The Inverse Spectral Problem for Jacobi-Type Pencils

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Abstract. In this paper we study the inverse spectral problem for Jacobi-type pencils. By a Jacobi-type pencil we mean the following pencil $J_5 - \lambda J_3$, where J_3 is a Jacobi matrix and J_5 is a semi-infinite real symmetric five-diagonal matrix with positive numbers on the second subdiagonal. In the case of a special perturbation of orthogonal polynomials on a finite interval the corresponding spectral function takes an explicit form.

Key words: operator pencil; recurrence relation; orthogonal polynomials; spectral function

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1 Introduction

The theory of operator pencils in Banach spaces has many applications and it is actively developing nowadays, see [7, 8, 10]. By operator pencils or operator polynomials one means polynomials of complex variable λ whose coefficients are linear bounded operators acting in a Banach space X

$$L(\lambda) = \sum_{j=0}^{m} \lambda^j A_j,$$

where $A_j \colon X \to X$, $j = 0, \ldots, m$. As it was noted in Rodman's book [10], there exist three broad topics in this theory which are the main foci of interest:

 (a) linearization, i.e., reduction to a linear polynomial;

 (b) various types of factorizations;

 (c) the problems of multiple completness of eigenvectors and generalized eigenvectors.

If X is a Hilbert space and all operators A_j are self-adjoint, the pencil $L(\lambda)$ is said to be self-adjoint. In the case $m = 1$ ($m = 2$) the pencil is called linear (respectively quadratic). For the recent progress on quadratic operator pencils and their applications we refer to the book of Möller and Pivovarchik [8].

Linear self-adjoint operator pencils has no spectral theory which could be compatible with the theory of single or commuting self-adjoint operators. The generalized eigenvalue problem

$$(A_0 + \lambda A_1)x = 0, \qquad x \in X,$$

is essentially more generic than the usual eigenvalue problem for a self-adjoint operator. We can explain this difference by the following astonishing result from the theory of linear matrix pencils.

Theorem 1.1 ([9, Theorem 15.2.1, p. 341])**.** *Any real square matrix B can be written as $B = AM^{-1}$ or $B = M^{-1}A$ where A and M are suitable symmetric matrices.*

Nevertheless, the general spectral theory for linear operator pencils has old and new contributions. In particular, it has applications to quotients of bounded operators and to the study of electron waveguides in graphene, see [5] and references therein.

Recently, Ben Amara, Vladimirov and Shkalikov investigated the following linear pencil of differential operators [2]

$$(py'')'' - \lambda(-y'' + cry) = 0. \tag{1.1}$$

The initial conditions are $y(0) = y'(0) = y(1) = y'(1) = 0$, or $y(0) = y'(0) = y'(1) = (py'')'(1) + \lambda\alpha y(1) = 0$. Here $p, r \in C[0, 1]$ are uniformly positive, while the parameters c and α are real. Equation (1.1) has several physical applications, including a motion of a partially fixed bar with additional constraints in the elasticity theory [2].

It turns out that equation (1.1) is related to certain generalized orthogonal polynomials, just as the Sturm–Liouville differential operator is related to the orthogonal polynomials on the real line (OPRL).

The theory of orthogonal polynomials has numerous old and new contributions and applications, see [4, 6, 11, 12, 13, 14]. This theory is closely related to spectral problems for Jacobi matrices. The direct and inverse spectral problems for Jacobi matrices (with matrix elements) were described, e.g., in [1, 3]. For various other types of spectral problems we refer to historical notes in [15]. Recall the following basic definition from [16].

Definition 1.2. A set $\Theta = (J_3, J_5, \alpha, \beta)$, where $\alpha > 0$, $\beta \in \mathbb{R}$, J_3 is a Jacobi matrix and J_5 is a semi-infinite real symmetric five-diagonal matrix with positive numbers on the second subdiagonal, is said to be *a Jacobi-type pencil* (*of matrices*).

From this definition we see that matrices J_3 and J_5 have the following form

$$J_3 = \begin{pmatrix} b_0 & a_0 & 0 & 0 & 0 & \cdots \\ a_0 & b_1 & a_1 & 0 & 0 & \cdots \\ 0 & a_1 & b_2 & a_2 & 0 & \cdots \\ \vdots & \vdots & \vdots & & \ddots \end{pmatrix}, \qquad a_k > 0, \quad b_k \in \mathbb{R}, \quad k \in \mathbb{Z}_+, \tag{1.2}$$

$$J_5 = \begin{pmatrix} \alpha_0 & \beta_0 & \gamma_0 & 0 & 0 & 0 & \cdots \\ \beta_0 & \alpha_1 & \beta_1 & \gamma_1 & 0 & 0 & \cdots \\ \gamma_0 & \beta_1 & \alpha_2 & \beta_2 & \gamma_2 & 0 & \cdots \\ 0 & \gamma_1 & \beta_2 & \alpha_3 & \beta_3 & \gamma_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots \end{pmatrix}, \qquad \alpha_n, \beta_n \in \mathbb{R}, \quad \gamma_n > 0, \quad n \in \mathbb{Z}_+. \tag{1.3}$$

With a Jacobi-type pencil of matrices Θ one associates a system of polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$, such that

$$p_0(\lambda) = 1, \qquad p_1(\lambda) = \alpha\lambda + \beta,$$

and

$$(J_5 - \lambda J_3)\vec{p}(\lambda) = 0, \tag{1.4}$$

where $\vec{p}(\lambda) = (p_0(\lambda), p_1(\lambda), p_2(\lambda), \dots)^{\mathrm{T}}$. Here the superscript T means the transposition. Polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ are said to be *associated to the Jacobi-type pencil of matrices* Θ.

Observe that for each system of orthonormal polynomials on the real line with $p_0 = 1$ one may choose J_3 to be the corresponding Jacobi matrix (which elements are the recurrence coefficients), $J_5 = J_3^2$, and α, β being the coefficients of p_1 $(p_1(\lambda) = \alpha\lambda + \beta)$.

One can rewrite relation (1.4) in the scalar form

$$\gamma_{n-2}p_{n-2}(\lambda) + (\beta_{n-1} - \lambda a_{n-1})p_{n-1}(\lambda) + (\alpha_n - \lambda b_n)p_n(\lambda)$$
$$+ (\beta_n - \lambda a_n)p_{n+1}(\lambda) + \gamma_n p_{n+2}(\lambda) = 0, \qquad n \in \mathbb{Z}_+, \tag{1.5}$$

where $p_{-2}(\lambda) = p_{-1}(\lambda) = 0$, $\gamma_{-2} = \gamma_{-1} = a_{-1} = \beta_{-1} = 0$.

