Reprints in Theory and Applications of Categories, No. 9, 2005, pp. 1–7.

CATEGORIES OF SPACES MAY NOT BE GENERALIZED SPACES AS EXEMPLIFIED BY DIRECTED GRAPHS

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Author commentary: When this paper was distributed at the 1986 international category theory meeting in Cambridge, its reception was mixed. So when Xavier Caicedo, the academic editor of the "Revista Colombiana de Matemáticas" proposed to publish it together with the proceedings of the 1983 Bogotà Workshop, I was pleased to accept; thanks to his continued generosity in granting copyright permission, it can now be reprinted in *TAC*.

The simple idea at the core of this paper has not yet been much pursued by workers in topos theory, even though I have tried in several later publications to point out its importance to various branches of mathematics, where those colleagues with greater knowledge and ability could, I believe, contribute.

Already in SGA4, Grothendieck had made a major advance on this problem, in a series of ten exercises for which he quite justly awarded himself "une médaille de chocolat". His construction, generalizing Giraud's gros topos of a topological space, is in terms of sites and has apparently not yet been assimilated well enough to suggest a corresponding invariant description.

In a broad sense, any topos over a base S can be conceived as a "generalized space"; even the basic facts that it may have a proper class of points, or that these points may form a category that does not reduce to a poset, do not prevent this imagination from being useful. For example, not only terminology such as "connected", taken from geometry, but even far-reaching constructions such as distributions, the "space" of distributions studied by Bunge and Carboni, and supports of distributions seen as singular coverings, studied by Bunge and Funk, are partly motivated by that exuberant generality. More precise results depend on limiting the generality, even on taking into explicit account some opposite qualitative distinctions within the generality.

The completeness theorems of Barr, Deligne, Diaconescu, Freyd, Joyal, Makkai, Reyes, and others, summed up by Johnstone in 1983 [J], in particular involve constructions showing that any S-topos (for example conceived as the infinitary positive theory of some kind of structure) has "enough" morphisms (= points, models) from other S -toposes of a very special kind, localic (i.e. with poset site) or sometimes groupoidal. Extending those methods, Tierney and Moerdijk, in collaboration with Joyal, showed by 1990 that every

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Thanks to Christoph Schubert for his expert transcription into TEX.

Received by the editors 2004-11-02.

Transmitted by M. Barr, R. Rosebrugh and R. F. C. Walters. Reprint published on 2005-03-23.

2000 Mathematics Subject Classification: 00A30, 14F20, 18B25, 18B30, 18B40, 55U40, 81P05.

Key words and phrases: topos of spaces, directed graphs, sheaves, idempotents, Mengen.

The article originally appeared in Revista Colombiana de Matemáticas, XX (1986) 179-186.

S-topos agrees with respect to certain cohomological invariants (not only with respect to internal logic) with special toposes deserving of being called generalized spaces in a more concrete sense.

That concrete sense was already much alive before category theory was made explicit: the invention of the cohomology of groups stemmed from discoveries by Hopf, Hurewicz, Steenrod, and others. Today those discoveries could be summed up by noting that both classical spaces and abstract groups are fully embedded in a single larger category (on which cohomology is defined) and participate there in mixed exact sequences embodying fundamental groups, universal covers, and even more particular geometric information. This single larger category could be taken to be the one of all S-toposes; however, a much smaller category suffices for that, namely the category of étendu (i.e. of those toposes having a site consisting entirely of monomorphisms). Grothendieck established the étendu as generalized spaces in a concrete sense, by qualitatively extending the classical work through the construction of more informative quotients of spaces.

But then, around 1960, Grothendieck made a remarkable pair of constructions. Workers in topos theory have yet to come to grips with the specific content of those two constructions, in spite of the 35 year development of a simplified general methodology. The petit étale topos of a scheme, brilliantly overcoming the lack of an inverse function theorem, is clearly a generalized space in a very concrete sense; yet it is not an étendu, and so it's specifically topos-theoretic particularity begs for clarification. On the other hand, the techniques of construction for analytic spaces [G2] involve embedding them in a single large topos qualitatively different from the sheaves on any concrete kind of generalized space.

A general topos incorporating all spaces of a given kind is not a new idea either. Since 1950 the topos of simplicial sets has been widely used, and in fact this combinatorial example served as a kind of model for the discussion in the present paper. That it has a connected object with distinct points concretely implies the axioms 1 and 2. The gros Zariski topos and other environments for algebraic and smooth geometry also enjoy that feature.

