On Complex Gamma-Function Integrals

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Abstract. It was observed recently that relations between matrix elements of certain operators in the $\mathrm{SL}(2,\mathbb{R})$ spin chain models take the form of multidimensional integrals derived by R.A. Gustafson. The spin magnets with $\mathrm{SL}(2,\mathbb{C})$ symmetry group and $\mathrm{L}_2(\mathbb{C})$ as a local Hilbert space give rise to a new type of Γ-function integrals. In this work we present a direct calculation of two such integrals. We also analyse properties of these integrals and show that they comprise the star-triangle relations recently discussed in the literature. It is also shown that in the quasi-classical limit these integral identities are reduced to the duality relations for Dotsenko–Fateev integrals.

Key words: Mellin–Barnes integrals; star-triangle relation

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1 Introduction

The multidimensional integrals derived by R.A. Gustafson [27, 28] together with their q- and elliptic analogues [27, 28, 29, 57, 58, 61] play an important role in different areas of physics and mathematics such as the theory of multi-variable orthogonal polynomials [62], Selberg type integrals and constant term identities [1, 22], and supersymmetric dualities in quantum field theory [60]. Recently, a new field – noncompact spin magnets – was added to this list [15].

Models of this type appear in gauge field theories and have been under intense investigations in the last two decades, see references [7, 8]. The mathematical description of such systems is well-developed and known as the quantum inverse scattering method(QISM) [20, 37, 38, 52, 53, 54]. Noncompact spin chains have an infinite-dimensional local Hilbert space and most conveniently can be analysed within the separation of variable (SoV) framework [53]. It was shown by Sklyanin that eigenfunctions of the entries of the monodromy matrix provide a suitable basis for solving the spectral problem for the spin chain Hamiltonian. Building such a basis for a generic spin chain is in itself a nontrivial problem and until recently the solution was available only for the spin chains with a rank-1 symmetry group. For the noncompact magnets of interest the corresponding eigenfunctions are known explicitly [9, 12, 14]. It should be mentioned here that recently there was a significant progress in constructing the SoV representation for compact chains of higher ranks, see [10, 24, 25, 34, 35, 40, 49].

In the $\mathrm{SL}(2,\mathbb{R})$ spin chin framework, Gustafson's integrals follow from identities for matrix elements of the shift operator [15]. Extension of this analysis to the $\mathrm{SL}(2,\mathbb{C})$ spin magnets leads to new integral identities [16, 17] which, up to the expected modification, are the exact replicas

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of the $SL(2,\mathbb{R})$ integrals. The analysis, however, essentially depends on the completeness of the SoV representation. Proof of completeness is a rather complicated problem, see, e.g., [13, 36, 51]. Completeness for the closed $SL(2,\mathbb{R})$ spin chain follows from that of the Toda chain [36] while for the $SL(2,\mathbb{C})$ magnets of the length $N > 2$ it is still an open question. A fruitful strategy seems to be to make use of Gustafson's integrals to prove completeness. To realize this one needs an independent derivation of the corresponding integrals. One of the purposes of this work is to provide such a derivation. We note also that complex gamma integrals were studied recently by V.F. Molchanov and Yu.A. Neretin [43] and Yu.A. Neretin [44, 45].

The paper is organized as follows: in Section 2, after setting the notations, we prove two integral identities which are direct analogs of Gustafson integrals associated with the classical $su(N)$ and $sp(N)$ Lie algebras [28]. We analyze the analytic properties of these integrals in Section 3 and derive two new integrals which are the $SL(2,\mathbb{C})$ versions of Gustafson integrals [27] generalizing the second Barnes lemma. In Section 4 it will be shown that the $N = 1, 2$ integrals take, after some rewriting, the form of the star-triangle relations derived in [5, 32, 33] as special limits of the elliptic star-triangle identity [6, 18, 56, 59]. Section 5 is devoted to the study of the quasi-classical limit of the Γ-integrals. We show that in this limit the integrals are equivalent to a special case of the duality relation [2] for Dotsenko–Fateev (DF) integrals [19]. In Section 6 we present an elementary proof of the above duality relation and give some evidence which suggest that similar duality relations hold for the Γ-integrals. Section 7 is reserved for a summary.

2 Gamma integrals

2.1 Definitions and basic properties

Let u, $\bar{u}$ be a pair of complex numbers of the form

$$u = \frac{n}{2} + \nu, \qquad \bar{u} = -\frac{n}{2} + \nu, \tag{2.1}$$

where n is an integer and ν is complex number. We will use the notations $[u] = u - \bar{u} = n$ for the discrete part and $\langle u \rangle = \nu$ for the continuous part and put $\|u\|^2 = -u\bar{u} = -\nu^2 + n^2/4$ so that for imaginary ν, $\|u\|^2 \geq 0$. The $\boldsymbol{\Gamma}$ function of the complex field $\mathbb{C}$ [23] is defined as

$$\boldsymbol{\Gamma}(u,\bar{u}) = \frac{\Gamma(u)}{\Gamma(1-\bar{u})} = \frac{\Gamma(n/2+\nu)}{\Gamma(1+n/2-\nu)} = (-1)^n \frac{\Gamma(-n/2+\nu)}{\Gamma(1-n/2-\nu)} = (-1)^{[u]}\boldsymbol{\Gamma}(\bar{u},u).$$

In what follows we will, for brevity, display only the first argument of the $\boldsymbol{\Gamma}$ function, i.e., $\boldsymbol{\Gamma}(u) \equiv \boldsymbol{\Gamma}(u,\bar{u})$. Hereafter the following functional relations will be useful

$$\boldsymbol{\Gamma}(u)\boldsymbol{\Gamma}(1-u) = (-1)^{[u]}, \qquad \boldsymbol{\Gamma}(u+1) = -u\bar{u}\boldsymbol{\Gamma}(u). \tag{2.2}$$

The $\boldsymbol{\Gamma}$ function appears in the generalization of Gustafson's integrals to the complex case. The corresponding integrals take the following form [16, 17]

$$\frac{1}{N!} \sum_{n_1,\ldots,n_N \in \mathbb{Z}+\frac{\sigma}{2}} \int_{-i\infty}^{i\infty} \frac{\prod\limits_{m=1}^{N+1} \prod\limits_{k=1}^{N} \boldsymbol{\Gamma}(z_m - u_k)\boldsymbol{\Gamma}(u_k + w_m)}{\prod\limits_{m<j} \boldsymbol{\Gamma}(u_m - u_j)\boldsymbol{\Gamma}(u_j - u_m)} \frac{d\nu_1}{2\pi i} \cdots \frac{d\nu_N}{2\pi i}$$

$$= \frac{\prod\limits_{k,j=1}^{N+1} \boldsymbol{\Gamma}(z_k + w_j)}{\boldsymbol{\Gamma}\left(\sum\limits_{k=1}^{N+1}(z_k + w_k)\right)}, \tag{2.3a}$$

$$\frac{1}{2^N N!} \sum_{n_1,\ldots,n_N \in \mathbb{Z}+\frac{\sigma}{2}} \int_{-i\infty}^{i\infty} \frac{\prod\limits_{k=1}^{N} \prod\limits_{m=1}^{2N+2} \mathbf{\Gamma}(z_m \pm u_k)}{\prod\limits_{k=1}^{N} \mathbf{\Gamma}(\pm 2u_k) \prod\limits_{k<j} \mathbf{\Gamma}(\pm u_k \pm u_j)} \frac{d\nu_1}{2\pi i} \cdots \frac{d\nu_N}{2\pi i}$$

$$= \frac{\prod\limits_{j<k} \mathbf{\Gamma}(z_j + z_k)}{\mathbf{\Gamma}\left(\sum\limits_{k=1}^{2N+2} z_k\right)}, \tag{2.3b}$$

where we put $\mathbf{\Gamma}(a \pm b) \equiv \mathbf{\Gamma}(a+b)\mathbf{\Gamma}(a-b)$, $\mathbf{\Gamma}(\pm a \pm b) \equiv \mathbf{\Gamma}(a+b)\mathbf{\Gamma}(a-b)\Gamma(-a+b)\mathbf{\Gamma}(-a-b)$. We will refer to the integrals in the first and second lines as $I_N^{(1)}$ and $I_N^{(2)}$, respectively.

The variables u_k, w_m, z_m have the form (2.1)

$$u_r = \frac{n_r}{2} + \nu_r, \qquad z_j = \frac{m_j}{2} + x_j, \qquad w_m = \frac{\ell_m}{2} + y_m$$

and similarly for the barred variables. However, n_r, m_i, ℓ_m are allowed take integer or half-integer values, simultaneously. Accordingly, the sums in (2.3) go over integers ($\sigma = 0$) or half-integers ($\sigma = 1/2$). For the first integral (2.3a) there is no difference between the integer/half-integer cases since they are related by the change of variables: $[u_r], [w_m], [z_j] \mapsto [u_r] + 1/2, [w_m] - 1/2, [z_j] + 1/2$ so that we will assume that the variables $[u_k], [w_m], [z_m]$ in the integral (2.3a) are integers.