Let us return to the differential equation (1.1) and explain how it is related to polynomials from (1.5). The idea is the same as in the classical case. We consider a partition of the interval $[0, 1]$

$$0 = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = 1,$$

with a uniform step h. Set $y_j := y(x_j)$, $p_j := p(x_j)$, $r_j := r(x_j)$, $0 \leq j \leq N$. Then we replace the derivatives in (1.1) by the corresponding discretizations $y'(x_j) \approx \frac{y_{j+1}-y_j}{h}$, $y''(x_j) \approx \frac{y_{j+1}-2y_j+y_{j-1}}{h^2}$. We obtain the following equation

$$p_{j-1}y_{j-2} - 2(p_j + p_{j-1})y_{j-1} + (p_{j+1}) + 4p_j + p_{j-1})y_j - 2(p_{j+1} + p_j)y_{j+1}$$
$$+ p_{j+1}y_{j+2} - \lambda h^2\big(y_{j-1} - \big(2 - h^2 cr_j\big)y_j + y_{j+1}\big) = 0.$$

Denote

$$\alpha_j = p_{j+1} + 4p_j + p_{j-1}, \qquad \beta_j = -2(p_{j+1} + p_j), \qquad \gamma_j = p_{j+1},$$
$$a_j = h^2, \qquad b_j = \big(-2 - h^2 cr_j\big)h^2.$$

Then we get

$$\gamma_{j-2}y_{j-2} + \beta_{j-1}y_{j-1} + \alpha_j y_j + \beta_j y_{j+1} + \gamma_j y_{j+2} + \lambda\big(a_{j-1}y_{j-1} + b_j y_j + a_j y_{j+1}\big) = 0.$$

If we set $\widetilde{\lambda} = -\lambda$, we obtain a relation of the form (1.5). Thus, relation (1.5) forms a discrete grid model for the equation (1.1). The boundary conditions $y(0) = 0$, $y'(0) = 0$ correspond to the convention $y_{-2} = y_{-1} = 0$. Instead of $[0, 1]$ we can consider $[0, +\infty)$. The conditions on the right end of the bar we replace by conditions on the left end which are controlled by α and β.

In this paper we shall introduce and study the direct and inverse spectral problems for Jacobi type pencils. We shall also consider a special perturbation of the class of orthonormal polynomials on a finite real interval. In that case it is possible to obtain a more convenient integral representation for the corresponding spectral function.

Notations. As usual, we denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z}_+$, the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively. By $\mathbb{P}$ we denote the set of all polynomials with complex coefficients.

By l_2 we denote the usual Hilbert space of all complex sequences $c = (c_n)_{n=0}^\infty = (c_0, c_1, c_2, \dots)^{\mathrm{T}}$ with the finite norm $\|c\|_{l_2} = \sqrt{\sum_{n=0}^\infty |c_n|^2}$. Here T means the transposition. The scalar product of two sequences $c = (c_n)_{n=0}^\infty$, $d = (d_n)_{n=0}^\infty \in l_2$ is given by $(c, d)_{l_2} = \sum_{n=0}^\infty c_n\overline{d_n}$. We denote $\vec{e}_m = (\delta_{n,m})_{n=0}^\infty \in l_2$, $m \in \mathbb{Z}_+$. By $l_{2,\mathrm{fin}}$ we denote the set of all finite vectors from l_2, i.e., vectors with all, but finite number, components being zeros. By $\mathrm{diag}(c_1, c_2, c_3, \dots)$ we mean a semi-infinite matrix with $c_j \in \mathbb{C}$ in j-th column and j-th row and zeros outside the main diagonal.

By $\mathfrak{B}(\mathbb{R})$ we denote the set of all Borel subsets of $\mathbb{R}$. If σ is a (non-negative) bounded measure on $\mathfrak{B}(\mathbb{R})$ then by L_σ^2 we denote a Hilbert space of all (classes of equivalences of) complex-valued functions f on $\mathbb{R}$ with a finite norm $\|f\|_{L_\sigma^2} = \sqrt{\int_\mathbb{R} |f(x)|^2 \mathrm{d}\sigma}$. The scalar product of two functions $f, g \in L_\sigma^2$ is given by $(f, g)_{L_\sigma^2} = \int_\mathbb{R} f(x)\overline{g(x)}\mathrm{d}\sigma$. By $[f]$ we denote the class of equivalence in L_σ^2 which contains the representative f. By $\mathcal{P}$ we denote a set of all (classes of equivalence which contain) polynomials in L_σ^2. As usual, we sometimes use the representatives instead of their classes in formulas. Let B be an arbitrary linear operator in L_σ^2 with the domain $\mathcal{P}$. Let $f(\lambda) \in \mathbb{P}$ be nonzero and of degree $d \in \mathbb{Z}_+$, $f(\lambda) = \sum_{k=0}^{d} d_k \lambda^k$, $d_k \in \mathbb{C}$. We set

$$f(B) = \sum_{k=0}^{d} d_k B^k, \qquad B^0 := E|_\mathcal{P}.$$

If $f \equiv 0$, then $f(B) := 0|_\mathcal{P}$.

If H is a Hilbert space then $(\cdot, \cdot)_H$ and $\|\cdot\|_H$ mean the scalar product and the norm in H, respectively. Indices may be omitted in obvious cases. For a linear operator A in H, we denote by $D(A)$ its domain, by $R(A)$ its range, by $\mathrm{Ker}\, A$ its null subspace (kernel), and A^* means the adjoint operator if it exists. If A is invertible then A^{-1} means its inverse. $\overline{A}$ means the closure of the operator, if the operator is closable. If A is bounded then $\|A\|$ denotes its norm. For a set $M \subseteq H$ we denote by $\overline{M}$ the closure of M in the norm of H. By $\mathrm{Lin}\, M$ we mean the set of all linear combinations of elements of M, and $\overline{\mathrm{span}}\, M := \overline{\mathrm{Lin}\, M}$. By $E = E_H$ ($0 = 0_H$) we denote the identity operator in H, i.e., $E_H x = x$, $x \in H$ (respectively the null operator in H, i.e., $0_H x = 0$, $x \in H$). If H_1 is a subspace of H, then $P_{H_1} = P_{H_1}^H$ is an operator of the orthogonal projection on H_1 in H.

2   Preliminaries

In this section, for the convenience of the reader, we recall basic definitions and results from [16]. Then we state the direct and inverse spectral problems for the Jacobi-type pencils. The direct problem will be solved immediately, while the inverse problem will be considered in the next section. Set

$$u_n := J_3 \vec{e}_n = a_{n-1}\vec{e}_{n-1} + b_n \vec{e}_n + a_n \vec{e}_{n+1},$$
$$w_n := J_5 \vec{e}_n = \gamma_{n-2}\vec{e}_{n-2} + \beta_{n-1}\vec{e}_{n-1} + \alpha_n \vec{e}_n + \beta_n \vec{e}_{n+1} + \gamma_n \vec{e}_{n+2}, \qquad n \in \mathbb{Z}_+.$$

Here and in what follows by $\vec{e}_k$ with negative k we mean (vector) zero. The following operator

$$Af = \frac{\zeta}{\alpha}(\vec{e}_1 - \beta\vec{e}_0) + \sum_{n=0}^{\infty} \xi_n w_n,$$

$$f = \zeta\vec{e}_0 + \sum_{n=0}^{\infty} \xi_n u_n \in l_{2,\mathrm{fin}}, \qquad \zeta, \xi_n \in \mathbb{C}, \tag{2.1}$$

with $D(A) = l_{2,\mathrm{fin}}$ is said to be *the associated operator for the Jacobi-type pencil* Θ. Notice that in the sums in (2.1) only finite number of ξ_n are nonzero. We shall always assume this in the case of elements from the linear span. In particular, we have

$$AJ_3\vec{e}_n = J_5\vec{e}_n, \qquad n \in \mathbb{Z}_+,$$

and therefore

$$AJ_3 = J_5.$$

As usual, the matrices J_3 and J_5 define linear operators with the domain $l_{2,\text{fin}}$ which we denote by the same letters.

