c}_{k-1,\ell-1} \\ &\quad - \frac{v}{\sqrt{m+\frac{1}{2}+\sigma}}\sum_{j=0}^{n-1}(-1)^{n-j}\sqrt{\frac{n!\Gamma(j+\frac{1}{2}+\tau)}{j!\Gamma(n+\frac{3}{2}+\tau)}}\hat{c}_{k-1,2j}. \qquad \blacksquare\end{aligned}$$

Corollary 4.5. *If $k > \ell$, then $\hat{c}_{k,\ell} = 0$.*

Proof. This follows by induction on ℓ using Lemmas 4.1, 4.3 and 4.4. $\blacksquare$

Remark 4.6. By Corollary 4.5, in Lemma 4.4, it is enough to consider the sum with j running from m to $n-1$.

Proposition 4.7. *If $k = 2m \le \ell = 2n$, then*

$$\hat{c}_{k,\ell} = (-1)^{m+n}s^{v/2}\sqrt{\frac{n!\Gamma(m+\frac{1}{2}+\sigma)}{m!\Gamma(n+\frac{1}{2}+\tau)}}\frac{\Gamma(n-m+v)}{(n-m)!\Gamma(v)},$$

and, if $k = 2m+1 \le \ell = 2n+1$, then

$$\hat{c}_{k,\ell} = (-1)^{m+n}s^{v/2}\sqrt{\frac{n!\Gamma(m+\frac{3}{2}+\sigma)}{m!\Gamma(n+\frac{3}{2}+\tau)}}\frac{\Gamma(n-m+v)}{(n-m)!\Gamma(v)}.$$

Proof. This is proved by induction on k. In turn, the case $k = 0$,

$$\hat{c}_{0,\ell} = (-1)^n s^{v/2}\sqrt{\frac{\Gamma(\frac{1}{2}+\sigma)}{n!\Gamma(n+\frac{1}{2}+\tau)}}\frac{\Gamma(n+v)}{\Gamma(v)}, \tag{4.4}$$

is proved by induction on ℓ. If $k=\ell=0$, (4.4) is (4.3). Given $\ell=2n>0$, assume that the result holds for $k=0$ and all $\ell'=2n'<\ell$. Then, by Lemma 4.2,

$$\begin{aligned}\hat{c}_{0,\ell}&=\frac{v}{\sqrt{n}}\sum_{j=0}^{n-1}(-1)^{n-j}\sqrt{\frac{(n-1)!\Gamma(j+\frac{1}{2}+\tau)}{j!\Gamma(n+\frac{1}{2}+\tau)}}(-1)^j s^{v/2}\sqrt{\frac{\Gamma(\frac{1}{2}+\sigma)}{j!\Gamma(j+\frac{1}{2}+\tau)}}\frac{\Gamma(j+v)}{\Gamma(v)}\\&=(-1)^n s^{v/2}\sqrt{\frac{(n-1)!\Gamma(\frac{1}{2}+\sigma)}{n\Gamma(n+\frac{1}{2}+\tau)}}\frac{v}{\Gamma(v)}\sum_{j=0}^{n-1}\frac{\Gamma(j+v)}{j!},\end{aligned}$$

obtaining (4.4) because

$$\frac{\Gamma(p+1+t)}{p!}=t\sum_{i=0}^{p}\frac{\Gamma(i+t)}{i!}\tag{4.5}$$

for all $p\in\mathbb{N}$ and $t\in\mathbb{R}\setminus(-\mathbb{N})$, as can be easily checked by induction on p.

Given $k>0$, assume that the result holds for all $k'<k$. If k is even, the statement follows directly from Lemma 4.3. If k is odd, by Lemma 4.4, Remark 4.6 and (4.5),

$$\begin{aligned}\hat{c}_{k,\ell}&=\frac{n+\frac{1}{2}+\sigma}{\sqrt{(m+\frac{1}{2}+\sigma)(n+\frac{1}{2}+\tau)}}(-1)^{m+n}s^{v/2}\sqrt{\frac{n!\Gamma(m+\frac{1}{2}+\sigma)}{m!\Gamma(n+\frac{1}{2}+\tau)}}\frac{\Gamma(n-m+v)}{(n-m)!\Gamma(v)}\\&\quad-\frac{v}{\sqrt{m+\frac{1}{2}+\sigma}}\sum_{j=m}^{n-1}(-1)^{n-j}\sqrt{\frac{n!\Gamma(j+\frac{1}{2}+\tau)}{j!\Gamma(n+\frac{3}{2}+\tau)}}\\&\quad\times(-1)^{m+j}s^{v/2}\sqrt{\frac{j!\Gamma(m+\frac{1}{2}+\sigma)}{m!\Gamma(j+\frac{1}{2}+\tau)}}\frac{\Gamma(j-m+v)}{(j-m)!\Gamma(v)}\\&=(-1)^{m+n}s^{v/2}\sqrt{\frac{n!\Gamma(m+\frac{1}{2}+\sigma)}{(m+\frac{1}{2}+\sigma)m!\Gamma(n+\frac{3}{2}+\tau)}}\frac{1}{\Gamma(v)}\\&\quad\times\left(\frac{\Gamma(n-m+v)(n+\frac{1}{2}+\sigma)}{(n-m)!}-v\sum_{i=0}^{n-m-1}\frac{\Gamma(i+v)}{i!}\right)\\&=(-1)^{m+n}s^{v/2}\sqrt{\frac{n!\Gamma(m+\frac{3}{2}+\sigma)}{m!\Gamma(n+\frac{3}{2}+\tau)}}\frac{\Gamma(n-m+v)}{(n-m)!\Gamma(v)}.\end{aligned}$$ ■

Remark 4.8. By (4.2), if k and ℓ are odd, then Corollary 4.5 and Proposition 4.7 also hold when $\sigma,\tau>-3/2$.

4.2 Case where $\sigma\neq\theta\neq\tau$ and $\sigma-\theta,\tau-\theta\notin-\mathbb{N}$

By (2.1) and (3.1),

$$\hat{c}_{0,0}=s^{v/2}\Gamma(\sigma+1/2)^{-1/2}\Gamma(\tau+1/2)^{-1/2}\Gamma(\theta+1/2)^{1/2}.\tag{4.6}$$

Lemma 4.9. *If $k=2m>0$, then*

$$\hat{c}_{k,0}=\frac{\sigma-\theta}{\sqrt{m}}\sum_{i=0}^{m-1}(-1)^{m-i}\sqrt{\frac{(m-1)!\Gamma(i+\frac{1}{2}+\sigma)}{i!\Gamma(m+\frac{1}{2}+\sigma)}}\hat{c}_{2i,0}.$$

Proof. By (2.2) and (2.8),

$$\hat{c}_{k,0} = \frac{1}{\sqrt{2ks}}\langle B'_\sigma\phi_{\sigma,k-1},\phi_{\tau,0}\rangle_\theta = \frac{1}{2\sqrt{ms}}\langle B'_\theta\phi_{\sigma,k-1},\phi_{\tau,0}\rangle_\theta + \frac{\theta-\sigma}{\sqrt{ms}}\langle x^{-1}\phi_{\sigma,k-1},\phi_{\tau,0}\rangle_\theta.$$

Here, by (2.3), (2.6) and (2.7),

$$\langle B'_\theta\phi_{\sigma,k-1},\phi_{\tau,0}\rangle_\theta = \langle\phi_{\sigma,k-1},B_\theta\phi_{\tau,0}\rangle_\theta = \langle\phi_{\sigma,k-1},B_\tau\phi_{\tau,0}\rangle_\theta = 0,$$

$$\langle x^{-1}\phi_{\sigma,k-1},\phi_{\tau,0}\rangle_\theta = -\sum_{i=0}^{m-1}(-1)^{m-i}\sqrt{\frac{(m-1)!\Gamma(i+\frac{1}{2}+\sigma)s}{i!\Gamma(m+\frac{1}{2}+\sigma)}}\hat{c}_{2i,0}. \qquad \blacksquare$$

Lemma 4.10. *If $k=2m>0$ and $\ell=2n>0$, then*

$$\hat{c}_{k,\ell} = \sqrt{\frac{n}{m}}\hat{c}_{k-1,\ell-1} + \frac{\sigma-\theta}{m}\sum_{i=0}^{m-1}(-1)^{m-i}\sqrt{\frac{m!\Gamma(i+\frac{1}{2}+\sigma)}{i!\Gamma(m+\frac{1}{2}+\sigma)}}\hat{c}_{2i,\ell}.$$

Proof. Like in the proof of Lemma 4.9,

$$\hat{c}_{k,\ell} = \frac{1}{2\sqrt{ms}}\langle B'_\theta\phi_{\sigma,k-1},\phi_{\tau,\ell}\rangle_\theta + \frac{\theta-\sigma}{\sqrt{ms}}\langle x^{-1}\phi_{\sigma,k-1},\phi_{\tau,\ell}\rangle_\theta,$$

Now, by (2.4), (2.6) and (2.7),

$$\langle B'_\theta\phi_{\sigma,k-1},\phi_{\tau,\ell}\rangle_\theta = \langle\phi_{\sigma,k-1},B_{1/2}\phi_{\tau,\ell}\rangle_\theta = \langle\phi_{\sigma,k-1},B_\tau\phi_{\tau,\ell}\rangle_\theta = 2\sqrt{ns}\hat{c}_{k-1,\ell-1},$$

$$\langle x^{-1}\phi_{\sigma,k-1},\phi_{\tau,\ell}\rangle_\theta = -\sum_{i=0}^{m-1}(-1)^{m-i}\sqrt{\frac{(m-1)!\Gamma(i+\frac{1}{2}+\sigma)s}{i!\Gamma(m+\frac{1}{2}+\sigma)}}\hat{c}_{2i,\ell}. \qquad \blacksquare$$

Lemma 4.11. *If $k=2m+1$ and $\ell=2n+1$, then*

$$\hat{c}_{k,\ell} = \frac{m+\frac{1}{2}+\theta}{\sqrt{(m+\frac{1}{2}+\sigma)(n+\frac{1}{2}+\tau)}}\hat{c}_{k-1,\ell-1} - \frac{\sigma-\theta}{\sqrt{n+\frac{1}{2}+\tau}}\sum_{i=0}^{m-1}(-1)^{m-i}\sqrt{\frac{m!\Gamma(i+\frac{1}{2}+\sigma)}{i!\Gamma(m+\frac{3}{2}+\sigma)}}\hat{c}_{2i,\ell-1}.$$