Johnstone's topological topos and my bornological topos (sheaves for finite coverings of countable sets) are intuitively also "general" in content (as opposed to "particular") and yet do not satisfy my axioms; but I share with Grothendieck the belief that a suitable development of tame topology will avoid phenomena such as Peano's space filling curves and the non-discreteness of anti-connected spaces. (According to a Joyal-Johnstone result in the 1992 book by Mac Lane and Moerdijk, the topos-theoretic match between continuous and combinatorial topology requires that the continuous interval be totally ordered, and it seems that a tame interval might satisfy that even though the classical one, taken as a site in its own right, does not.)

It was fortunate that the simple enterprise of clarifying the role of two kinds of graphs provided sufficient illustration of the basic distinction here discussed. A feature apparent in this example, namely that the "sheaves" on a particular space B in a general topos are contained in a "particular" topos (obtained by collapsing idempotents in a site), I later found in many other examples in geometry.

Abstract. Axioms are proposed for the distinctive internal connectedness of a topos that models all spaces of a "general" combinatorial, algebraic, or smooth kind. It is shown that sheaves on any particular space, like representations of any particular group, do not satisfy these axioms. For each object B in a general topos, a topos of the "opposite" or "particular" kind (containing sheaves and unramified coverings of B) is constructed. All these features are illustrated by the simple example of reflexive directed graphs.

It has long been recognized [G1], [L] that even within geometry (that is, even apart from their algebraic/logical role) toposes come in (at least) two varieties: as spaces (possibly generalized, treated via the category of sheaves of discrete sets), or as categories of spaces (analytic [G2], topological [J], combinatorial, etc.). The success of theorems [J'] which approximate toposes by generalized spaces has perhaps obscured the role of the second class of toposes, though some explicit knowledge of it is surely necessary for a reasonable axiomatic understanding of toposes of C^∞ spaces or of the topos of simplicial sets. Perhaps some of the confusion is due to the lack of a stabilized definition of morphism appropriate to categories of spaces in the way that "geometric morphisms" are appropriate to generalized spaces.

There are certain properties which a topos of spaces often has; a wise selection of these should serve as an axiomatic definition of the subject. While we have not achieved that goal yet, we list some important properties and show that these properties cannot be true for a "generalized space" of the localic or groupoid kind.

We consider a topos $\mathcal{E}$ defined over another topos $\mathcal{S}$. The latter need not be the category of abstract sets, though it will often be Boolean. In many cases it is instructive to think of $\mathcal{S}$ as *derived from* $\mathcal{E}$ (rather than the other way around), as Cantor derived "cardinal numbers" (= abstract sets) from "Mengen" (= sets with topological or similar structure, as they arise in geometry and analysis). Indeed $\mathcal{S}$ can be viewed as a sheaf topos in $\mathcal{E}$, for an essential topology:

Axiom 0. $\mathcal{E} \longrightarrow \mathcal{S}$ is local; $\Gamma^* \dashv \Gamma_* \dashv \Gamma^!$.

The Γ^* may be considered as the inclusion of *discrete* spaces $\mathcal{S}$ into "all" spaces $\mathcal{E}$, whereas the sheaf inclusion $\Gamma^!$ may be considered as the inclusion of *codiscrete* or chaotic spaces into $\mathcal{E}$; that these inclusions have the same domain category $\mathcal{S}$ may be summed up in Hegelian fashion by "pure becoming is identical with non-becoming".[1]

Of course, there are some spatial toposes which satisfy axiom 0, although they are extremely special since Γ_* is the fiber-functor for a canonically-defined extremal point of $\mathcal{E}$; for example, the Zariski spectrum of a *local* ring does admit such a point $\Gamma^!$. On the other hand, the topos of G-sets for a groupoid G cannot satisfy axiom 0.

Our further axioms will be stated in terms of a further left adjoint $\Pi_0 = \Gamma_!$ assigning to each space a discrete space of components.

[1] That is, completely random motion, as a category in itself, is indistinguishable from immobility, as a category in itself, even though they are of course completely different (except for 0, 1) as subcategories of the category of spaces (= frames for continuous motion).