For an arbitrary non-zero polynomial $f(\lambda) \in \mathbb{P}$ of degree $d \in \mathbb{Z}_+$, $f(\lambda) = \sum\limits_{k=0}^{d} d_k \lambda^k$, $d_k \in \mathbb{C}$, we set $f(A) = \sum\limits_{k=0}^{d} d_k A^k$. Here $A^0 := E|_{l_{2,\text{fin}}}$. For $f(\lambda) \equiv 0$, we set $f(A) = 0|_{l_{2,\text{fin}}}$. The following relations hold [16]

$$\vec{e}_n = p_n(A)\vec{e}_0, \qquad n \in \mathbb{Z}_+, \tag{2.2}$$

$$\big(p_n(A)\vec{e}_0, p_m(A)\vec{e}_0\big)_{l_2} = \delta_{n,m}, \qquad n, m \in \mathbb{Z}_+. \tag{2.3}$$

Denote by $\{r_n(\lambda)\}_{n=0}^{\infty}$, $r_0(\lambda) = 1$, the system of polynomials satisfying

$$J_3 \vec{r}(\lambda) = \lambda \vec{r}(\lambda), \qquad \vec{r}(\lambda) = (r_0(\lambda), r_1(\lambda), r_2(\lambda), \dots)^{\mathrm{T}}. \tag{2.4}$$

These polynomials are orthonormal on the real line with respect to a (possibly non-unique) non-negative finite measure σ on $\mathfrak{B}(\mathbb{R})$ (Favard's theorem). Consider the following operator

$$U \sum_{n=0}^{\infty} \xi_n \vec{e}_n = \left[\sum_{n=0}^{\infty} \xi_n r_n(x) \right], \qquad \xi_n \in \mathbb{C}, \tag{2.5}$$

which maps $l_{2,\text{fin}}$ onto $\mathcal{P}$. Here by $\mathcal{P}$ we denote a set of all (classes of equivalence which contain) polynomials in L_σ^2. Denote

$$\mathcal{A} = \mathcal{A}_\sigma = UAU^{-1}. \tag{2.6}$$

The operator $\mathcal{A} = \mathcal{A}_\sigma$ is said to be *the model representation in L_σ^2 of the associated operator A.*

Theorem 2.1 ([16]). *Let $\Theta = (J_3, J_5, \alpha, \beta)$ be a Jacobi-type pencil. Let $\{r_n(\lambda)\}_{n=0}^{\infty}$, $r_0(\lambda) = 1$, be a system of polynomials satisfying (2.4) and σ be their (arbitrary) orthogonality measure on $\mathfrak{B}(\mathbb{R})$. The associated polynomials $\{p_n(\lambda)\}_{n=0}^{\infty}$ satisfy the following relations*

$$\int_{\mathbb{R}} p_n(\mathcal{A})(1)\overline{p_m(\mathcal{A})(1)}\mathrm{d}\sigma = \delta_{n,m}, \qquad n, m \in \mathbb{Z}_+, \tag{2.7}$$

where $\mathcal{A}$ is the model representation in L_σ^2 of the associated operator A.

We have now recalled all basic notations and results which we shall need in the present paper. We introduce the following definition.

Definition 2.2. Let $\Theta = (J_3, J_5, \alpha, \beta)$ be a Jacobi-type pencil and $\{p_n(\lambda)\}_{n=0}^{\infty}$ be the associated polynomials to Θ. A sesquilinear functional $S(u, v)$, $u, v \in \mathbb{P}$, satisfying the following relation

$$S(p_n, p_m) = \delta_{n,m}, \qquad n, m \in \mathbb{Z}_+,$$

is said to be *the spectral function of the Jacobi-type pencil Θ.*

In a usual manner, we may introduce the corresponding direct and inverse spectral problems. *The direct spectral problem for a Jacobi-type pencil Θ* consists in searching for answers on the following questions:

1) Does the spectral function exist?

2) If the spectral function exists, is it unique?

3) If the spectral function exists, how to find it (or them)?

By relation (2.3) we see that the spectral function always exists. By the linearity it follows that the spectral function is unique. It has the following representation

$$S(u, v) = \left(u(A)\vec{e}_0, v(A)\vec{e}_0\right)_{l_2}, \qquad u, v \in \mathbb{P}, \tag{2.8}$$

where A is the associated operator for Θ. By (2.8) we obtain that

$$\overline{S(u, v)} = S(v, u), \qquad u, v \in \mathbb{P}, \tag{2.9}$$

and

$$S(u, u) \geq 0, \qquad u \in \mathbb{P}. \tag{2.10}$$

Moreover, the following integral representation holds

$$S(u, v) = \int_{\mathbb{R}} u(\mathcal{A}_\sigma)(1)\overline{v(\mathcal{A}_\sigma)(1)}\mathrm{d}\sigma, \qquad u, v \in \mathbb{P}. \tag{2.11}$$

Thus, the direct spectral problem for Θ is solved in full.

The inverse spectral problem for a Jacobi-type pencil Θ consists in searching for answers on the following questions:

(a) Is it possible to reconstruct the Jacobi-type pencil $\Theta = (J_3, J_5, \alpha, \beta)$ using its spectral function? If it is possible, what is the procedure of the reconstruction?

(b) What are necessary and sufficient conditions for a sesquilinear functional $\sigma(u, v)$, $u, v \in \mathbb{P}$, to be the spectral function of a Jacobi-type pencil $\Theta = (J_3, J_5, \alpha, \beta)$?

Questions (a) and (b) will be investigated in the next section.

3   The inverse spectral problem for a Jacobi-type pencil

An answer to the question (b) of the inverse spectral problem is given by the following theorem.