Proof. By (2.2),

$$\hat{c}_{k,\ell} = \frac{1}{2\sqrt{(n+\frac{1}{2}+\tau)s}}\langle\phi_{\sigma,k},B'_\tau\phi_{\tau,\ell-1}\rangle_\theta,$$

where, by (2.8),

$$\begin{aligned}\langle\phi_{\sigma,k},B'_\tau\phi_{\tau,\ell-1}\rangle_\theta &= \langle\phi_{\sigma,k},B'_\theta\phi_{\tau,\ell-1}\rangle_\theta = \langle B_\theta\phi_{\sigma,k},\phi_{\tau,\ell-1}\rangle_\theta\\ &= \langle B_\sigma\phi_{\sigma,k},\phi_{\tau,\ell-1}\rangle_\theta + 2(\theta-\sigma)\langle x^{-1}\phi_{\sigma,k},\phi_{\tau,\ell-1}\rangle_\theta.\end{aligned}$$

Hence, by (2.4) and (2.6),

$$\begin{aligned}\hat{c}_{k,\ell} &= \sqrt{\frac{m+\frac{1}{2}+\sigma}{n+\frac{1}{2}+\tau}}\hat{c}_{k-1,\ell-1} - \frac{\sigma-\theta}{\sqrt{n+\frac{1}{2}+\tau}}\sum_{i=0}^{m}(-1)^{m-i}\sqrt{\frac{m!\Gamma(i+\frac{1}{2}+\sigma)}{i!\Gamma(m+\frac{3}{2}+\sigma)}}\hat{c}_{2i,\ell-1}\\ &= \frac{m+\frac{1}{2}+\theta}{\sqrt{(n+\frac{1}{2}+\tau)(m+\frac{1}{2}+\sigma)}}\hat{c}_{k-1,\ell-1}\\ &\quad - \frac{\sigma-\theta}{\sqrt{n+\frac{1}{2}+\tau}}\sum_{i=0}^{m-1}(-1)^{m-i}\sqrt{\frac{m!\Gamma(i+\frac{1}{2}+\sigma)}{i!\Gamma(m+\frac{3}{2}+\sigma)}}\hat{c}_{2i,\ell-1}.\end{aligned} \qquad \blacksquare$$

Proposition 4.12. *If $k = 2m$ and $\ell = 2n$, then*

$$\hat{c}_{k,\ell} = (-1)^{m+n} s^{v/2} \sqrt{\frac{m!n!\Gamma(\frac{1}{2}+\theta)}{\Gamma(m+\frac{1}{2}+\sigma)\Gamma(n+\frac{1}{2}+\tau)}} \times \sum_{p=0}^{\min\{m,n\}} \frac{\Gamma(m-p+\sigma-\theta)\Gamma(n-p+\tau-\theta)}{(m-p)!(n-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)},$$

and, if $k = 2m+1$ and $\ell = 2n+1$, then

$$\hat{c}_{k,\ell} = (-1)^{m+n} s^{v/2} \sqrt{\frac{m!n!\Gamma(\frac{1}{2}+\theta)}{\Gamma(m+\frac{3}{2}+\sigma)\Gamma(n+\frac{3}{2}+\tau)}} \times \sum_{p=0}^{\min\{m,n\}} \frac{(1+p)\Gamma(m-p+\sigma-\theta)\Gamma(n-p+\tau-\theta)}{(m-p)!(n-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)}.$$

Proof. The result is proved by induction on k and ℓ. First, consider the case $\ell = 0$. When $k = \ell = 0$, the result is given by (4.6). Now, take any $k = 2m > 0$, and assume that the result holds for all $\hat{c}_{k',0}$ with $k' = 2m' < k$. Then, by Lemma 4.9 and (4.5),

$$\begin{aligned}\hat{c}_{k,0} &= \frac{\sigma-\theta}{\sqrt{m}} \sum_{i=0}^{m-1} (-1)^{m-i} \sqrt{\frac{(m-1)!\Gamma(i+\frac{1}{2}+\sigma)}{i!\Gamma(m+\frac{1}{2}+\sigma)}} \\ &\quad \times (-1)^i s^{v/2} \sqrt{\frac{\Gamma(\frac{1}{2}+\theta)}{i!\Gamma(i+\frac{1}{2}+\sigma)\Gamma(\frac{1}{2}+\tau)}} \frac{\Gamma(i+\sigma-\theta)}{\Gamma(\sigma-\theta)} \\ &= (-1)^m s^{v/2} \sqrt{\frac{m!}{\Gamma(m+\frac{1}{2}+\sigma)\Gamma(\frac{1}{2}+\tau)}} \frac{\sigma-\theta}{m} \sum_{i=0}^{m-1} \frac{\Gamma(i+\sigma-\theta)}{i!\Gamma(\sigma-\theta)} \\ &= (-1)^m s^{v/2} \sqrt{\frac{\Gamma(\frac{1}{2}+\theta)}{m!\Gamma(m+\frac{1}{2}+\sigma)\Gamma(\frac{1}{2}+\tau)}} \frac{\Gamma(m+\sigma-\theta)}{\Gamma(\sigma-\theta)}.\end{aligned}$$

From the case $\ell = 0$, the result also follows for the case $k = 0$ by (4.1).

Now, take $k = 2m > 0$ and $\ell = 2n > 0$, and assume that the result holds for all $\hat{c}_{k',\ell'}$ with $k' < k$ and $\ell' \leq \ell$. By Lemma 4.10,

$$\begin{aligned}\hat{c}_{k,\ell} &= \sqrt{\frac{n}{m}} (-1)^{m+n-2} s^{v/2} \sqrt{\frac{(m-1)!(n-1)!\Gamma(\frac{1}{2}+\theta)}{\Gamma(m+\frac{1}{2}+\sigma)\Gamma(n+\frac{1}{2}+\tau)}} \\ &\quad \times \sum_{q=0}^{\min\{m-1,n-1\}} \frac{(1+q)\Gamma(m-1-q+\sigma-\theta)\Gamma(n-1-q+\tau-\theta)}{(m-1-q)!(n-1-q)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)} \\ &\quad + \frac{\sigma-\theta}{m} \sum_{i=0}^{m-1} (-1)^{m-i} \sqrt{\frac{m!\Gamma(i+\frac{1}{2}+\sigma)}{i!\Gamma(m+\frac{1}{2}+\sigma)}} (-1)^{i+n} s^{v/2} \sqrt{\frac{i!n!\Gamma(\frac{1}{2}+\theta)}{\Gamma(i+\frac{1}{2}+\sigma)\Gamma(n+\frac{1}{2}+\tau)}} \\ &\quad \times \sum_{p=0}^{\min\{i,n\}} \frac{\Gamma(i-p+\sigma-\theta)\Gamma(n-p+\tau-\theta)}{(i-p)!(n-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)} \\ &= (-1)^{m+n} s^{v/2} \frac{1}{m} \sqrt{\frac{m!n!\Gamma(\frac{1}{2}+\theta)}{\Gamma(m+\frac{1}{2}+\sigma)\Gamma(n+\frac{1}{2}+\tau)}} \\ &\quad \times \Bigg(\sum_{q=0}^{\min\{m-1,n-1\}} \frac{(1+q)\Gamma(m-1-q+\sigma-\theta)\Gamma(n-1-q+\tau-\theta)}{(m-1-q)!(n-1-q)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)}\end{aligned}$$

$$+ (\sigma-\theta)\sum_{i=0}^{m-1}\sum_{p=0}^{\min\{i,n\}}\frac{\Gamma(i-p+\sigma-\theta)\Gamma(n-p+\tau-\theta)}{(i-p)!(n-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)}\Bigg).$$

Then the desired expression for $\hat{c}_{k,\ell}$ follows because

$$\sum_{q=0}^{\min\{m-1,n-1\}}\frac{(1+q)\Gamma(m-1-q+\sigma-\theta)\Gamma(n-1-q+\tau-\theta)}{(m-1-q)!(n-1-q)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)}$$
$$=\sum_{p=0}^{\min\{m,n\}}\frac{p\Gamma(m-p+\sigma-\theta)\Gamma(n-p+\tau-\theta)}{(m-p)!(n-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)},$$

and, by (4.5),

$$(\sigma-\theta)\sum_{i=0}^{m-1}\sum_{p=0}^{\min\{i,n\}}\frac{\Gamma(i-p+\sigma-\theta)\Gamma(n-p+\tau-\theta)}{(i-p)!(n-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)}$$
$$=(\sigma-\theta)\sum_{p=0}^{\min\{m-1,n\}}\sum_{j=0}^{m-1-p}\frac{\Gamma(j+\sigma-\theta)\Gamma(n-p+\tau-\theta)}{j!(n-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)}$$
$$=\sum_{p=0}^{\min\{m,n\}}\frac{(m-p)\Gamma(m-p+\sigma-\theta)\Gamma(n-p+\tau-\theta)}{(m-p)!(n-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)}. \tag{4.7}$$

Finally, take $k=2m+1$ and $\ell=2n+1$, and assume that the result holds for all $\hat{c}_{k',\ell'}$ with $k'<k$ and $\ell'<\ell$. By Lemma 4.11,

$$\hat{c}_{k,\ell}=\frac{(n+1)(-1)^{m+n}s^{v/2}}{\sqrt{(m+\frac{1}{2}+\sigma)(n+\frac{1}{2}+\tau)}}\sqrt{\frac{m!n!\Gamma(\frac{1}{2}+\theta)}{\Gamma(m+\frac{1}{2}+\sigma)\Gamma(n+\frac{1}{2}+\tau)}}$$
$$\times\sum_{p=0}^{\min\{m,n\}}\frac{\Gamma(m-p+\sigma-\theta)\Gamma(n-p+\tau-\theta)}{(m-p)!(n-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)}$$
$$+\frac{\sigma-\theta}{\sqrt{n+\frac{1}{2}+\tau}}\sum_{i=0}^{m-1}(-1)^{m-i}\sqrt{\frac{m!\Gamma(i+\frac{1}{2}+\sigma)}{i!\Gamma(m+\frac{3}{2}+\sigma)}}$$
$$\times(-1)^{i+n}s^{v/2}\sqrt{\frac{i!n!\Gamma(\frac{1}{2}+\theta)}{\Gamma(i+\frac{1}{2}+\sigma)\Gamma(n+\frac{1}{2}+\tau)}}\sum_{p=0}^{\min\{i,n\}}\frac{\Gamma(i-p+\sigma-\theta)\Gamma(n-p+\tau-\theta)}{(i-p)!(n-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)}$$
$$=(-1)^{m+n}s^{v/2}\sqrt{\frac{m!n!\Gamma(\frac{1}{2}+\theta)}{\Gamma(m+\frac{3}{2}+\sigma)\Gamma(n+\frac{3}{2}+\tau)}}$$
$$\times\Bigg(\sum_{p=0}^{\min\{m,n\}}\frac{(m+1)\Gamma(m-p+\sigma-\theta)\Gamma(n-p+\tau-\theta)}{(m-p)!(n-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)}$$
$$-(\sigma-\theta)\sum_{i=0}^{m-1}\sum_{p=0}^{\min\{i,n\}}\frac{\Gamma(i-p+\sigma-\theta)\Gamma(n-p+\tau-\theta)}{(i-p)!(n-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)}\Bigg).$$

Then we get the stated expression for $\hat{c}_{k,\ell}$ using (4.7) again. ■

Remark 4.13. By (4.2), if k and ℓ are odd, then Proposition 4.12 also holds when $\sigma,\tau>-3/2$.

