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A HOMOTOPY DOUBLE GROUPOID OF A HAUSDORFF SPACE

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ABSTRACT. We associate to a Hausdorff space, X, a double groupoid, $\boldsymbol{\rho}_2^{\square}(X)$, the *homotopy double groupoid* of X. The construction is based on the geometric notion of *thin square*. Under the equivalence of categories between small 2-categories and double categories with connection given in [BM] the homotopy double groupoid corresponds to the *homotopy* 2-*groupoid*, $\mathbf{G}_2(X)$, constructed in [HKK]. The cubical nature of $\boldsymbol{\rho}_2^{\square}(X)$ as opposed to the globular nature of $\mathbf{G}_2(X)$ should provide a convenient tool when handling 'local-to-global' problems as encountered in a generalised van Kampen theorem and dealing with tensor products and enrichments of the category of compactly generated Hausdorff spaces.

Introduction

We associate to a Hausdorff space, X, a double groupoid, $\boldsymbol{\rho}_2^{\square}(X)$, called the *homotopy double groupoid* of X. The construction is based on the geometric notion of *thin square* extending the notion of *thin relative homotopy* introduced in [HKK]. Roughly speaking, a thin square is a continuous map from the unit square of the real plane into X which factors through a tree. More precisely, the homotopy double groupoid is a double groupoid with connection which, under the equivalence of categories between small 2-categories and double categories with connection given in [BM], corresponds to the *homotopy* 2-*groupoid*, $\mathbf{G}_2(X)$, of X constructed in [HKK]. We make use of the properties of pushouts of trees in the category of Hausdorff spaces investigated in [HKK]. The construction of the 2-cells of the homotopy double groupoid is based on a suitable *cubical* approach to the notion of *thin* 3-*cube* whereas the construction of the 2-cells of the homotopy 2-groupoid can be interpreted by means of a *globular* notion of thin 3-cube.

Why double groupoids with connection?

The homotopy double groupoid of a space and the related homotopy 2-groupoid are constructed directly from the cubical singular complex and so remain close to geometric intuition in an almost classical way. Unlike in the globular 2-groupoid approach, however, the resulting structure remains cubical and in particular is symmetrical with respect to the two directions. Cubes subdivide neatly so complicated pasting arguments can be avoided in this context. Composition as against subdivision is easy to handle. The 'geometry' is near to the surface and naturally leads to the algebra.

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We see this as an elegant model for the 2-type of a space. Elegance often goes with efficiency. The algebra fits better than that of the related model in 'local-to-global' situations such as encountered in a generalised van Kampen theorem as has already proved true in the relative case of the homotopy double groupoid of a triple of spaces of Brown-Higgins [BH 1]. Similarly the tensor product structure of cubical sets transfers to our model and this can then be applied to get information on function spaces and a double groupoid enrichment for the category of compactly generated Hausdorff spaces in which the geometry of higher homotopies is more transparent than, say, in the related 2-groupoid enrichment.

1. The singular cubical set of a topological space

We shall be concerned with the low dimensional part (up to dimension 3) of the singular cubical set

$$R^{\square}(X) = (R^{\square}_n(X), \partial_i^-, \partial_i^+, \varepsilon_i)$$

of a topological space X. We recall the definition (cf. [BH 2,BH 1]).

For $n \geqslant 0$ let

$$R^{\square}_n(X) = \mathbf{Top}(I^n, X)$$

denote the set of *singular n–cubes* in X, i.e. continuous maps $I^n \longrightarrow X$, where $I = [0, 1]$ is the unit interval of real numbers. We shall identify $R^{\square}_0(X)$ with the set of points of X. For $n = 1, 2, 3$ a singular n–cube will be called a *path*, resp. *square*, resp. *cube*, in X.

The *face maps*

$$\partial_i^-, \partial_i^+ : R^{\square}_n(X) \longrightarrow R^{\square}_{n-1}(X) \quad (i = 1, \dots, n)$$

resp. *degeneracy maps*

$$\varepsilon_i : R^{\square}_{n-1}(X) \longrightarrow R^{\square}_n(X) \quad (i = 1, \dots, n)$$

are given by

$$\begin{aligned}
\partial_i^-(a)(s_1, \dots, s_{n-1}) &= a(s_1, \dots, s_{i-1}, 0, s_{i+1}, \dots, s_{n-1}),\\
\partial_i^+(a)(s_1, \dots, s_{n-1}) &= a(s_1, \dots, s_{i-1}, 1, s_{i+1}, \dots, s_{n-1}),
\end{aligned}$$

resp.

$$\varepsilon_i(b)(s_1, \dots, s_n) = b(s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$$

where $a : I^n \longrightarrow X$, resp. $b : I^{n-1} \longrightarrow X$, are continuous maps.

The face and degeneracy maps satisfy the usual cubical relations (cf. (1.1) of [BH 2], (5.1) of [KP]).

If $a \in R^{\square}_n(X)$, then

$$\partial(a) = (\partial_1^-(a), \partial_1^+(a), \dots, \partial_n^-(a), \partial_n^+(a))$$

will be called the *boundary* of a.

A path $a \in R_1^{\square}(X)$ will sometimes be denoted a_s by abuse of language and will be called a *path from* a_0 *to* a_1, denoted $a : a_0 \simeq a_1$.

