The Asymptotic Expansion of Kummer Functions for Large Values of the a-Parameter, and Remarks on a Paper by Olver*

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Abstract. It is shown that a known asymptotic expansion of the Kummer function $U(a,b,z)$ as a tends to infinity is valid for z on the full Riemann surface of the logarithm. A corresponding result is also proved in a more general setting considered by Olver (1956).

Key words: Kummer functions; asymptotic expansions

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1 Introduction

Recently, the author collaborated on a project [1] investigating the maximal domain in which an integral addition theorem for the Kummer function $U(a,b,z)$ due to Magnus [2, 3] is valid. In this work it is important to know the asymptotic expansion of $U(a,b,z)$ as a tends to infinity. Such an expansion is well-known, and, for instance, can be found in Slater's book [8]. Slater's expansion is in terms of modified Bessel functions $K_\nu(z)$, and it is derived from a paper by Olver [5]. However, there are two problems when we try to use the known result. As Temme [9] pointed out, there is an error in Slater's expansion. Moreover, in all known results the range of validity for the variable z is restricted to certain sectors in the z-plane.

The purpose of this paper is two-fold. Firstly, we correct the error in [8], and we show that the corrected expansion based on [5] agrees with the result in [9] which was obtained in an entirely different way. Secondly, we show that the asymptotic expansion of $U(a,b,z)$ as a tends to infinity is valid for z on the full Riemann surface of the logarithm. This is somewhat surprising because often the range of validity of asymptotic expansions is restricted by Stokes' lines. Olver's results in [5] are valid for a more general class of functions (containing confluent hypergeometric functions as a special case.) He introduces a restriction on $\arg z$, and on [5, p. 76] he writes "In the case of the series with the basis function K_μ we establish the asymptotic property in the range $|\arg z| \le \frac{3}{2}\pi$. It is, in fact, unlikely that the valid range exceeds this ...". However, we show in this paper that the restriction $|\arg z| \le \frac{3}{2}\pi$ can be removed at least under an additional assumption (2.4).

In Section 2 of this paper we review the results that we need from Olver [5]. We discuss these results in Section 3. In Section 4 we prove that Olver's asymptotic expansion holds on the full Riemann surface of the logarithm. Sections 5, 6 and 7 deal with extensions to more general values of parameters. In Section 8 we specialize to asymptotic expansions of Kummer functions. In Section 9 we make the connection to Temme [9].

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2 Olver's work

Olver [5, (7.3)] considers the differential equation

$$w''(z) = \frac{1}{z}w'(z) + \left(u^2 + \frac{\mu^2-1}{z^2} + f(z)\right) w(z). \tag{2.1}$$

The function $f(z)$ is even and analytic in a simply-connected domain D containing 0. It is assumed that $\Re\mu \geq 0$. The goal is to find the asymptotic behavior of solutions of (2.1) as $0 < u \to \infty$.

Olver [5, (7.4)] starts with a formal solution to (2.1) of the form

$$w(z) = z\mathcal{Z}_\mu(uz)\sum_{s=0}^{\infty}\frac{A_s(z)}{u^{2s}} + \frac{z}{u}\mathcal{Z}_{\mu+1}(uz)\sum_{s=0}^{\infty}\frac{B_s(z)}{u^{2s}},$$

where either $\mathcal{Z}_\mu = I_\mu$, $\mathcal{Z}_{\mu+1} = I_{\mu+1}$ or $\mathcal{Z}_\mu = K_\mu$, $\mathcal{Z}_{\mu+1} = -K_{\mu+1}$ are modified Bessel functions. The functions $A_s(z) = A_s(\mu, z)$, $B_s(z) = B_s(\mu, z)$ are defined by $A_0(z) = 1$, and then recursively, for $s \geq 0$,

$$2B_s(z) = -A_s'(z) + \int_0^z \left(f(t)A_s(t) - \frac{2\mu+1}{t}A_s'(t)\right) dt, \tag{2.2}$$

$$2A_{s+1}(z) = \frac{2\mu+1}{z}B_s(z) - B_s'(z) + \int f(z)B_s(z)dz. \tag{2.3}$$

The integral in (2.3) denotes an arbitrary antiderivative of $f(z)B_s(z)$. The functions $A_s(z)$, $B_s(z)$ are analytic in D, and they are even and odd, respectively.

If the domain D is unbounded, Olver [5, p. 77] requires that $f(z) = O(|z|^{-1-\alpha})$ as $|z| \to \infty$, where $\alpha > 0$. In our application to the confluent hypergeometric equation in Section 8 the function $f(z) = z^2$ does not satisfy this condition. Therefore, throughout this paper, we will take

$$D = \{z\colon |z| < R_0\}, \tag{2.4}$$

where R_0 is a positive constant. Olver [5, p. 77] introduces various subdomains D', D_1, D_2 of D. We may choose $D' = \{z\colon |z| \leq R\}$, where $0 < R < R_0$. The domain D_1 comprises those points z in D' which can be joined to the origin by a contour which lies in D' and does not cross either the imaginary axis, or the line through z parallel to the imaginary axis. For our special D' the contour can be taken as the line segment connecting z and 0, so $D_1 = D'$. The domain D_1 appears in Olver [5, Theorem D(i)]. According to this theorem, (2.1) has a solution $W_1(u, z)$ of the form

$$W_1(u,z) = zI_\mu(uz)\left(\sum_{s=0}^{N-1}\frac{A_s(z)}{u^{2s}} + g_1(u,z)\right) + \frac{z}{u}I_{\mu+1}(uz)\left(\sum_{s=0}^{N-1}\frac{B_s(z)}{u^{2s}} + zh_1(u,z)\right), \tag{2.5}$$

where

$$|g_1(u,z)| + |h_1(u,z)| \leq \frac{K_1}{u^{2N}} \qquad \text{for} \quad 0 < |z| \leq R, \quad u \geq u_1. \tag{2.6}$$

Remarks 2.1.

1. The parameter μ is considered fixed. We may write $W_1(u, \mu, z)$ to indicate the dependence of W_1 on μ.

2. Every solution $w(z)$ of (2.1) is defined on the Riemann surface of the logarithm over D. Note that there is no restriction on $\arg z$ in (2.6), see [5, p. 76].
3. The precise statement is this: for every positive integer N there are functions g_1, h_1 and positive constants K_1, u_1 (independent of u, z) such that (2.5), (2.6) hold.
4. The functions $A_s(z)$, $B_s(z)$ are not uniquely determined because of the free choice of integration constants in (2.3). Even if we make a definite choice of these integration constants, the solution $W_1(u,z)$ is not uniquely determined by (2.5), (2.6). For example, one can replace $W_1(u,z)$ by $(1+e^{-u})W_1(u,z)$.
5. Olver's construction of $W_1(u,z)$ is independent of N but may depend on R. In our application to the confluent hypergeometric differential equation we have $f(z)=z^2$. Then R can be any positive number but $W_1(u,z)$ may depend on the choice of R.
6. Olver has the term $\frac{z}{1+|z|}$ in place of z in front of h_1 in (2.5) but since we assume $|z|\le R$ this makes no difference.

For the definition of D_2 we suppose that a is an arbitrary point of the sector $|\arg a|<\frac{1}{2}\pi$ and $\epsilon>0$. Then D_2 comprises those points $z\in D'$ for which $|\arg z|\le\frac{3}{2}\pi$, $\Re z\le\Re a$, and a contour can be found joining z and a which satisfies the following conditions:

(i) it lies in D',

(ii) it lies wholly to the right of the line through z parallel to the imaginary axis,

(iii) it does not cross the negative imaginary axis if $\frac{1}{2}\pi\le\arg z\le\frac{3}{2}\pi$, and does not cross the positive imaginary axis if $-\frac{3}{2}\pi\le\arg z\le-\frac{\pi}{2}$,

(iv) it lies outside the circle $|t|=\epsilon|z|$.

In our special case $D'=\{z\colon |z|\le R\}$ we choose $a=R$. If $0\le\arg z\le\frac{3}{2}\pi$ and $0<|z|\le R$, we choose the contour starting at z moving in positive direction parallel to the imaginary axis until we hit the circle $|t|=R$. Then we move clockwise along the circle $|t|=R$ towards a. Taking into account condition (iv), we see that D_2 is the set of points z with $-\frac{3}{2}\pi+\delta\le z\le\frac{3}{2}\pi-\delta$, $0<|z|\le R$, where $\delta>0$. The domain D_2 appears in Olver [5, Theorem D(ii)]. According to this theorem, (2.1) has a solution $W_2(u,z)$ of the form

$$W_2(u,z)=zK_\mu(uz)\left(\sum_{s=0}^{N-1}\frac{A_s(z)}{u^{2s}}+g_2(u,z)\right)\\ -\frac{z}{u}K_{\mu+1}(uz)\left(\sum_{s=0}^{N-1}\frac{B_s(z)}{u^{2s}}+zh_2(u,z)\right), \tag{2.7}$$

where

$$|g_2(u,z)|+|h_2(u,z)|\le\frac{K_2}{u^{2N}}\qquad\text{for}\quad 0<|z|\le R,\quad |\arg z|\le\frac{3}{2}\pi-\delta,\quad u\ge u_2. \tag{2.8}$$

Note that in (2.8) there is a restriction on $\arg z$.