5 The sesquilinear form $\mathfrak{t}'$

Consider the notation of Section 4. Since $x^{-1}\mathcal{S}_{\text{odd}} = \mathcal{S}_{\text{ev}}$, a sesquilinear form $\mathfrak{t}'$ in $L^2_{\sigma,\tau}$, with $\mathsf{D}(\mathfrak{t}') = \mathcal{S}$, is defined by

$$\mathfrak{t}'(\phi,\psi) = \langle \phi_{\text{ev}}, x^{-1}\psi_{\text{odd}}\rangle_\theta = \langle x\phi_{\text{ev}}, \psi_{\text{odd}}\rangle_{\theta-1}.$$

Note that $\mathfrak{t}'$ is neither symmetric nor bounded from the left. The goal of this section is to study $\mathfrak{t}'$, and use it to prove Theorem 1.3.

Let $c'_{k,\ell} = \mathfrak{t}'(\phi_{\sigma,k},\phi_{\tau,\ell})$. Clearly, $c'_{k,\ell} = 0$ if k is odd or ℓ is even.

5.1 Case where $\sigma = \theta = \tau$

In this case, we have $v = 0$.

Proposition 5.1. *For $k = 2m$ and $\ell = 2n+1$, if $k > \ell$ ($m > n$), then $c'_{k,\ell} = 0$, and, if $k < \ell$ ($m \le n$), then*

$$c'_{k,\ell} = (-1)^{n-m}s^{1/2}\sqrt{\frac{n!\Gamma(m+\frac{1}{2}+\sigma)}{m!\Gamma(n+\frac{3}{2}+\sigma)}}.$$

Proof. This follows from (2.6) since $\hat{c}_{k,\ell} = \delta_{k,\ell}$ in this case. ∎

Proposition 5.2. *There is some $\omega = \omega(\sigma,\tau) > 0$ so that, for $k = 2m$ and $\ell = 2n+1$,*

$$|c'_{k,\ell}| \preccurlyeq s^{1/2}(m+1)^{-\omega}(n+1)^{-\omega}.$$

Proof. We can assume that $m \le n$ according to Proposition 5.1. Moreover

$$|c'_{k,\ell}| \preccurlyeq s^{1/2}(m+1)^{\sigma/2-1/4}(n+1)^{-\sigma/2-1/4}$$

for all $m \le n$ by Proposition 5.1 and Lemma 3.12. Therefore the result follows using Lemma 3.14, reversing the roles of m and n, because $-\sigma/2 - 1/4 < -u/2 < 0$. ∎

5.2 Case where $\sigma = \theta \ne \tau$ and $\tau - \sigma \notin -\mathbb{N}$

Recall that $v = \tau - \sigma$ in this case. Moreover $c'_{k,\ell} = 0$ if $k > \ell$ by (2.6) and Corollary 4.5.

Proposition 5.3. *For $k = 2m < \ell = 2n+1$ ($m \le n$),*

$$c'_{k,\ell} = (-1)^{m+n}s^{(1+v)/2}\sqrt{\frac{n!\Gamma(m+\frac{1}{2}+\sigma)}{m!\Gamma(n+\frac{3}{2}+\tau)}}\frac{\Gamma(n-m+1+v)}{(n-m)!\Gamma(1+v)}.$$

Proof. By (2.6), Corollary 4.5, Proposition 4.7 and (4.5),

$$\begin{aligned} c'_{k,\ell} &= s^{1/2}\sum_{j=m}^{n}(-1)^{n-j}\sqrt{\frac{n!\Gamma(j+\frac{1}{2}+\tau)}{j!\Gamma(n+\frac{3}{2}+\tau)}}(-1)^{m+j}s^{v/2}\sqrt{\frac{j!\Gamma(m+\frac{1}{2}+\sigma)}{m!\Gamma(j+\frac{1}{2}+\tau)}}\frac{\Gamma(j-m+v)}{(j-m)!\Gamma(v)} \\ &= (-1)^{m+n}s^{(1+v)/2}\sqrt{\frac{n!\Gamma(m+\frac{1}{2}+\sigma)}{m!\Gamma(n+\frac{3}{2}+\tau)}}\frac{1}{\Gamma(v)}\sum_{i=0}^{n-m}\frac{\Gamma(i+v)}{i!} \\ &= (-1)^{m+n}s^{(1+v)/2}\sqrt{\frac{n!\Gamma(m+\frac{1}{2}+\sigma)}{m!\Gamma(n+\frac{3}{2}+\tau)}}\frac{\Gamma(n-m+1+v)}{(n-m)!\Gamma(1+v)}. \end{aligned}$$

∎

Proposition 5.4. *If* $\sigma - 1 < \tau < \sigma + 1, 2\sigma + \frac{1}{2}$, *then there is some* $\omega = \omega(\sigma, \tau) > 0$ *so that, for* $k = 2m < \ell = 2n + 1$,

$$|c'_{k,\ell}| \preccurlyeq s^{(1+v)/2}(m+1)^{-\omega}(n+1)^{-\omega}.$$

Proof. By Proposition 5.3 and Lemma 3.12,

$$|c'_{k,\ell}| \preccurlyeq s^{(1+v)/2}(m+1)^{\sigma/2-1/4}(n+1)^{-\tau/2-1/4}(n-m+1)^{v}.$$

Then the result follows by Lemma 3.14, interchanging the roles of m and n, using the condition of Theorem 1.3(a). ■

5.3 Case where $\sigma \neq \theta = \tau$ and $\sigma - \theta \notin -\mathbb{N}$

Recall that $v = \sigma - \tau$ in this case.

Proposition 5.5. *For* $k = 2m$ *and* $\ell = 2n + 1$,

$$c'_{k,\ell} = (-1)^{m+n}s^{(1+v)/2}\sqrt{\frac{m!n!}{\Gamma(m+\frac{1}{2}+\sigma)\Gamma(n+\frac{3}{2}+\tau)}}\sum_{j=0}^{n}\frac{\Gamma(j+\frac{1}{2}+\tau)\Gamma(m-j+v)}{j!(m-j)!\Gamma(v)}.$$

Proof. By (2.6), Corollary 4.5, Proposition 4.7 and (4.1),

$$\begin{aligned}
c'_{k,\ell} &= s^{1/2}\sum_{j=0}^{n}(-1)^{n-j}\sqrt{\frac{n!\Gamma(j+\frac{1}{2}+\tau)}{j!\Gamma(n+\frac{3}{2}+\tau)}}(-1)^{j+m}s^{v/2}\sqrt{\frac{m!\Gamma(j+\frac{1}{2}+\tau)}{j!\Gamma(m+\frac{1}{2}+\sigma)}}\frac{\Gamma(m-j+v)}{(m-j)!\Gamma(v)}\\
&= (-1)^{m+n}s^{(1+v)/2}\sqrt{\frac{m!n!}{\Gamma(m+\frac{1}{2}+\sigma)\Gamma(n+\frac{3}{2}+\tau)}}\frac{\Gamma(j+\frac{1}{2}+\tau)\Gamma(m-j+v)}{j!(m-j)!\Gamma(v)}.
\end{aligned}$$ ■

Define the following subsets of $\mathbb{R}^4$:

- $\mathfrak{S}_1$ is the set of points $(\alpha, \beta, \gamma, \delta)$ such that:

$$\begin{aligned}
&\gamma \geq 0,\ \delta > -1 \Longrightarrow \alpha+\gamma, \alpha+\beta+\gamma+\delta+1 < 0,\\
&\gamma \geq 0,\ \delta \leq -1 \Longrightarrow \alpha+\gamma, \alpha+\beta+\gamma < 0,\\
&\gamma < 0,\ \delta > -1 \Longrightarrow \alpha+\gamma, \alpha+\beta+\delta+1, \alpha+\beta+\gamma+\delta+1 < 0,\\
&\gamma < 0,\ \delta \leq -1 \Longrightarrow \alpha+\beta, \alpha+\gamma, \alpha+\beta+\gamma < 0.
\end{aligned}$$

- $\mathfrak{S}_2$ be the set of points $(\alpha, \beta, \gamma, \delta)$ such that:

$$\begin{aligned}
&\gamma \geq 0,\ \delta > -\tfrac{1}{2} \Longrightarrow \alpha+\gamma, \alpha+\beta+\gamma+\delta+1 < 0,\\
&\gamma \geq 0,\ \delta \leq -\tfrac{1}{2} \Longrightarrow \alpha+\gamma, \alpha+\beta+\gamma+\tfrac{1}{2} < 0,\\
&-\tfrac{1}{2} < \gamma < 0,\ \delta > -\tfrac{1}{2} \Longrightarrow \begin{cases}\alpha+\gamma, \alpha+\beta+\delta+1, \alpha+\beta+\gamma+\delta+1 < 0, \text{ or}\\ \alpha+\gamma+\frac{1}{2}, \alpha+\beta+\gamma+\delta+1 < 0,\end{cases}\\
&-\tfrac{1}{2} < \gamma < 0,\ \delta \leq -\tfrac{1}{2} \Longrightarrow \begin{cases}\alpha+\gamma, \alpha+\beta+\frac{1}{2}, \alpha+\beta+\gamma+\frac{1}{2} < 0, \text{ or}\\ \alpha+\gamma+\frac{1}{2}, \alpha+\beta+\gamma+\frac{1}{2} < 0,\end{cases}\\
&\gamma = -\tfrac{1}{2},\ \delta > -\tfrac{1}{2} \Longrightarrow \alpha, \alpha+\beta+\delta+\tfrac{1}{2} < 0,\\
&\gamma = -\tfrac{1}{2},\ \delta \leq -\tfrac{1}{2} \Longrightarrow \alpha, \alpha+\beta < 0,\\
&\gamma < -\tfrac{1}{2},\ \delta > -\tfrac{1}{2} \Longrightarrow \alpha+\gamma+\tfrac{1}{2}, \alpha+\beta+\delta+\tfrac{1}{2}, \alpha+\beta+\gamma+\delta+1 < 0,\\
&\gamma < -\tfrac{1}{2},\ \delta \leq -\tfrac{1}{2} \Longrightarrow \alpha+\gamma+\tfrac{1}{2}, \alpha+\beta, \alpha+\beta+\gamma+\tfrac{1}{2} < 0.
\end{aligned}$$