Theorem 3.1. *A sesquilinear functional* $S(u, v)$, $u, v \in \mathbb{P}$, *satisfying relations* (2.9), (2.10), *is the spectral function of a Jacobi-type pencil if and only if it admits the following integral representation*

$$S(u, v) = \int_{\mathbb{R}} u(\mathcal{A})(1)\overline{v(\mathcal{A})(1)}\mathrm{d}\sigma, \qquad u, v \in \mathbb{P}, \tag{3.1}$$

where σ *is a non-negative measure on* $\mathfrak{B}(\mathbb{R})$ *with all finite power moments,*

$$\int_{\mathbb{R}} \mathrm{d}\sigma = 1, \qquad \int_{\mathbb{R}} |g(x)|^2 \mathrm{d}\sigma > 0,$$

for any non-zero complex polynomial g, *and* $\mathcal{A}$ *is a linear operator in* L_σ^2 *with the following properties:*

(i) $D(\mathcal{A}) = \mathcal{P}$;

(ii) *for each* $k \in \mathbb{Z}_+$ *it holds*

$$\mathcal{A}x^k = \xi_{k,k+1}x^{k+1} + \sum_{j=0}^{k} \xi_{k,j}x^j, \tag{3.2}$$

where $\xi_{k,k+1} > 0$, $\xi_{k,j} \in \mathbb{R}$, $0 \leq j \leq k$;

(iii) *the operator $\mathcal{A}\Lambda_0$ is symmetric, here by Λ_0 we denote the operator of the multiplication by an independent variable in L^2_σ restricted to $\mathcal{P}$.*

Proof. *Necessity.* Let $\Theta = (J_3, J_5, \alpha, \beta)$ be a Jacobi-type pencil and A be the associated operator of Θ. Define polynomials $\{r_n(\lambda)\}$, the measure σ, the operators U and $\mathcal{A} = \mathcal{A}_\sigma$ as in the introduction, see (2.4), (2.5), (2.6). Let $S(u,v)$ be the spectral function of the pencil Θ. It has an integral representation (2.11). It remains to check that σ and $\mathcal{A}$ possess all the required properties in the statement of the theorem.

Since σ is generated by the Jacobi matrix J_3, it has all required properties as in the statement of the theorem. The operator $\mathcal{A}_\sigma$ is linear and defined on $\mathcal{P}$.

Lemma 3.2. *The associated operator A of a Jacobi-type pencil Θ has the following properties:*

1) *for each $n \in \mathbb{Z}_+$ it holds*

$$\vec{e}_n = d_{n,n} A^n \vec{e}_0 + \sum_{j=0}^{n-1} d_{n,j} A^j \vec{e}_0, \qquad (3.3)$$

where $d_{n,n} > 0$, $d_{n,j} \in \mathbb{R}$, $0 \le j \le n - 1$;

2) *for each $n \in \mathbb{Z}_+$ it holds*

$$A^n \vec{e}_0 = c_{n,n} \vec{e}_n + \sum_{j=0}^{n-1} c_{n,j} \vec{e}_j, \qquad (3.4)$$

where $c_{n,n} > 0$, $c_{n,j} \in \mathbb{R}$, $0 \le j \le n - 1$;

3) *for each $n \in \mathbb{Z}_+$ it holds*

$$A\vec{e}_n = f_{n,n+1} \vec{e}_{n+1} + \sum_{j=0}^{n} f_{n,j} \vec{e}_j, \qquad (3.5)$$

where $f_{n,n+1} > 0$, $f_{n,j} \in \mathbb{R}$, $0 \le j \le n$.

Proof of Lemma 3.2. Let $p_n(\lambda) = \sum\limits_{j=0}^{n} d_{n,j}\lambda^j$, $d_{n,n} > 0$, $d_{n,j} \in \mathbb{R}$, be the associated polynomials of the pencil. By (2.2) we conclude that relation (3.3) holds.

Let us check relation (3.4) by the induction argument. For the cases $n = 0, 1$ relation (3.4) obviously holds, see the definition of A. Assume that (3.4) holds for $n = 1, 2, \ldots, r$, $r \in \mathbb{N}$. By (3.3) with $n = r + 1$ we may write

$$A^{r+1}\vec{e}_0 = \frac{1}{d_{r+1,r+1}} \vec{e}_{r+1} - \frac{1}{d_{r+1,r+1}} \sum_{j=0}^{r} d_{r+1,j} A^j \vec{e}_0$$

$$= \frac{1}{d_{r+1,r+1}} \vec{e}_{r+1} + \sum_{j=0}^{r} \sum_{m=0}^{j} \frac{(-1)d_{r+1,j} c_{j,m}}{d_{r+1,r+1}} \vec{e}_m.$$

Finally, in order to prove relation (3.5) we apply the operator A to the both sides of relation (3.3) and use (3.4)

$$A\vec{e}_n = d_{n,n} A^{n+1} \vec{e}_0 + \sum_{j=0}^{n-1} d_{n,j} A^{j+1} \vec{e}_0$$

$$= d_{n,n} c_{n+1,n+1} \vec{e}_{n+1} + \sum_{j=0}^{n} d_{n,n} c_{n+1,j} \vec{e}_j + \sum_{j=0}^{n-1} \sum_{k=0}^{j+1} d_{n,j} c_{j+1,k} \vec{e}_k, \qquad n \in \mathbb{Z}_+.$$

Lemma 3.2 is proved. $\blacksquare$

Return to the proof of Theorem 3.1. Choose an arbitrary $k \in \mathbb{Z}_+$. Observe that

$$x^k = \sum_{j=0}^{k} g_{k,j} r_j(x), \qquad g_{k,j} \in \mathbb{R}, \quad g_{k,k} > 0.$$

By (3.5) we may write

$$
\begin{aligned}
\mathcal{A}_\sigma [x^k] &= \sum_{j=0}^{k} g_{k,j} U A \vec{e}_j = \sum_{j=0}^{k} g_{k,j} \left(f_{j,j+1}[r_{j+1}] + \sum_{l=0}^{j} f_{j,l}[r_l] \right) \\
&= g_{k,k} f_{k,k+1}[r_{k+1}] + \left[\sum_{j=0}^{k-1} g_{k,j} f_{j,j+1} r_{j+1} + \sum_{j=0}^{k} g_{k,j} \sum_{l=0}^{j} f_{j,l} r_l \right] \\
&= \left[g_{k,k} f_{k,k+1} \widetilde{\mu_{k+1,k+1}} x^{k+1} + g_{k,k} f_{k,k+1} \sum_{m=0}^{k} \widetilde{\mu_{k+1,m}} x^m \right. \\
&\quad \left. + \sum_{j=0}^{k-1} g_{k,j} f_{j,j+1} r_{j+1} + \sum_{j=0}^{k} g_{k,j} \sum_{l=0}^{j} f_{j,l} r_l \right],
\end{aligned}
\tag{3.6}
$$

where we set $r_l(\lambda) = \sum_{j=0}^{l} \widetilde{\mu}_{l,j} \lambda^j$, $\widetilde{\mu}_{l,l} > 0$, $\mu_{l,j} \in \mathbb{R}$. Since we get a real polynomial of degree $k+1$ on the right-hand side of (3.6) and it has a positive leading coefficient $g_{k,k} f_{k,k+1} \widetilde{\mu_{k+1,k+1}}$, then relation (3.2) is proved.

It remains to verify (iii). For arbitrary $n, m \in \mathbb{Z}_+$ we may write

$$
\begin{aligned}
(\mathcal{A}_\sigma \Lambda_0 [r_n(x)], [r_m(x)]) &= \left(U A U^{-1} [a_{n-1} r_{n-1}(x) + b_n r_n(x) + a_n r_{n+1}(x)], [r_m(x)] \right) \\
&= (A J_3 \vec{e}_n, \vec{e}_m) = (J_5 \vec{e}_n, \vec{e}_m) = (\vec{e}_n, J_5 \vec{e}_m) \\
&= (\vec{e}_n, A(a_{m-1} \vec{e}_{m-1} + b_m \vec{e}_m + a_m \vec{e}_{m+1})) \\
&= ([r_n(x)], \mathcal{A}_\sigma \Lambda_0 [r_m(x)]).
\end{aligned}
\tag{3.7}
$$

By the linearity we conclude that $\mathcal{A}_\sigma \Lambda_0$ is symmetric.