In the rest of this paper we choose the functions $A_s(z)$ such that

$$A_s(0)=0\qquad\text{if}\quad s\ge 1. \tag{2.9}$$

Then the functions $A_s(z)$, $B_s(z)$ are uniquely determined.

3 Properties of solutions W_1 and W_2

The differential equation (2.1) has a regular singularity at $z=0$ with exponents $1\pm\mu$. Substituting $x=z^2$ we obtain an equation which has a regular singularity at $x=0$ with exponents $\frac{1}{2}(1\pm\mu)$. Therefore, for every μ which is not a negative integer, (2.1) has a unique solution $W_+(z)=W_+(u,\mu,z)$ of the form

$$W_+(z)=z^{1+\mu}\sum_{n=0}^\infty c_n z^{2n},$$

where the c_n are determined by $c_0=1$, and

$$4n(\mu+n)c_n=u^2c_{n-1}+\sum_{j=0}^{n-1} f_j c_{n-1-j} \qquad \text{for} \quad n\ge 1$$

when

$$f(z)=\sum_{n=0}^\infty f_n z^{2n}.$$

If μ is not an integer, then $W_+(u,\mu,z)$ and $W_+(u,-\mu,z)$ form a fundamental system of solutions of (2.1). If $\Re\mu\ge 0$, there is a solution $W_-(z)$ linearly independent of $W_+(z)$ such that

$$W_-(z)=z^{1-\mu}p\big(z^2\big)+d\ln z W_+(z),$$

where p is a power series and d is a suitable constant. If $\mu\ne 0$ we choose $p(0)=1$. If μ is not an integer then $d=0$.

Lemma 3.1. *Suppose $\Re\mu\ge 0$. There is a function $\alpha(u)$ such that*

$$W_1(u,z)=\alpha(u)W_+(u,z),$$

and, for every $N=1,2,3,\dots,$

$$\alpha(u)=\frac{2^{-\mu}u^\mu}{\Gamma(\mu+1)}\left(1+O\left(\frac{1}{u^{2N}}\right)\right) \qquad \textit{as} \quad 0<u\to\infty. \tag{3.1}$$

Proof. There are functions $\alpha_+(u)$, $\alpha_-(u)$ such that

$$W_1(u,z)=\alpha_+(u)W_+(u,z)+\alpha_-(u)W_-(u,z). \tag{3.2}$$

Suppose $\Re\mu>0$. Then (3.2) implies

$$\lim_{z\to 0^+} z^{\mu-1}W_1(u,z)=\alpha_-(u). \tag{3.3}$$

We use [7, (10.30.1)]

$$\lim_{z\to 0} I_\nu(z)z^{-\nu}=\frac{2^{-\nu}}{\Gamma(\nu+1)}.$$

Then (2.5), (2.6) give

$$\lim_{z\to 0^+} z^{\mu-1}W_1(u,z)=0. \tag{3.4}$$

It follows from (3.3), (3.4) that $\alpha_-(u)=0$.

Now suppose that $\Re\mu = 0$, $\mu \neq 0$. Then we argue as before but instead of $z \to 0^+$ we approach 0 along a spiral $z = re^{\pm ir}$, $0 < r \to 0$, when $\pm\Im\mu > 0$. Then along this spiral $z^{2\mu} \to 0$. We obtain again that $\alpha_-(u) = 0$. In a similar way, we also show that $\alpha_-(u) = 0$ when $\mu = 0$.

Therefore, (3.2) gives

$$\lim_{z\to 0^+} z^{-\mu-1} W_1(z,u) = \alpha_+(u)$$

and, from (2.5), (2.6), (2.9)

$$\lim_{z\to 0^+} z^{-\mu-1} W_1(u,z) = \frac{2^{-\mu}u^\mu}{\Gamma(\mu+1)}\left(1 + O\left(\frac{1}{u^{2N}}\right)\right)$$

which implies (3.1) with $\alpha(u) = \alpha_+(u)$. ■

Let us define

$$W_3(u,\mu,z) = \frac{2^{-\mu}u^\mu}{\Gamma(\mu+1)} W_+(u,\mu,z).$$

Then Lemma 3.1 gives

$$W_3(u,z) = \tilde\alpha(u) W_1(u,z), \qquad \text{where} \quad \tilde\alpha(u) = 1 + O\left(\frac{1}{u^{2N}}\right).$$

Therefore, W_3 admits the asymptotic expansion (2.5), (2.6), so we can replace W_1 by W_3. Note that in contrast to W_1, W_3 is a uniquely defined function which is identified as a (Floquet) solution of (2.1) and not by its asymptotic behavior as $u \to \infty$.

Unfortunately, it seems impossible to replace W_2 by an easily identifiable solution of (2.1). However, we will now prove several useful properties of W_2.

Lemma 3.2. *Suppose that $\Re\mu \geq 0$. There is a function $\beta(u)$ such that*

$$W_2\big(u, ze^{\pi i}\big) - e^{\pi i(1-\mu)} W_2(u,z) = \beta(u) W_3(u,z), \tag{3.5}$$

and, for every $N = 1, 2, 3, \ldots$,

$$\beta(u) = \pi i\left(1 + O\left(\frac{1}{u^{2N}}\right)\right) \qquad \text{as} \quad 0 < u \to \infty. \tag{3.6}$$

Proof. We set $\lambda_\pm = e^{\pi i(1\pm\mu)}$. Equation (2.1) has a fundamental system of solutions W_+, W_- such that

$$W_+\big(ze^{\pi i}\big) = \lambda_+ W_+(z), \qquad W_-\big(ze^{\pi i}\big) = \lambda_- W_-(z) + \rho W_+(z).$$

Let $w(z) = c_+W_+(z) + c_-W_-(z)$ be any solution of (2.1). Then

$$w\big(ze^{\pi i}\big) - \lambda_- w(z) = ((\lambda_+ - \lambda_-)c_+ + \rho c_-)W_+(z).$$

If we apply this result to $w = W_2$ we see that there is a function $\beta(u)$ such that (3.5) holds.

Let $z > 0$ and set $z_1 = ze^{\pi i}$. We use (2.7) for z_1 in place of z, and [7, (10.34.2)]

$$K_\nu\big(ze^{\pi i m}\big) = e^{-\pi i\nu m} K_\nu(z) - \pi i \frac{\sin(\pi\nu m)}{\sin(\pi\nu)} I_\nu(z) \tag{3.7}$$

with $m=1$. Then

$$W_2(u,z_1)=z\left(\lambda_- K_\mu(uz)+\pi i I_\mu(uz)\right)\left(\sum_{s=0}^{N-1}\frac{A_s(z)}{u^{2s}}+g_2(u,z_1)\right)$$
$$+\frac{z}{u}\left(-\lambda_- K_{\mu+1}(uz)+\pi i I_{\mu+1}(uz)\right)\left(\sum_{s=0}^{N-1}\frac{B_s(z)}{u^{2s}}+zh_2(u,z_1)\right).$$

Using (2.7) a second time, we find that

$$W_2(u,z_1)-\lambda_- W_2(u,z)=\pi i z I_\mu(uz)\left(\sum_{s=0}^{N-1}\frac{A_s(z)}{u^{2s}}+g_2(u,z_1)\right)$$
$$+\pi i\frac{z}{u}I_{\mu+1}(uz)\left(\sum_{s=0}^{N-1}\frac{B_s(z)}{u^{2s}}+zh_2(u,z_1)\right)$$
$$+\lambda_- zK_\mu(uz)(g_2(u,z_1)-g_2(u,z))-\lambda_-\frac{z^2}{u}K_{\mu+1}(uz)(h_2(u,z_1)-h_2(u,z)).$$

We now expand the right-hand side of (3.5) using (2.5), and compare the expansions. Setting $z=R$ and dividing by $RI_\mu(uR)$, we obtain

$$(\beta(u)-\pi i)\left(1+O\left(\frac{1}{u}\right)\right)=O\left(\frac{1}{u^{2N}}\right)\quad\text{as } 0<u\to\infty,$$

where we used [7, (10.40.1)]

$$I_\nu(x)=\frac{e^x}{\sqrt{2\pi x}}\left(1+O\left(\frac{1}{x}\right)\right)\qquad\text{as}\quad 0<x\to\infty, \tag{3.8}$$

and [7, (10.40.2)]

$$K_\nu(x)=\sqrt{\frac{\pi}{2x}}e^{-x}\left(1+O\left(\frac{1}{x}\right)\right)\qquad\text{as}\quad 0<x\to\infty. \tag{3.9}$$

This proves (3.6). ∎

Lemma 3.3.

(a) *If* $\Re\mu>0$ *then, for every* $N=1,2,3,\dots,$ *we have*

$$\limsup_{z\to 0^+}\left|z^{\mu-1}W_2(u,z)-\Gamma(\mu)2^{\mu-1}u^{-\mu}\left(1-2\mu\sum_{s=0}^{N-1}\frac{B_s'(0)}{u^{2s+2}}\right)\right|=O\left(\frac{u^{-\mu}}{u^{2N+2}}\right) \tag{3.10}$$

as $0<u\to\infty$.