In particular, $(\alpha,\beta,\gamma,\delta)\in\mathfrak{S}_1$ if

$$\alpha+\beta,\alpha+\gamma,\alpha+\beta+\gamma,\alpha+\beta+\delta+1,\alpha+\beta+\gamma+\delta+1<0.$$

Lemma 5.6. *If $(\alpha,\beta,\gamma,\delta)\in\mathfrak{S}_1\cup\mathfrak{S}_2$, then there is some $\omega>0$ such that, for all naturals $m\ge n$,*

$$(m+1)^\alpha(n+1)^\beta\sum_{p=0}^{n}(m-p+1)^\gamma(p+1)^\delta\preccurlyeq(m+1)^{-\omega}(n+1)^{-\omega}.$$

Proof. For all $\epsilon>0$,

$$\begin{aligned}\sum_{p=0}^{n}(p+1)^\delta=\sum_{q=1}^{n+1}q^\delta&\le\begin{cases}\int_1^{n+2}x^\delta dx & \text{if } \delta\ge0\\ 1+\int_1^{n+1}x^\delta dx & \text{if } \delta<0\end{cases}\\ &\preccurlyeq\begin{cases}(n+1)^{\delta+1} & \text{if } \delta>-1\\ 1+\ln(n+1) & \text{if } \delta=-1\\ 1 & \text{if } \delta<-1\end{cases}\preccurlyeq\begin{cases}(n+1)^{\delta+1} & \text{if } \delta>-1,\\ (n+1)^\epsilon & \text{if } \delta=-1,\\ 1 & \text{if } \delta<-1.\end{cases}\end{aligned}\tag{5.1}$$

Hence, using that

$$\sum_{p=0}^{n}(m-p+1)^\gamma(p+1)^\delta\preccurlyeq\begin{cases}(m+1)^\gamma\sum_{p=0}^{n}(p+1)^\delta & \text{if } \gamma\ge0,\\ (m-n+1)^\gamma\sum_{p=0}^{n}(p+1)^\delta & \text{if } \gamma<0,\end{cases}$$

we get

$$\begin{aligned}&(m+1)^\alpha(n+1)^\beta\sum_{p=0}^{n}(m-p+1)^\gamma(p+1)^\delta\\ &\quad\preccurlyeq\begin{cases}(m+1)^{\alpha+\gamma}(n+1)^{\beta+\delta+1} & \text{if } \gamma\ge0 \text{ and } \delta>-1,\\ (m+1)^{\alpha+\gamma}(n+1)^{\beta+\epsilon} & \text{if } \gamma\ge0 \text{ and } \delta=-1,\\ (m+1)^{\alpha+\gamma}(n+1)^{\beta} & \text{if } \gamma\ge0 \text{ and } \delta<-1,\\ (m+1)^{\alpha}(n+1)^{\beta+\delta+1}(m-n+1)^\gamma & \text{if } \gamma<0 \text{ and } \delta>-1,\\ (m+1)^{\alpha}(n+1)^{\beta+\epsilon}(m-n+1)^\gamma & \text{if } \gamma<0 \text{ and } \delta=-1,\\ (m+1)^{\alpha}(n+1)^{\beta}(m-n+1)^\gamma & \text{if } \gamma<0 \text{ and } \delta<-1,\end{cases}\end{aligned}$$

for all $\epsilon>0$. Then the result follows when $(\alpha,\beta,\gamma,\delta)\in\mathfrak{S}_1$ by Lemma 3.14.

On the other hand, for all $\epsilon>0$,

$$\begin{aligned}\sum_{p=0}^{n}(m-p+1)^\gamma=\sum_{q=m-n+1}^{m+1}q^\gamma&\le\begin{cases}\int_{m-n+1}^{m+2}x^\gamma dx & \text{if } \gamma\ge0\\ (m-n+1)^\gamma+\int_{m-n+1}^{m+1}x^\gamma dx & \text{if } \gamma<0\end{cases}\\ &\preccurlyeq\begin{cases}(m+1)^{\gamma+1} & \text{if } \gamma>-1\\ 1+\ln(m+1) & \text{if } \gamma=-1\\ (m-n+1)^\gamma & \text{if } \gamma<-1\end{cases}\preccurlyeq\begin{cases}(m+1)^{\gamma+1} & \text{if } \gamma>-1,\\ (m+1)^\epsilon & \text{if } \gamma=-1,\\ (m-n+1)^\gamma & \text{if } \gamma<-1.\end{cases}\end{aligned}\tag{5.2}$$

The following gives a better estimate when $\gamma \geq 0$, and an alternative estimate when $-1 < \gamma < 0$:

$$\sum_{p=0}^{n}(m-p+1)^{\gamma} \leq \begin{cases}(m+1)^{\gamma}(n+1) & \text{if } \gamma \geq 0,\\ (m-n+1)^{\gamma}(n+1) & \text{if } -1<\gamma<0.\end{cases} \tag{5.3}$$

Now, using the Cauchy–Schwartz inequality

$$\sum_{p=0}^{n}(m-p+1)^{\gamma}(p+1)^{\delta} \leq \left(\sum_{p=0}^{n}(m-p+1)^{2\gamma}\right)^{\frac{1}{2}}\left(\sum_{p=0}^{n}(p+1)^{2\delta}\right)^{\frac{1}{2}},$$

and applying (5.1) with 2δ, and (5.2) and (5.3) with 2γ, we obtain

$$\begin{aligned}&(m+1)^{\alpha}(n+1)^{\beta}\sum_{p=0}^{n}(m-p+1)^{\gamma}(p+1)^{\delta}\\ &\preccurlyeq \begin{cases}(m+1)^{\alpha+\gamma}(n+1)^{\beta+\delta+1} & \text{if } \gamma\geq 0,\ \delta>-\frac{1}{2},\\ (m+1)^{\alpha+\gamma}(n+1)^{\beta+\frac{1}{2}+\epsilon} & \text{if } \gamma\geq 0,\ \delta=-\frac{1}{2},\\ (m+1)^{\alpha+\gamma}(n+1)^{\beta+\frac{1}{2}} & \text{if } \gamma\geq 0,\ \delta<-\frac{1}{2},\\ \left.\begin{array}{l}(m+1)^{\alpha}(n+1)^{\beta+\delta+1}(m-n+1)^{\gamma}\\ (m+1)^{\alpha+\gamma+\frac{1}{2}}(n+1)^{\beta+\delta+\frac{1}{2}}\end{array}\right\} & \text{if } -\frac{1}{2}<\gamma<0,\ \delta>-\frac{1}{2},\\ \left.\begin{array}{l}(m+1)^{\alpha}(n+1)^{\beta+\frac{1}{2}+\epsilon}(m-n+1)^{\gamma}\\ (m+1)^{\alpha+\gamma+\frac{1}{2}}(n+1)^{\beta+\epsilon}\end{array}\right\} & \text{if } -\frac{1}{2}<\gamma<0,\ \delta=-\frac{1}{2},\\ \left.\begin{array}{l}(m+1)^{\alpha}(n+1)^{\beta+\frac{1}{2}}(m-n+1)^{\gamma}\\ (m+1)^{\alpha+\gamma+\frac{1}{2}}(n+1)^{\beta}\end{array}\right\} & \text{if } -\frac{1}{2}<\gamma<0,\ \delta<-\frac{1}{2},\\ (m+1)^{\alpha+\epsilon}(n+1)^{\beta+\delta+\frac{1}{2}} & \text{if } \gamma=-\frac{1}{2},\ \delta>-\frac{1}{2},\\ (m+1)^{\alpha+\epsilon}(n+1)^{\beta+\epsilon} & \text{if } \gamma=\delta=-\frac{1}{2},\\ (m+1)^{\alpha+\epsilon}(n+1)^{\beta} & \text{if } \gamma=-\frac{1}{2},\ \delta<-\frac{1}{2},\\ (m+1)^{\alpha}(n+1)^{\beta+\delta+\frac{1}{2}}(m-n+1)^{\gamma+\frac{1}{2}} & \text{if } \gamma<-\frac{1}{2},\ \delta>-\frac{1}{2},\\ (m+1)^{\alpha}(n+1)^{\beta+\epsilon}(m-n+1)^{\gamma+\frac{1}{2}} & \text{if } \gamma<-\frac{1}{2},\ \delta=-\frac{1}{2},\\ (m+1)^{\alpha}(n+1)^{\beta}(m-n+1)^{\gamma+\frac{1}{2}} & \text{if } \gamma<-\frac{1}{2},\ \delta\leq-\frac{1}{2},\end{cases}\end{aligned}$$

for all $\epsilon > 0$. So the result also holds when $(\alpha,\beta,\gamma,\delta)\in\mathfrak{S}_2$ by Lemma 3.14. ■

Remark 5.7. Lemma 5.6 could be slightly improved by using also that

$$\sum_{p=0}^{n}(m-p+1)^{\gamma}(n+1)^{\delta} \preccurlyeq \begin{cases}(n+1)^{\delta}\sum_{p=0}^{n}(m-p+1)^{\gamma} & \text{if } \delta\geq 0,\\ \sum_{p=0}^{n}(m-p+1)^{\gamma} & \text{if } \delta<0,\end{cases}$$

and estimating $\sum_{p=0}^{n}(m-p+1)^{\gamma} = \sum_{q=m-n+1}^{m+1} q^{\gamma}$ like in the proof. But this would have no consequences in our application (Section 7).