The integration contours in (2.3) separate the series of " $\pm$ " poles due to the $\mathbf{\Gamma}$ functions in the numerators. The poles are located at

$$\nu_{rj}^{I,+}(p) = \frac{1}{2}|n_r - m_j| + x_j + p, \qquad \nu_{rj}^{I,-}(p) = -\frac{1}{2}|n_r + \ell_j| - y_j - p, \qquad p \geq 0, \tag{2.4}$$

where $r \in \{1, N\}$, $j \in \{1, N+1\}$ for the first integral and

$$\nu_{rj}^{II,+}(p) = \frac{1}{2}|n_r - m_j| + x_j + p, \qquad \nu_{rj}^{II,-}(p) = -\frac{1}{2}|n_r + m_j| - x_j - p, \qquad p \geq 0,$$

where $r \in \{1, N\}$, $j \in \{1, 2N+2\}$, for the second one.

Let us discuss now the convergence properties of the integrals (2.3). Since the integrands are meromorphic functions and contours of integration avoid poles it is sufficient to analyse the region of large $u_r = n_r/2 + i\nu_r$, $\|u_r\|^2 = \nu_r^2 + n_r^2/4 \to \infty$, only. With the help of (2.2) we simplify the denominators in the integrals (2.3) as follows

$$\prod_{1 \leq i < k \leq N} \frac{1}{\mathbf{\Gamma}(u_i - u_k)\mathbf{\Gamma}(u_k - u_i)} = (-1)^{(N+1)\sum_k [u_k]} \prod_{1 \leq i < k \leq N} \|u_i - u_k\|^2 \tag{2.5}$$

and

$$\prod_{k=1}^{N} \frac{1}{\mathbf{\Gamma}(\pm 2u_k)} \prod_{1 \leq i < m \leq N} \frac{1}{\mathbf{\Gamma}(\pm u_i \pm u_m)}$$

$$= \varkappa_N 4^N \prod_{k=1}^{N} \|u_k\|^2 \prod_{1 \leq i < m \leq N} \|u_i - u_m\|^2 \|u_i + u_m\|^2, \tag{2.6}$$

where $\varkappa_N = 1$ for the integer case and $\varkappa_N = (-1)^{N(N+1)/2}$ for the half-integer case. Finally, taking into account that for large u

$$\mathbf{\Gamma}(z - u)\mathbf{\Gamma}(u + w) = (-1)^{[z-u]} \frac{\mathbf{\Gamma}(u + w)}{\mathbf{\Gamma}(u - z)}$$

$$= (-1)^{[z-u]} u^{z+w-1} (-\bar{u})^{\bar{z}+\bar{w}-1} \big(1 + O\left(1/\|u\| \right) \big) \tag{2.7}$$

we conclude that the integrals (2.3) converge absolutely provided

$$\sum_{j=1}^{N+1} \mathrm{Re}(x_j + y_j) < 1 \qquad \text{and} \qquad \sum_{j=1}^{2N+2} \mathrm{Re}(x_j) < 1, \tag{2.8}$$

respectively. From now on we assume that these conditions are satisfied.

2.2   Determinant representation

In this subsection we present the integrals (2.3) as determinants of 1-dimensional integrals. Such a representation will be useful in what follows.[1] For its derivation let us denote by $\mathcal{Q}(u|z,w)$ the function

$$\mathcal{Q}(u|z,w) = \prod_{k=1}^{N+1} (-1)^{[u]} \mathbf{\Gamma}(z_k - u)\mathbf{\Gamma}(u + w_k)$$

and by $\mathcal{Q}_{ik}(z,w)$ its Mellin moments

$$\mathcal{Q}_{ik}(z,w) = \int \mathcal{D}u\, u^{i-1}(-\bar{u})^{k-1}\mathcal{Q}(u|z,w), \qquad i,k = 1,\ldots,N. \tag{2.9}$$

Here we introduced a short-hand notation for the integration measure

$$\int \mathcal{D}u \equiv \sum_{n=-\infty}^{\infty} \int_{-i\infty}^{i\infty} \frac{\mathrm{d}\nu}{2\pi i}.$$

Let $\boldsymbol{\mathcal{Q}}_N(z,w)$ be the following $N \times N$ matrix constructed from the Mellin moments

$$\boldsymbol{\mathcal{Q}}_N(z,w) = \begin{pmatrix} \mathcal{Q}_{11}(z,w) & \cdots & \mathcal{Q}_{1N}(z,w) \\ \vdots & \ddots & \vdots \\ \mathcal{Q}_{N1}(z,w) & \cdots & \mathcal{Q}_{NN}(z,w) \end{pmatrix}. \tag{2.10}$$

Rewriting the product on the r.h.s. of equation (2.5) as the product of two Vandermonde determinants

$$\prod_{1 \le i < k \le N} \|u_i - u_k\|^2 = \Delta(u)\Delta(-\bar{u}), \qquad \Delta(u) = \det_{1 \le i,j \le N} u_j^{i-1}$$

and taking into account the symmetry of the integrand in (2.3a) with respect to the permutations $u_i \leftrightarrow u_j$ we bring the first integral into the determinant form

$$I_N^{(1)} = \int \mathcal{D}u_1 \cdots \mathcal{D}u_N \Delta(-\bar{u}) \prod_{k=1}^{N} \mathcal{Q}(u_k|z,w) u_k^{k-1} = \det \boldsymbol{\mathcal{Q}}_N(z,w). \tag{2.11}$$

Proceeding in a similar way one gets the determinant representation for the second integral as follows

$$I_N^{(2)} = \varkappa_N \det \widetilde{\boldsymbol{\mathcal{Q}}}_N(z) = \varkappa_N \det \begin{vmatrix} \widetilde{\mathcal{Q}}_{11}(z) & \cdots & \widetilde{\mathcal{Q}}_{1N}(z) \\ \vdots & \ddots & \vdots \\ \widetilde{\mathcal{Q}}_{N1}(z) & \cdots & \widetilde{\mathcal{Q}}_{NN}(z) \end{vmatrix},$$

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[1]The determinant representation for elliptic hypergeometric integrals was constructed in [47].

where $\varkappa_N$ is a phase factor, see equation (2.6),

$$\widetilde{\mathcal{Q}}_{ik}(z) = 2 \int \mathcal{D}u\, u^{2i-1}(-\bar{u})^{2k-1} \widetilde{\mathcal{Q}}(u,z) \qquad \text{and} \qquad \widetilde{\mathcal{Q}}(u,z) = \prod_{j=1}^{2N+2} \mathbf{\Gamma}(z_j \pm u).$$

Let us note here that the conditions (2.8) are equivalent to the requirement of absolute convergence of the Mellin moments $\mathcal{Q}_{NN}(z,w)$ and $\widetilde{\mathcal{Q}}_{NN}(z)$, respectively.

2.3   Proof of identities (2.3)

Calculating the integral (2.3a) we will assume that the parameters z_k, w_k satisfy the conditions

$$\sum_{k=1}^{N+1} \mathrm{Re}(z_k + w_k) < 1 \qquad \text{and} \qquad \sum_{k=1}^{N+1} \mathrm{Re}(\bar{z}_k + \bar{w}_k) < 1. \tag{2.12}$$

These conditions imply the condition (2.8), but do not follow from it and will be removed at the end of the calculation.

By virtue of (2.8) the integrals (2.9) are all absolutely convergent. We evaluate the integral over ν by closing the contour in the left half-plane and then computing the sum over the residues. Recalling that $u = n/2 + \nu$ and $\bar{u} = -n/2 + \nu$ one gets

$$\begin{aligned}
M_{ik}(n) &= \int_{-i\infty}^{i\infty} u^{i-1}(-\bar{u})^{k-1} \mathcal{Q}(u|z,w) \frac{d\nu}{2\pi i} \\
&= (-1)^{(N+1)n} \sum_{j=1}^{N+1} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\,\bar{p}_j!} (-w_j - p)^{i-1}(\bar{w}_j + \bar{p}_j)^{k-1} \\
&\quad \times \prod_{m=1}^{N+1} \frac{\Gamma(z_m + w_j + p)}{\Gamma(1 - \bar{z}_m - \bar{w}_j - \bar{p}_j)} \prod_{m \neq j}^{N+1} \frac{\Gamma(w_m - w_j - p)}{\Gamma(1 - \bar{w}_m + \bar{w}_j + \bar{p}_j)},
\end{aligned}$$

where $\bar{p}_j = p + n + \ell_j$.