(b) *If* $\Re\mu=0$, $\mu\neq 0$, (3.10) *holds when we replace* $z^{\mu-1}W_2(u,z)$ *by*

$$z^{\mu-1}W_2(u,z)-\Gamma(-\mu)2^{-\mu-1}u^\mu z^{2\mu}.$$

(c) *If* $\mu=0$ *then*

$$\limsup_{z\to 0^+}\left|\frac{W_2(u,z)}{z\ln z}+1\right|=O\left(\frac{1}{u^{2N}}\right).$$

Proof. Suppose that $\Re\mu > 0$. Then we use [7, (10.30.2)]

$$\lim_{x\to 0^+} x^\nu K_\nu(x) = \Gamma(\nu)2^{\nu-1} \qquad \text{for} \quad \Re\nu > 0.$$

It follows that

$$\lim_{z\to 0^+} z^\mu K_\mu(uz) = \Gamma(\mu)2^{\mu-1}u^{-\mu}, \tag{3.11}$$

$$\lim_{z\to 0^+} \frac{1}{u} z^{\mu+1} K_{\mu+1}(uz) = \Gamma(\mu+1)2^{\mu}u^{-\mu-2}. \tag{3.12}$$

Using (2.7), (2.9), (3.11), (3.12), we obtain

$$\begin{aligned}\limsup_{z\to 0^+} &\left| z^{\mu-1}W_2(u,z) - \Gamma(\mu)2^{\mu-1}u^{-\mu}\left(1 - 2\mu\sum_{s=0}^{N-1}\frac{B_s'(0)}{u^{2s+2}}\right)\right| \\ &\le \limsup_{z\to 0^+}\left|\Gamma(\mu)2^{\mu-1}u^{-\mu}g_2(u,z) - \Gamma(\mu+1)2^\mu u^{-\mu-2}h_2(u,z)\right|.\end{aligned}$$

Now (2.8) gives (3.10) with $N-1$ in place of N. If $\Re\mu = 0$, $\mu \ne 0$, then we use [5, (9.7)]

$$K_\mu(x) = \Gamma(\mu)2^{\mu-1}x^{-\mu} + \Gamma(-\mu)2^{-\mu-1}x^\mu + o(1) \qquad \text{as} \quad 0 < x \to 0$$

and argue similarly. If $\mu = 0$ we use [7, (10.30.3)]

$$\lim_{x\to 0^+}\frac{K_0(x)}{\ln x} = -1. \qquad \blacksquare$$

Theorem 3.4. *Suppose that $\Re\mu \ge 0$ and μ is not an integer. There are functions $\gamma(u)$, $\delta(u)$ such that*

$$W_2(u,z) = \gamma(u)W_3(u,\mu,z) + \delta(u)W_3(u,-\mu,z), \tag{3.13}$$

and, for every $N = 1,2,3,\dots,$

$$\gamma(u) = -\frac{\pi}{2\sin(\pi\mu)}\left(1 + O\left(\frac{1}{u^{2N}}\right)\right), \tag{3.14}$$

$$\delta(u) = \frac{\pi}{2\sin(\pi\mu)}\left(1 - 2\mu\sum_{s=0}^{N-1}\frac{B_s'(0)}{u^{2s+2}} + O\left(\frac{1}{u^{2N+2}}\right)\right). \tag{3.15}$$

Proof. Since μ is not an integer, $W_3(u,\mu,z)$ and $W_3(u,-\mu,z)$ are linearly independent so (3.13) holds for some suitable functions γ, δ. From (3.13) we get

$$W_2\big(u, ze^{\pi i}\big) - e^{\pi i(1-\mu)}W_2(u,z) = \gamma(u)\big(e^{\pi i(1+\mu)} - e^{\pi i(1-\mu)}\big)W_3(u,z).$$

Comparing with Lemma 3.2, we find $-2i\gamma(u)\sin(\pi\mu) = \beta(u)$. Now (3.6) gives (3.14).

Suppose that $\Re\mu > 0$. Then (3.13) yields

$$\lim_{z\to 0^+} z^{\mu-1}W_2(u,z) = \delta(u)\frac{2^\mu u^{-\mu}}{\Gamma(1-\mu)}.$$

Using Lemma 3.3(a) we obtain

$$\Gamma(\mu)2^{\mu-1}u^{-\mu}\left(1 - 2\mu\sum_{s=0}^{N-1}\frac{B_s'(0)}{u^{2s+2}} + O\left(\frac{1}{u^{2N+2}}\right)\right) = \delta(u)\frac{2^\mu u^{-\mu}}{\Gamma(1-\mu)}.$$

Applying the reflection formula for the Gamma function, we obtain (3.15). If $\Re\mu = 0$, $\mu \ne 0$, the proof of (3.15) is similar. $\blacksquare$

4 Removal of restriction on arg z

Using $\beta(u)$ from Lemma 3.2 we define

$$W_4(u,z)=\frac{\pi i}{\beta(u)}W_2(u,z).$$

Then we have

$$W_4\big(u,ze^{\pi i}\big)=e^{\pi i(1-\mu)}W_4(u,z)+\pi i W_3(u,z). \tag{4.1}$$

Moreover, (3.6) shows that W_4 shares the asymptotic expansion (2.7), (2.8) with W_2. From (4.1) we obtain

$$W_4\big(u,ze^{\pi i m}\big)=e^{\pi i(1-\mu)m}W_4(u,z)+\pi i\frac{\sin(\pi(\mu+1)m)}{\sin(\pi(\mu+1))}W_3(u,z) \tag{4.2}$$

for every integer m. We will use (4.2) and the asymptotic expansions (2.5), (2.7) for $|\arg z|\le\frac12\pi$ to prove that in (2.8) we can remove the restriction on $\arg z$ completely.

Theorem 4.1. *Suppose that $\Re\mu\ge 0$. For every $N=1,2,3,\dots$, $W_2(u,z)$ can be written as the right-hand side of (2.7), and (2.8) holds without a restriction on $\arg z$:*

$$|g_2(u,z)|+|h_2(u,z)|\le\frac{K_2}{u^{2N}}\qquad \textit{for}\quad 0<|z|\le R,\quad u\ge u_2.$$

Proof. Without loss of generality we replace W_2 by W_4. We assume that $|\arg z|\le\frac12\pi$, $0<|z|\le R$, $u>0$, m is an integer and $z_1:=ze^{\pi i m}$. We insert (2.5), (2.7) on the right-hand side of (4.2). Using (3.7) we obtain

$$W_4(u,z_1)=z_1K_\mu(uz_1)\sum_{s=0}^{N-1}\frac{A_s(z_1)}{u^{2s}}-\frac{z_1}{u}K_{\mu+1}(uz_1)\sum_{s=0}^{N-1}\frac{B_s(z_1)}{u^{2s}}+f(u,z), \tag{4.3}$$

where

$$f=E_1g_2+E_2h_2+E_3g_1+E_4h_1,$$

with

$$E_1(u,z)=e^{-\pi i(\mu+1)m}zK_\mu(uz),\qquad E_2(u,z)=-e^{-\pi i(\mu+1)m}\frac{z^2}{u}K_{\mu+1}(uz),$$

$$E_3(u,z)=\pi i\frac{\sin(\pi(\mu+1)m)}{\sin(\pi(\mu+1))}zI_\mu(uz),\qquad E_4(u,z)=\pi i\frac{\sin(\pi(\mu+1)m)}{\sin(\pi(\mu+1))}\frac{z^2}{u}I_{\mu+1}(uz).$$

We will construct functions $G_j(u,z)$ and $H_j(u,z)$ such that

$$E_j(u,z)=z_1K_\mu(uz_1)G_j(u,z)-\frac{z_1^2}{u}K_{\mu+1}(uz_1)H_j(u,z)$$

for $j=1,2,3,4$. Then (4.3) becomes

$$\begin{aligned}W_4(u,z_1)&=z_1K_\mu(uz_1)\left(\sum_{s=0}^{N-1}\frac{A_s(z_1)}{u^{2s}}+g_3(u,z)\right)\\&\qquad-\frac{z_1}{u}K_{\mu+1}(uz_1)\left(\sum_{s=0}^{N-1}\frac{B_s(z_1)}{u^{2s}}+z_1h_3(u,z)\right),\end{aligned} \tag{4.4}$$

where

$$g_3 = G_1 g_2 + G_2 h_2 + G_3 g_1 + G_4 h_1, \qquad h_3 = H_1 g_2 + H_2 h_2 + H_3 g_1 + H_4 h_1.$$

We now use [7, (10.28.2)]

$$K_\mu(x) I_{\mu+1}(x) + K_{\mu+1}(x) I_\mu(x) = \frac{1}{x}. \tag{4.5}$$

From (4.5) and the relation

$$I_\mu\big(z e^{\pi i m}\big) = e^{\pi i \mu m} I_\mu(z) \tag{4.6}$$

we obtain

$$u z_1 K_\mu(u z_1) e^{\pi i(\mu+1)m} I_{\mu+1}(uz) + u z_1 K_{\mu+1}(u z_1) e^{\pi i \mu m} I_\mu(uz) = 1.$$