If $a : a_0 \simeq a_1$ and $b : b_0 \simeq b_1$ are paths such that $a_1 = b_0$, then we denote by

$$a + b : a_0 \simeq b_1$$

their *concatenation*, i.e.

$$(a+b)(s) = \begin{cases} a(2s), & \text{if } 0 \leqslant s \leqslant \frac{1}{2} \\ b(2s-1), & \text{if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases} . \tag{1.1}$$

If x is a point of X, then $\varepsilon_1(x) \in R_1^{\square}(X)$, denoted e_x, is the *constant path* at x, i.e.

$$e_x(s) = x \text{ for all } s \in I.$$

If $a : x \simeq y$ is a path in X, we denote by $-a : y \simeq x$ the *path reverse to* a, i.e.

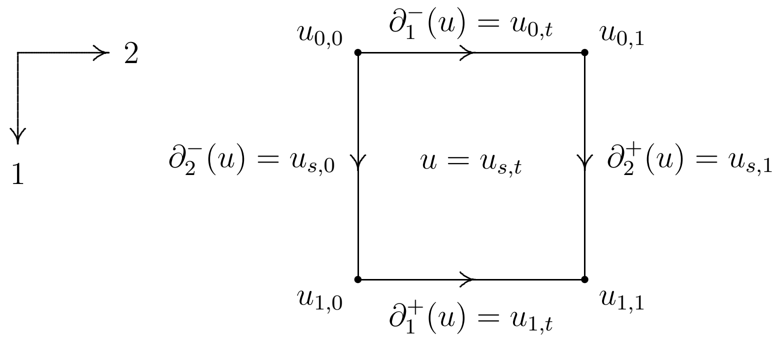
$$(-a)(s) = a(1-s) \text{ for } s \in I.$$

If x is a point of X, we write $\odot_x$ or simply $\odot$ for the doubly degenerate square $\varepsilon_2\varepsilon_1(x) = \varepsilon_1\varepsilon_1(x)$, i.e.

$$\odot_x(s,t) = x \text{ for all } s,t \in I.$$

A square $u \in R_2^{\square}(X)$ will sometimes be denoted $u_{s,t}$ by abuse of language. Then $\partial_1^-(u), \partial_1^+(u), \partial_2^-(u), \partial_2^+(u)$ correspond to $u_{0,t}, u_{1,t}, u_{s,0}, u_{s,1}$. Similarly, a cube $U \in R_3^{\square}(X)$ will sometimes be denoted by $U_{r,s,t}$.

A square $u \in R_2^{\square}(X)$ may be illustrated by the following picture.



Besides the boundary, $\dot{u} = \partial(u)$, of a square $u \in R_2^{\square}(X)$, we will have to consider its 0-*skeleton*, $\ddot{u}$, defined by

$$\ddot{u} = (u_{0,0},\ u_{0,1},\ u_{1,0},\ u_{1,1}) .$$

If $u \in R_2^{\square}(X)$ is a square, then we denote by $\overset{\leftrightarrow}{u}$ its *reflexion*, i.e.

$$\overset{\leftrightarrow}{u}_{s,t} = u_{t,s} \text{ for } s,t \in I.$$

In the set of squares $R_2^\square(X)$ we have two partial compositions $+_1$ (*vertical composition*) and $+_2$ (*horizontal composition*). If $u, v \in R_2^\square(X)$ are squares such that $\partial_1^+(u) = \partial_1^-(v)$, then $u +_1 v$ is defined by

$$(u +_1 v)(s,t) = \begin{cases} u(2s,t), & \text{if } 0 \leqslant s \leqslant \frac{1}{2} \\ v(2s-1,t), & \text{if } \frac{1}{2} \leqslant s \leqslant 1 \end{cases}. \tag{1.2}$$

If $u, w \in R_2^\square(X)$ are squares such that $\partial_2^+(u) = \partial_2^-(w)$, then $u +_2 w$ is defined by

$$(u +_2 w)(s,t) = \begin{cases} u(s,2t), & \text{if } 0 \leqslant t \leqslant \frac{1}{2} \\ w(s,2t-1), & \text{if } \frac{1}{2} \leqslant t \leqslant 1 \end{cases}. \tag{1.3}$$

Similarly, in the set of cubes $R_3^\square(X)$ we have three partial compositions $+_1, +_2, +_3$.

If $u \in R_2^\square(X)$ is a square, then $-_1u, -_2u$ are defined by

$$(-_1u)(s,t) = u(1-s,t) \text{ resp. } (-_2u)(s,t) = u(s,1-t).$$

The obvious properties of vertical and horizontal composition of squares are listed in [BH 2,BH 1]. In particular we have the following *interchange property.*

Let $u, u', w, w' \in R_2^\square(X)$ be squares, then

$$(u +_2 w) +_1 (u' +_2 w') = (u +_1 u') +_2 (w +_1 w'), \tag{1.4}$$

whenever both sides are defined.