The first observation is that since the summand vanishes for $\bar{p}_j < 0$, only the poles at $\nu_{rj}^{\mathrm{I},-}(p)$ given in (2.4), contribute to the integral and, second, that under the assumptions (2.12) the sum over p converges uniformly on n. Therefore, to evaluate $\sum_n M_{ik}(n)$ we swap the summation over n and p, change the summation variable from n to $\bar{p}_j$ and finally suppress the index j in $\bar{p}_j$: $\bar{p}_j \mapsto \bar{p}$

$$\sum_n M_{ik}(n) = \sum_{j=1}^{N+1} \sum_{p=0}^{\infty} \sum_{n=-\infty}^{\infty} (\ldots) \mapsto \sum_{j=1}^{N+1} \sum_{p=0}^{\infty} \sum_{\bar{p}_j=-\infty}^{\infty} (\ldots) = \sum_{j=1}^{N+1} \sum_{p=0}^{\infty} \sum_{\bar{p}=0}^{\infty} (\ldots).$$

The final answer can be written in the form

$$\begin{aligned}
\mathcal{Q}_{ik}(z,w) &= \sum_{j=1}^{N+1} (-1)^{(N+1)[w_j]} \left(\prod_{m=1}^{N+1} \frac{\Gamma(z_m + w_j)}{\Gamma(1 - \bar{z}_m - \bar{w}_j)} \right) \left(\prod_{\substack{m=1 \\ m \neq j}}^{N+1} \frac{\Gamma(w_m - w_j)}{\Gamma(1 - \bar{w}_m + \bar{w}_j)} \right) \\
&\quad \times \left(\sum_{p=0}^{\infty} (-w_j - p)^{i-1} \prod_{m=1}^{N+1} \frac{(z_m + w_j)_p}{(1 - w_m + w_j)_p} \right) \\
&\quad \times \left(\sum_{\bar{p}=0}^{\infty} (\bar{w}_j + \bar{p})^{k-1} \prod_{m=1}^{N+1} \frac{(\bar{z}_m + \bar{w}_j)_{\bar{p}}}{(1 - \bar{w}_m + \bar{w}_j)_{\bar{p}}} \right), \tag{2.13}
\end{aligned}$$

where $(a)_p$ is the Pochhammer symbol. Thus $\mathcal{Q}_{ik} = \sum\limits_{j=1}^{N+1} \mathcal{Q}_{ik}(j) = \sum\limits_{j=1}^{N+1} \mathcal{Q}_i(j)\bar{\mathcal{Q}}_k(j)$. Hence the determinant can be represented as follows

$$
\det \mathcal{Q} = \sum_{j_1=1}^{N+1} \cdots \sum_{j_N=1}^{N+1} \det \|\mathcal{Q}_{ik}(j_k)\| = \sum_{\sigma \in S_{N+1}} \det \|\mathcal{Q}_{ik}(j_{\sigma(k)}\|
$$
$$
= \sum_{\sigma \in S_{N+1}} \left(\prod_{k=1}^{N} \bar{\mathcal{Q}}_k(j_{\sigma(k)}) \right) \det \|\mathcal{Q}_i(j_{\sigma(k)})\|,
$$

where we take into account that $\det \|Q_{ik}(j_k)\| = 0$ whenever $j_m = j_n$ for $n \neq m$. Then making use of (2.13) one can bring (2.11) into the following form

$$
I_N^{(1)} = \frac{1}{N!} \sum_{\sigma \in S_{N+1}} (-1)^{(N+1)\sum\limits_{s=1}^{N}[w_{\sigma(s)}]} \prod_{k=1}^{N} \left(\prod_{j=1}^{N+1} \frac{\Gamma(z_j + w_{\sigma(k)})}{\Gamma(1 - \bar{z}_j - \bar{w}_{\sigma(k)})} \prod_{\substack{m=1 \\ m \neq \sigma(k)}}^{N+1} \frac{\Gamma(w_m - w_{\sigma(k)})}{\Gamma(1 - \bar{w}_m + \bar{w}_{\sigma(k)})} \right)
$$
$$
\times \sum_{p_1,\ldots,p_N=0}^{\infty} \left(\prod_{k<m} (w_{\sigma(k)} + p_k - w_{\sigma(m)} - p_m) \right) \prod_{m=1}^{N+1} \prod_{k=1}^{N} \frac{(z_m + w_{\sigma(k)})_{p_k}}{(1 - w_m + w_{\sigma(k)})_{p_k}}
$$
$$
\times \sum_{\bar{p}_1,\ldots,\bar{p}_N=0}^{\infty} \left(\prod_{k<m} (\bar{w}_{\sigma(m)} + \bar{p}_m - \bar{w}_{\sigma(k)} - \bar{p}_k) \right) \prod_{m=1}^{N+1} \prod_{k=1}^{N} \frac{(\bar{z}_m + \bar{w}_{\sigma(k)})_{\bar{p}_k}}{(1 - \bar{w}_m + \bar{w}_{\sigma(k)})_{\bar{p}_k}}.
$$

The infinite sums over $\{p\}$, $\{\bar{p}\}$ can be evaluated by Milne's U(n) Gauss summation [42], see also [28, equation (5.8)] and [50],

$$
\sum_{p_1,\ldots,p_N=0}^{\infty} \left(\prod_{1 \le k < m \le N} (\alpha_{\sigma(k)} + p_k - \alpha_{\sigma(m)} - p_m) \right) \prod_{i=1}^{N+1} \prod_{k=1}^{N} \frac{(\beta_i + \alpha_{\sigma(k)})_{p_k}}{(1 - \alpha_i + \alpha_{\sigma(k)})_{p_k}}
$$
$$
= \frac{\Gamma\left(1 - \sum\limits_{k=1}^{N+1}(\alpha_k + \beta_k)\right) \prod\limits_{i=1}^{N} \Gamma(1 + \alpha_{\sigma(i)} - \alpha_{\sigma(N+1)})}{\prod\limits_{i=1}^{N+1} \Gamma(1 - \beta_i - \alpha_{\sigma(N+1)})}. \tag{2.14}
$$

Using (6.5) we obtain the following representation for the integral $I_N^{(1)}$

$$
I_N^{(1)} = \frac{1}{N!} \sum_{\sigma \in S_{N+1}} (-1)^{(N+1)\sum\limits_{s=1}^{N}[w_{\sigma(s)}]} \prod_{k=1}^{N} \left(\prod_{i=1}^{N+1} \frac{\Gamma(z_i + w_{\sigma(k)})}{\Gamma(1 - \bar{z}_i - \bar{w}_{\sigma(k)})} \prod_{\substack{m=1 \\ m \neq \sigma(k)}}^{N+1} \frac{\Gamma(w_m - w_{\sigma(k)})}{\Gamma(1 - \bar{w}_m + \bar{w}_{\sigma(k)})} \right)
$$
$$
\times \prod_{1 \le m < j \le N} (w_{\sigma(m)} - w_{\sigma(j)}) \Gamma\left(1 - \sum_{k=1}^{N+1}(z_k + w_k)\right) \frac{\prod\limits_{k=1}^{N} \Gamma(1 + w_{\sigma(k)} - w_{\sigma(N+1)})}{\prod\limits_{k=1}^{N+1} \Gamma(1 - z_k - w_{\sigma(N+1)}))}
$$
$$
\times \prod_{1 \le m < j \le N} (\bar{w}_{\sigma(j)} - \bar{w}_{\sigma(m)}) \Gamma\left(1 - \sum_{k=1}^{N+1}(\bar{z}_k + \bar{w}_k)\right) \frac{\prod\limits_{k=1}^{N} \Gamma(1 + \bar{w}_{\sigma(k)} - \bar{w}_{\sigma(N+1)})}{\prod\limits_{k=1}^{N+1} \Gamma(1 - \bar{z}_k - \bar{w}_{\sigma(N+1)})}.
$$

After some simplifications this can be written as

$$I_N^{(1)} = \frac{\Gamma\left(1 - \sum_{k=1}^{N+1}(\bar{z}_k + \bar{w}_k)\right)}{\Gamma\left(\sum_{k=1}^{N+1}(z_k + w_k)\right)} \prod_{i,k=1}^{N+1} \frac{\Gamma(z_i + w_k)}{\Gamma(1 - \bar{z}_i - \bar{w}_k)} \frac{R_N}{N! \sin \pi \left(\sum_{k=1}^{N+1}(z_k + w_k)\right)},$$

where

$$R_N = \sum_{\sigma \in S_{N+1}} \frac{\prod_{k=1}^{N+1} \sin \pi(z_{\sigma(k)} + w_{\sigma(N+1)})}{\prod_{k=1}^{N} \sin \pi(w_{\sigma(N+1)} - w_{\sigma(k)})}$$

$$\times (-1)^{(N+1)\sum_{s=1}^{N}[w_{\sigma(s)}]} \prod_{1 \leq k < j \leq N} \frac{\sin \pi(\bar{w}_{\sigma(j)} - \bar{w}_{\sigma(k)})}{\sin \pi(w_{\sigma(j)} - w_{\sigma(k)})}. \tag{2.15}$$

Taking into account that $w_k - \bar{w}_k$ is an integer one finds that the last product in (2.15) yields $(-1)^{(N-1)\sum_{s=1}^{N}[w_{\sigma(s)}]}$ which cancels the second factor in (2.15). In the last step we use the [28, Lemma 5.10] which states that

$$\sum_{\sigma \in S_{N+1}} \frac{\prod_{k=1}^{N+1} \sin \pi(\beta_k + \alpha_{\sigma(N+1)})}{\prod_{k=1}^{N} \sin \pi(\alpha_{\sigma(N+1)} - \alpha_{\sigma(k)})} = N! \sin \pi \sum_{k=1}^{N+1}(\alpha_k + \beta_k).$$

It results in

$$R_N = N! \sin \pi \sum_{k=1}^{N+1}(z_k + w_k),$$

so that we get the required result for $I_N^{(1)}$

$$I_N^{(1)} = \frac{\prod_{k,j=1}^{N+1} \mathbf{\Gamma}(z_k + w_j)}{\mathbf{\Gamma}\left(\sum_{k=1}^{N+1}(z_k + w_k)\right)}.$$

Finally by analytic continuation in the ν_k the assumptions (2.12) can be relaxed to the condition (2.8).