Therefore, we can choose

$$G_1(u,z) = uz K_\mu(uz) I_{\mu+1}(uz), \qquad H_1(u,z) = -u^2 K_\mu(uz) I_\mu(uz).$$

We set

$$l_0(x) = \ln \frac{1 + 2|x|}{|x|}, \qquad l_\mu(x) = 1 \qquad \text{if} \quad \mu \neq 0,$$

and note the estimates [5, (9.12)]

$$|I_\mu(x) K_\mu(x)| \le \frac{C l_\mu(x)}{1 + |x|}, \qquad |I_{\mu+1}(x) K_\mu(x)| \le \frac{C|x| l_\mu(x)}{1 + |x|^2}, \tag{4.7}$$

$$|I_{\mu+1}(x) K_{\mu+1}(x)| \le \frac{C}{1 + |x|}, \qquad |I_\mu(x) K_{\mu+1}(x)| \le \frac{C}{|x|} \tag{4.8}$$

valid when $|\arg x| \le \frac{1}{2}\pi$ with C independent of x. At this point we assume that $\mu \neq 0$ (the case $\mu = 0$ is mentioned at the end of the proof). The estimates (4.7) give

$$|G_1(u,z)| \le C, \qquad |H_1(u,z)| \le C u^2. \tag{4.9}$$

Similarly, we choose

$$G_2(u,z) = -z^2 K_{\mu+1}(uz) I_{\mu+1}(uz), \qquad H_2(u,z) = uz K_{\mu+1}(uz) I_\mu(uz).$$

The estimates (4.8) give

$$|G_2(u,z)| \le C|z|^2, \qquad |H_2(u,z)| \le C. \tag{4.10}$$

It follows from (3.7) that

$$E_3(u,z) = -E_1(u,z) + z_1 K_\mu(u z_1), \qquad E_4(u,z) = -E_2(u,z) - \frac{z_1^2}{u} K_{\mu+1}(u z_1).$$

Therefore, we can choose

$$G_3(u,z) = 1 - G_1(u,z), \qquad H_3(u,z) = -H_1(u,z),$$
$$G_4(u,z) = -G_2(u,z), \qquad H_4(u,z) = 1 - H_2(u,z).$$

From (4.9), (4.10), we get

$$|G_3(u,z)| \le C+1, \qquad |H_3(u,z)| \le Cu^2, \tag{4.11}$$
$$|G_4(u,z)| \le C|z|^2, \qquad |H_4(u,z)| \le C+1. \tag{4.12}$$

The estimates (4.9), (4.10), (4.11), (4.12) give

$$|g_3(u,z)| \le C|g_2(u,z)| + C|z|^2|h_2(u,z)| + (C+1)|g_1(u,z)| + C|z|^2|h_1(u,z)|,$$
$$|h_3(u,z)| \le Cu^2|g_2(u,z)| + C|h_2(u,z)| + Cu^2|g_1(u,z)| + (C+1)|h_1(u,z)|.$$

Since we assumed that

$$|g_1(u,z)| + |h_1(u,z)| + |g_2(u,z)| + |h_2(u,z)| \le \frac{K}{u^{2N}}$$

for $|\arg z| \le \frac{1}{2}\pi$, $0 < |z| \le R$, $u \ge u_0$, the expansion (4.4) has the desired form with N replaced by $N-1$.

Suppose $\mu = 0$. We use [7, (10.31.2)]

$$K_0(x) = -\left(\ln\left(\frac{1}{2}x\right) + \gamma\right) I_0(x) + \frac{\frac{1}{4}x^2}{(1!)^2} + \left(1+\frac{1}{2}\right)\frac{\left(\frac{1}{4}x^2\right)^2}{(2!)^2} + \cdots. \tag{4.13}$$

It follows from (4.13) that there exist positive constants $r > 0$, $D > 0$ such that

$$\frac{|K_0(x)|}{|K_0(xe^{\pi i m})|} \le D \qquad \text{for} \quad 0 < |x| \le r, \quad |\arg x| \le \frac{1}{2}\pi, \quad m \in \mathbb{Z}.$$

Then we set

$$G_1(u,z) = \frac{K_0(uz)}{K_0(uz_1)}, \qquad H_1(u,z) = 0 \qquad \text{if} \quad 0 < |uz| \le r$$

with G_1 and H_1 the same as before when $|uz| > r$. The estimates (4.9) are valid with a suitable constant C. The rest of the proof is unchanged. This completes the proof of the theorem. ■

5 Extension to complex u

So far we considered only $0 < u \to \infty$. Now we set $u = te^{i\theta}$, where $t > 0$ and $\theta \in \mathbb{R}$. In (2.1) we substitute $z = e^{-i\theta}x$, $\tilde{w}(x) = w(z)$. Then we obtain the differential equation

$$\frac{d^2}{dx^2}\tilde{w}(x) = \frac{1}{x}\frac{d}{dx}\tilde{w}(x) + \left(t^2 + \frac{\mu^2-1}{x^2} + e^{-2i\theta} f\left(e^{-i\theta}x\right)\right)\tilde{w}(x). \tag{5.1}$$

Assuming $\Re\mu \ge 0$, we can apply Olver's theory to this equation, and obtain functions $\tilde{W}_1(t,x)$ and $\tilde{W}_2(t,x)$. Since we assumed that $f(z)$ is analytic in the disk $\{z\colon |z| < R_0\}$, the new function $\tilde{f}(x) = e^{-2i\theta} f(e^{-i\theta}x)$ is analytic in the same disk. Therefore, the domains D_1, D_2 are the same as before. The functions $\tilde{A}_s(x)$, $\tilde{B}_s(x)$ that appear in place of $A_s(z)$, $B_s(z)$ satisfy

$$\tilde{A}_s(x) = e^{-2si\theta}A_s(z), \qquad \tilde{B}_s(x) = e^{-(2s+1)i\theta}B_s(z),$$

so

$$\frac{\tilde{A}_s(x)}{t^{2s}} = \frac{A_s(z)}{u^{2s}}, \qquad \frac{\tilde{B}_s(x)}{t^{2s+1}} = \frac{B_s(z)}{u^{2s+1}}.$$

Therefore, the functions $e^{-i\theta}\tilde{W}_1(t,x)$ and $e^{-i\theta}\tilde{W}_2(t,x)$ have the asymptotic expansions (2.5), (2.6) and (2.7), (2.8) with (t,x) replacing (u,z).

Let $\tilde{W}_3(t,\mu,x)$ be the function W_3 for the differential equation (5.1). Then

$$W_3\big(te^{i\theta},\mu,e^{-i\theta}x\big) = e^{-i\theta}\tilde{W}_3(t,\mu,x).$$

It follows that $W_3(u,\mu,z)$ can be expanded in the form of the right-hand side of (2.5), and (2.6) holds for $0<|z|\le R$ and $u=te^{i\theta}$ for any fixed real θ.

We would like to connect $\tilde{W}_2$ to W_2 in a similar manner but this is not possible at this point because $W_2(u,z)$ is only defined for $u>0$, and so we cannot substitute $u=te^{i\theta}$.

6 Properties of A_s, B_s

For any $\mu\in\mathbb{C}$ we consider the solution $A_s(z)=A_s(\mu,z)$, $B_s(z)=B_s(\mu,z)$ of the recursion (2.2), (2.3) which is uniquely determined by $A_0(z)=1$ and (2.9). The following lemma is mentioned by Olver [4, p. 327], [5, p. 81, line 6].

Lemma 6.1. *Let $\hat{A}_s(z)$, $\hat{B}_s(z)$ be any solution of (2.2), (2.3) with $\hat{A}_0(z)=1$. Then, for all $s\ge 0$,*

$$\hat{A}_s(z) = \sum_{r=0}^{s} A_r(z)\hat{A}_{s-r}(0), \qquad \hat{B}_s(z) = \sum_{r=0}^{s} B_r(z)\hat{A}_{s-r}(0). \tag{6.1}$$

Proof. Let us denote the right-hand sides of equations (6.1) by $A_s^*(z)$, $B_s^*(z)$, respectively. It is easy to show that $A_s^*(z)$, $B_s^*(z)$ is a solution of (2.2), (2.3). Since $A_0^*(z)=1$ and $A_s^*(0)=\hat{A}_s(0)$, this solution must agree with $\hat{A}_s(z)$, $\hat{B}_s(z)$. ∎

We now define $a_0(z)=1$ and, for $s\ge 0$,

$$a_{s+1}(z) := A_{s+1}(-\mu,z) + \frac{2\mu}{z}B_s(-\mu,z), \tag{6.2}$$

$$b_s(z) := B_s(-\mu,z). \tag{6.3}$$

Theorem 6.2. *The functions $a_s(z)$, $b_s(z)$ satisfy (2.2), (2.3) with A_s, B_s replaced by a_s, b_s, respectively, and, for all $s\ge 0$,*

$$a_s(z) = A_s(\mu,z) + 2\mu\sum_{r=0}^{s-1} A_r(\mu,z)B'_{s-1-r}(-\mu,0), \tag{6.4}$$

$$b_s(z) = B_s(\mu,z) + 2\mu\sum_{r=0}^{s-1} B_r(\mu,z)B'_{s-1-r}(-\mu,0). \tag{6.5}$$

Proof. We have

$$2A_{s+1}(-\mu,z) = \frac{-2\mu+1}{z}B_s(-\mu,z) - B'_s(-\mu,z) + \int f(z)B_s(-\mu,z)dz.$$

We add $\frac{4\mu}{z}B_s(-\mu,z)$ on both sides and get

$$2a_{s+1}(z) = \frac{2\mu+1}{z}b_s(z) - b'_s(z) + \int f(z)b_s(z)dz. \tag{6.6}$$

This is (2.3) for $a_s(z)$, $b_s(z)$.