More generally, we have an interchange property for rectangular decomposition of squares. In more detail, for positive integers m, n let $\varphi_{m,n} : I^2 \longrightarrow [0,m] \times [0,n]$ be the homeomorphism $(s,t) \longmapsto (ms, nt)$. An $m \times n$ *subdivision* of a square $u : I^2 \longrightarrow X$ is a factorisation $u = u' \circ \varphi_{m,n}$; its *parts* are the squares $u_{ij} : I^2 \longrightarrow X$ defined by

$$u_{ij}(s,t) = u'(s+i-1, t+j-1).$$

We then say that u is the *composite* of the squares u_{ij}, and we use matrix notation $u = [u_{ij}]$. Note that we can also write

$$u +_1 v = \begin{bmatrix} u \\ v \end{bmatrix}, \ u +_2 w = \begin{bmatrix} u & w \end{bmatrix}$$

and that the two sides of the interchange property (1.4) can be written

$$\begin{bmatrix} u & w \\ u' & w' \end{bmatrix}.$$

Finally, we define two functions, called *connections*,

$$\Gamma^-, \Gamma^+ : R_1^\square(X) \longrightarrow R_2^\square(X) \tag{1.5}$$

as follows. Let $a \in R_1^{\square}(X)$ be a path, $a : x \simeq y$, then let

$$\Gamma^-(a)(s,t) = a(\max(s,t));\ \Gamma^+(a)(s,t) = a(\min(s,t)).$$

The following formulae are obvious.

$$\begin{aligned}
&\partial_1^-\Gamma^-(a) = \partial_2^-\Gamma^-(a) = a;\ \ \partial_1^+\Gamma^-(a) = \partial_2^+\Gamma^-(a) = e_y;\\
&\partial_1^-\Gamma^+(a) = \partial_2^-\Gamma^+(a) = e_x;\ \partial_1^+\Gamma^+(a) = \partial_2^+\Gamma^+(a) = a;\\
&\Gamma^-(e_x) = \Gamma^+(e_x) = \odot_x.
\end{aligned} \tag{1.6}$$

The full structure of $R^{\square}(X)$ as a *cubical complex with connections and compositions* has been exhibited in [ABS].

2. Double tracks, 2-tracks

2.1. DEFINITION. *Let $a, a' : x \simeq y$ be paths in X. Then a* relative homotopy $u : a \simeq a' : x \simeq y$ *between a and a' is a square $u \in R_2^{\square}(X)$ such that*

$$\partial(u) = (a, a', e_x, e_y)\ .$$

2.2. DEFINITION.

(1) Let $u : I^2 \longrightarrow X$ be a square in X. Then the homotopy class of u relative to the boundary $\dot{I}^2$ of I will be called a double track *and denoted by $\{u\}$.*

(2) If $u : a \simeq a' : x \simeq y$ is a relative homotopy, then the double track $\{u\}$ will be called a 2-track *from a to a' and denoted by $\{u\} : a \Longrightarrow a'$.*

Explicitly, we have the following conditions.

(1′) Let u be a square in X with boundary $\partial(u) = (a, a', b, b')$ and let u' be another square. Then $\{u\} = \{u'\}$ if and only if $\partial(u) = \partial(u')$ and there exists a 3-cube $U \in R_3^{\square}(X)$ such that

$$\partial(U) = (u, u', \varepsilon_1(a), \varepsilon_1(a'), \varepsilon_1(b), \varepsilon_1(b'))\ .$$

(2′) Let $u, u' : a \simeq a' : x \simeq y$ be relative homotopies. Then $\{u\} = \{u'\}$ if and only if there is a 3–cube $U \in R_3^{\square}(X)$ such that

$$\partial(U) = (u, u', \varepsilon_1(a), \varepsilon_1(a'), \odot_x, \odot_y).$$

If $u : a \simeq a' : x \simeq y$ and $v : a' \simeq a'' : x \simeq y$ are relative homotopies, vertical pasting (1.2) of u and v induces a vertical pasting operation on 2–tracks, denoted $\{u\} +_1 \{v\} : a \Longrightarrow a''$, yielding a groupoid structure (with identities, $\{\varepsilon_1(a)\}$, denoted 0 or 0_a) on the set $\prod_2 X(x,y)$ of 2–tracks between paths in X from x to y. Similarly if $u : a \simeq a' : x \simeq y$

and $w : b \simeq b' : y \simeq z$, horizontal pasting (1.3) of homotopies induces a horizontal pasting operation on 2–tracks

$$(\{u\}, \{w\}) \longmapsto \{u\} +_2 \{w\} : a + b \Longrightarrow a' + b',$$

satisfying the *interchange law*

$$(\{u\} +_2 \{w\}) +_1 (\{u'\} +_2 \{w'\}) = (\{u\} +_1 \{u'\}) +_2 (\{w\} +_1 \{w'\}).$$

3. Thin squares

The following definition is crucial for the construction of the homotopy double groupoid. Recall first that if K and L are (finite) simplicial complexes then a map $\Phi : |K| \longrightarrow |L|$ between the underlying spaces of K resp. L is PWL (*piecewise linear*) if there exist subdivisions of K and L relative to which Φ is simplicial. Also by a *tree* we mean the underlying space $|K|$ of a finite 1-connected 1-dimensional simplicial complex K.