One sees that the crucial point in the proof of (2.3a) is the factorization of the double sum arising after evaluating the integral over ν by residue theorem into the product of two infinite sums, see equation (2.13). We remark that this factorization was first noticed by Ismagilov [30, 31], see also [45]. After this property is established the further analysis follows along the lines of [28].

The proof of the identity (2.3b) proceeds along the same lines so we will not go into details and only highlight the essential differences. First, we assume that the parameters z_k satisfy the conditions

$$\sum_{k=1}^{2N+2} \mathrm{Re}(z_k) < 1 \qquad \text{and} \qquad \sum_{k=1}^{2N+2} \mathrm{Re}(\bar{z}_k) < 1. \tag{2.16}$$

After evaluating the integrals over ν using residue calculus, and carrying out some minor simplifications, we obtain

$$
\begin{aligned}
I_N^{(2)} = \frac{2^N \varkappa_N}{N!}(-1)^N \sum_\pi &\Bigg(\Bigg\{ \sum_{y_1,\ldots,y_N=0}^\infty \Bigg(\prod_{j=1}^N (z_{\pi(j)}+y_j) \prod_{1\le i<j\le N} \big(z_{\pi(i)}+y_i \pm (z_{\pi(j)}+y_j)\big)\Bigg) \\
&\times \prod_{k=1}^N \frac{(-1)^{y_k}}{y_k!}\Gamma(2z_{\pi(k)}+y_k) \prod_{\substack{j=1 \\ j\neq\pi(k)}}^{2N+2} \Gamma(z_j \pm (z_{\pi(k)}+y_k)) \Bigg\} \\
&\times \Bigg\{ \sum_{\bar y_1,\ldots,\bar y_N=0}^\infty \Bigg(\prod_{j=1}^N (\bar z_{\pi(j)}+\bar y_j) \prod_{1\le i<j\le N} \big(\bar z_{\pi(i)}+\bar y_i \pm (\bar z_{\pi(j)}+\bar y_j)\big)\Bigg) \\
&\times \prod_{k=1}^N \frac{1}{\bar y_k!}\frac{1}{\Gamma(1-2\bar z_{\pi(k)}-\bar y_k)} \prod_{\substack{j=1 \\ j\neq\pi(k)}}^{2N+2} \frac{1}{\Gamma(1-\bar z_j \pm (\bar z_{\pi(k)}+\bar y_k))} \Bigg\} \Bigg),
\end{aligned}
$$

where π is an injective map from $\{1,\ldots,N\}$ to $\{1,\ldots,2N+2\}$. The sums over y_j, $\bar y_j$ can be evaluated with the help of the following hypergeometric series summation formula, see [26, 28]

$$
\begin{aligned}
&\sum_{y_1,\ldots,y_n=-\infty}^\infty \Bigg\{ \prod_{j=1}^n \frac{z_j+y_j}{z_j} \prod_{1\le i<j\le n} \frac{z_i+y_i \pm (z_j+y_j)}{z_i \pm z_j} \prod_{i=1}^{2n+2}\prod_{j=1}^n \frac{\Gamma(w_i \pm (z_j+y_j))}{\Gamma(w_i \pm z_j)} \Bigg\} \\
&= \frac{\Gamma\left(1-\sum_{k=1}^{2n+2} w_i\right) \prod_{i=1}^{2n+2}\prod_{j=1}^n \Gamma(1-w_i \pm z_j)}{\prod_{j=1}^N \Gamma(1\pm 2z_j) \prod_{1\le i<j\le n}\Gamma(1\pm z_i \pm z_j) \prod_{1\le i<j\le 2n+2}\Gamma(1-w_i-w_j)}.
\end{aligned}
$$

Collecting all factors we obtain

$$
I_N^{(2)} = \frac{\prod_{1\le i<j\le 2N+2} \mathbf{\Gamma}(z_i+z_j)}{\mathbf{\Gamma}\left(1-\sum_k z_k\right)} \times T_N,
$$

where

$$
T_N = \frac{(-1)^N}{2^N N!} \frac{\prod_{1\le i<j\le 2N+2} \sin\pi(z_i+z_j)}{\sin\pi\left(\sum_k z_k\right)} \sum_\pi \frac{\prod_{1\le i<j\le N}\sin^2\pi(z_{\pi(i)}\pm z_{\pi(j)}) \prod_{j=1}^N \sin\pi(2z_{\pi(j)})}{\prod_{j=1}^N \prod_{\substack{i=1 \\ i\neq\pi(j)}}^{2N+2} \sin\pi(z_i \pm z_{\pi(j)})}.
$$

One can show that $T_N = 1$ by taking the limit $q \to 1$ of the identities in [28, equations (7.11) and (7.12)]. Finally, the conditions (2.16) can be relaxed to (2.8) by analytic continuation in the ν_k.

3   Limiting cases

In this section we derive two more integrals which are replicas of the integrals [27, equations (3.2) and (5.4)]. We show that in the complex case these integrals are intrinsically related to the integrals (2.3). This property is not seen in the $\mathrm{SL}(2,\mathbb{R})$ setup.

We start our analysis with the integral (2.3a) and introduce the variables, complex ζ and integer η, as follows

$$\zeta + \frac{\eta}{2} = \sum_{k=1}^{N+1}(z_k + w_k), \qquad \zeta - \frac{\eta}{2} = \sum_{k=1}^{N+1}(\bar{z}_k + \bar{w}_k). \tag{3.1}$$

The r.h.s. of (2.3a) is a meromorphic function of ζ with poles located at the points $\zeta_p = |1 + \eta/2| + p$, $p \in \mathbb{N}_+$. For $\eta = 0$ and ζ close to 1 the r.h.s. of (2.3a) takes the form

$$I_N^{(1)} = \frac{1}{1-\zeta}\prod_{k,j=1}^{N+1}\mathbf{\Gamma}(z_k + w_j) + \cdots, \tag{3.2}$$

where z_k, w_k obey the constraint $\sum_{k=1}^{N+1}(z_k + w_k) = \sum_{k=1}^{N+1}(\bar{z}_k + \bar{w}_k) = 1$. At the same time, only the element $\mathcal{Q}_{NN}$ of the matrix (2.10) becomes singular at this point. The corresponding integral diverges at large ν and n as $\zeta \to 1$. Indeed, taking into account equation (2.7) one finds

$$\mathcal{Q}(u|z,w) = (-1)^{\sum_k[z_k]}u^{\zeta-N-1}(-\bar{u})^{\zeta-N-1}\big(1 + O(1/\|u\|)\big).$$

Thus for $\zeta \to 1$

$$\mathcal{Q}_{N,N}(z,w) \simeq (-1)^{\sum_k[z_k]}\frac{1}{\pi}\int_{r>\Lambda}\frac{1}{r^{4-2\zeta}}\mathrm{d}x\mathrm{d}y + \cdots = \frac{1}{1-\zeta}(-1)^{\sum_k[z_k]} + \cdots \tag{3.3}$$

and therefore

$$I_N^{(1)} \underset{\zeta\to 1}{=} \mathcal{Q}_{NN}(z,w) \times \det \widehat{\boldsymbol{\mathcal{Q}}}_{N-1}(z,w) + \text{finite terms.}$$