Equation (2.2) is true for $a_s(z)$, $b_s(z)$ when $s=0$. Suppose $s\ge 1$. We have

$$2B_s'(-\mu,z)=-A_s''(-\mu,z)+f(z)A_s(-\mu,z)+\frac{2\mu-1}{z}A_s'(-\mu,z).$$

Using the definitions of $a_s(z)$, $b_s(z)$ we get

$$2b_s'(z)=-a_s''(z)+f(z)a_s(z)-\frac{2\mu+1}{z}a_s'(z)+\frac{4\mu}{z}a_s'(z)+G, \tag{6.7}$$

where

$$G:=\frac{d^2}{dz^2}\left(\frac{2\mu}{z}b_{s-1}(z)\right)-f(z)\frac{2\mu}{z}b_{s-1}(z)-\frac{2\mu-1}{z}\frac{d}{dz}\left(\frac{2\mu}{z}b_{s-1}(z)\right).$$

In (6.7) we replace $\frac{4\mu}{z}a_s'(z)$ through (6.6). Then we obtain

$$2b_s'(z)=-a_s''(z)+f(z)a_s(z)-\frac{2\mu+1}{z}a_s'(z)+H+G, \tag{6.8}$$

where

$$H:=\frac{2\mu}{z}\left[\frac{d}{dz}\left(\frac{2\mu+1}{z}b_{s-1}(z)\right)-b_{s-1}''(z)+f(z)b_{s-1}(z)\right].$$

By direct computation, we show $H+G=0$ for any function $b_{s-1}(z)$. Therefore, by integrating (6.8) noting that $a_s(z)$ is even and $b_s(z)$ is odd, we obtain (2.2) for $a_s(z)$, $b_s(z)$.

We now get (6.4), (6.5) from Lemma 6.1. ■

Using multiplication of formal series, we can write (6.4), (6.5) as

$$F(u,-\mu)\sum_{s=0}^\infty\frac{A_s(z)}{u^{2s}}=\sum_{s=0}^\infty\frac{a_s(z)}{u^{2s}}, \tag{6.9}$$

$$F(u,-\mu)\sum_{s=0}^\infty\frac{B_s(z)}{u^{2s}}=\sum_{s=0}^\infty\frac{b_s(z)}{u^{2s}}, \tag{6.10}$$

where

$$F(u,\mu)=1-2\mu\sum_{s=0}^\infty\frac{B_s'(\mu,0)}{u^{2s+2}}.$$

We differentiate (6.5) with respect to z and set $z=0$. Then we find

$$B_s'(-\mu,0)=B_s'(\mu,0)+2\mu\sum_{r=0}^{s-1}B_r'(\mu,0)B_{s-1-r}'(-\mu,0),$$

or, equivalently,

$$F(u,\mu)F(u,-\mu)=1. \tag{6.11}$$

In particular, it follows that

$$F(u,\mu)\sum_{s=0}^\infty\frac{a_s(z)}{u^{2s}}=\sum_{s=0}^\infty\frac{A_s(z)}{u^{2s}}, \tag{6.12}$$

$$F(u,\mu)\sum_{s=0}^\infty\frac{b_s(z)}{u^{2s}}=\sum_{s=0}^\infty\frac{B_s(z)}{u^{2s}}. \tag{6.13}$$

7 Asymptotic expansion of W_3 when $\Re\mu < 0$

In Section 3 we saw that $W_3(u, \mu, z)$ can be written as the right-hand side of (2.5), and (2.6) holds. However, this was proved only when $\Re\mu \geq 0$. Now we remove this restriction.

Theorem 7.1. *Suppose that $\mu \in \mathbb{C}$ is not a negative integer, and $u = te^{i\theta}$ with $t > 0$, $\theta \in \mathbb{R}$. Then $W_3(u, \mu, z)$ can be written as the right-hand side of (2.5) and, for each $R > 0$ and $N \geq 1$, there are constants L_1 and t_1 such that*

$$|g_1(u, z)| + |h_1(u, z)| \leq \frac{L_1}{t^{2N}} \qquad \text{for} \quad 0 < |z| \leq R, \quad t \geq t_1.$$

Proof. In Sections 3 and 5 we proved this statement for $\Re\mu \geq 0$. Therefore, it will be sufficient to treat $W_3(u, -\mu, z)$ with $\Re\mu > 0$. By the considerations in Section 5, it is sufficient to consider $\theta = 0$, so $u > 0$. Suppose $|\arg z| \leq \frac{1}{2}\pi$, $0 < |z| \leq R$. By (3.13), we have

$$c\delta(u)W_3(u, -\mu, z) = cW_2(u, \mu, z) - c\gamma(u)W_3(u, \mu, z), \tag{7.1}$$

where $c = \frac{2}{\pi}\sin(\pi\mu)$. On the right-hand side of (7.1) we insert the expansions (2.5) for W_3 and (2.7) for W_2. Taking into account (3.14), we can expand $-c\gamma(u)W_3(u, \mu, z)$ the same way as W_3. Then using [7, (10.27.4)]

$$K_\nu(x) = \frac{\pi}{2\sin(\pi\nu)} \left(I_{-\nu}(x) - I_\nu(x)\right), \tag{7.2}$$

we obtain

$$c\delta(u)W_3(u, -\mu, z) = zI_{-\mu}(uz) \sum_{s=0}^{N-1} \frac{A_s(z)}{u^{2s}} + \frac{z}{u} I_{-\mu-1}(uz) \sum_{s=0}^{N-1} \frac{B_s(z)}{u^{2s}} + f(u, z), \tag{7.3}$$

where

$$f = E_1 g_2 + E_2 h_2 + E_3 g_1 + E_4 h_1$$

with

$$E_1(u, z) = czK_\mu(uz), \qquad E_2(u, z) = -c\frac{z^2}{u}K_{\mu+1}(uz),$$

$$E_3(u, z) = zI_\mu(uz), \qquad E_4(u, z) = \frac{z^2}{u}I_{\mu+1}(uz).$$

We will construct functions $G_j(u, z)$ and $H_j(u, z)$ such that

$$E_j(u, z) = zI_{-\mu}(uz)G_j(u, z) + \frac{z^2}{u}I_{1-\mu}(uz)H_j(u, z)$$

for $j = 1, 2, 3, 4$. Also using [7, (10.29.1)]

$$I_{\nu-1}(x) - I_{\nu+1}(x) = \frac{2\nu}{x}I_\nu(x), \tag{7.4}$$

(7.3) becomes

$$\begin{aligned} c\delta(u)W_3(u, -\mu, z) &= zI_{-\mu}(uz) \left(\sum_{s=0}^{N-1} \frac{\tilde{A}_s(z)}{u^{2s}} + g_3(u, z) \right) \\ &\quad + \frac{z}{u} I_{1-\mu}(uz) \left(\sum_{s=0}^{N-1} \frac{B_s(z)}{u^{2s}} + zh_3(u, z) \right), \end{aligned} \tag{7.5}$$

where

$$\tilde{A}_0(z) = 1, \qquad \tilde{A}_s(z) = A_s(z) - \frac{2\mu}{z} B_{s-1}(z) \qquad \text{for} \quad s = 1, \dots, N-1,$$

and

$$\begin{aligned} g_3 &= -\frac{2\mu}{z} B_{N-1}(z) u^{-2N} + G_1 g_2 + G_2 h_2 + G_3 g_1 + G_4 h_1, \\ h_3 &= H_1 g_2 + H_2 h_2 + H_3 g_1 + H_4 h_1. \end{aligned}$$

The identities (4.5) and [7, (10.29.1)]

$$K_{\nu-1}(x) - K_{\nu+1}(x) = -\frac{2\nu}{x} K_\nu(x) \tag{7.6}$$

give

$$uz I_{-\mu}(uz) \left(K_{\mu+1}(uz) - \frac{2\mu}{uz} K_\mu(uz) \right) + uz I_{1-\mu}(uz) K_\mu(uz) = 1.$$

Therefore, we can choose

$$\begin{aligned} G_3(u, z) &= uz \left(K_{\mu+1}(uz) - \frac{2\mu}{uz} K_\mu(uz) \right) I_\mu(uz), \\ H_3(u, z) &= u^2 K_\mu(uz) I_\mu(uz). \end{aligned}$$