Axiom 1. $\mathcal{E} \longrightarrow \mathcal{S}$ is *essential*, that is $\Gamma_! \dashv \Gamma^*$ exists, but moreover we require that it preserves finite products

$$\Gamma_!(X \times Y) \xrightarrow{\sim} \Gamma_!(X) \times \Gamma_!(Y)$$
$$\Gamma_!(1) \xrightarrow{\sim} 1$$

for all X, Y in $\mathcal{E}$.

The axiom is necessary for the naive construction of the homotopic passage from quantity to quality; namely, it insures that (not only Γ_* but also) $\Gamma_!$ is a closed functor, thus inducing a second way of associating an $\mathcal{S}$-enriched category to each $\mathcal{E}$-enriched category

$$\mathcal{E}\text{-cat} \xrightarrow{[\ ]} \mathcal{S}\text{-cat}.$$

For example, $\mathcal{E}$ itself as an $\mathcal{E}$-enriched category gives rise to a homotopy category in which

$$[\mathcal{E}](X, Y) = \Gamma_!(Y^X).$$

This product-preserving property of $\Gamma_!$ is well-known to be false in the group case, where $\Gamma_!(G \times G) = n$, where $n = \#G$, whereas $\Gamma_!(G) = 1$. Again, it *can* hold for some (extremely special) spaces: For a topos $\mathcal{E}$ localic over $\mathcal{S}$, $\Gamma_!$ is left exact if only it preserves products, and hence there is again a canonically defined point, at the opposite extreme; for example, the Zariski spectrum of an integral domain admits a product preserving $\Gamma_!$. If $\mathcal{S}$ is an "exponential variety" in $\mathcal{E}$, then $\Gamma_!(\Gamma^*(A) \times Y) \xrightarrow{\sim} A \times \Gamma_!(Y)$ which is, however, only a fragment of our axiom 1. It is at this point that the constructions of generalized spaces which "cover" a given topos insofar as the "internal logic" is concerned, fail to preserve the structure of a "topos of spaces". (For covering as an "exponential variety" would preserve our axiom 2).

Axiom 2. $\Gamma_!(\Omega) = 1$, where Ω is the truth-value object in the topos $\mathcal{E}$ of spaces.

Since Ω has the structure of a monoid with zero, in the presence of axiom 1 its being connected (axiom 2) implies its being contractible in that

$$[\mathcal{E}](X, \Omega) = 1$$

for all X in $\mathcal{E}$, and hence that $X \longrightarrow \Omega^X$ is a natural embedding of every space into a contractible space; moreover, any retract, such as Ω_j for a topology j, (for example the Boolean algebra $\Omega_{\neg\neg}$) is also contractible. Of course axiom 2 cannot be true of a Boolean topos since $\Gamma_!$ preserves any sum such as $1 + 1$.

PROPOSITION 1. Axioms 1 and 2 cannot both be true for a localic topos $\mathcal{E}$ over sets $\mathcal{S}$.

Proof. Axiom 1 implies that $\Gamma_!$ preserves pullbacks in the localic case. In any case there is a pullback diagram

$$\begin{tikzcd} 2 \arrow[r] \arrow[d] & 1 \arrow[d] \\ \Omega \arrow[r] & \Omega \end{tikzcd}$$

in $\mathcal{E}$ where $2 = 1 + 1$. Thus applying $\Gamma_!$ we get an impossible pullback diagram

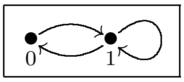
$$\begin{tikzcd} 2 \arrow[r] \arrow[d] & 1 \arrow[d] \\ 1 \arrow[r] & 1 \end{tikzcd}$$

in $\mathcal{S}$. ∎

The above axioms (incomplete though they may be) enable us to make some rather sharp distinctions. For example, there are (at least) two distinct toposes commonly referred to as "the category of directed graphs" and even commonly considered to be more or less of the same value since, for example, the notion of "free category" generated by either kind of graph makes sense. The two are

$$\mathcal{S}^{\Delta_1^{\mathrm{op}}} \qquad \mathcal{S}^{\cdot\rightrightarrows\cdot}$$