Sufficiency. Suppose that a sesquilinear functional $S(u, v)$, $u, v \in \mathbb{P}$, satisfying relations (2.9), (2.10), is given. Assume that it has the integral representation (3.1) where σ is a non-negative probability measure on $\mathfrak{B}(\mathbb{R})$ with all finite moments, $\int_{\mathbb{R}} |g(x)|^2 d\sigma > 0$, for any non-zero complex polynomial g, and $\mathcal{A}$ is a linear operator in L^2_σ with properties (i)–(iii). By condition (ii) and the induction argument it can be directly verified that for each $n \in \mathbb{Z}_+$ it holds

$$\mathcal{A}^n[1] = a_{n,n}[x^n] + \sum_{j=0}^{n-1} a_{n,j} [x^j],$$

where $a_{n,n} > 0$, $a_{n,j} \in \mathbb{R}$, $0 \leq j \leq n-1$.

Suppose that $S(u, u) = 0$ for a complex polynomial u. Then

$$0 = \|u(\mathcal{A})(1)\|^2_{L^2_\sigma}.$$

Assume that u is nonzero, $u(\lambda) = \sum_{k=0}^{n} d_k \lambda^k$, $d_n \neq 0$, $d_k \in \mathbb{C}$, $n \in \mathbb{Z}_+$. Observe that

$$u(\mathcal{A})(1) = \left(\sum_{k=0}^{n} d_k \mathcal{A}^k \right) [1] = d_n \mathcal{A}^n[1] + \sum_{k=0}^{n-1} d_k \mathcal{A}^k[1]$$

$$= d_n \left(a_{n,n}[x^n] + \left[\sum_{j=0}^{n-1} a_{n,j} x^j\right] \right) + \sum_{k=0}^{n-1} d_k \sum_{j=0}^{k} a_{k,j}[x^j]$$

$$= \left[d_n a_{n,n} x^n + d_n \sum_{j=0}^{n-1} a_{n,j} x^j + \sum_{k=0}^{n-1} \sum_{j=0}^{k} d_k a_{k,j} x^j \right] =: [r(x)].$$

We have a nonzero polynomial r (of degree n) with $\|r\|_{L^2_\sigma} = 0$. This contradicts to our assumptions on the measure σ. Consequently, $u \equiv 0$.

The functional S defines an inner product on $\mathbb{P}$. Thus, the complex vector space $\mathbb{P}$ becomes a space H with a scalar product. It is a normed space with the norm $\|p\| = \sqrt{S(p,p)}$. We shall not need its completion. Set

$$\mathbf{p}_0(\lambda) = \frac{1}{\|1\|_H} = 1, \qquad \mathbf{p}_1(\lambda) = \frac{\lambda - S(x,1)1}{\|\lambda - S(x,1)1\|_H}.$$

Let us check that $\mathbf{p}_1(\lambda)$ is well-defined. Denote by $\{s_k\}_{k=0}^{\infty}$ the power moments of σ

$$s_k = \int_{\mathbb{R}} x^k d\sigma, \qquad k \in \mathbb{Z}_+.$$

Let

$$\Delta_n := \det(s_{k+l})_{k,l=0}^n, \qquad n \in \mathbb{Z}_+, \qquad \Delta_{-1} := 1,$$

be the corresponding Hankel determinants. Observe that

$$S(x,1) = \int \mathcal{A}(1) d\sigma = \int (\xi_{0,1}\lambda + \xi_{0,0}) d\sigma = \xi_{0,1}s_1 + \xi_{0,0} \in \mathbb{R},$$

$$S(x,x) = \int \mathcal{A}(1)\overline{\mathcal{A}(1)} d\sigma = \int (\xi_{0,1}\lambda + \xi_{0,0})^2 d\sigma = \xi_{0,1}^2 s_2 + 2\xi_{0,1}\xi_{0,0}s_1 + \xi_{0,0}^2.$$

We may write

$$\|\lambda - S(x,1)1\|_H^2 = S(\lambda - S(x,1)1, \lambda - S(x,1)1) = S(\lambda,\lambda) - (S(\lambda,1))^2$$
$$= \xi_{0,1}^2 (s_2 - s_1^2) = \xi_{0,1}^2 \Delta_1 > 0,$$

where we have used our assumptions on the measure σ. Then

$$\mathbf{p}_1(\lambda) = \frac{\lambda - \xi_{0,1}s_1 - \xi_{0,0}}{\xi_{0,1}\sqrt{\Delta_1}}.$$

Set

$$\alpha = \frac{1}{\xi_{0,1}\sqrt{\Delta_1}}, \qquad \beta = -\frac{\xi_{0,1}s_1 + \xi_{0,0}}{\xi_{0,1}\sqrt{\Delta_1}}. \tag{3.8}$$

Let $\{\mathbf{r}_n(\lambda)\}_{n=0}^{\infty}$ be orthonormal polynomials with respect to the measure σ (having positive leading coefficients). Denote by J_3 the corresponding Jacobi matrix (formed by the recurrence coefficients of $\mathbf{r}_n$). Denote

$$J_5 = (g_{m,n})_{m,n=0}^{\infty}, \qquad g_{m,n} := (\mathcal{A}\Lambda_0[\mathbf{r}_n(\lambda)], [\mathbf{r}_m(\lambda)])_{L^2_\sigma}.$$

By condition (iii) of the theorem we conclude that J_5 is a symmetric semi-infinite matrix. Let

$$\mathbf{r}_n(\lambda) = \sum_{k=0}^n \eta_{n,k} \lambda^k, \qquad \eta_{n,k} \in \mathbb{R}, \quad \eta_{n,n} > 0, \quad n \in \mathbb{Z}_+.$$

For an arbitrary $n \in \mathbb{Z}_+$ by condition (ii) we may write

$$\mathcal{A}\Lambda_0[\mathbf{r}_n(\lambda)] = \sum_{k=0}^{n} \eta_{n,k}\mathcal{A}[\lambda^{k+1}] = \left[\eta_{n,n}\xi_{n+1,n+2}\lambda^{n+2} + d_{n+1}(\lambda)\right], \tag{3.9}$$

where $d_{n+1}(\lambda)$ is a zero polynomial or a polynomial with real coefficients, $\deg p \leq n+1$. Then

$$g_{m,n} = 0, \qquad m, n \in \mathbb{Z}_+, \quad m > n+2,$$

and

$$g_{n+2,n} = \eta_{n,n}\xi_{n+1,n+2}\left(\left[\lambda^{n+2}\right], [r_{n+2}(\lambda)]\right) = \frac{\eta_{n,n}\xi_{n+1,n+2}}{\eta_{n+2,n+2}} > 0, \qquad n \in \mathbb{Z}_+.$$

By (3.9) we also see that $g_{m,n}$ are real numbers. We conclude that J_5 is real five-diagonal and it has positive numbers on the second sub-diagonal.