The estimates (4.7), (4.8) give

$$|G_3(u, z)| \le C_3, \qquad |H_3(u, z)| \le D_3 u^2. \tag{7.7}$$

Similarly, we choose

$$\begin{aligned} G_4(u, z) &= z^2 \left(K_{\mu+1}(uz) - \frac{2\mu}{uz} K_\mu(z) \right) I_{\mu+1}(uz), \\ H_4(u, z) &= uz K_\mu(uz) I_{\mu+1}(uz), \end{aligned}$$

and estimate

$$|G_4(u, z)| \le C_4 |z|^2, \qquad |H_4(u, z)| \le D_4. \tag{7.8}$$

It follows from (7.2), (7.4) that

$$\begin{aligned} E_1(u, z) &= z I_{-\mu}(uz) - E_3(u, z), \\ E_2(u, z) &= \frac{z^2}{u} \left(-\frac{2\mu}{uz} I_{-\mu}(uz) + I_{-\mu+1}(uz) \right) - E_4(u, z). \end{aligned}$$

Therefore, we can choose

$$\begin{aligned} G_1(u, z) &= 1 - G_3(u, z), & H_1(u, z) &= -H_3(u, z), \\ G_2(u, z) &= -\frac{2\mu}{u^2} - G_4(u, z), & H_2(u, z) &= 1 - H_4(u, z). \end{aligned}$$

From (7.7), (7.8), we get

$$|G_1(u, z)| \le C_1, \qquad |H_1(u, z)| \le D_1 u^2, \tag{7.9}$$

$$|G_2(u, z)| \le C_2 (1 + |z|^2), \qquad |H_2(u, z)| \le D_2. \tag{7.10}$$

Since we know that

$$|g_1(u,z)| + |h_1(u,z)| + |g_2(u,z)| + |h_2(u,z)| \le \frac{K}{u^{2N}}$$

for $|\arg z| \le \frac{1}{2}\pi$, $0 < |z| \le R$, $u \ge u_0$, the estimates (7.7), (7.8), (7.9), (7.10) give

$$|g_3(u,z)| + |h_3(u,z)| \le \frac{L}{u^{2N-2}} \qquad \text{if} \quad |\arg z| \le \frac{1}{2}\pi, \quad 0 < |z| \le R, \quad u \ge u_3.$$

Now we divide both sides of (7.5) by $c\delta(u)$ and use (3.15), (6.12), (6.13) (with μ replaced by $-\mu$). Then we obtain the desired expansion of $W_3(u,-\mu,z)$ for $\Re\mu < 0$ and $|\arg z| \le \frac{1}{2}\pi$, $0 < |z| \le R$. The restriction on $\arg z$ is easily removed using (4.6) and $W_3(e^{\pi i m}z) = e^{\pi i(\mu+1)m}W_3(z)$. ■

8 Application to the confluent hypergeometric equation

The confluent hypergeometric differential equation

$$xv''(x) + (b-x)v'(x) - av(x) = 0$$

has solutions $M(a,b,x)$ and $U(a,b,x)$. Substituting $x = z^2$, $w = e^{-\frac{1}{2}z^2}z^b v$ we obtain the differential equation

$$w''(z) = \frac{1}{z}w'(z) + \left(u^2 + \frac{\mu^2-1}{z^2} + z^2\right)w(z), \tag{8.1}$$

where

$$a = \frac{1}{4}u^2 + \frac{1}{2}b, \qquad \mu = b - 1. \tag{8.2}$$

Equation (8.1) agrees with (2.1) when $f(z) = z^2$. Let A_s, B_s be defined as in Section 2 for $f(z) = z^2$. In this case, $A_s(z)$, $B_s(z)$ are polynomials. Throughout this section, we assume that a, b, u, μ satisfy (8.2).

The function $M(a,b,x)$ is given by a power series in x and $M(a,b,0) = 1$. Therefore, the function W_3 associated with (8.1) is given by

$$W_3(u,\mu,z) = \frac{2^{1-b}u^{b-1}}{\Gamma(b)}e^{-\frac{1}{2}z^2}z^b M\big(a,b,z^2\big). \tag{8.3}$$

Theorem 7.1 implies the following theorem.

Theorem 8.1. *Suppose that $b \in \mathbb{C}$ is not 0 or a negative integer, $u = te^{i\theta}$ with $t > 0$, $\theta \in \mathbb{R}$, and $N \ge 1$, $R > 0$. Then we can write*

$$\begin{aligned} &\frac{2^{1-b}u^{b-1}}{\Gamma(b)}e^{-\frac{1}{2}z^2}z^b M\big(\tfrac{1}{4}u^2 + \tfrac{1}{2}b, b, z^2\big) \\ &\quad = zI_{b-1}(uz)\left(\sum_{s=0}^{N-1}\frac{A_s(z)}{u^{2s}} + g_1(u,z)\right) + \frac{z}{u}I_b(uz)\left(\sum_{s=0}^{N-1}\frac{B_s(z)}{u^{2s}} + zh_1(u,z)\right), \end{aligned} \tag{8.4}$$

where

$$|g_1(u,z)| + |h_1(u,z)| \le \frac{L_1}{t^{2N}} \qquad \textit{for} \quad 0 < |z| \le R, \quad t \ge t_1.$$

and L_1, t_1 are positive constants independent of z and u (but possibly depending on b, θ, N, R). There is no restriction on $\arg z$. The polynomials $A_s(z)$, $B_s(z)$ appearing in (8.4) are determined by the recursion (2.2), (2.3) with $f(z) = z^2$ and the conditions $A_0(z) = 1$, $A_s(0) = 0$ for $s \ge 1$.

Suppose that $\Re b \geq 1$. Let $W_2(u,z)$ be the function associated with equation (8.1) which satisfies (2.7), (2.8). There are functions $\beta_1(u)$, $\beta_2(u)$ such that

$$W_2(u,z) = \beta_1(u)e^{-\frac{1}{2}z^2}z^b M\big(a,b,z^2\big) + \beta_2(u)e^{-\frac{1}{2}z^2}z^b U\big(a,b,z^2\big). \tag{8.5}$$

The determination of $\beta_1(u)$, $\beta_2(u)$ is not obvious. It is in this part of the analysis where there is an error in [8]. Slater [8, p. 79] derives $\beta_2(u) \sim \Gamma(a)2^{b-2}u^{1-b}$, and claims "we can take $\beta_1(u) = 0$" without proof. When comparing with [8], note that our $\beta_2(u)$ is denoted by $1/\beta_2(u)$ in [8]. Actually, the stated formula for $\beta_2(u)$ is correct but it is only the leading term of the required full asymptotic expansion given in the following lemma.

Lemma 8.2. *Suppose $\Re b \geq 1$. For every $N = 1,2,3,\dots,$ as $0 < u \to \infty$,*

$$\beta_2(u) = \Gamma(a)2^{b-2}u^{1-b}\left(1 + 2(1-b)\sum_{s=0}^{N-1}\frac{B_s'(0)}{u^{2s+2}} + O\left(\frac{1}{u^{2N+2}}\right)\right). \tag{8.6}$$

Proof. Suppose $\Re b > 1$. Then [7, (13.2.18)]

$$\lim_{z\to 0^+} z^{2b-2}U\big(a,b,z^2\big) = \frac{\Gamma(b-1)}{\Gamma(a)}$$

and (8.5) give

$$\lim_{z\to 0^+} z^{b-2}W_2(u,z) = \beta_2(u)\frac{\Gamma(b-1)}{\Gamma(a)}.$$

Comparing with (3.10), we obtain (8.6).

If $\Re b = 1$, $b \neq 1$, the proof is similar using Lemma 3.3(b) and [7, (13.2.18)]

$$U(a,b,x) = \frac{\Gamma(b-1)}{\Gamma(a)}x^{1-b} + \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + O(x) \qquad \text{as} \quad x \to 0^+.$$

If $b = 1$ we use Lemma 3.3(c) and [7, (13.2.19)]

$$\lim_{x\to 0^+}\frac{U(a,1,x)}{\ln x} = -\frac{1}{\Gamma(a)}. \qquad \blacksquare$$

We cannot show that $\beta_1(u) = 0$ but we can prove that $|\beta_1(u)|$ is very small as $u \to \infty$. To this end we need the following lemma.