3.1. Definition. *(1) A square $u : I^2 \longrightarrow X$ in a topological space X is* thin *if there is a factorisation of u*

(3.2) $$u : I^2 \xrightarrow{\Phi_u} J_u \xrightarrow{p_u} X,$$

where J_u is a tree and Φ_u is PWL on the boundary $\dot{I}^2$ of I^2.
(2) A double track is thin *if it has a thin representative.*

3.3. Example. Let $a \in R_1^\square(X)$ be a path. Then the degeneracies $\varepsilon_1(a)$, $\varepsilon_2(a)$ and the connections $\Gamma^-(a)$, $\Gamma^+(a)$ are thin. This is shown by the factorisations

$$\varepsilon_1(a) : I^2 \xrightarrow{pr_2} I \xrightarrow{a} X; \ \varepsilon_2(a) : I^2 \xrightarrow{pr_1} I \xrightarrow{a} X$$

resp.

$$\Gamma^-(a) : I^2 \xrightarrow{\max} I \xrightarrow{a} X; \ \Gamma^+(a) : I^2 \xrightarrow{\min} I \xrightarrow{a} X,$$

where pr_i denotes the projection onto the i-th coordinate.

3.4. Remark. Explicitly, the condition on Φ_u in Definition 3.1 means

(i) there is a finite 1-connected, 1-dimensional simplicial complex L_u such that $J_u = |L_u|$,

(ii) there is a subdivision of $\dot{I}^2$ and a subdivision of L_u on which $\Phi_u| \dot{I}^2 : \dot{I}^2 \longrightarrow L_u$ is simplicial.

Then it is not hard to see that, without loss of generality, we may assume that the vertices of $\dot{I}^2$ are elements of the subdivision of $\dot{I}^2$. Furthermore, by a common refinement type argument, one can show that in Definition 3.1 the condition

($*$) $$\Phi_u \textit{ is } \mathrm{PWL} \textit{ on the boundary of } I^2$$

is equivalent to the condition

$$(**)\qquad \Phi_u \textit{ is } \mathrm{PWL} \textit{ on each edge of the boundary of } I^2.$$

Finally, it may happen in (3.2) that a segment of the subdivision of $\dot{I}^2$ is mapped constantly under u but is mapped bijectively under Φ_u onto a segment of the subdivision of L_u. Then p_u will map this segment constantly. By contracting the segment to a point we see that, without changing the double track of u, we may assume that Φ_u is a relative homotopy if u is a relative homotopy. It follows that a 2-track is thin in the sense of Definition 3.1 (2) if and only if it is thin in the sense of Definition 2.1 (2) of [HKK].

In order to be able to make use of a central lemma of [HKK] dealing with pushouts of trees, from now on, we assume that

$$X \textit{ is a Hausdorff space.}$$

3.5. PROPOSITION. *The class $\mathcal{T}^2$ of thin squares in X is closed under vertical and horizontal composition of squares.*

PROOF. We restrict ourselves to horizontal composition $+_2$. Let $u, w \in R_2^\square(X)$ be thin squares such that $\partial_2^+(u) = \partial_2^-(w)$. Then we have to show that $u +_2 w$ is thin. Let

$$I^2 \xrightarrow{\Phi_u} J_u \xrightarrow{p_u} X \text{ resp. } I^2 \xrightarrow{\Phi_w} J_w \xrightarrow{p_w} X$$

be factorisations of u resp. w according to the definition of thin squares. Consider the diagram

$$(3.6)\qquad \begin{tikzcd} I^2 \arrow[r, "\Phi_u"] & J_u \arrow[dr, "\pi_u"] \arrow[drr, "p_u"] & & \\ I \arrow[u, "\eta_2^+"] \arrow[ur, "\partial_2^+\Phi_u"] \arrow[dr, "\partial_2^-\Phi_w"'] \arrow[d, "\eta_2^-"'] & & J \arrow[r, "p"] & X \\ I^2 \arrow[r, "\Phi_w"'] & J_w \arrow[ur, "\pi_w"] \arrow[urr, "p_w"'] & & \end{tikzcd}$$

where $\eta_2^+(s) = (s, 1)$ and $\eta_2^-(s) = (s, 0)$. The central quadrilateral is supposed to be a pushout in the category of Hausdorff spaces. Then, by Lemma 2.3 of [HKK], J is a tree and π_u, π_w are PWL. Let p be the induced map, let $\varphi_u = \pi_u\Phi_u$, $\varphi_w = \pi_w\Phi_w$. Then φ_u, φ_w are PWL on $\dot{I}^2$ and

$$I^2 \xrightarrow{\Phi} J \xrightarrow{p} X$$

with $\Phi = \varphi_u +_2 \varphi_w$ gives the desired factorisation of $u +_2 w$. ∎

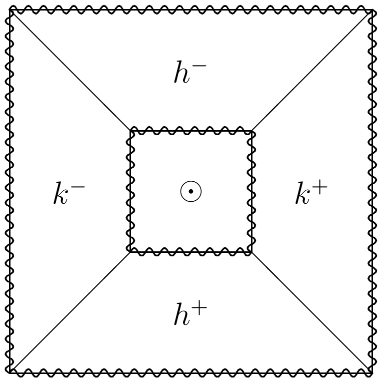
We conclude this section by two filling lemmas. For the definition of a box and a filler in a cubical set, we refer the reader to Definition I. 5.3 of [KP], (see also Definition (2.2) of [K]).