Consider a Jacobi-type pencil $\widetilde{\Theta} = (J_3, J_5, \alpha, \beta)$. Let $\{p_n(\lambda)\}_{n=0}^{\infty}$ be the associated polynomials to $\widetilde{\Theta}$. For the pencil $\widetilde{\Theta}$ we define the standard objects from the introduction. The polynomials $r_n(\lambda)$ (see (2.4)) coincide with $\mathbf{r}_n(\lambda)$. We may also choose the given σ as the orthogonality measure (the orthogonality measure can be non-unique). The operators A, U and $\mathcal{A}_\sigma$ we define in the standard way, see (2.1), (2.5), (2.6). We only can not use the brief notation $\mathcal{A}$ for $\mathcal{A}_\sigma$, since $\mathcal{A}$ already denotes the operator in the integral representation of S.

For the pencil $\widetilde{\Theta}$ we may apply our arguments in the proved Necessity. By relation (3.7) we may write

$$(\mathcal{A}_\sigma\Lambda_0[r_n], [r_m])_{L^2_\sigma} = (J_5\vec{e}_n, \vec{e}_m)_{l_2} = g_{m,n} = (\mathcal{A}\Lambda_0[r_n], [r_m])_{L^2_\sigma}, \qquad n, m \in \mathbb{Z}_+.$$

Here the last equality follows by the definition of the matrix J_5. Therefore

$$\mathcal{A}_\sigma[\lambda p(\lambda)] = \mathcal{A}[\lambda p(\lambda)], \qquad \forall\, p \in \mathbb{P}. \tag{3.10}$$

Notice that

$$\mathcal{A}_\sigma[1] = UA\vec{e}_0 = \frac{1}{\alpha}U(\vec{e}_1 - \beta\vec{e}_0) = \left[\frac{1}{\alpha}(r_1(\lambda) - \beta)\right]. \tag{3.11}$$

The orthonormal polynomial $r_1(\lambda)$ has the following form

$$r_1(\lambda) = \frac{1}{\sqrt{\Delta_1}}(\lambda - s_1). \tag{3.12}$$

By (3.11), (3.12) and (3.8) we conclude that

$$\mathcal{A}_\sigma[1] = [\xi_{0,1}\lambda + \xi_{0,0}].$$

On the other hand, by property (ii) we have $\mathcal{A}[1] = [\xi_{0,1}\lambda + \xi_{0,0}]$. Therefore

$$\mathcal{A}_\sigma[1] = \mathcal{A}[1]. \tag{3.13}$$

Relations (3.10) and (3.13) show that $\mathcal{A}_\sigma = \mathcal{A}$. Comparing relations (2.11) and (3.1) we see that the spectral function of $\widetilde{\Theta}$ coincides with S. ∎

Theorem 3.1 provides characteristic properties for the model representation $\mathcal{A}$ of the associated operator of a Jacobi-type pencil. It is seen that these properties are close to the properties of the multiplication operator Λ_0. Of course, Λ_0 itself satisfies properties (i)–(iii).

Corollary 3.3. *Let σ be a non-negative measure on $\mathfrak{B}(\mathbb{R})$ with all finite power moments, $\int_{\mathbb{R}} d\sigma = 1$, $\int_{\mathbb{R}} |g(x)|^2 d\sigma > 0$, for any non-zero complex polynomial g. A linear operator $\mathcal{A}$ in L^2_σ is a model representation in L^2_σ of the associated operator of a Jacobi-type pencil if and only if properties (i)–(iii) of Theorem 3.1 hold.*

Proof. It follows directly from our constructions in the proof of Theorem 3.1. ∎

Let $\Theta = (J_3, J_5, \alpha, \beta)$ be a Jacobi-type pencil and $\mathcal{A}$ be a model representation in L^2_σ of the associated operator of Θ. By the latter corollary we conclude that $\mathcal{A}\Lambda_0$ is symmetric

$$(\mathcal{A}\Lambda_0[u(\lambda)], [v(\lambda)])_{L^2_\sigma} = ([u(\lambda)], \mathcal{A}\Lambda_0[v(\lambda)])_{L^2_\sigma}, \qquad u, v \in \mathbb{P}. \tag{3.14}$$

Suppose that the orthogonality measure σ is supported inside a finite real segment $[a, b]$, $0 < a < b < +\infty$, i.e., $\sigma(\mathbb{R}\backslash[a, b]) = 0$. In this case the operator Λ of the multiplication by an independent variable has a bounded inverse on the whole L^2_σ. By (3.14) we may write

$$\left(\Lambda^{-1}\mathcal{A}[\lambda u(\lambda)], [\lambda v(\lambda)]\right)_{L^2_\sigma} = \left(\Lambda^{-1}[\lambda u(\lambda)], \mathcal{A}[\lambda v(\lambda)]\right)_{L^2_\sigma}, \qquad u, v \in \mathbb{P}.$$

Denote $\mathcal{P}_0 = \Lambda\mathcal{P}$ and $\mathcal{A}_0 = \mathcal{A}|_{\mathcal{P}_0}$. Then

$$\left(\Lambda^{-1}\mathcal{A}_0 f, g\right)_{L^2_\sigma} = \left(\Lambda^{-1}f, \mathcal{A}_0 g\right)_{L^2_\sigma}, \qquad f, g \in \mathcal{P}_0.$$

Thus, in this case $\mathcal{A}_0$ *is symmetric with respect to the form* $(\Lambda^{-1}\cdot, \cdot)_{L^2_\sigma}$. Analogous arguments were used in the theory of operator pencils, see [7, Chapter IV, p. 163].

Example 3.4. Let $a_n = 1$, $b_n = c$, $c > 2$, $\alpha_n \in \mathbb{R}$, $\beta_n = 0$, $\gamma_n = 1$, $n \in \mathbb{Z}_+$. Define J_3 and J_5 by (1.2), (1.3) with the above parameters. Consider a Jacobi-type pencil $\Theta = (J_3, J_5, \alpha, \beta)$, with arbitrary $\alpha > 0$ and $\beta \in \mathbb{R}$. In this case

$$r_n(x) = U_n\left(\frac{x-c}{2}\right), \qquad n \in \mathbb{Z}_+,$$

where $U_n(t) = \frac{\sin((n+1)\arccos t)}{\sqrt{1-t^2}}$ is Chebyshev's polynomial of the second kind. The orthonormality relations for $r_n(x)$ have the following form

$$\int_{c-2}^{c+2} r_n(\lambda)r_m(\lambda)\sqrt{1 - \left(\frac{\lambda-c}{2}\right)^2}\frac{1}{\pi}d\lambda = \delta_{n,m}, \qquad n, m \in \mathbb{Z}_+.$$

By the recurrence relation (1.5) we calculate

$$p_2(\lambda) = \alpha\lambda^2 + (c+\beta)\lambda - \alpha_0.$$

Since $(c+\beta)^2 + 4\alpha\alpha_0$ can be made zero or negative by a proper choice of α_0, then we have a non-classical case. Thus, this case is worthy of an additional investigation which will be done elsewhere.