Here $\widehat{\boldsymbol{\mathcal{Q}}}_{N-1}$ is the main $N-1$ minor of the matrix $\boldsymbol{\mathcal{Q}}_N$. Comparing the residues at $\zeta = 1$ in equation (3.3) and (3.2) we obtain the following identity

$$\det \widehat{\boldsymbol{\mathcal{Q}}}_{N-1}(z,w) = (-1)^{\sum_k[z_k]}\prod_{k,j=1}^{N+1}\mathbf{\Gamma}(z_k + w_j), \tag{3.4}$$

which holds provided $\sum_{k=1}^{N+1}(z_k + w_k) = \sum_{k=1}^{N+1}(\bar{z}_k + \bar{w}_k) = 1$. Rewriting the l.h.s. of (3.4) in an integral form one obtains

$$\frac{1}{(N-1)!}\int \mathcal{D}u_1 \cdots \mathcal{D}u_{N-1}\prod_{p=1}^{N-1}(-1)^{n_p}\frac{\prod_{j=1}^{N+1}\prod_{k=1}^{N-1}\mathbf{\Gamma}(z_j - u_k)\mathbf{\Gamma}(u_k + w_j)}{\prod_{k<j}\mathbf{\Gamma}(u_k - u_j)\mathbf{\Gamma}(u_j - u_k)}$$

$$= (-1)^{\sum_m[z_m]}\prod_{k,j=1}^{N+1}\mathbf{\Gamma}(z_k + w_j). \tag{3.5}$$

This integral is an analog of the integral [27, equation (3.2)]. Indeed, replacing the variable z_{N+1} by $z_{N+1} = 1 - \sum_{k=1}^{N}z_k - \sum_{m=1}^{N+1}w_m$ and using the relation (2.2) one can bring equation (3.5) into the form which is a replica of [27, equation (3.2)]

$$\frac{1}{(N-1)!}\int \mathcal{D}u_1 \cdots \mathcal{D}u_{N-1}\frac{\prod_{k=1}^{N-1}\prod_{m=1}^{N}\prod_{j=1}^{N+1}\mathbf{\Gamma}(z_m - u_k)\mathbf{\Gamma}(u_k + w_j)}{\prod_{m=1}^{N-1}\mathbf{\Gamma}(\gamma + u_m)\prod_{k<j}\mathbf{\Gamma}(u_k - u_j)\mathbf{\Gamma}(u_j - u_k)}$$

$$= \frac{\prod\limits_{k=1}^{N} \prod\limits_{j=1}^{N+1} \boldsymbol{\Gamma}(z_k + w_j)}{\prod\limits_{j=1}^{N+1} \boldsymbol{\Gamma}(\gamma - w_j)}, \tag{3.6}$$

where $\gamma = \sum\limits_{k=1}^{N} z_k + \sum\limits_{m=1}^{N+1} w_m$.

The analysis of the second integral, equation (2.3b), proceeds along the same lines so we give only a brief account. Similar to (3.1) we define variables, η and ζ, by $\zeta + \eta/2 = \sum\limits_{k=1}^{2N+2} z_k$. The l.h.s. and r.h.s. of equation (2.3b) have a pole at $\zeta = 1$ (for $\eta = 0$). Comparing the corresponding residues we get

$$\frac{1}{2^{N-1}(N-1)!} \int_{\pm} \mathcal{D}u_1 \cdots \mathcal{D}u_{N-1} \frac{\prod\limits_{k=1}^{N-1} \prod\limits_{j=1}^{2N+2} \boldsymbol{\Gamma}(z_j \pm u_k)}{\prod\limits_{k=1}^{N-1} \boldsymbol{\Gamma}(\pm 2u_k) \prod\limits_{j<k} \boldsymbol{\Gamma}(\pm u_k \pm u_j)}$$
$$= \pm \prod\limits_{1 \le j < k \le 2N+2} \boldsymbol{\Gamma}(z_j + z_k), \tag{3.7}$$

where $\sum\limits_{k=1}^{2N+2} z_k = \sum\limits_{k=1}^{2N+2} \bar{z}_k = 1$ and the subscript $\pm$ at the integral sign indicates that the sum goes over either integer n (plus) or half-integer n (minus). Introducing the variable $\gamma = \sum\limits_{k=1}^{2N+1} z_k$, one can rewrite this integral in the form identical to the integral [27, equation (5.4)]

$$\frac{2^{1-N}}{(N-1)!} \int_{\pm} \mathcal{D}u_1 \cdots \mathcal{D}u_{N-1} \frac{\prod\limits_{k=1}^{N-1} \prod\limits_{j=1}^{2N+1} \boldsymbol{\Gamma}(z_j \pm u_k)}{\prod\limits_{k=1}^{N-1} \boldsymbol{\Gamma}(\gamma \pm u_k)\boldsymbol{\Gamma}(\pm 2u_k) \prod\limits_{j<k} \boldsymbol{\Gamma}(\pm u_k \pm u_j)}$$
$$= \frac{\prod\limits_{1 \le j < k \le 2N+1} \boldsymbol{\Gamma}(z_j + z_k)}{\prod\limits_{k=1}^{2N+3} \boldsymbol{\Gamma}(\gamma - z_k)}. \tag{3.8}$$

Thus in the SL$(2, \mathbb{C})$ setup the integrals (3.5), (3.6) and (3.7), (3.8) are intrinsically related to the integrals (2.3). For $N = 2$ the relation (3.8) was derived by Sarkissian and Spiridonov [55].

4   Star-triangle relation

In this section we show that the star-triangle relations with the Boltzmann weights given by a product of Γ-functions [5, 32, 33] follow in a rather straightforward way from the integrals (2.3). The star-triangle relation underlies exact solvability of various two-dimensional lattice models, see references [3, 4] for a review. Here we recall such a relation inherent to the noncompact SL$(2, \mathbb{C})$ spin chain magnets [12].

Let $s_\alpha(z) \equiv s_{\alpha,\bar{\alpha}}(z, \bar{z})$ be a function of the complex variables $z = x + \mathrm{i}y$, $\bar{z} \equiv z^* = x - \mathrm{i}y$,

$$\boldsymbol{s}_\alpha(z) = [z]^{-\alpha} \equiv z^{-\alpha}\bar{z}^{-\bar{\alpha}}, \qquad [\alpha] = \alpha - \bar{\alpha} \in \mathbb{Z}.$$

The function $s_\alpha(z)$ is a single valued function on the complex plane and in the physics literature it is usually called a propagator.[2] It satisfies two identities:

- the chain relation

$$\frac{1}{\pi}\int \boldsymbol{s}_{\alpha_1}(z_1-z)\boldsymbol{s}_{\alpha_2}(z-z_2)\mathrm{d}^2z = \frac{\boldsymbol{\Gamma}(1-\alpha_1)\boldsymbol{\Gamma}(1-\alpha_2)}{\boldsymbol{\Gamma}(2-\alpha_1-\alpha_2)}\boldsymbol{s}_{\alpha_1+\alpha_2-1}(z_1-z_2), \qquad (4.1)$$

- the star-triangle relation

$$\frac{1}{\pi}\int \prod_{k=1}^{3}\boldsymbol{s}_{\alpha_k}(z_k-z)\mathrm{d}^2z$$

$$= \left(\prod_{k=1}^{3}\boldsymbol{\Gamma}(1-\alpha_k)\right)\boldsymbol{s}_{1-\alpha_1}(z_2-z_3)\boldsymbol{s}_{1-\alpha_2}(z_3-z_1)\boldsymbol{s}_{1-\alpha_3}(z_1-z_2), \qquad (4.2)$$

which holds provided $\alpha_1+\alpha_2+\alpha_3 = \bar\alpha_1+\bar\alpha_2+\bar\alpha_3 = 2$.

In fact these two relations are equivalent: equation (4.2) is reduced to equation (4.1) in the limit $z_3 \to \infty$ and, vice versa, equation (4.2) can be derived from equation (4.1) by a $\mathrm{SL}(2,\mathbb{C})$ transformation. The relation (4.2) underlies the integrability of the noncompact $\mathrm{SL}(2,\mathbb{C})$ spin chain magnets.

In [5, 32, 33] new solutions of the star-triangle relation have been derived from the elliptic star-triangle relation [6, 18, 56, 59]. Below we show that these star-triangle relations can be derived from the integrals (2.3).

First we consider the relation associated with the integral (2.3a). To this end we define the propagator

$$\boldsymbol{S}_\alpha(u) = (-1)^{[\alpha/2+u]}\boldsymbol{\Gamma}\left(\frac{1-\alpha}{2}+u\right)\boldsymbol{\Gamma}\left(\frac{1-\alpha}{2}-u\right)$$

$$= (-1)^{[\alpha/2+u]}\frac{\Gamma\left(\frac{1-\alpha}{2}+u\right)\Gamma\left(\frac{1-\alpha}{2}-u\right)}{\Gamma\left(\frac{1+\bar\alpha}{2}-\bar u\right)\Gamma\left(\frac{1+\bar\alpha}{2}+\bar u\right)}. \qquad (4.3)$$

The variables u and α have the form

$$u = n/2 + i\nu, \qquad \bar u = -n/2 + i\nu, \qquad \alpha = m + \sigma, \qquad \bar\alpha = -m + \sigma,$$

where $[u] = n$ and $\frac{1}{2}[\alpha] = m$ are either both integer or half-integer numbers and $\langle u \rangle = \nu$, $\langle \alpha \rangle = \sigma$ are real complex numbers, respectively. Under these conditions the arguments of the $\boldsymbol{\Gamma}$-functions in (4.3) have the form (2.1). Slightly abusing the terminology we call the u (α) even or odd depending on the character of n (m) and refer to this property as parity. Also, in order to avoid possible misunderstanding due to our agreement to indicate only the "holomorphic" arguments of functions, $f(\alpha) \equiv f(\alpha,\bar\alpha)$, we accept that, whenever $\bar x$ is not explicitly defined, $x+\alpha \equiv (x+\alpha, x+\bar\alpha)$, e.g., $\boldsymbol{\Gamma}(1/2+z) \equiv \boldsymbol{\Gamma}(1/2+z, 1/2+\bar z)$.