Lemma 8.3. *Let $b \in \mathbb{C}$, $\Re x > 0$, and $\epsilon > 0$. There is a constant Q independent of a such that*

$$|\Gamma(a)U(a,b,x)| \leq Q \qquad \text{if} \quad \Re a \geq \epsilon.$$

Proof. We use the integral representation [7, (13.4.4)]

$$\Gamma(a)U(a,b,x) = \int_0^\infty e^{-xt}t^{a-1}(1+t)^{b-a-1}dt.$$

Therefore, if $\Re a \geq \epsilon$,

$$\begin{aligned}|\Gamma(a)U(a,b,x)| &\leq \int_0^\infty e^{-\Re x t}\left(\frac{t}{1+t}\right)^{\Re a-\epsilon}\left(\frac{t}{1+t}\right)^{\epsilon-1}(1+t)^{\Re b-2}dt\\ &\leq \int_0^\infty e^{-\Re x t}\left(\frac{t}{1+t}\right)^{\epsilon-1}(1+t)^{\Re b-2}dt =: Q. \qquad \blacksquare\end{aligned}$$

Lemma 8.4. *Suppose $\Re b \geq 1$. For every $q < R$ we have $\beta_1(u) = O(e^{-qu})$ as $0 < u \to \infty$.*

Proof. In the following let $0 < z \leq R$ (and b) be fixed. By Lemmas 8.2, 8.3, there is a constant $C_1 > 0$ such that, for sufficiently large $u > 0$,

$$\left|\beta_2(u)e^{-\frac{1}{2}z^2}z^b U\left(a,b,z^2\right)\right| \leq C_1\left|u^{1-b}\right|Q. \tag{8.7}$$

Using (3.8) we get from Theorem 8.1 with $N = 1$, for some constant $C_2 > 0$,

$$\left|e^{-\frac{1}{2}z^2}z^b M\left(a,b,z^2\right)\right| \geq C_2\left|u^{\frac{1}{2}-b}\right|e^{zu}. \tag{8.8}$$

Similarly, (3.9), (2.7), (2.8) yield

$$|W_2(u,z)| \leq C_3 u^{-\frac{1}{2}}e^{-zu}. \tag{8.9}$$

Substituting (8.7), (8.8) and (8.9) in (8.5). we find

$$|\beta_1(u)| \leq \frac{C_3}{C_2}\left|u^{b-1}\right|e^{-2zu} + \frac{C_1 Q}{C_2}u^{\frac{1}{2}}e^{-zu}.$$

If we choose $z = R$, we obtain the desired estimate. ∎

Lemma 8.5. *Suppose $\Re b \geq 1$. For every $N = 1, 2, 3, \ldots,$ the function*

$$\beta_2(u)e^{-\frac{1}{2}z^2}z^b U\left(a,b,z^2\right)$$

can be written in the form of the right-hand side of (2.7), and (2.8) holds with R replaced by $\frac{1}{3}R$.

Proof. Let

$$L(u,z) := \beta_1(u)e^{-\frac{1}{2}z^2}z^b M\left(a,b,z^2\right).$$

Applying Theorem 8.1 and Lemma 8.4, we estimate, for $0 < |z| \leq R$,

$$|L(u,z)| \leq C_1 e^{-qu}|z|\left(|I_\mu(uz)| + u^{-1}|I_{\mu+1}(uz)|\right), \tag{8.10}$$

where $q < R$ will be chosen later. We use the estimate

$$|I_\nu(x)| \leq C_2 e^{|x|} \qquad \text{for} \quad |\arg x| \leq \frac{3}{2}\pi \tag{8.11}$$

provided that $\Re\nu \geq 0$. This inequality follows from [5, (9.2), (9.3)]. Therefore, (8.10) yields

$$|L(u,z)| \leq C_3|z|e^{-qu+\frac{1}{3}Ru} \qquad \text{for} \quad 0 < |z| \leq \frac{1}{3}R, \quad |\arg z| \leq \frac{3}{2}\pi, \quad u \geq u_0. \tag{8.12}$$

Using (4.5), we have

$$L(u,z) = zK_\mu(uz)g(u,z) - \frac{z}{u}K_{\mu+1}(uz)zh(u,z),$$

where

$$g(u,z) = uI_{\mu+1}(uz)L(u,z), \qquad h(u,z) = -\frac{u^2}{z}I_\mu(uz)L(u,z).$$

From (8.11), (8.12), we get, for $0 < |z| \leq \frac{1}{3}R$, $|\arg z| \leq \frac{3}{2}\pi$,

$$|g(u,z)| \leq C_2 C_3 R u e^{u(\frac{2}{3}R-q)}, \qquad |h(u,z)| \leq C_2 C_3 u^2 e^{u(\frac{2}{3}R-q)}.$$

By (8.5), we can write $\beta_2(u)e^{-\frac{1}{2}z^2}z^b U(a,b,z^2)$ as the right-hand side of (2.7) with g_2 replaced by $g_2 - g$ and h_2 replaced by $h_2 - h$. If we choose $q = \frac{5}{6}R$, g and h become exponentially small as $u \to \infty$, and the theorem is proved. ∎

Lemma 8.6. *Suppose $\Re b \geq 1$. For all $N = 1, 2, 3, \ldots,$ we have, as $0 < u \to \infty$,*

$$\frac{\beta_2(u) 2^b u^{1-b}}{\Gamma(1+a-b)} = 1 + O\left(\frac{1}{u^{2N}}\right). \tag{8.13}$$

Moreover, for all $b \in \mathbb{C}$ and all $N = 1, 2, 3, \ldots,$ we have, as $0 < u \to \infty$,

$$\frac{\Gamma(1+a-b)}{\Gamma(a)} 2^{2-2b} u^{2b-2} = 1 + 2(1-b) \sum_{s=0}^{N-1} \frac{B_s'(0)}{u^{2s+2}} + O\left(\frac{1}{u^{2N+2}}\right). \tag{8.14}$$

Proof. We set

$$T(u,z) := \beta_2(u) e^{-\frac{1}{2}z^2} z^b U\big(a, b, z^2\big).$$

Using [7, (13.2.12)]

$$U\big(a, b, xe^{2i\pi}\big) = e^{-2\pi i b} U(a,b,x) + \frac{2\pi i e^{-\pi i b}}{\Gamma(b)\Gamma(1+a-b)} M(a,b,x)$$

and (8.3) we obtain

$$T(u, ze^{i\pi}) - e^{-\pi i b} T(u,z) = \beta_2(u) \frac{\pi i 2^b u^{1-b}}{\Gamma(1+a-b)} W_3(u,z).$$

Now we argue as in the proof of Lemma 3.2 (applying Lemma 8.5 twice) and arrive at (8.13). If $\Re b \geq 1$ the asymptotic formula (8.14) follows from (8.13) and Lemma 8.2. If $\Re b < 1$ we use (6.11). ∎

Theorem 8.7. *Suppose that $b \in \mathbb{C}$, $N \geq 1$ and $R > 0$. Then we can write*

$$\begin{aligned} &\Gamma\big(1 + \tfrac{1}{4}u^2 - \tfrac{1}{2}b\big) 2^{-b} u^{b-1} e^{-\frac{1}{2}z^2} z^b U\big(\tfrac{1}{4}u^2 + \tfrac{1}{2}b, b, z^2\big) \\ &\qquad = z K_{b-1}(uz) \left(\sum_{s=0}^{N-1} \frac{A_s(z)}{u^{2s}} + g_2(u,z)\right) - \frac{z}{u} K_b(uz) \left(\sum_{s=0}^{N-1} \frac{B_s(z)}{u^{2s}} + z h_2(u,z)\right), \end{aligned} \tag{8.15}$$

where

$$|g_2(u,z)| + |h_2(u,z)| \leq \frac{K_2}{u^{2N}} \qquad \text{for} \quad 0 < |z| \leq R, \quad u \geq u_2, \tag{8.16}$$

and K_2, u_2 are constants independent of z and u. There is no restriction on $\arg z$. The polynomials $A_s(z)$, $B_s(z)$ appearing in (8.15) are determined by the recursion (2.2), (2.3) with $f(z) = z^2$ and the conditions $A_0(z) = 1$, $A_s(0) = 0$ for $s \geq 1$.

Alternatively, we have

$$\begin{aligned} &\Gamma\big(\tfrac{1}{4}u^2 + \tfrac{1}{2}b\big) 2^{b-2} u^{1-b} e^{-\frac{1}{2}z^2} z^b U\big(\tfrac{1}{4}u^2 + \tfrac{1}{2}b, b, z^2\big) \\ &\qquad = z K_{b-1}(uz) \left(\sum_{s=0}^{N-1} \frac{a_s(z)}{u^{2s}} + g_2(u,z)\right) - \frac{z}{u} K_b(uz) \left(\sum_{s=0}^{N-1} \frac{b_s(z)}{u^{2s}} + z h_2(u,z)\right), \end{aligned} \tag{8.17}$$

where again (8.16) holds. The polynomials $a_s(z)$, $b_s(z)$ are defined by (6.2), (6.3).

Proof. We denote

$$V(u,\mu,z) := \Gamma(1+a-b)2^{-b}u^{b-1}e^{-\frac{1}{2}z^2}z^bU\big(a,b,z^2\big).$$

Then we have

$$V(u,-\mu,z) = \Gamma(a)2^{b-2}u^{1-b}e^{-\frac{1}{2}z^2}z^bU\big(a,b,z^2\big)$$

which follows from [7, (13.2.40)]

$$U(a,b,x) = x^{1-b}U(1+a-b,2-b,x).$$

For any $b \in \mathbb{C}$, (6.9), (6.10), (6.12), (6.13), (8.14) show that the expansions (8.15) and (8.17) are equivalent. We will prove (8.15) and (8.17) for $\Re\mu \geq 0$ and $\Re\mu < 0$, respectively.