3.7. LEMMA. *Let*

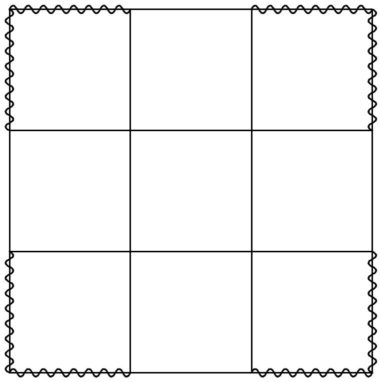
$$\gamma = (-, \odot, k^-, k^+, h^-, h^+)$$

be a $(3, 1, 0)$*-box in* $R^\square(X)$ *such that the reflexions of* k^-, k^+, h^-, h^+ *are all thin relative homotopies. Then there exists a filler* $\lambda \in R_3^\square(X)$ *of* γ *such that* $\partial_1^- \lambda$ *is thin.*

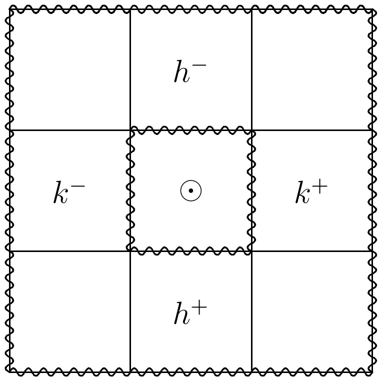
SKETCH OF PROOF. Retracting I^3 onto the given faces via the usual affine projection from a suitable point outside the cube we obtain a cube $H \in R_3^\square(X)$ such that $\partial_1^-(H)$ can be illustrated by the following picture where a twiddled line denotes a constant path.



Let $w : I^2 \longrightarrow I^2$ be the square indicated by the picture



where the four corners are appropriate connections and the other parts are mapped by affine transformations. Let $U : I^3 \longrightarrow I^2$ be the obvious homotopy joining $U_{0,s,t} = w$ and $U_{1,s,t} = \mathrm{id}_{I^2}$. Then $K = \partial_1^-(H) \circ U : I^3 \longrightarrow X$ is a cube such that $\partial_1^+(K) = \partial_1^-(H)$ and $\partial_1^-(K)$ can be described by the following picture



where the four corners are appropriate connections. By assumption and by 3.3 all the components of $\partial_1^-(K)$ are thin, hence $\partial_1^-(K)$ is thin by Proposition 3.5.

Furthermore, the restriction of K to $I \times \dot{I}^2$ is constant. Hence we have an induced map $\bar{K} : I^3/\sim \longrightarrow X$ where

$$(r, s, t) \sim (1, s, t) \text{ for each } (s, t) \in \dot{I}^2 .$$

Pasting $\bar{K}$ and H appropriately, we obtain the desired filler λ. ∎

3.8. Lemma. *Let $u, v : I^2 \longrightarrow X$ be thin squares such that*

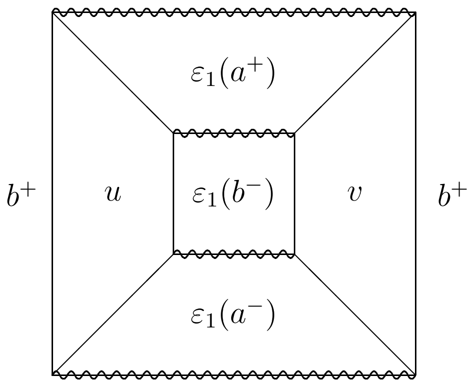
$$\partial(u) = (a^-, a^+, b^-, b^+) = \partial(v) .$$

Then there exists a filler $\lambda \in R_3^\square(X)$ of the $(3,3,1)$-box

$$\gamma = (u, v, \varepsilon_1(a^-), \varepsilon_1(a^+), \varepsilon_1(b^-), -)$$

such that $\partial_3^+\lambda$ is thin.

Sketch of Proof. Applying the standard retraction argument as in the proof of Lemma 3.7 we obtain a filler $\lambda \in R_3^\square(X)$ of γ such that $\partial_3^+\lambda$ can be illustrated by the following picture.



Let $w : I^2 \longrightarrow X$ denote the square indicated by the figure

$$b^+ \quad \begin{array}{|c|c|c|}\hline & \varepsilon_1(a^+) & \\ \hline u & \varepsilon_1(b^-) & v \\ \hline & \varepsilon_1(a^-) & \\ \hline \end{array} \quad b^+$$

where the four corners are appropriate connections. It follows that w is thin. Let

$$w : I^2 \xrightarrow{\Phi_w} J_w \xrightarrow{p_w} X$$

be a factorisation of w as in Definition 3.1(1). Precomposition with a suitable deformation $I^2 \longrightarrow I^2$ yields a factorisation of $\partial_3^+\lambda$ which shows that $\partial_3^+\lambda$ is thin. ■

4. The homotopy double groupoid of a Hausdorff space

In this section we associate with a Hausdorff space X the structure of a double groupoid, $\boldsymbol{\rho}_2^{\square}(X)$, which will be called the *homotopy double groupoid* of X. More precisely, we shall introduce the structure of what has been called 'special double groupoid with special connection' in [BS] and more recently *edge symmetric double groupoid with connection* [BM].