The propagator $\boldsymbol{S}_\alpha$ inherits many properties of $\boldsymbol{s}_\alpha$:

- for even(odd) α the propagator is an even(odd) function of u

$$\boldsymbol{S}_\alpha(-u) = (-1)^{[\alpha]}\boldsymbol{S}_\alpha(u),$$

- for imaginary $\langle \alpha \rangle = \sigma$, $\boldsymbol{S}_\alpha(u)(\boldsymbol{S}_\alpha(u))^\dagger = 1$, i.e., the propagator reduces to a phase factor, while for $[\alpha] = 0$ and $\langle \alpha \rangle$ real, $\boldsymbol{S}_\alpha(u)$ is real and positive.

---

[2]Let us stress here that $[z]^\alpha$ denotes the power function while $[\alpha]$ without any superscript stands for the "integer" part of α, $[\alpha] = \alpha - \bar\alpha$. We hope that this slightly unfortunate notation does not lead to confusion.

The chain relation for the propagator $\boldsymbol{S}$ follows from the integral (2.3a) for $N = 1$. Namely, making the substitution

$$z_1 = (1 - \alpha_1)/2 + z, \qquad w_1 = (1 - \alpha_1)/2 - z,$$
$$z_2 = (1 - \alpha_2)/2 + w, \qquad w_2 = (1 - \alpha_2)/2 - w$$

in (2.3a) one obtains

$$\sum_{n=-\infty}^{\infty} \int_{-i\infty}^{i\infty} \boldsymbol{S}_{\alpha_1}(z - u)\boldsymbol{S}_{\alpha_2}(u - w)\frac{\mathrm{d}\nu}{2\pi\mathrm{i}} = \frac{\boldsymbol{\Gamma}(1 - \alpha_1)\boldsymbol{\Gamma}(1 - \alpha_2)}{\boldsymbol{\Gamma}(2 - \alpha_1 - \alpha_2)}\boldsymbol{S}_{\alpha_1 + \alpha_2 - 1}(z - w), \qquad (4.4)$$

where $u = n/2 + \nu$ and sum goes over integers. The parity of the α_1 and z (α_2 and w) is always the same. The integral (4.4) is well defined provided $\mathrm{Re}\langle\alpha_k\rangle < 1$, $k = 1, 2$ and $\mathrm{Re}\langle\alpha_1 + \alpha_2\rangle > 1$: the poles of the $\boldsymbol{\Gamma}$ functions in the integral (4.4) are separated by the integration contour if $\mathrm{Re}\langle\alpha_k\rangle < 1$ and the integral converges at large u if $\mathrm{Re}\langle\alpha_1 + \alpha_2\rangle > 1$.

The star-triangle relation for $\boldsymbol{S}_\alpha$ can be obtained from the integral identity (3.5) for $N = 2$. Let us make the following substitution

$$z_i \mapsto \frac{1 - \alpha_i}{2} + z_i, \qquad \bar{z}_i \mapsto \frac{1 - \bar{\alpha}_i}{2} + \bar{z}_i, \qquad w_i \mapsto \frac{1 - \alpha_i}{2} - z_i, \qquad \bar{w}_i \mapsto \frac{1 - \bar{\alpha}_i}{2} - \bar{z}_i$$

in that equation. Taking into account that the condition $\sum_k(z_k + w_k) = 1$ gives rise to the following restriction on the indices: $\sum_k \alpha_k = \sum_k \bar{\alpha}_k = 2$, one derives after some algebra

$$\sum_{n=-\infty}^{\infty} \int_{-i\infty}^{i\infty} \prod_{k=1}^{3} \boldsymbol{S}_{\alpha_k}(z_k - u)\frac{\mathrm{d}\nu}{2\pi\mathrm{i}}$$
$$= \left[\prod_{k=1}^{3} \boldsymbol{\Gamma}(1 - \alpha_k)\right] \boldsymbol{S}_{1-\alpha_1}(z_2 - z_3)\boldsymbol{S}_{1-\alpha_2}(z_3 - z_1)\boldsymbol{S}_{1-\alpha_3}(z_1 - z_2). \qquad (4.5)$$

Here, again, the α_k and z_k are even or odd simultaneously. For the special choice of the parameters, $\alpha_k = \bar{\alpha}_k$, this relation coincides with the star-triangle relation [5, equation (22)].

Proceeding with the second integral (2.3b) we define the propagator as the product of four $\boldsymbol{\Gamma}$ functions

$$\boldsymbol{\mathcal{D}}_\alpha(z_1, z_2) = \boldsymbol{\Gamma}\left(\frac{1 - \alpha}{2} \pm z_1 \pm z_2\right) = \frac{\Gamma\left(\frac{1-\alpha}{2} \pm z_1 \pm z_2\right)}{\Gamma\left(\frac{1+\bar{\alpha}}{2} \pm \bar{z}_1 \pm \bar{z}_2\right)}.$$

The requirement for the arguments of $\boldsymbol{\Gamma}$ functions, $\frac{1-\alpha}{2} \pm z_1 \pm z_2$, to be integers imposes obvious restrictions on the relative parity of all variables. Obviously, the propagator $\boldsymbol{\mathcal{D}}_\alpha(z_1, z_2)$ is an even function of z_1, z_2 and invariant under the $z_1 \leftrightarrow z_2$ permutation. Therefore, for each variable $z_k = n_k/2 + \nu_k$ one can restrict n_k to positive (negative) values. Note that unlike the previous case the propagator $\boldsymbol{\mathcal{D}}_\alpha$ is not shift invariant. Also, up to a phase factor depending on the parity of arguments, $\boldsymbol{\mathcal{D}}_\alpha(z_1, z_2) \sim \boldsymbol{S}_\alpha(z_1 - z_2)\boldsymbol{S}_\alpha(z_1 + z_2)$.

The chain relation for the propagator $\boldsymbol{\mathcal{D}}_\alpha$ follows from the identity (2.3b). Indeed, after the substitution $z_{1(2)} = (1 - \alpha_1)/2 \pm z$, $z_{3(4)} = (1 - \alpha_2)/2 \pm w$ the integral (2.3b) takes the form

$$2\int_{\pm} \mathcal{D}u\|u\|^2\boldsymbol{\mathcal{D}}_{\alpha_1}(z, u)\boldsymbol{\mathcal{D}}_{\alpha_2}(u, w) = \pm\frac{\boldsymbol{\Gamma}(1 - \alpha_1)\boldsymbol{\Gamma}(1 - \alpha_2)}{\boldsymbol{\Gamma}(2 - \alpha_1 - \alpha_2)}\boldsymbol{\mathcal{D}}_{\alpha_1 + \alpha_2 - 1}(z, w).$$

Here the subscripts $\pm$ indicate that the sum goes over all integers ("+") or half-integers ("−") and we also recall that for $u = n/2 + i\nu$, $\|u\|^2 = \nu^2 + n^2/4$.

Next, substituting $z_{2i-1} = (1-\alpha_i)/2 + z_i$, $z_{2i} = (1-\alpha_i)/2 - z_i$, for $i = 1, 2, 3$ in equation (3.8) for $N = 2$ one gets

$$2 \int_{\pm} \mathcal{D}u \|u\|^2 \prod_{k=1}^{3} \boldsymbol{\mathcal{D}}_{\alpha_k}(z_k, u)$$
$$= \left(\prod_{k=1}^{3} \boldsymbol{\Gamma}(1 - \alpha_k) \right) \boldsymbol{\mathcal{D}}_{1-\alpha_1}(z_2, z_3) \boldsymbol{\mathcal{D}}_{1-\alpha_2}(z_1, z_3) \boldsymbol{\mathcal{D}}_{1-\alpha_3}(z_1, z_2), \tag{4.6}$$

where $\sum_k \alpha_k = \sum_k \bar{\alpha}_k = 2$. As was mentioned earlier the parity of all variables have to be coordinated so that (4.6) encompasses four different identities:

(1) All α_k are integer / have positive parity

    (a) z_k, u are even,
    (b) z_k, u are odd,

(2) α_1 and α_2, α_3 are even and odd, respectively

    (a) u and z_1 are even and z_2, z_3 are odd,
    (b) u and z_1 are odd and z_2, z_3 are even.

Variant (1a) corresponds to the star-triangle relation obtained by Kels [32, equation (17)]. An extension of the star-triangle relation of [32] which follows from the relation (3.8) with $N = 2$ was also considered by Sarkissian and Spiridonov [55].

In Cases (1a) and (1b) the functions $\boldsymbol{\mathcal{D}}_\alpha(z_1, z_2)$ are real if $0 < \alpha = \bar{\alpha} < 1$: $\boldsymbol{\mathcal{D}}_\alpha(z_1, z_2) > 0$ for even z_1, z_2 (Case (1a)) and $-\boldsymbol{\mathcal{D}}_\alpha(z_1, z_2) > 0$ for odd z_1, z_2 (Case (1b)). In both cases these functions can be interpreted as the Boltzmann weights of lattice integrable models, for more details see references [32, 33].