Proposition 5.8. *If $(\sigma,\tau)\in\mathfrak{J}_1\cup\mathfrak{J}_2$, then there is some $\omega=\omega(\sigma,\tau,\theta)>0$ so that, for $k=2m$ and $\ell=2n+1$,*

$$|c'_{k,\ell}| \preccurlyeq s^{(1+v)/2}(m+1)^{-\omega}(n+1)^{-\omega}.$$

Proof. By Proposition 5.5 and Lemma 3.12,

$$|c'_{k,\ell}| \preccurlyeq s^{(1+v)/2}(m+1)^{1/4-\sigma/2}(n+1)^{-1/4-\tau/2}\sum_{j=0}^{n}(m-j+1)^{\tau-1/2}(j+1)^{v-1}.$$

Then the result follows by Lemma 5.6 since $(\sigma,\tau)\in\mathfrak{J}_1\cup\mathfrak{J}_2$ means that $(\alpha,\beta,\gamma,\delta)\in\mathfrak{S}_1\cup\mathfrak{S}_2$ for $\alpha=1/4-\sigma/2$, $\beta=-1/4-\tau/2$, $\gamma=\tau-1/2$ and $\delta=v-1$. ∎

5.4 Case where $\sigma\neq\theta=\tau+1$ and $\sigma-\tau-1\notin-\mathbb{N}$

Note that $v=\sigma-\tau-2$ in this case. Moreover

$$c'_{k,\ell}=\langle\phi_{\sigma,k},x^{-1}\phi_{\tau,\ell}\rangle_{\tau+1}=\langle x\phi_{\sigma,k},\phi_{\tau,\ell}\rangle_{\tau}=\langle\phi_{\tau,\ell},x\phi_{\sigma,k}\rangle_{\tau} \tag{5.4}$$

for $k=2m$ and $\ell=2n+1$ (Remark 1.4(iii)).

Proposition 5.9. *Let $k=2m$ and $\ell=2n+1$. If $k+1<\ell$ $(m<n)$, then $c'_{k,\ell}=0$. If $k+1\geq\ell$ $(m\geq n)$, then*

$$c'_{k,\ell}=(-1)^{m+n}s^{(v+2)/2}\sqrt{\frac{m!\Gamma(n+\frac{3}{2}+\tau)}{n!\Gamma(m+\frac{1}{2}+\sigma)}}\frac{\Gamma(m-n+v+1)}{(m-n)!\Gamma(v+1)}.$$

Proof. By (2.5) and (5.4),

$$c'_{k,\ell}=\sqrt{\frac{m+\frac{1}{2}+\sigma}{s}}\hat{c}_{\tau,\sigma,\tau,k+1,\ell}+\sqrt{\frac{m}{s}}\hat{c}_{\tau,\sigma,\tau,k-1,\ell}. \tag{5.5}$$

So $c'_{k,\ell}=0$ if $k+1<\ell$ by Corollary 4.5. When $k+1=\ell$ $(m=n)$, by (5.5) and Proposition 4.7,

$$c'_{k,\ell}=\sqrt{\frac{m+\frac{1}{2}+\sigma}{s}}s^{(v+2)/2}\sqrt{\frac{\Gamma(n+\frac{3}{2}+\tau)}{\Gamma(m+\frac{3}{2}+\sigma)}}=s^{(v+1)/2}\sqrt{\frac{\Gamma(n+\frac{3}{2}+\tau)}{\Gamma(m+\frac{1}{2}+\sigma)}}.$$

When $k-1\geq\ell$ $(m>n)$, by (5.5) and Proposition 4.7,

$$\begin{aligned}
c'_{k,\ell}&=\sqrt{\frac{m+\frac{1}{2}+\sigma}{s}}(-1)^{m+n}s^{(v+2)/2}\sqrt{\frac{m!\Gamma(n+\frac{3}{2}+\tau)}{n!\Gamma(m+\frac{3}{2}+\sigma)}}\frac{\Gamma(m-n+v+2)}{(m-n)!\Gamma(v+2)}\\
&\quad+\sqrt{\frac{m}{s}}(-1)^{m+n-1}s^{(v+2)/2}\sqrt{\frac{(m-1)!\Gamma(n+\frac{3}{2}+\tau)}{n!\Gamma(m+\frac{1}{2}+\sigma)}}\frac{\Gamma(m-n+v+1)}{(m-1-n)!\Gamma(v+2)}\\
&=(-1)^{m+n}s^{(v+1)/2}\sqrt{\frac{m!\Gamma(n+\frac{3}{2}+\tau)}{n!\Gamma(m+\frac{1}{2}+\sigma)}}\frac{\Gamma(m-n+v+1)}{(m-1-n)!\Gamma(v+2)}\left(\frac{m-n+v+1}{m-n}-1\right)\\
&=(-1)^{m+n}s^{(v+1)/2}\sqrt{\frac{m!\Gamma(n+\frac{3}{2}+\tau)}{n!\Gamma(m+\frac{1}{2}+\sigma)}}\frac{\Gamma(m-n+v+1)}{(m-n)!\Gamma(v+1)}.
\end{aligned}$$

∎

Proposition 5.10. *If $\tau<\frac{3\sigma}{2}-\frac{9}{4},\sigma-\frac{5}{3}$, then there is some $\omega=\omega(\tau,\sigma)>0$ so that, for $k=2m$ and $\ell=2n+1$,*

$$|c'_{k,\ell}|\preccurlyeq s^{(v+1)/2}(m+1)^{-\omega}(n+1)^{-\omega}.$$

Proof. By Proposition 5.9, we can assume that $k+1\geq\ell$ $(m\geq n)$, and, in this case, using also Lemma 3.12, we get

$$|c'_{k,\ell}|\preccurlyeq s^{(v+1)/2}(m+1)^{1/4-\sigma/2}(n+1)^{1/4+\tau/2}(m-n+1)^{-v}.$$

Then the result follows using Lemma 3.14. ∎

5.5 Case where $\sigma \neq \theta \neq \tau$ and $\sigma - \theta, \tau - \theta \notin -\mathbb{N}$

Proposition 5.11. *For $k = 2m$ and $\ell = 2n + 1$,*

$$c'_{k,\ell} = (-1)^{m+n} s^{(1+v)/2} \sqrt{\frac{m!n!\Gamma(\frac{1}{2}+\theta)}{\Gamma(m+\frac{1}{2}+\sigma)\Gamma(n+\frac{3}{2}+\tau)}} \times \sum_{p=0}^{\min\{m,n\}} \frac{\Gamma(m-p+\sigma-\theta)\Gamma(n-p+1+\tau-\theta)}{(m-p)!(n-p)!\Gamma(\sigma-\theta)\Gamma(1+\tau-\theta)}.$$

Proof. By (2.6) and Proposition 4.12,

$$\begin{aligned} c'_{k,\ell} &= s^{1/2} \sum_{j=m}^{n} (-1)^{n-j} \sqrt{\frac{n!\Gamma(j+\frac{1}{2}+\tau)}{j!\Gamma(n+\frac{3}{2}+\tau)}} (-1)^{m+j} s^{v/2} \sqrt{\frac{m!j!\Gamma(\frac{1}{2}+\theta)}{\Gamma(m+\frac{1}{2}+\sigma)\Gamma(j+\frac{1}{2}+\tau)}} \\ &\quad \times \sum_{p=0}^{\min\{m,j\}} \frac{\Gamma(m-p+\sigma-\theta)\Gamma(j-p+\tau-\theta)}{(m-p)!(j-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)} \\ &= (-1)^{m+n} s^{(1+v)/2} \sqrt{\frac{m!n!\Gamma(\frac{1}{2}+\theta)}{\Gamma(m+\frac{1}{2}+\sigma)\Gamma(n+\frac{3}{2}+\tau)}} \\ &\quad \times \sum_{j=m}^{n} \sum_{p=0}^{\min\{m,j\}} \frac{\Gamma(m-p+\sigma-\theta)\Gamma(j-p+\tau-\theta)}{(m-p)!(j-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)}. \end{aligned}$$

But, by (4.5),

$$\begin{aligned} &\sum_{j=m}^{n} \sum_{p=0}^{\min\{m,j\}} \frac{\Gamma(m-p+\sigma-\theta)\Gamma(j-p+\tau-\theta)}{(m-p)!(j-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)} \\ &\quad = \sum_{p=0}^{\min\{m,n\}} \sum_{j=p}^{n} \frac{\Gamma(m-p+\sigma-\theta)\Gamma(j-p+\tau-\theta)}{(m-p)!(j-p)!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)} \\ &\quad = \sum_{p=0}^{\min\{m,n\}} \sum_{i=0}^{n-p} \frac{\Gamma(m-p+\sigma-\theta)\Gamma(i+\tau-\theta)}{(m-p)!i!\Gamma(\sigma-\theta)\Gamma(\tau-\theta)} \\ &\quad = \sum_{p=0}^{\min\{m,n\}} \frac{\Gamma(m-p+\sigma-\theta)\Gamma(n-p+1+\tau-\theta)}{(m-p)!(n-p)!\Gamma(\sigma-\theta)\Gamma(1+\tau-\theta)}. \end{aligned}$$

■

Lemma 5.12. *If $(\alpha,\beta,\gamma,\delta), (\beta,\alpha,\delta,\gamma) \in \mathfrak{S}_1 \cup \mathfrak{S}_2$, then there is some $\omega > 0$ such that, for all $m, n \in \mathbb{N}$,*

$$(m+1)^{\alpha}(n+1)^{\beta} \sum_{p=0}^{\min\{m,n\}} (m-p+1)^{\gamma}(n-p+1)^{\delta} \preccurlyeq (m+1)^{-\omega}(n+1)^{-\omega}.$$

Proof. Since (m,α,γ) and (n,β,δ) play symmetric roles, we consider only the case where $m \geq n$. Then the result follows like Lemma 5.6 because $\sum_{p=0}^{n}(n-p+1)^{\delta} = \sum_{q=1}^{n+1} q^{\delta}$. ■

Remark 5.13. In particular, the conditions of Lemma 5.12 are satisfied if

$$\begin{aligned} &\alpha+\beta, \alpha+\gamma, \beta+\delta < 0, \\ &\alpha+\beta+\gamma+1, \alpha+\beta+\delta+1, \alpha+\beta+\gamma+\delta+1 < 0. \end{aligned}$$

Proposition 5.14. *If* $(\sigma,\tau,\theta)\in(\mathfrak{K}_1\cup\mathfrak{K}_2)\cap(\mathfrak{K}'_1\cup\mathfrak{K}'_2)$*, then there is some* $\omega=\omega(\sigma,\tau,\theta)>0$ *so that, for* $k=2m$ *and* $\ell=2n+1$*,*

$$|c'_{k,\ell}|\preccurlyeq s^{(1+v)/2}(m+1)^{-\omega}(n+1)^{-\omega}.$$

Proof. By Proposition 5.11 and Lemma 3.12,

$$|c'_{k,\ell}|\preccurlyeq s^{(1+v)/2}(m+1)^{1/4-\sigma/2}(n+1)^{-1/4-\tau/2} \times\sum_{p=0}^{\min\{m,n\}}(m-p+1)^{\sigma-\theta-1}(n-p+1)^{\tau-\theta}.$$

Then the result follows by Lemma 5.12, since $(\sigma,\tau,\theta)\in(\mathfrak{K}_1\cup\mathfrak{K}_2)\cap(\mathfrak{K}'_1\cup\mathfrak{K}'_2)$ means that $(\alpha,\beta,\gamma,\delta),(\beta,\alpha,\delta,\gamma)\in\mathfrak{S}_1\cup\mathfrak{S}_2$ for $\alpha=1/4-\sigma/2$, $\beta=-1/4-\tau/2$, $\gamma=\sigma-\theta-1$ and $\delta=\tau-\theta$. ■

5.6 Proof of Theorem 1.3

Assume the conditions of Theorem 1.3. Let $\mathfrak{j}_{\sigma,\tau}$ be the positive definite symmetric sesquilinear form in $L^2_{\sigma,\tau}$, with domain $\mathcal{S}$, defined by $\mathfrak{j}_{\sigma,\tau}(\phi,\psi)=\langle J_{\sigma,\tau}\phi,\psi\rangle_{\sigma,\tau}$.