We briefly recall some of the basic facts about double groupoids in the above sense. In the first place, a double groupoid, D, consists of a triple of groupoid structures

$$(D_2, D_1, \partial_1^-, \partial_1^+, +_1, \varepsilon_1),\ (D_2, D_1, \partial_2^-, \partial_2^+, +_2, \varepsilon_2)$$
$$(D_1, D_0, \partial_1^-, \partial_1^+, +, \varepsilon)$$

as partly shown in the diagram

$$\begin{tikzcd} D_2 \arrow[r, shift left, "\partial_2^-"] \arrow[r, shift right, "\partial_2^+"'] \arrow[d, shift right, "\partial_1^-"'] \arrow[d, shift left, "\partial_1^+"] & D_1 \arrow[d, shift right, "\partial_1^-"'] \arrow[d, shift left, "\partial_1^+"] \\ D_1 \arrow[r, shift left, "\partial_1^-"] \arrow[r, shift right, "\partial_1^+"'] & D_0. \end{tikzcd}$$

The elements of D_2 resp. D_1 are called *squares* resp. *edges*. The compositions, $+_1$, resp. $+_2$, are referred to as *vertical* resp. *horizontal composition* of squares. The axioms for a double groupoid include the usual relations of a 2-dimensional cubical set and the *interchange law.* The interchange law allows one to use matrix notation for composites of squares, as indicated in section 1, in an arbitrary double groupoid.

The data for the homotopy double groupoid, $\boldsymbol{\rho}_2^{\square}(X)$, will be denoted by

$$(\rho_2^{\square}(X), \mathbf{G}(X), \partial_1^-, \partial_1^+, +_1, \varepsilon_1), \ (\rho_2^{\square}(X), \mathbf{G}(X), \partial_2^-, \partial_2^+, +_2, \varepsilon_2)$$
$$(\mathbf{G}(X), X, \partial_1^-, \partial_1^+, +, \varepsilon).$$

Here $\mathbf{G}(X)$ denotes the *path groupoid* of X of [HKK]. We recall the definition. The objects of $\mathbf{G}(X)$ are the points of X. The morphisms of $\mathbf{G}(X)$ are the equivalence classes of paths in X with respect to the following relation $\sim_T$.

4.1. Definition. *Let $a, a' : x \simeq y$ be paths in X. Then a is* thinly equivalent *to a', denoted $a \sim_T a'$, if there is a thin relative homotopy between a and a' (cf. (2.1), (3.1) (1)).*

We note that $\sim_T$ is an equivalence relation. Reflexivity and symmetry are clear, transitivity follows from Proposition 3.5. We use $\langle a \rangle : x \simeq y$ to denote the $\sim_T$ class of a path $a : x \simeq y$ and call $\langle a \rangle$ the *semitrack* of a. The groupoid structure of $\mathbf{G}(X)$ is induced by concatenation,+, of paths. Here one makes use of the fact that if $a : x \simeq x', \ a' : x' \simeq x'', \ a'' : x'' \simeq x'''$ are paths then there are canonical thin relative homotopies

$$(a + a') + a'' \simeq a + (a' + a'') : x \simeq x''' \ (\textit{rescale})$$
$$a + e_{x'} \simeq a : x \simeq x'; \ e_x + a \simeq a : x \simeq x' \ (\textit{dilation})$$
$$a + (-a) \simeq e_x : x \simeq x \ (\textit{cancellation}).$$

(Explicit formulae are given on pp. 47,48 of [Spa].)

The source and target maps of $\mathbf{G}(X)$ are given by

$$\partial_1^- \langle a \rangle = x, \ \partial_1^+ \langle a \rangle = y,$$

if $\langle a \rangle : x \simeq y$ is a semitrack. Identities and inverses are given by $\varepsilon(x) = \langle e_x \rangle$ resp. $-\langle a \rangle = \langle -a \rangle$.

In order to construct $\rho_2^{\square}(X)$ we define a relation on the set of squares, $R_2^{\square}(X)$. We introduce the notion of cubically thin homotopy.

4.2. Definition. *Let u, u' be squares in X with common vertices, i.e. $\ddot{u} = \ddot{u}'$.*
(1) A cubically thin homotopy *$U : u \equiv_T^{\square} u'$ between u and u' is a cube $U \in R_3^{\square}(X)$ such that*

(i) U is a homotopy between u and u', i.e.

$$\partial_1^-(U) = u, \quad \partial_1^+(U) = u',$$

(ii) U is rel. vertices of I^2, i.e.

$$\partial_2^-\partial_2^-(U), \quad \partial_2^-\partial_2^+(U), \quad \partial_2^+\partial_2^-(U), \ \partial_2^+\partial_2^+(U)$$

are constant,

(iii) the faces $\partial_i^\alpha(U)$ are thin for $\alpha = \pm 1,\ i = 1, 2$ (cf. (3.1) (1)).

(2) The square u is cubically T-equivalent *to u', denoted $u \equiv_T^\square u'$ if there is a cubically thin homotopy between u and u'.*

Remark. For the definition of the relation $\equiv_T^\square$ condition 4.2 (1) (iii) in the definition of a cubically thin homotopy can be replaced by

(iii′) the 2-tracks of the faces $\partial_i^\alpha(U)$ are thin for $\alpha = \pm 1,\ i = 1, 2$.