5 Quasi-classical limit

Let us consider the identities (4.4), (4.5) when the external variables become large. We replace the variables $z_k(w_k)$ in (4.4), (4.5) by $Lz_k = L(x_k + iy_k)$ and take the limit $L \to \infty$. The variables x_k, y_k can be considered in this limit as continuous ones, so that $z_k \in \mathbb{C}$, $\bar{z}_k = z_k^*$. It is easy to check that the leading contribution to the integrals (4.4), (4.5) comes from the region where $u_k \sim L$, so that we replace $u_k \mapsto Lu_k$ as well. Moreover in this limit $\boldsymbol{S}_\alpha$ turns into $\boldsymbol{s}_\alpha$:

$$L^{2\langle \alpha \rangle} \boldsymbol{S}_\alpha(Lz) \underset{L \to \infty}{\mapsto} z^{-\alpha} \bar{z}^{-\bar{\alpha}} = \boldsymbol{s}_\alpha(z). \tag{5.1}$$

The sum over n in $\int \mathcal{D}u$ can be replaced by the integral so that

$$\frac{1}{L^2} \sum_{n=-\infty}^{\infty} \int_{-i\infty}^{i\infty} \frac{d\nu}{2\pi i} \underset{L \to \infty}{\mapsto} \frac{1}{\pi} \int d^2 u, \tag{5.2}$$

where $u = u_x + iu_y$ and $d^2 z = du_x du_y$. Taking into account equations (5.1) and (5.2) one checks that in the limit $L \to \infty$ equations (4.4), (4.5) become the chain and the star-triangle relations, equations (4.1) and (4.2), respectively.

In what follows we study the quasi-classical limit the integrals (2.3a) and (3.5) for general N. First of all we rewrite these identities in term of the propagator $\boldsymbol{S}_\alpha$. After the change of variables

$$z_k \mapsto \frac{1 - \alpha_k}{2} + z_k, \qquad w_k \mapsto \frac{1 - \alpha_k}{2} - z_k, \qquad k = 1, \ldots, N + 1.$$

Equation (2.3a) takes the following form

$$
\frac{1}{N!} \int \mathcal{D}u_1 \cdots \mathcal{D}u_N \prod_{1 \le i < j \le N} \|u_i - u_j\|^2 \prod_{k=1}^{N} \prod_{m=1}^{N+1} \boldsymbol{S}_{\alpha_m}(z_m - u_k)
$$
$$
= (-1)^{\sum_{m=1}^{N} m[\alpha_{m+1}]} \frac{\prod_{k=1}^{N+1} \boldsymbol{\Gamma}(1-\alpha_k)}{\boldsymbol{\Gamma}\left(N+1-\sum_{k=1}^{N+1}\alpha_k\right)} \prod_{1\le i<k\le N+1} \boldsymbol{S}_{\alpha_i+\alpha_k-1}(z_i - z_k).
$$

Similarly, the identity (3.5) can be represented as follows

$$
\frac{1}{(N-1)!} \int \mathcal{D}u_1 \cdots \mathcal{D}u_{N-1} \prod_{1 \le i < j \le N-1} \|u_i - u_j\|^2 \prod_{k=1}^{N} \prod_{m=1}^{N+1} \boldsymbol{S}_{\alpha_m}(z_m - u_k)
$$
$$
= (-1)^{\sum_{m=1}^{N} m[\alpha_{m+1}]} \left(\prod_{k=1}^{N+1} \boldsymbol{\Gamma}(1-\alpha_k)\right) \prod_{1\le i<k\le N+1} \boldsymbol{S}_{\alpha_i+\alpha_k-1}(z_i - z_k),
$$

and where $\sum_k \alpha_k = \sum_k \bar{\alpha}_k = N$.

In the quasi-classical limit these identities are reduced to the following two-dimensional integrals with power functions

$$
\frac{1}{N!} \int \prod_{1 \le i < j \le N} |u_i - u_j|^2 \prod_{k=1}^{N} \prod_{m=1}^{N+1} [z_m - u_k]^{-\alpha_m} d^2u_1 \cdots d^2u_N
$$
$$
= \pi^N (-1)^{\sum_{m=1}^{N} m[\alpha_{m+1}]} \frac{\prod_{k=1}^{N+1} \boldsymbol{\Gamma}(1-\alpha_k)}{\boldsymbol{\Gamma}\left(N+1-\sum_{k=1}^{N+1}\alpha_k\right)} \prod_{1\le i<k\le N+1} [z_i - z_k]^{1-\alpha_i-\alpha_k} \tag{5.3}
$$

and

$$
\frac{1}{(N-1)!} \int \prod_{1 \le i < j \le N-1} |u_i - u_j|^2 \prod_{k=1}^{N} \prod_{m=1}^{N+1} [z_m - u_k]^{-\alpha_m} d^2u_1 \cdots d^2u_{N-1}
$$
$$
= \pi^{N-1} (-1)^{\sum_{m=1}^{N} m[\alpha_{m+1}]} \left(\prod_{k=1}^{N+1} \boldsymbol{\Gamma}(1-\alpha_k)\right) \prod_{1\le i<k\le N+1} [z_i - z_k]^{1-\alpha_1-\alpha_k}. \tag{5.4}
$$

It can be checked that in the quasi-classical limit the propagator $\mathcal{D}_\alpha(z,w)$ turns into $[z^2 - w^2]^{-\alpha}$. Therefore equations (2.3b) and equation (3.7) do not produce new identities in this limit but reduce to the integrals (5.3) and (5.4) after an appropriate change of variables.

One can take a different point of view on the integrals (2.3a) and (2.3b) and consider them as the "quantized" version of the integrals (5.3). For $\alpha_k = \bar{\alpha}_k$ the integral (5.3) is the special case of the duality relation [2] for the Dotsenko–Fateev (DF) integrals [19] considered in the next section. Therefore one can hope that there exists a "quantized" version of the duality relation in a general situation.

6   Dotsenko–Fateev integrals

In this section we give an elementary proof of the following relation

$$\frac{1}{\pi^n n!} \int \prod_{i<k}^{n}[y_i - y_k] \prod_{i=1}^{n} \prod_{j=1}^{n+m+1} [y_i - z_j]^{-\alpha_j} \mathrm{d}^2 y_1 \cdots \mathrm{d}^2 y_n = \frac{(-1)^{\sum\limits_{k=1}^{n+m} k[\alpha_{k+1}]} \prod\limits_{j=1}^{n+m+1} \mathbf{\Gamma}(1 - \alpha_j)}{\mathbf{\Gamma}\left(1 + n - \sum\limits_{j=1}^{n+m+1} \alpha_j\right)}$$

$$\times \prod_{i<j}^{n+m+1} [z_j - z_i]^{1-\alpha_i-\alpha_j} \frac{1}{\pi^m m!} \int \prod_{i<k}^{m}[u_i - u_k] \prod_{i=1}^{m} \prod_{j=1}^{n+m+1} [u_i - z_j]^{-1+\alpha_j} \mathrm{d}^2 u_1 \cdots \mathrm{d}^2 u_m, \quad (6.1)$$

which is, provided the parameters satisfy the constraint $\alpha_k = \bar{\alpha}_k$, the duality relation [2, 21] for the DF integrals [19], see also [46]. For $m = 0$ this identity coincides with (5.3).

To evaluate the integral on the l.h.s. of (6.1) we go over to the variables $x_k = x_k(y_1, \ldots, y_n)$ which are essentially the elementary symmetric functions

$$\prod_{i=1}^{n}(y_i + t) = x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} + t^n.$$

Every point in x-space has $n!$ preimages in y-space and the Jacobian of the transformation is $J = \left|\frac{\partial x_k}{\partial y_i}\right|^2 = \prod_{i<j}^{n}[y_i - y_k]$. Introducing the notation $t_j = -z_j$ one gets for the integral on the l.h.s. of (6.1)

$$\pi^{-n} \int \prod_{j=1}^{n+m+1} \left[x_1 + x_2 t_j + x_3 t_j^2 + \cdots + x_n t_j^{n-1} + t_j^n\right]^{-\alpha_j} \mathrm{d}^2 x_1 \cdots \mathrm{d}^2 x_n. \qquad (6.2)$$