We can state the following *moment problem for Jacobi-type pencils*: find a non-negative measure σ on $\mathfrak{B}(\mathbb{R})$ with all finite power moments, $\int_{\mathbb{R}} d\sigma = 1$, $\int_{\mathbb{R}} |g(x)|^2 d\sigma > 0$, for any non-zero complex polynomial g, and a linear operator $\mathcal{A}$ in L^2_σ with properties (i)–(iii) of Theorem 3.1 such that

$$\int_{\mathbb{R}} \mathcal{A}^m(1)\overline{\mathcal{A}^n(1)}d\sigma = s_{m,n}, \qquad m, n \in \mathbb{Z}_+,$$

where $\{s_{m,n}\}_{m,n=0}^\infty$ is a prescribed set of real numbers (called moments).

As usual, there appear three important questions: 1) the solvability of the moment problem; 2) the uniqueness of a solution (the determinateness); 3) a description of all solutions. This moment problem will be studied elsewhere.

4   A special perturbation of orthogonal polynomials on a finite interval

If we look at the orthogonality relations (2.7) or at the representation of the spectral function (2.11), we can see that we need to calculate a polynomial of the operator $\mathcal{A}$. However, we do not have at hand any functional calculus for the operator $\mathcal{A}$. It remains to calculate the powers of $\mathcal{A}$ recurrently. It would be helpful to omit this procedure and to have a more transparent relation. It is possible to do this in the following special case.

Consider a Jacobi-type pencil $\Theta = (J_3, J_5, \alpha, \beta)$, with J_3, J_5 defined by (1.2), (1.3), satisfying the following conditions

$$\sigma(\mathbb{R}\backslash[-c,c]) = 0, \qquad 0 < c < 1, \tag{4.1}$$

$$J_5 = aJ_3^2 + bJ_3 + d\,\mathrm{diag}(1,0,0,0,\dots), \qquad a > 0, \quad b,d \in \mathbb{R},$$

$$\alpha = \frac{1}{aa_0}, \qquad \beta = -\frac{b_0}{a_0} - \frac{b}{aa_0}. \tag{4.2}$$

Here σ, as before (see the introduction), is the orthogonality measure for polynomials $\{r_n(x)\}_{n=0}^\infty$ (related to J_3).

Introduce other related objects from the introduction: the space L_σ^2, the operators A, U and $\mathcal{A} = UAU^{-1}$. Observe that

$$U J_3 U^{-1}[p(x)] = [xp(x)], \qquad p \in \mathbb{P}.$$

For an arbitrary $n \in \mathbb{Z}_+$ we may write

$$\begin{aligned}
\mathcal{A}[xr_n(x)] &= UAU^{-1}UJ_3U^{-1}[r_n(x)] = UAJ_3\vec{e}_n = UJ_5\vec{e}_n \\
&= U\big(aJ_3^2 + bJ_3 + d\,\mathrm{diag}(1,0,0,0,\dots)\big)\vec{e}_n \\
&= aUJ_3U^{-1}UJ_3\vec{e}_n + bUJ_3\vec{e}_n + dU\,\mathrm{diag}(1,0,0,0,\dots)\vec{e}_n \\
&= \big[(ax+b)xr_n(x) + d(r_n, r_0)_{L_\sigma^2}\big].
\end{aligned}$$

By the linearity we get

$$\mathcal{A}[xp(x)] = \big[(ax+b)xp(x) + d(p,1)_{L_\sigma^2}\big], \qquad p \in \mathbb{P}.$$

Moreover, we have

$$\mathcal{A}[1] = UAU^{-1}U\vec{e}_0 = UA\vec{e}_0 = U\frac{1}{\alpha}(\vec{e}_1 - \beta\vec{e}_0) = \left[\frac{1}{\alpha}(r_1(x) - \beta)\right].$$

Choose an arbitrary polynomial $q(x) \in \mathbb{P}$. We may write $q(x) = x\frac{q(x)-q(0)}{x} + q(0)$. Then

$$\begin{aligned}
\mathcal{A}[q(x)] &= \mathcal{A}\left[x\frac{q(x)-q(0)}{x}\right] + \mathcal{A}[q(0)] \\
&= \left[(ax^2 + bx)\frac{q(x)-q(0)}{x} + d\left(\frac{q(x)-q(0)}{x}, 1\right)_{L_\sigma^2} + \frac{q(0)}{\alpha}(r_1(x) - \beta)\right] \\
&= \left[(ax+b)q(x) + d\left(\frac{q(x)-q(0)}{x}, 1\right)_{L_\sigma^2}\right].
\end{aligned}$$

Denote

$$s_k := \int_{\mathbb{R}} x^k \mathrm{d}\sigma, \qquad k \in \mathbb{Z}_+, \qquad s_{-1} := 0.$$

Observe that

$$\mathcal{A}\left[\sum_{k=0}^{\infty} c_k x^k\right] = \sum_{k=0}^{\infty} c_k \mathcal{A}[x^k] = \sum_{k=0}^{\infty} c_k\left[ax^{k+1} + bx^k + ds_{k-1}\right], \qquad c_k \in \mathbb{C},$$

where all but finite number of c_k are zeros (i.e., $\mathcal{A}$ acts on polynomials).

Observe that the operator $\mathcal{A}$ can be unbounded: the corresponding example will be considered below. Thus, there appears a question: how to simplify the calculation of $u(\mathcal{A})$ for a complex polynomial u in the integral representation (2.11)? Is there any (at least) polynomial calculus for $\mathcal{A}$? The answer is affirmative. Consider the following transformation from L_σ^2 to l_2

$$G\left[\sum_{k=0}^{\infty} c_k x^k\right] = \sum_{k=0}^{\infty} c_k \vec{e}_k, \qquad c_k \in \mathbb{C}.$$

Here all but finite number of c_k are zeros. This will be assumed in what follows when dealing with operators on polynomials. Since σ satisfies conditions for the integral representation (3.1) (see the Necessity of the proof of Theorem 3.1), then $\int_{\mathbb{R}} |g(x)|^2 d\sigma > 0$, for any non-zero complex polynomial g. This shows that two different polynomials can not belong to the same class of the equivalence in L_σ^2. Thus, the operator G is well-defined. It is a linear operator with $D(G) = \mathcal{P}$ and $R(G) = l_{2,\text{fin}}$. Moreover, G is invertible, since the polynomial is uniquely determined by its coefficients. Set

$$\mathbf{A} = G\mathcal{A}G^{-1}.$$

Then $\mathbf{A}$ is a linear operator in l_2 with $D(\mathbf{A}) = l_{2,\text{fin}}$. Consider the following shift operator on the whole l_2

$$Sx = \sum_{k=0}^{\infty} c_k \vec{e}_{k+1}, \qquad x \in l_2, \quad x = (c_k)_{k=0}^{\infty}, \quad c_k \in \mathbb{C}.$$

Notice that S is linear and $\|Sx\| = \|x\|$, $\forall\, x \in l_2$. We may write

$$\mathbf{A}\left(\sum_{k=0}^{\infty} c_k \vec{e}_k\right) = (aS + bE)\sum_{k=0}^{\infty} c_k \vec{e}_k + \left(d\sum_{k=0}^{\infty} c_k s_{k-1}\right)\vec{e}_0.$$