Proposition 5.15. *For any* $\epsilon>0$*, there is some* $E=E(\epsilon,\sigma,\tau,\theta)>0$ *such that, for all* $\phi\in\mathcal{S}$*,*

$$|\mathfrak{t}'(\phi)|\le\epsilon s^{(v-1)/2}\mathfrak{j}_{\sigma,\tau}(\phi)+Es^{(1+v)/2}\|\phi\|^2_{\sigma,\tau}.$$

Proof. This follows from Propositions 5.2, 5.4, 5.8, 5.10 and 5.14 using the arguments of the proof of Proposition 3.17. ■

Proof of Theorem 1.3. This is analogous to the proof of Theorem 1.1. Thus some details and the bibliographic references are omitted.

Let $\mathfrak{t}_{\sigma,\tau}$ be the positive definite sesquilinear form in $L^2_{\sigma,\tau}$, with $\mathsf{D}(\mathfrak{t}_{\sigma,\tau})=\mathcal{S}$, defined by $\mathfrak{t}_\sigma$ on $\mathcal{S}_{\rm ev}$ and $\mathfrak{t}_\tau$ on $\mathcal{S}_{\rm odd}$, and vanishing on $\mathcal{S}_{\rm ev}\times\mathcal{S}_{\rm odd}$. The adjoint of $|x|^{2(\theta-\sigma)}x^{-1}\colon\mathcal{S}_{\rm odd}\to\mathcal{S}_{\rm ev}$, as a densely defined operator of $L^2_{\tau,\rm odd}$ to $L^2_{\sigma,\rm ev}$, is given by $|x|^{2(\theta-\tau)}x^{-1}$, with the appropriate domain. Then the symmetric sesquilinear form $\mathfrak{v}=\mathfrak{j}_{\sigma,\tau}+\xi\mathfrak{t}_{\sigma,\tau}+2\eta\,\Re\mathfrak{t}'$ in $L^2_{\sigma,\tau}$, with $\mathsf{D}(\mathfrak{v})=\mathcal{S}$, is given by the right hand side of (1.3). Using Propositions 3.17 and 5.15, for any $\epsilon>0$, there are some $C=C(\epsilon,\sigma,\tau,u)>0$ and $E=E(\epsilon,\sigma,\tau,\theta)>0$ such that, for all $\phi\in\mathcal{S}$,

$$\begin{aligned}&|(\xi\mathfrak{t}_{\sigma,\tau}+2\eta\,\Re\mathfrak{t}')(\phi)|\\&\quad\le\epsilon\left(\xi s^{u-1}+2|\eta|s^{(1+v)/2}\right)\mathfrak{j}_{\sigma,\tau}(\phi)+\left(\xi Cs^u+2|\eta|Es^{(1+v)/2}\right)\|\phi\|^2_{\sigma,\tau}.\end{aligned}\tag{5.6}$$

Then, taking ϵ so that $\epsilon(\xi s^{u-1}+2|\eta|s^{(v+1)/2})<1$, since $\mathfrak{j}_{\sigma,\tau}$ is closable and positive definite, it follows that $\mathfrak{v}$ is sectorial and closable, and $\mathsf{D}(\bar{\mathfrak{v}})=\mathsf{D}(\overline{\mathfrak{j}_{\sigma,\tau}})$; in particular, $\mathfrak{v}$ is bounded from below because it is also symmetric. Therefore $\bar{\mathfrak{v}}$ is induced by a self-adjoint operator $\mathcal{V}$ in $L^2_{\sigma,\tau}$ with $\mathsf{D}(\mathcal{V}^{1/2})=\mathsf{D}(\bar{\mathfrak{v}})$. Thus $\mathcal{S}$ is a core of $\bar{\mathfrak{v}}$ and $\mathcal{V}^{1/2}$. By Proposition 3.18 and since $\mathfrak{t}'(\phi)=0$ for all $\phi\in\mathcal{S}_{\rm ev/odd}$, there is some $D=D(\sigma,\tau,u)>0$ such that

$$\begin{aligned}\mathfrak{v}(\phi_{\sigma,k})&\ge(2k+1+2\sigma)s+\xi Ds^u(k+1)^{-u}&&\text{if } k \text{ is even},\\ \mathfrak{v}(\phi_{\tau,k})&\ge(2k+1+2\tau)s+\xi Ds^u(k+1)^{-u}&&\text{if } k \text{ is odd}.\end{aligned}$$

Therefore $\mathcal{V}$ has a discrete spectrum satisfying the first inequality of (1.4); in particular $\mathcal{V}$ and $\bar{\mathfrak{v}}$ are positive definite. The second inequality of (1.4) holds because

$$\bar{\mathfrak{v}}(\phi)\le\big(1+\epsilon\big(\xi s^{u-1}+2|\eta|s^{(1+v)/2}\big)\big)\overline{\mathfrak{j}_{\sigma,\tau}}(\phi)+\xi Cs^u+2|\eta|Es^{(1+v)/2}\|\phi\|^2_{\sigma,\tau}$$

for all $\phi\in\mathsf{D}(\bar{\mathfrak{v}})$ by (5.6) and since $\mathcal{S}$ is a core of $\bar{\mathfrak{v}}$ and $\overline{\mathfrak{j}_{\sigma,\tau}}$. ■

6 Operators induced on $\mathbb{R}_+$

Let $\mathcal{S}_{\mathrm{ev/odd},+} = \{\phi|_{\mathbb{R}_+} \mid \phi \in \mathcal{S}_{\mathrm{ev/odd}}\}$. For $c, d > -1/2$, let $L^2_{c,+} = L^2(\mathbb{R}_+, x^{2c}dx)$ and $L^2_{c,d,+} = L^2_{c,+} \oplus L^2_{d,+}$, whose scalar products are denoted by $\langle\ ,\ \rangle_c$ and $\langle\ ,\ \rangle_{c,d}$, respectively. For $c_1, c_2, d_1, d_2 \in \mathbb{R}$, let

$$P_0 = H - 2c_1 x^{-1}\frac{d}{dx} + c_2 x^{-2}, \qquad Q_0 = H - 2d_1 \frac{d}{dx} x^{-1} + d_2 x^{-2}.$$

Morever let $\xi > 0$ and $\eta, \theta \in \mathbb{R}$.

Corollary 6.1. *If $a^2 + (2c_1 - 1)a - c_2 = 0$, $0 < u < 1$ and $\sigma := a + c_1 > u - 1/2$, then there is a positive self-adjoint operator $\mathcal{P}$ in $L^2_{c_1,+}$ satisfying the following:*

(i) $x^a\mathcal{S}_{\mathrm{ev},+}$ is a core of $\mathcal{P}^{1/2}$ and, for all $\phi, \psi \in x^a\mathcal{S}_{\mathrm{ev},+}$,

$$\langle \mathcal{P}^{1/2}\phi, \mathcal{P}^{1/2}\psi\rangle_{c_1} = \langle P_0\phi, \psi\rangle_{c_1} + \xi\langle x^{-u}\phi, x^{-u}\psi\rangle_{c_1}.$$

(ii) $\mathcal{P}$ has a discrete spectrum. Let $\lambda_0 \le \lambda_2 \le \cdots$ be its eigenvalues, repeated according to their multiplicity. There is some $D = D(\sigma, u) > 0$, and, for each $\epsilon > 0$, there is some $C = C(\epsilon, \sigma, u) > 0$ so that (1.2) holds for all $k \in 2\mathbb{N}$.

Corollary 6.2. *If $b^2 + (2d_1 + 1)b - d_2 = 0$, $0 < u < 1$ and $\tau := b + d_1 > u - 3/2$, then there is a positive self-adjoint operator $\mathcal{Q}$ in $L^2_{d_1,+}$ satisfying the following:*

(i) $x^b\mathcal{S}_{\mathrm{odd},+}$ is a core of $\mathcal{Q}^{1/2}$ and, for all $\phi, \psi \in x^b\mathcal{S}_{\mathrm{odd},+}$,

$$\langle \mathcal{Q}^{1/2}\phi, \mathcal{Q}^{1/2}\psi\rangle_{d_1} = \langle Q_0\phi, \psi\rangle_{d_1} + d_3\langle x^{-u}\phi, x^{-u}\psi\rangle_{d_1}.$$

(ii) $\mathcal{Q}$ has a discrete spectrum. Let $\lambda_1 \le \lambda_3 \le \cdots$ be its eigenvalues, repeated according to their multiplicity. There is some $D = D(\tau, u) > 0$, and, for each $\epsilon > 0$, there is some $C = C(\epsilon, \tau, u) > 0$ so that (1.2) holds for all $k \in 2\mathbb{N} + 1$, with τ instead of σ.