Then using the momentum space representation for the propagators

$$[z]^{-\alpha} = \frac{1}{\pi} \mathrm{i}^{\alpha - \bar{\alpha}} \mathbf{\Gamma}(1 - \alpha) \int \mathrm{e}^{\mathrm{i}(pz + \bar{z}\bar{p})} [p]^{\alpha - 1} \mathrm{d}^2 p$$

in (6.2) and carrying out all integrals over x_k one gets

$$\frac{\mathrm{i}^{\sum_j (\alpha_j - \bar{\alpha}_j)}}{\pi^{m+1}} \prod_{k=1}^{n+m+1} \mathbf{\Gamma}(1 - \alpha_k) \prod_{i=1}^{n+m+1} \int \mathrm{d}^2 p_i [p_i]^{\alpha_k - 1} \prod_{k=1}^{n} \delta^{(2)}\left(\sum_j p_j t_j^{k-1}\right) \mathrm{e}^{\mathrm{i}\sum_j (p_j t_j^n + \bar{p}_j \bar{t}_j^n)}$$

$$= \frac{\prod\limits_{k=1}^{n+m+1} \mathbf{\Gamma}(1 - \alpha_k)}{\pi^m \mathbf{\Gamma}\left(n + 1 - \sum_j \alpha_j\right)} \prod_{i=1}^{n+m+1} \int \frac{\mathrm{d}^2 p_i}{[p_i]^{1-\alpha_i}} \prod_{k=1}^{n} \delta^{(2)}\left(\sum_j p_j t_j^{k-1}\right) \delta^{(2)}\left(\sum_j p_j t_j^n - 1\right).$$

In order to get the last δ-function one represents $\mathrm{e}^{\mathrm{i}\sum_j (p_j t_j^n + \bar{p}_j \bar{t}_j^n)}$ as $\int \mathrm{e}^{\mathrm{i}(\lambda + \bar{\lambda})} \delta^{(2)}(\lambda - \sum_j p_j t_j^n) d\lambda$ and rescales $p_j \to \lambda p_j$. The delta functions cut out a m-dimensional subspace in the $(n+m+1)$-dimensional space defined by the linear equations

$$\sum_{j=1}^{n+m+1} p_j t_j^{k-1} = 0, \qquad k = 1, \ldots, n \qquad \text{and} \qquad \sum_{j=1}^{n+m+1} p_j t_j^n = 1. \qquad (6.3)$$

The solution depends on m free variables, $u_1, \ldots, u_m$, and takes the form

$$p_j(u_1, \ldots, u_m) = \lambda_j (u_1 + t_j) \cdots (u_m + t_j), \qquad (6.4)$$

where $\lambda_j = \prod_{k \neq j} (t_j - t_k)^{-1}$, $j = 1, \ldots, n + m + 1$. That p_j, (6.4), satisfy equations (6.3) follows from an identity due to Milne [41, Lemma 1.33], see also [48, equation (2.3)],

$$\sum_{k=1}^{N} \frac{(b_1 - t_k) \cdots (b_N - t_k)}{t_k \prod_{j \neq k} (t_j - t_k)} = \frac{b_1 \cdots b_N}{t_1 \cdots t_N} - 1. \tag{6.5}$$

To see it is enough to put $N = m + n + 1$, $b_k = -u_k$ for $k \leq m$ and $b_k = -u$, for $k > m$ in the above identity and compare the coefficients at the powers u^k, $k = 0, \ldots, n$ on both sides of (6.5).

Making the change of variables

$$p_j = \lambda_j (u_1 + t_j) \cdots (u_m + t_j) + s_j, \qquad j = 1, \ldots, n + 1,$$
$$p_j = \lambda_j (u_1 + t_j) \cdots (u_m + t_j), \qquad j = n + 2, \ldots, n + m + 1,$$

and taking into account that

$$\prod_{k=1}^{n} \delta^2 \left(\sum_{j=1}^{n+m+1} p_j t_j^{k-1} \right) \delta^2 \left(\sum_{j=1}^{n+m+1} p_j t_j^{n} - 1 \right) = \prod_{k=1}^{n+1} \delta^2 \left(\sum_{j=1}^{n+1} s_j t_j^{k-1} \right) = \frac{\prod_{j=1}^{n+1} \delta^2 (s_j)}{\prod_{1 \leq i < j \leq n+1} [t_i - t_j]}$$

and

$$\prod_{j=1}^{n+m+1} \mathrm{d}^2 p_j = \left(\prod_{n+2 \leq i < j \leq n+m+1} [t_i - t_j] \prod_{k=n+2}^{n+m+1} [\lambda_k] \right) \frac{1}{m!} \prod_{j=1}^{m} \mathrm{d}^2 u_j \prod_{j=1}^{n+1} \mathrm{d}^2 s_j$$

one gets after some algebra

$$\frac{\prod_{k=1}^{n+m+1} \mathbf{\Gamma}(1 - \alpha_k)}{\mathbf{\Gamma}\left(n + 1 - \sum_j \alpha_j\right)} \left(\prod_{j<k} (-1)^{\alpha_k - \bar{\alpha}_k} [t_j - t_k]^{1 - \alpha_k - \alpha_j} \right)$$
$$\times \frac{1}{\pi^m m!} \prod_{j=1}^{n+m+1} \int \prod_{k=1}^{m} (u_k + t_j)^{\alpha_j - 1} \mathrm{d}^2 u_1 \cdots \mathrm{d}^2 u_m.$$

It coincides with the r.h.s. of equation (6.1) after changing $t_j \to -z_j$. We have learned from the discussions with Litvinov that similar proof of the duality relation (6.1) is presented in his lectures on conformal field theory [39].

Finally, we make a conjecture that the duality relation (6.1) admits a generalization to the "quantized" cases. Namely, these relations take the form

$$\frac{1}{n!} \int \mathcal{D} u_1 \cdots \mathcal{D} u_n \prod_{i<j} \|u_i - u_j\|^2 \prod_{i=1}^{n} \prod_{j=1}^{n+m+1} (-1)^{[u_i]} \mathbf{\Gamma}(z_j - u_i) \mathbf{\Gamma}(u_i + w_j)$$
$$= \frac{\prod_{i,j=1}^{n+m+1} \mathbf{\Gamma}(z_i + w_j)}{\mathbf{\Gamma}\left(\sum_j (z_j + w_j) - m\right)} (-1)^{m \sum_j [z_j - w_j]} \frac{1}{m!} \int \mathcal{D} u_1 \cdots \mathcal{D} u_m$$
$$\times \prod_{i<j} \|u_i - u_j\|^2 \prod_{i=1}^{m} \prod_{j=1}^{n+m+1} (-1)^{[u_i]} \mathbf{\Gamma}(z'_j - u_i) \mathbf{\Gamma}(w'_j + u_i), \tag{6.6}$$

where $z'_j = 1/2 - w_j$, $w'_j = 1/2 - z_j$ and

$$\frac{2^n}{n!} \int_\pm \mathcal{D}u_1 \cdots \mathcal{D}u_n \|u_k\|^2 \prod_{1 \le i < j \le n} \|u_i \pm u_j\|^2 \prod_{i=1}^{n} \prod_{j=1}^{2(n+m+1)} \boldsymbol{\Gamma}(z_j \pm u_i)$$

$$= \varkappa_{n+m} \frac{\displaystyle\prod_{i<j}^{2(n+m+1)} \boldsymbol{\Gamma}(z_i + z_j)}{\boldsymbol{\Gamma}\left(\sum_j z_j - m\right)} \frac{2^m}{m!} \int_\pm \mathcal{D}u_1 \cdots \mathcal{D}u_m \|u_k\|^2$$

$$\times \prod_{1 \le i < j \le m} \|u_i \pm u_j\|^2 \prod_{i=1}^{m} \prod_{j=1}^{2(n+m+1)} \boldsymbol{\Gamma}(1/2 - z_j \pm u_i), \tag{6.7}$$

where $\varkappa_k = 1, (-1)^{k(k+1)/2}$ for the integer and half-integer cases, respectively. For $m = 0$ these integrals are equivalent to the integrals (2.3) and in the quasi-classical limit both of them reproduce the duality relation (6.1). For $n = m = 1$ the relations (6.6) and (6.7) follow from the star-triangle relations (4.5) and (4.6). For few first n and m the integrals (6.6), (6.7) go through numerical tests. Closing this section we note that quite similar duality relations were observed recently in the so-called conformal fishnet model [11].

7   Summary

In [16, 17] a generalization of Gustafson integrals to the complex case have been obtained. The derivation of these integrals rely on the completeness of the SoV representation for the $\mathrm{SL}(2, \mathbb{C})$ magnets, that is not yet proven. In this work we presented a direct calculation of two $\boldsymbol{\Gamma}$ function integrals. We expect that these integral identities will be helpful in proving the completeness of the SoV representation for the $\mathrm{SL}(2, \mathbb{C})$ spin chains.

The complex integrals are, up to appropriate modification of the Γ function and integration measure, exact copies of the integrals obtained by R.A. Gustafson [28]. However, the analytic properties of these integrals are different. We have shown that several, apparently distinct in the $\mathrm{SL}(2, \mathbb{R})$ context integrals are intrinsically related to each other in the $\mathrm{SL}(2, \mathbb{C})$ formulation.

We have also shown that the complex $\boldsymbol{\Gamma}$ integrals for the lowest N underlie the star-triangle identities derived in references [5, 32, 33] and in the quasi-classical limit are reduced to the special ($m = 0$) case of the duality relation for the DF integrals [2, 19, 21]. We also conjecture an appropriate modification of these duality relations to hold for the integrals with complex $\boldsymbol{\Gamma}$ functions.

Acknowledgements

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