Here all but finite number of $c_k \in \mathbb{C}$ are zeros. This will be assumed in the sequel for operators on $l_{2,\text{fin}}$. By condition (4.1) we see that

$$|s_n| = \left|\int_{\mathbb{R}} x^n \chi_{[-c,c]}(x)d\sigma\right| \leq \int_{\mathbb{R}} \left|x^n \chi_{[-c,c]}(x)\right|d\sigma \leq c^n, \qquad n \in \mathbb{Z}_+,$$

where $\chi_{[-c,c]}(x)$ is the characteristic function of the segment $[-c,c]$. Therefore $\vec{s} := \sum_{k=1}^{\infty} s_{k-1}\vec{e}_k$ belongs to l_2. Consequently, we may write

$$\mathbf{A}w = (aS + bE)w + d(w, \vec{s})_{l_2}\vec{e}_0, \qquad w \in l_{2,\text{fin}}.$$

Set

$$\widehat{\mathbf{A}}w = (aS + bE)w + d(w, \vec{s})_{l_2}\vec{e}_0, \qquad w \in l_2.$$

The linear operator $\widehat{\mathbf{A}}$ is an extension of $\mathbf{A}$. Observe that the operator $\widehat{\mathbf{A}}$ is *bounded* and

$$\left\|\widehat{\mathbf{A}}\right\| \leq a + |b| + |d|\|\vec{s}\|_{l_2}.$$

We can apply Riesz's calculus for $\widehat{\mathbf{A}}$. For an arbitrary polynomial $u \in \mathbb{P}$ we may write

$$G\big(u(\mathcal{A})[1]\big) = u(\widehat{\mathbf{A}})\vec{e}_0 = -\frac{1}{2\pi i}\int_\gamma u(z)R_z(\widehat{\mathbf{A}})\mathrm{d}z \cdot \vec{e}_0 = -\frac{1}{2\pi i}\int_\gamma u(z)\big(R_z(\widehat{\mathbf{A}})\vec{e}_0\big)\mathrm{d}z, \qquad (4.3)$$

where γ is a circle centered at zero with a radius ρ bigger then $a + |b| + |d|\|\vec{s}\|_{l_2}$, and $R_z(\widehat{\mathbf{A}}) = (\widehat{\mathbf{A}} - zE)^{-1}$. Here the last integral converges in the norm of l_2.

Let us calculate $R_z(\widehat{\mathbf{A}})\vec{e}_0 =: \vec{f}$, $z \in \gamma$. We may write

$$\vec{e}_0 = \big(\widehat{\mathbf{A}} - zE\big)\vec{f} = (aS + bE)\vec{f} + d(\vec{f}, \vec{s})_{l_2}\vec{e}_0 - z\vec{f}.$$

Then

$$\frac{1}{a}\big(1 - d(\vec{f}, \vec{s})_{l_2}\big)\vec{e}_0 = \left(S - \left(\frac{z-b}{a}\right)E\right)\vec{f}.$$

Since $z \in \gamma$, then $\left|\frac{z-b}{a}\right| > 1$ and

$$\vec{f} = \frac{1 - d(\vec{f}, \vec{s})_{l_2}}{a}\left(S - \left(\frac{z-b}{a}\right)E\right)^{-1}\vec{e}_0. \qquad (4.4)$$

Denote $\vec{u} = \sum_{k=0}^{\infty} u_k \vec{e}_k := \big(S - \big(\frac{z-b}{a}\big)E\big)^{-1}\vec{e}_0$. Then

$$\vec{e}_0 = \left(S - \left(\frac{z-b}{a}\right)E\right)\vec{u} = S\vec{u} + \left(\frac{b-z}{a}\right)\vec{u}.$$

For the components of $\vec{u}$ we obtain the following equations

$$\left(\frac{b-z}{a}\right)u_0 = 1, \qquad u_{n-1} + \left(\frac{b-z}{a}\right)u_n = 0, \qquad n \in \mathbb{N}.$$

Then

$$\vec{u} = \sum_{k=0}^{\infty}(-1)\left(\frac{a}{z-b}\right)^{k+1}\vec{e}_k.$$

By (4.4) we may write

$$\vec{f} = \tau \sum_{k=0}^{\infty}(-1)\left(\frac{a}{z-b}\right)^{k+1}\vec{e}_k, \qquad \tau \in \mathbb{C},$$

where

$$\tau = \frac{1 - d(\vec{f}, \vec{s})_{l_2}}{a}.$$

Then

$$a\tau = 1 - d\left(\tau\sum_{k=0}^{\infty}(-1)\left(\frac{a}{z-b}\right)^{k+1}\vec{e}_k, \vec{s}\right)_{l_2},$$

and

$$\tau = \frac{1}{a + d\sum_{k=0}^{\infty}(-1)\big(\frac{a}{z-b}\big)^{k+1}s_{k-1}}.$$

Denote

$$s(z) := \sum_{k=0}^{\infty} (-1) \left(\frac{a}{z-b} \right)^{k+1} s_{k-1}, \qquad z \in \gamma,$$

$$\vec{v}(z) := \sum_{k=0}^{\infty} (-1) \left(\frac{a}{z-b} \right)^{k+1} \vec{e}_k, \qquad z \in \gamma.$$

Then

$$R_z(\widehat{\mathbf{A}}) \vec{e}_0 = \vec{f} = \frac{1}{a + ds(z)} \vec{v}(z), \qquad z \in \gamma.$$

Thus, we obtained a transparent expression for $R_z(\widehat{\mathbf{A}}) \vec{e}_0$. It can be used in relation (4.3) and it gives a transparent expression for $u(\mathcal{A})[1]$.

Example 4.1. Let Θ be a Jacobi-type pencil with $\alpha = \beta = \sqrt{2}$, $a_k = \sqrt{2}$, $b_k = 2$, $k \in \mathbb{Z}_+$, $\alpha_n = \beta_n = 0$, $\gamma_n = 1$, $n \in \mathbb{Z}_+$, and J_3, J_5 have form (1.2), (1.3). This Jacobi type pencil was considered in [16] and explicit formulas for the associated polynomials $p_n(\lambda)$ were obtained.

Let κ be an arbitrary positive number bigger than $2 + 2\sqrt{2}$. Consider a Jacobi-type pencil $\widehat{\Theta} = (\widehat{J}_3, \widehat{J}_5, \widehat{\alpha}, \widehat{\beta})$, where

$$\widehat{J}_3 = \frac{1}{\kappa} J_3, \qquad \widehat{J}_5 = \frac{1}{\kappa} J_5, \qquad \widehat{\alpha} = \alpha, \qquad \widehat{\beta} = \beta.$$

Notice that the associated polynomials and the associated operators for Θ and $\widehat{\Theta}$ are the same. It was shown in [16] that the associated operator for Θ is unbounded. Thus, the associated operator for the pencil $\widehat{\Theta}$ is unbounded, as well. The pencil $\widehat{\Theta}$ satisfies conditions (4.1), (4.2) with

$$c = \frac{2 + 2\sqrt{2}}{\kappa}, \qquad a = \frac{1}{2}\kappa, \qquad b = -2, \qquad d = \frac{1}{\kappa}.$$

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