Corollary 6.3. *Under the conditions of Corollaries 6.1 and 6.2, if moreover the conditions of Theorem 1.3 are satisfied with some $\theta > -1/2$, then there is a positive self-adjoint operator $\mathcal{W}$ in $L^2_{c_1,d_1,+}$ satisfying the following:*

(i) $x^a\mathcal{S}_{\mathrm{ev},+} \oplus x^b\mathcal{S}_{\mathrm{odd},+}$ is a core of $\mathcal{W}^{1/2}$, and, for $\phi = (\phi_1, \phi_2)$ and $\psi = (\psi_1, \psi_2)$ in $x^a\mathcal{S}_{\mathrm{ev},+} \oplus x^b\mathcal{S}_{\mathrm{odd},+}$,

$$\begin{aligned}\langle \mathcal{W}^{1/2}\phi, \mathcal{W}^{1/2}\psi\rangle_{c_1,d_1} &= \langle (P_0 \oplus Q_0)\phi, \psi\rangle_{c_1,d_1} + \xi\langle x^{-u}\phi, x^{-u}\psi\rangle_{c_1,d_1} \\ &\quad + \eta\left(\langle x^{-a-b-1}\phi_2, \psi_1\rangle_\theta + \langle \phi_1, x^{-a-b-1}\psi_2\rangle_\theta\right). \end{aligned} \tag{6.1}$$

(ii) $\mathcal{W}$ has a discrete spectrum. Its eigenvalues form two groups, $\lambda_0 \le \lambda_2 \le \cdots$ and $\lambda_1 \le \lambda_3 \le \cdots$, repeated according to their multiplicity, such that there is some $D = D(\sigma, \tau, u) > 0$, and, for each $\epsilon > 0$, there are some $C = C(\epsilon, \sigma, \tau, u) > 0$ and $E = E(\epsilon, \sigma, \tau) > 0$ so that (1.4) holds for all $k \in \mathbb{N}$.

These corollaries follow directly from Theorems 1.1 and 1.3 because the given conditions on a and b characterize the cases where P_0 and Q_0 correspond to $|x|^a U_{\sigma,\mathrm{ev}}|x|^{-a}$ and $|x|^b U_{\tau,\mathrm{odd}}|x|^{-b}$, respectively, via the isomorphisms $|x|^a\mathcal{S}_{\mathrm{ev}} \to x^a\mathcal{S}_{\mathrm{ev},+}$ and $|x|^b\mathcal{S}_{\mathrm{odd}} \to x^b\mathcal{S}_{\mathrm{odd},+}$ defined by restriction [3, Theorem 1.4 and Section 5]. In fact, Corollaries 6.1 and 6.2 are equivalent because, if $c_1 = d_1 + 1$ and $c_2 = d_2$, then $Q_0 = xP_0x^{-1}$ and $x\colon L^2_{c_1,+} \to L^2_{d_1,+}$ is a unitary operator.

Remarks 1.2(ii) and 3.20 have obvious versions for these corollaries. In particular, $\mathcal{P} = \overline{P}$, $\mathcal{Q} = \overline{Q}$ and $\mathcal{W} = \overline{W}$, where $P = P_0 + \xi x^{-2u}$, $Q = Q_0 + \xi x^{-2u}$ and

$$W = \begin{pmatrix} P & \eta x^{2(\theta-\sigma)+a-b-1} \\ \eta x^{2(\theta-\tau)+b-a-1} & Q \end{pmatrix} = \begin{pmatrix} P & \eta x^{2(\theta-c_1)-a-b-1} \\ \eta x^{2(\theta-d_1)-a-b-1} & Q \end{pmatrix},$$

with $\mathsf{D}(P) = \bigcap_{m=0}^{\infty} \mathsf{D}(\mathcal{P}^m)$, $\mathsf{D}(Q) = \bigcap_{m=0}^{\infty} \mathsf{D}(\mathcal{Q}^m)$ and $\mathsf{D}(W) = \bigcap_{m=0}^{\infty} \mathsf{D}(\mathcal{W}^m)$. According to Remark 1.4(iii), we can write (6.1) as

$$\begin{aligned}\langle \mathcal{W}^{1/2}\phi, \mathcal{W}^{1/2}\psi\rangle_{c_1,d_1} &= \langle (P_0 \oplus Q_0)\phi, \psi\rangle_{c_1,d_1} + \xi\langle x^{-u}\phi, x^{-u}\psi\rangle_{c_1,d_1} \\ &\quad + \eta\big(\langle x^{-a-b+1}\phi_2, \psi_1\rangle_{\theta'} + \langle \phi_1, x^{-a-b+1}\psi_2\rangle_{\theta'}\big),\end{aligned}$$

and we have

$$W = \begin{pmatrix} P & \eta x^{2(\theta'-c_1)-a-b+1} \\ \eta x^{2(\theta'-d_1)-a-b+1} & Q \end{pmatrix}.$$

7 Application to the Witten's perturbation on strata

Let M be a Riemannian n-manifold. Let d, δ and Δ denote the de Rham derivative and coderivative, and the Laplacian, with domain the graded space $\Omega_0(M)$ of compactly supported differential forms, and let $L^2\Omega(M)$ be the graded Hilbert space of square integrable differential forms. Any closed extension $\mathbf{d}$ of d in $L^2\Omega(M)$, defining a complex ($\mathbf{d}^2 = 0$), is called an *ideal boundary condition* (*i.b.c.*) of d, which defines a self-adjoint extension $\boldsymbol{\Delta} = \mathbf{d}^*\mathbf{d} + \mathbf{d}\mathbf{d}^*$ of Δ, called the *Laplacian* of $\mathbf{d}$. There always exists a minimum/maximum i.b.c., $d_{\min} = \overline{d}$ and $d_{\max} = \delta^*$, whose Laplacians are denoted by $\Delta_{\min/\max}$. We get corresponding cohomologies $H_{\min/\max}(M)$, and versions of Betti numbers and Euler characteristic, $\beta^i_{\min/\max}$ and $\chi_{\min/\max}$. These are quasi-isometric invariants; in particular, $H_{\max}(M)$ is the usual L^2 cohomology. If M is complete, then there is a unique i.b.c., but these concepts become interesting in the non-complete case. For instance, if M is the interior of a compact manifold with non-empty boundary, then $d_{\min/\max}$ is defined by taking relative/absolute boundary conditions. Given $s > 0$ and $f \in C^\infty(M)$, the above ideas can be considered as well for the Witten's perturbations $d_s = e^{-sf}de^{sf} = d + s\,df\wedge$, with formal adjoint $\delta_s = e^{sf}\delta e^{-sf} = \delta - s\,df\lrcorner$ and Laplacian Δ_s. In fact, this theory can be considered for any elliptic complex.

On the other hand, let us give a rough idea of the concept of *stratified space*. It is a Hausdorff, locally compact and second countable space A with a partition into C^∞ manifolds (*strata*) satisfying certain conditions. An order on the family of strata is defined so that $X \le Y$ means that $X \subset \overline{Y}$. With this order relation, the maximum length of chains of strata is called the *depth* of A. Then we continue describing A by induction on depth A, as well as its the group $\operatorname{Aut}(A)$ of its *automorphisms*. If depth $A = 0$, then A is just a C^∞ manifold, whose *automorphisms* are its diffeomorphisms. Now, assume that depth $A > 0$, and the descriptions are given for lower depth. Then it is required that each stratum X has an open neighborhood T (a *tube*) that is a fiber bundle whose typical fiber is a cone $c(L) = (L\times[0,\infty))/(L\times\{0\})$ and structural group $c(\operatorname{Aut}(L))$, where L is a compact stratification of lower depth (the *link* of X), and $c(\operatorname{Aut}(L))$ consists of the homeomorphisms $c(\phi)$ of $c(L)$ induced by the maps $\phi \times \mathrm{id}$ on $L \times [0,\infty)$ ($\phi \in \operatorname{Aut}(L)$). The point $* = L \times \{0\} \in c(L)$ is called the *vertex*. An *automorphism* of A is a homeomorphism that restricts to diffeomorphisms between the strata, and whose restrictions to their tubes are fiber bundle homomorphisms. This completes the description because the depth is locally finite by the local compactness.

The local trivializations of the tubes can be considered as "stratification charts", giving a local description of the form $\mathbb{R}^m \times c(L)$. Via these charts, a stratum M of A corresponds, either to $\mathbb{R}^m \times \{*\} \equiv \mathbb{R}^m$, or to $\mathbb{R}^m \times N \times \mathbb{R}_+$ for some stratum N of L. The concept of *general adapted metric* on M is defined by induction on the depth. It is any Riemannian metric in the case of depth zero. For positive depth, a Riemannian metric g on M is called a *general adapted metric* if, on each local chart as above, g is quasi-isometric, either to the flat Euclidean metric g_0 if M corresponds to $\mathbb{R}^m$, or to $g_0 + x^{2u}\tilde{g} + (dx)^2$ if M corresponds to $\mathbb{R}^m \times N \times \mathbb{R}_+$, where $\tilde{g}$ is a general adapted metric on N, x is the canonical coordinate of $\mathbb{R}_+$, and $u > 0$ depends on M and each stratum $X < M$, whose tube is considered to define the chart. This assignment $X \mapsto u$ is called the *type* of the metric. We omit the term "general" when we take $u = 1$ for all strata.

Assuming that A is compact, it is proved in [4] that, for certain class of general adapted metrics g on a stratum M of A with numbers $u \le 1$, the Laplacian $\Delta_{\min/\max}$ has a discrete spectrum, its eigenvalues satisfy a weak version of the Weyl's asymptotic formula, and the method of Witten is extended to get Morse inequalities involving the numbers $\beta^i_{\min/\max}$ and another numbers $\nu^i_{\min/\max}$ defined by the local data around the "critical points" of a version of Morse functions on M; here, the "critical points" live in the metric completion of M. This is specially important in the case of a stratified pseudo-manifold A with regular stratum M, where $H_{\max}(M)$ is the intersection homology with perversety depending on the type of the metric [12, 13]. Again, we proceed by induction on the depth to prove these assertions. In the case of depth zero, these properties hold because we are in the case of closed manifolds. Now, assume that the depth is positive, and these properties hold for lower depth. Via a globalization procedure and a version of the Künneth formula, the computations boil down to the case of the Witten's perturbation d_s for a stratum $M = N \times (0,\infty)$ of a cone $c(L)$ with an adapted metric $g = x^{2u}\tilde{g} + (dx)^2$, where we consider the "Morse function" $f = \pm x^2/2$.