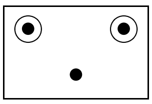
where Δ_1 is the three-element monoid of all order-preserving endomaps of the two-element linearly ordered set [1]; splitting the idempotents shows that Γ_* is essentially representable and hence $\Gamma^!$, the notion of codiscrete graph, exists for $\mathcal{S}^{\Delta_1^{\mathrm{op}}}$, though not for $\mathcal{S}^{\cdot\rightrightarrows\cdot}$. However, the one-dimensional simplicial sets $\mathcal{S}^{\Delta_1^{\mathrm{op}}}$ and the "irreflexive" graphs $\mathcal{S}^{\cdot\rightrightarrows\cdot}$ differ already in regard to axiom 1: the functor $\Gamma_!$ is in either case just the coequalizer of the structural maps, but, as is well-known, reflexive coequalizers preserve products, whereas irreflexive coequalizers do not. The subobject classifier for $\mathcal{S}^{\Delta_1^{\mathrm{op}}}$ has five elements



and is obviously connected. A similar statement is true for $\mathcal{S}^{\cdot\rightrightarrows\cdot}$ but the foregoing remarks are sufficient to show the following.

PROPOSITION 2. The topos $\mathcal{S}^{\Delta_1^{\mathrm{op}}}$ satisfies the axioms 0, 1, 2 for a "topos of spaces", whereas the topos $\mathcal{S}^{\cdot\rightrightarrows\cdot}$ of diagram schemes does not satisfy 0 or 1.

In fact, at least two arguments can be given to show that $\mathcal{S}^{\cdot\rightrightarrows\cdot}$ definitely belongs to the other variety of toposes, namely that it is in fact a simple example of a generalized space. For one thing, the category $\mathcal{S}^{\cdot\rightrightarrows\cdot}$ of irreflexive graphs is an étendue; in fact, it is locally localic in an illuminating manner: Consider the space



which has three points and five open sets. A sheaf on this space consists of a set E of global sections, two sets V_0, V_1 of sections over the two open points, and two restriction maps $E \longrightarrow V_0$, $E \longrightarrow V_1$.

If we consider the two-element group acting on the space by interchanging the two open points, we can take the "quotient" (descent) in the 2-category of toposes by the equivalence relation associated to this action; this has the effect of forcing $V_0 = V_1$, but allowing the two restrictions $E \rightrightarrows V$ to remain different. Conversely, there is an object A in $\mathcal{S}^{\cdot\rightrightarrows\cdot}$ such that $\mathcal{S}^{\cdot\rightrightarrows\cdot}/A$ is (the topos of sheaves on) the three point space, showing explicitly the local homeomorphism of the two toposes.

Another aspect of the status of $\mathcal{S}^{\cdot\rightrightarrows\cdot}$ as a generalized space is revealed by its relationship to the category of spaces $\mathcal{S}^{\Delta_1^{\mathrm{op}}}$. If $\mathcal{E}$ over $\mathcal{S}$ is a topos of "spaces", then each object B of $\mathcal{E}$ should be capable of serving as a domain of variation in its own right; in particular it should have sense to speak of abstract sets varying over B, giving rise to a topos $\mathcal{S}(B)$ (usually a subcategory of $\mathcal{E}/B$), which should be an example of a generalized space ("should be" since we don't yet have axioms strong enough to capture the special nature of generalized spaces, yet general enough to include the classical petit étale example!). In case $B \in \mathcal{E} = \mathcal{S}^{\Delta_1^{\mathrm{op}}}$ is a graph, one reasonable definition of

$$\mathcal{E}/B \supset \mathcal{S}(B)$$

is simply to take all $E \longrightarrow B$ which have discrete fibers in the sense that

$$\begin{tikzcd}
\Gamma^*\Gamma_*\, E \arrow[r] \arrow[d] & E \arrow[d] \\
\Gamma^*\Gamma_*\, B \arrow[r] & B
\end{tikzcd}$$

is a pullback. These might be called "B-partite graphs" generalizing the bipartite graphs which arise as the special case where

$$B = \boxed{\bullet \rightleftarrows \bullet}$$

The toposes $\mathcal{S}(B)$ are *all* étendues, and behave with excellent functorial comportment under morphisms $B \longrightarrow B'$ in the topos of spaces $\mathcal{E} = \mathcal{S}^{\Delta_1^{\mathrm{op}}}$; thus they seem to embody well one idea of the generalized spaces associated to objects of $\mathcal{E}$.

PROPOSITION 3. If $L = \boxed{\bullet \circlearrowleft}$ is the object of $\mathcal{S}^{\Delta_1^{\mathrm{op}}}$ obtained by identifying the two points of the representable object $\Delta[1]$, then irreflexive graphs may be identified with L-partite graphs:

$$\mathcal{S}^{\cdot\rightrightarrows\cdot} \simeq \mathcal{S}(L).$$

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