Suppose $\Re\mu \geq 0$. Then (8.15), (8.16) follow from Lemmas 8.5 and 8.6 when $|\arg z| \leq \frac{3}{2}\pi - \delta$. Since the function $V(u,\mu,z)$ is independent of R we can replace $\frac{1}{3}R$ by R. By Theorem 4.1, we can remove the restriction on $\arg z$. Note that in the proof of Theorem 4.1 we only used that $W_2(u,z)$ solves (2.1) and admits the asymptotic expansions (2.7), (2.8). Therefore, we can apply the theorem to the function $V(u,\mu,z)$ in place of $W_2(u,z)$.

Now suppose that $\Re\mu < 0$. Then, using the expansion we just proved,

$$\begin{aligned} V(u,-\mu,z) = zK_{-\mu}(uz)&\left(\sum_{s=0}^{N-1}\frac{A_s(-\mu,z)}{u^{2s}} + g_2(u,z)\right) \\ &- \frac{z}{u}K_{-\mu+1}(uz)\left(\sum_{s=0}^{N-1}\frac{B_s(-\mu,z)}{u^{2s}} + zh_2(u,z)\right). \end{aligned}$$

Using (6.2), (6.3), (7.6) and $K_\nu(x) = K_{-\nu}(x)$, we obtain (8.17), (8.16). ■

So far we considered only asymptotic expansions of $U(a,b,z^2)$ as $0 < u \to \infty$. Now we set $u = te^{i\theta}$, where $t > 0$ and $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$. Using the notation of Section 5, we have

$$e^{-i\theta}W_2(t,x) = \beta_1(u)e^{-\frac{1}{2}z^2}z^bM\big(a,b,z^2\big) + \beta_2(u)e^{-\frac{1}{2}z^2}z^bU\big(a,b,z^2\big).$$

It is easy to see that Lemma 8.2 remains valid. Since we allow $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$, $a = \frac{1}{4}u^2 + \frac{1}{2}b$ may have negative real part. We need a modification of Lemma 8.3.

Lemma 8.8. *Let $b \in \mathbb{C}$, $-\pi < \arg x < 0$, $|\arg(a-1)| \leq \pi - \delta$ for some $\delta > 0$. Then there is a constant Q independent of a such that*

$$|\Gamma(a)U(a,b,x)| \leq Q.$$

Proof. We use the integral representation [7, (13.4.14)]

$$\big(e^{2\pi i(a-1)} - 1\big)\Gamma(a)U(a,b,x) = \int_C e^{-xt}t^{a-1}(1+t)^{b-a-1}dt,$$

where the contour C starts at $+\infty i$ and follows the positive imaginary axis, then describes a loop around 0 in positive direction and returns to $+\infty i$. The argument of t starts at $\frac{1}{2}\pi$ and increases to $\frac{5}{2}\pi$. It will be sufficient to estimate $\Gamma(a)U(a,b,x)$ in the sector $\frac{1}{2}\pi \leq \arg(a-1) \leq \alpha_0$, where $\frac{1}{2}\pi < \alpha_0 < \pi$. The loop is chosen so that $w = \frac{t}{1+t}$ describes the circle $|w| = \cos\theta_0$, where $\theta_0 \in (0, \frac{1}{2}\pi)$ is the unique solution of the equation

$$\cos\theta_0 = e^{\theta_0 \tan\alpha_0}.$$

Then one obtains $|w^{a-1}| \leq 1$ on the contour C which implies the desired estimate. ■

The proofs of Lemma 8.6 and Theorem 8.7 can be easily modified to give the desired asymptotic expansions for $u = te^{i\theta}$ as $0 < t \to \infty$ for fixed $\theta \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$. In (8.16) we now have $u = te^{i\theta}$, $t \geq t_2$ and $0 < |z| \leq R$.

9 Comparison with Temme [9]

It is known [7, (5.11.13)] that, as $z \to \infty$, $|\arg z| \le \pi - \delta$,

$$\frac{\Gamma(z+r)}{\Gamma(z+s)} \sim z^{r-s} \sum_{n=0}^{\infty} \binom{r-s}{n} B_n^{(r-s+1)}(r) \frac{1}{z^n}, \tag{9.1}$$

where the generalized Bernoulli polynomials $B_n^{(\ell)}(x)$ are defined by the Maclaurin expansion

$$\left(\frac{t}{e^t-1}\right)^{\ell} e^{xt} = \sum_{n=0}^{\infty} B_n^{(\ell)}(x) \frac{t^n}{n!}.$$

We apply (9.1) with $z = \frac{1}{4}u^2$, $0 < u \to \infty$, and $r = 1 - \frac{1}{2}b$, $s = \frac{1}{2}b$. Then we obtain with $a = \frac{1}{4}u^2 + \frac{1}{2}b$,

$$\frac{\Gamma(1+a-b)}{\Gamma(a)} 2^{2-2b} u^{2b-2} \sim \sum_{n=0}^{\infty} \frac{d_n}{u^{2n}},$$

where

$$d_n = 4^n \binom{1-b}{n} B_n^{(2-b)} \left(1 - \frac{1}{2}b\right).$$

We notice that

$$\left(\frac{t}{e^t-1}\right)^{2-b} e^{(1-\frac{1}{2}b)t} = \left(\frac{\frac{1}{2}t}{\sinh \frac{1}{2}t}\right)^{2-b}$$

is an even function of t. Therefore, $d_n = 0$ for odd n.

It follows from (8.14) that

$$B_n'(0) = \frac{1}{2}\frac{1}{1-b} d_{n+1},$$

and then from (6.2), (6.3)

$$a_n(0) = \tilde{d}_n, \qquad b_n'(0) = -\frac{1}{2}\frac{1}{1-b}\tilde{d}_{n+1}, \tag{9.2}$$

where $\tilde{d}_n$ is obtained from d_n by replacing b by $2-b$, that is,

$$\tilde{d}_n = 4^n \binom{b-1}{n} B_n^{(b)} \left(\frac{1}{2}b\right).$$

Temme [9, (3.22)] obtained the asymptotic expansion of (8.17) involving polynomials $a_n^{\dagger}(z)$, $b_n^{\dagger}(z)$ in place of $a_n(z)$, $b_n(z)$. The polynomials $a_n^{\dagger}(z)$, $b_n^{\dagger}(z)$ as follows. Introduce the function

$$f(s,z) = e^{z^2\mu(s)} \left(\frac{\frac{1}{2}s}{\sinh \frac{1}{2}s}\right)^{b}, \qquad \mu(s) = \frac{1}{s} - \frac{1}{e^s-1} - \frac{1}{2},$$

and its Maclaurin expansion

$$f(s,z) = \sum_{k=0}^{\infty} c_k(z) s^k.$$

Then recursively, set $c_k^{(0)} = c_k$ and

$$c_k^{(n+1)} = 4\big(z^2 c_{k+2}^{(n)} + (1-b+k)c_{k+1}^{(n)}\big),$$

where $k \geq 0$ and $n \geq 0$. Then set

$$a_n^\dagger = c_0^{(n)}, \qquad b_n^\dagger = -2zc_1^{(n)}.$$

Theorem 9.1. *For every $n = 0, 1, 2, \dots,$ we have $a_n = a_n^\dagger$ and $b_n = b_n^\dagger$.*

Proof. The function f satisfies the partial differential equation

$$4\frac{\partial f}{\partial s} = \frac{\partial^2 f}{\partial z^2} + \frac{1}{z}\left(2b - 1 - 4\frac{z^2}{s}\right)\frac{\partial f}{\partial z} - z^2 f.$$

This implies

$$4(k+1)c_{k+1} + 4zc_{k+1}' = c_k'' + \frac{2b-1}{z}c_k' - z^2 c_k, \qquad ()' = \frac{d}{dz}. \tag{9.3}$$

By induction on n one can show that (9.3) is also true with c_k replaced by $c_k^{(n)}$ for any $n = 0, 1, 2, \dots$. If we use this extended equation with $k = 0$ and $k = 1$, then we obtain (2.2), (2.3) with $a_s^\dagger$, $b_s^\dagger$ in place of A_s, B_s, respectively.

When $z = 0$, we have

$$a_n^\dagger(0) = c_0^{(n)}(0) = 4^n(1-b)_n c_n(0) = 4^n\frac{(1-b)_n}{n!}B_n^{(b)}\left(\frac{1}{2}b\right).$$

Comparing with (9.2) and using that $c_n(0) = 0$ for odd n, we find, for all n,

$$a_n^\dagger(0) = a_n(0). \tag{9.4}$$

Since both a_n, b_n and $a_n^\dagger, b_n^\dagger$ solve (2.2), (2.3), (9.4) implies that $a_n^\dagger = a_n$, $b_n^\dagger = b_n$ for all n. ■

10 Concluding remark

In this paper we started from Olver's paper [5], added some results, and then applied them to the confluent hypergeometric functions. A referee pointed out that Chapter 12 of Olver's book [6] contains a reworked version of [5] also involving error bounds. It would be interesting to start from this book chapter and derive results analogous to the ones obtained in the present paper. However, in contrast to [5] the book chapter assumes that μ is positive while in our original problem [1] μ is complex. Therefore, an extension of the results in [6, Chapter 12] to complex μ would be required to obtain results for the confluent hypergeometric functions in full generality.

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