Let $\tilde{d}_{\min/\max}$, $\tilde{\delta}_{\min/\max}$ and $\widetilde{\Delta}_{\min/\max}$ denote the operators defined as above for N with $\tilde{g}$. Take differential forms $0 \neq \gamma \in \ker \widetilde{\Delta}_{\min/\max}$, of degree r, and $0 \neq \alpha, \beta \in \mathsf{D}(\widetilde{\Delta}_{\min/\max})$, of degrees r and $r-1$, with $\tilde{d}_{\min/\max}\beta = \mu\alpha$ and $\tilde{\delta}_{\min/\max}\alpha = \mu\beta$ for some $\mu > 0$. Since $\widetilde{\Delta}_{\min/\max}$ is assumed to have a discrete spectrum, $L^2\Omega(N)$ has a complete orthonormal system consisting of forms of these types. Correspondingly, there is a "direct sum splitting" of d_s into the following two types of subcomplexes:

$$C_0^\infty(\mathbb{R}_+)\,\gamma \xrightarrow{d_{s,r}} C_0^\infty(\mathbb{R}_+)\,\gamma \wedge dx,$$
$$C_0^\infty(\mathbb{R}_+)\,\beta \xrightarrow{d_{s,r-1}} C_0^\infty(\mathbb{R}_+)\,\alpha + C_0^\infty(\mathbb{R}_+)\,\beta \wedge dx \xrightarrow{d_{s,r}} C_0^\infty(\mathbb{R}_+)\,\alpha \wedge dx.$$

Forgetting the differential form part, they can be considered as two types of simple elliptic complexes of legths one and two,

$$C_0^\infty(\mathbb{R}_+) \xrightarrow{d_{s,r}} C_0^\infty(\mathbb{R}_+),$$
$$C_0^\infty(\mathbb{R}_+) \xrightarrow{d_{s,r-1}} C_0^\infty(\mathbb{R}_+) \oplus C_0^\infty(\mathbb{R}_+) \xrightarrow{d_{s,r}} C_0^\infty(\mathbb{R}_+).$$

Let $\kappa = (n-2r-1)u/2$. In the complex of length one, $d_{s,r}$ is a densely defined operator of $L^2_{\kappa,+}$ to $L^2_{\kappa,+}$, we have

$$d_{s,r} = \frac{d}{dx} \pm sx, \qquad \delta_{s,r} = -\frac{d}{dx} - \kappa x^{-1} \pm sx,$$

and the corresponding components of the Laplacian are

$$\Delta_{s,r} = H - 2\kappa x^{-1}\frac{d}{dx} \mp s(1+2\kappa), \qquad \Delta_{s,r+1} = H - 2\kappa \frac{d}{dx} x^{-1} \mp s(-1+2\kappa).$$

Up to the constant terms, these operators are of the form already considered in [3], without the term with x^{-2u}, and the spectrum of $\Delta_{s,\min/\max,r}$ and $\Delta_{s,\min/\max,r+1}$ is well known.

Table 1. Self-adjoint extensions of $\Delta_{s,r-1}$ and $\Delta_{s,r+1}$.

a	σ	condition	b	τ	condition
0	$\kappa+u$	$\kappa>-\frac{1}{2}$	0	κ	$\kappa>u-\frac{3}{2}$
$1-2(\kappa+u)$	$1-\kappa-u$	$\kappa<\frac{3}{2}-2u$	$1-2\kappa$	$-1-\kappa$	$\kappa<\frac{1}{2}-u$

Table 2. Self-adjoint extensions of $\Delta_{s,r}$.

a	b	σ	τ	θ	condition
0	0	κ	$\kappa+u$	κ	$\kappa>u-\frac{1}{2}$
$1-2\kappa$	$-1-2(\kappa+u)$	$1-\kappa$	$-1-\kappa-u$	$-\kappa-u$	$\kappa<\frac{1}{2}-2u$
0	$-1-2(\kappa+u)$	κ	$-1-\kappa-u$	$-\frac{1}{2}-u$	impossible
$1-2\kappa$	0	$1-\kappa$	$\kappa+u$	$\frac{1}{2}$	$-\frac{1+u}{2}<\kappa<\frac{1-u}{2}$ or $\kappa=-\frac{1}{2}-u,\frac{1}{2}$

In the complex of length two, $d_{s,r-1}$ is a densely defined operator of $L^2_{\kappa+u,+}$ to $L^2_{\kappa,+}\oplus L^2_{\kappa+u,+}$, $d_{s,r}$ is a densely defined operator of $L^2_{\kappa,+}\oplus L^2_{\kappa+u,+}$ to $L^2_{\kappa,+}$, we have

$$d^{\pm}_{s,r-1}=\begin{pmatrix}\mu\\ \frac{d}{d\rho}\pm s\rho\end{pmatrix},\qquad \delta^{\pm}_{s,r-1}=\begin{pmatrix}\mu\rho^{-2u} & -\frac{d}{d\rho}-2(\kappa+u)\rho^{-1}\pm s\rho\end{pmatrix},$$

$$d^{\pm}_{s,r}=\begin{pmatrix}\frac{d}{d\rho}\pm s\rho & -\mu\end{pmatrix},\qquad \delta^{\pm}_{s,r}=\begin{pmatrix}-\frac{d}{d\rho}-2\kappa\rho^{-1}\pm s\rho\\ -\mu\rho^{-2u}\end{pmatrix},$$

and the corresponding components of the Laplacian are

$$\Delta_{s,r-1}=H-2(\kappa+u)x^{-1}\frac{d}{dx}+\mu^2x^{-2u}\mp s(1+2(\kappa+u)),$$
$$\Delta_{s,r+1}=\Delta_s\equiv H-2\kappa\frac{d}{dx}x^{-1}+\mu^2x^{-2u}\mp s(-1+2\kappa),$$
$$\Delta_{s,r}\equiv\begin{pmatrix}A & -2\mu ux^{-1}\\ -2\mu ux^{-2u-1} & B\end{pmatrix},$$

where

$$A=H-2\kappa x^{-1}\frac{d}{dx}+\mu^2x^{-2u}\mp s(1+2\kappa),$$
$$B=H-2(\kappa+u)\frac{d}{dx}x^{-1}+\mu^2x^{-2u}\mp s(-1+2(\kappa+u)).$$

Up to the constant terms, $\Delta_{s,r-1}$ and A are of the form of P, and $\Delta_{s,r+1}$ and B are of the form of Q, in Section 6. In the case $u=1$, these operators were studied in [3]. Thus assume that $u<1$. Then, according to Corollaries 6.1–6.3, we get self-adjoint extensions of $\Delta_{s,r-1}$, $\Delta_{s,r+1}$ and $\Delta_{s,r}$ as indicated in Tables 1 and 2, where the conditions are determined by the hypotheses; indeed most possibilities of the hypothesis are needed. With further analysis [4] and some more restrictions on u, the maximum and minimum Laplacians can be given by appropriate choices of these operators, depending on the values of κ. Moreover the eigenvalue estimates of these corollaries play a key role in this research.

If A is a stratified pseudo-manifold, our restrictions on u allow to get enough metrics to represent all intersection cohomologies of A with perversity less or equal than the lower middle perversity, according to [12, 13].

Acknowledgements

The first author was partially supported by MICINN, Grant MTM2011-25656.

References

[1] Álvarez López J., Calaza M., Application of the method of Bonan–Clark to the generalized Hermite polynomials, arXiv:1101.5022.

[2] Álvarez López J., Calaza M., Witten's perturbation on strata, *Asian J. Math.* to appear, arXiv:1205.0348.

[3] Álvarez López J., Calaza M., Embedding theorems for the Dunkl harmonic oscillator, *SIGMA* **10** (2014), 004, 16 pages, arXiv:1301.4196.

[4] Álvarez López J., Calaza M., Witten's perturbation on strata with generalized adapted metrics, in preparation.

[5] Brasselet J.P., Hector G., Saralegi M., $\mathcal{L}^2$-cohomologie des espaces stratifiés, *Manuscripta Math.* **76** (1992), 21–32.

[6] Dunkl C.F., Reflection groups and orthogonal polynomials on the sphere, *Math. Z.* **197** (1988), 33–60.

[7] Dunkl C.F., Differential-difference operators associated to reflection groups, *Trans. Amer. Math. Soc.* **311** (1989), 167–183.

[8] Dunkl C.F., Integral kernels with reflection group invariance, *Canad. J. Math.* **43** (1991), 1213–1227.

[9] Dunkl C.F., Symmetric functions and B_N-invariant spherical harmonics, *J. Phys. A: Math. Gen.* **35** (2002), 10391–10408, math.CA/0207122.

[10] Genest V.X., Vinet L., Zhedanov A., The singular and the 2 : 1 anisotropic Dunkl oscillators in the plane, *J. Phys. A: Math. Theor.* **46** (2013), 325201, 17 pages, arXiv:1305.2126.

[11] Kato T., Perturbation theory for linear operators, *Classics in Mathematics*, Springer-Verlag, Berlin, 1995.

[12] Nagase M., L^2-cohomology and intersection homology of stratified spaces, *Duke Math. J.* **50** (1983), 329–368.

[13] Nagase M., Sheaf theoretic L^2-cohomology, in Complex Analytic Singularities, *Adv. Stud. Pure Math.*, Vol. 8, North-Holland, Amsterdam, 1987, 273–279.

[14] Nowak A., Stempak K., Imaginary powers of the Dunkl harmonic oscillator, *SIGMA* **5** (2009), 016, 12 pages, arXiv:0902.1958.

[15] Nowak A., Stempak K., Riesz transforms for the Dunkl harmonic oscillator, *Math. Z.* **262** (2009), 539–556, arXiv:0802.0474.

[16] Plyushchay M., Hidden nonlinear supersymmetries in pure parabosonic systems, *Internat. J. Modern Phys. A* **15** (2000), 3679–3698, hep-th/9903130.

[17] Reed M., Simon B., Methods of modern mathematical physics. IV. Analysis of operators, Academic Press, New York – London, 1978.

[18] Rosenblum M., Generalized Hermite polynomials and the Bose-like oscillator calculus, in Nonselfadjoint Operators and Related Topics (Beer Sheva, 1992), *Oper. Theory Adv. Appl.*, Vol. 73, Editors A. Feintuch, I. Gohberg, Birkhäuser, Basel, 1994, 369–396, math.CA/9307224.

[19] Rösler M., Generalized Hermite polynomials and the heat equation for Dunkl operators, *Comm. Math. Phys.* **192** (1998), 519–542, q-alg/9703006.

[20] Rösler M., Dunkl operators: theory and applications, in Orthogonal Polynomials and Special Functions (Leuven, 2002), *Lecture Notes in Math.*, Vol. 1817, Editors E. Koelink, W. Van Assche, Springer, Berlin, 2003, 93–135, math.CA/0210366.

[21] Szegő G., Orthogonal polynomials, *American Mathematical Society Colloquium Publications*, Vol. 23, 4th ed., Amer. Math. Soc., Providence, R.I., 1975.

[22] van Diejen J.F., Vinet L. (Editors), Calogero–Moser–Sutherland models, *CRM Series in Mathematical Physics*, Springer-Verlag, New York, 2000.

[23] Yang L.M., A note on the quantum rule of the harmonic oscillator, *Phys. Rev.* **84** (1951), 788–790.