The obvious proof makes use of composition of cubes and a dilation argument together with 3.3 and Proposition 3.5. ∎

4.3. Proposition. *The relation $\equiv_T^\square$ is an equivalence relation on $R_2^\square(X)$.*

Proof. The only problem is transitivity: the fact that $U +_1 V$ is cubically thin if U and V are and $\partial_1^+(U) = \partial_1^-(V)$ follows from Proposition 3.5.

If $u \in R_2^\square(X)$ we write $\{u\}_T^\square$, or simply $\{u\}_T$, for the equivalence class of u with respect to $\equiv_T^\square$. We denote the set of equivalence classes $R_2^\square(X)/\equiv_T^\square$ by $\rho_2^\square(X)$.

In view of the relation $\equiv_T^\square$ of (4.2)(2) the face maps

$$\partial_i^\alpha : R_2^\square(X) \longrightarrow R_1^\square(X) \quad (i = 1, 2; \alpha = \pm 1)$$

of the singular cubical set $R^\square(X)$ induce maps

$$\partial_i^\alpha : \rho_2^\square(X) \longrightarrow \mathbf{G}(X)$$

by the formula

$$\partial_i^\alpha\{u\}_T = \langle \partial_i^\alpha u \rangle$$

for $\{u\}_T \in \rho_2^\square(X)$.

In a similar way, the degeneracy maps $\varepsilon_i : R_1^\square(X) \longrightarrow R_2^\square(X)$ $(i = 1, 2)$ induce maps

$$\varepsilon_i : \mathbf{G}(X) \longrightarrow \rho_2^\square(X)$$

by the formula

$$\varepsilon_i\langle a \rangle = \{\varepsilon_i(a)\}_T$$

for $\langle a \rangle \in \mathbf{G}(X)$.

It follows that the cubical relations of the singular cubical set $R^\square(X)$ endow $\boldsymbol{\rho}_2^\square(X)$ with the structure of a 2-dimensional cubical set. ∎

The following proposition describes the effect of composing a thin square and an arbitrary square.

4.4. Proposition. *If $u, v \in R_2^\square(X)$ are squares such that $\partial_2^-(u) = \partial_2^+(v)$ and v is thin, then*

$$v +_2 u \equiv_T^\square u \ ,$$

and similarly for the other three cases.

Proof. For $0 \leqslant s \leqslant 1$, let $v_s : I \longrightarrow X$ be given by $v_s(t) = v(s,t)$. We define $\Lambda : I^3 \longrightarrow X$ by

$$\Lambda(r,s,t) = \Gamma^-(v_s)(r,t)$$

where Γ^- is the connection defined in (1.5). Then for the boundary of Λ we have

$$\partial(\Lambda) = (v, \varepsilon_2\partial_2^+(v),\ \Gamma^-(\partial_1^- v),\ \Gamma^-(\partial_1^+ v),\ \overleftrightarrow{v}, \varepsilon_1\partial_2^+(v)) \ ,$$

where $\overleftrightarrow{v}$ is the reflexion of v.

Since v is assumed to be thin and degeneracies and connections are thin (cf. 3.3), it follows that Λ is a cubically thin equivalence

$$\Lambda : v \equiv_T^\square \varepsilon_2\partial_2^+(v) \ .$$

If H denotes the constant equivalence

$$\varepsilon_1(u) : u \equiv_T^\square u \ ,$$

then $\Lambda +_3 H$ is defined and we have

$$\Lambda +_3 H : v +_2 u \equiv_T^\square \varepsilon_2\partial_2^+(v) +_2 u \ ,$$

furthermore

$$\varepsilon_2\partial_2^+(v) +_2 u \equiv_T^\square u$$

by a dilation type argument. It follows that

$$v +_2 u \equiv_T^\square u \ .$$

■

Next, we show how vertical, resp. horizontal, composition in $\boldsymbol{\rho}_2^\square(X)$ is induced by vertical, resp. horizontal, composition in $R_2^\square(X)$.

For $i = 1,2$ let $D_2^i(X) = R_2^\square(X) \sqcap_i R_2^\square(X)$ be given by the pullback

$$\begin{tikzcd} D_2^i(X) \arrow[r] \arrow[d] & R_2^\square(X) \arrow[d, "\partial_i^-"] \\ R_2^\square(X) \arrow[r, "\partial_i^+"'] & R_1^\square(X) \end{tikzcd} \ .$$

Thus $D_2^i(X)$ is the domain of the composition $+_i : D_2^i(X) \longrightarrow R_2^\square(X)$ (cf. (1.2), (1.3)). Define $\Delta_2^i(X)$ to be the quotient of $D_2^i(X)$ by the relation

$$(u,v) \equiv (u',v')$$

if there is a pair of cubes (H, K) such that

$$\partial_3^- H = u,\ \partial_3^- K = v,\ \partial_3^+ H = u',\ \partial_3^+ K = v',\ \partial_i^+ H = \partial_i^- K$$

and the reflexions of the remaining faces of H, K are thin relative homotopies.

For $(u, v) \in D_2^i(X)$ let $\{(u, v)\} \in \Delta_2^i(X)$ denote the equivalence class of (u, v) with respect to $\equiv$.

In the following proposition $D_2^i(X)$ is compared with the domain $\rho_2^\square(X) \sqcap_i \rho_2^\square(X)$ of the composition $+_i$ to be defined in $\boldsymbol{\rho}_2^\square(X)$.

4.5. Proposition. *There is a natural bijection*

$$\eta : \Delta_2^i(X) \longrightarrow \rho_2^\square(X) \sqcap_i \rho_2^\square(X).$$

Proof. The function η is induced by the two projections $\Delta_2^i(X) \longrightarrow \rho_2^\square(X)$ defined by the projections $D_2^i(X) \longrightarrow R_2^\square(X)$, i.e. for $(u, v) \in D_2^i(X)$ we have

$$\eta\{(u, v)\} = (\{u\}_T, \{v\}_T)\ .$$

We define an inverse ξ to η. For convenience, we assume $i = 2$. Let

$$(\{u\}_T, \{v\}_T) \in \rho_2^\square(X) \sqcap_2 \rho_2^\square(X).$$

Then there is a thin relative homotopy $h : \partial_2^+(u) \simeq \partial_2^-(v)$. So $u +_2 (\overleftrightarrow{h} +_2 v)$ is defined.

Let

$$\xi(\{u\}_T, \{v\}_T) = \{(u, \overleftrightarrow{h} +_2 v)\}. \tag{4.6}$$

We have to prove that ξ is well defined.

The idea is borrowed from Proposition 3 of [BH 1] where it is shown that the homotopy double groupoid of a triple of spaces inherits two well defined partial compositions in dimension 2 (see also Proposition 6.2 of [B 4]).

Let u', v', h' be alternative choices in (4.6). Then there are cubes $U, V \in R_3^\square(X)$ such that

$$\partial_3^-(U) = u,\ \partial_3^+(U) = u',\ \partial_3^-(V) = v,\ \partial_3^+(V) = v'$$

and the reflexions of the remaining faces of U and V are thin relative homotopies. By Lemma 3.7 we obtain a cube $H \in R_3^\square(X)$ such that

$$\partial(H) = (\partial_1^-(H),\ \odot,\ \partial_2^+(U),\ \partial_2^-(V),\ \overleftrightarrow{h},\ \overleftrightarrow{h'})$$

and $\partial_1^-(H)$ is thin. The pair of cubes $(U, H +_2 V)$ shows that

$$(u, \overleftrightarrow{h} +_2 v) \equiv (u', \overleftrightarrow{h'} +_2 v')$$

as required.

It is clear that $\xi\eta = 1$. From Proposition 4.4 it follows that $\eta\xi = 1$. ∎

REMARK. The inverse ξ to η can also be described by the formula

$$\xi(\{u\}_T,\ \{v\}_T) = \{(u +_2 \overset{\leftrightarrow}{h}, v)\}. \tag{4.7}$$

4.8. THEOREM. *The compositions* $+_1, +_2$ *on* $R_2^{\square}(X)$ *are inherited by* $\rho_2^{\square}(X)$ *making* $\boldsymbol{\rho}_2^{\square}(X)$ *into a double groupoid.*

SKETCH OF PROOF. In order to define $+_2$ on $\rho_2^{\square}(X)$, let $(\{u\}_T, \{v\}_T) \in \rho_2^{\square}(X) \sqcap_2 \rho_2^{\square}(X)$. Then

$$\{u\}_T +_2 \{v\}_T = \{u' +_2 v'\}_T \tag{4.9}$$

for any choice $u' \in \{u\}_T, v' \in \{v\}_T$ such that $\partial_2^+(u) = \partial_2^-(v)$. Then $+_2$ is well defined on $\rho_2^{\square}(X)$ by definition of the relation $\equiv$.

In particular, we have

$$\{u\}_T +_2 \{v\}_T = \{[u\ \overset{\leftrightarrow}{h}\ v]\}_T \tag{4.10}$$

if $h : \partial_2^+(u) \simeq \partial_2^-(v)$ is a thin relative homotopy.

The definition of $+_1$ is similar. In particular, for $(\{u\}_T, \{w\}_T) \in \rho_2^{\square}(X) \sqcap_1 \rho_2^{\square}(X)$ we have

$$\{u\}_T +_1 \{w\}_T = \{\begin{bmatrix} u \\ k \\ w \end{bmatrix}\}_T, \tag{4.11}$$

if $k : \partial_1^+(u) \simeq \partial_1^-(w)$ is a thin relative homotopy.

Then $+_1$ and $+_2$ induce two groupoid structures on $\rho_2^{\square}(X)$ with identities $\{\varepsilon_1(a)\}_T$, resp. $\{\varepsilon_2(a)\}_T$, and inverses given by

$$-_1\{u\}_T = \{-_1 u\}_T, \text{ resp. } -_2\{u\}_T = \{-_2 u\}_T\ .$$

Furthermore the face maps $\rho_2^{\square}(X) \longrightarrow \mathbf{G}(X)$ and degeneracy maps $\mathbf{G}(X) \longrightarrow \rho_2^{\square}(X)$ are morphisms of groupoids.

To verify the interchange law, suppose $\bar{u}, \bar{v}, \bar{w}, \bar{z} \in \rho_2^{\square}(X)$, where $\bar{u} = \{u\}_T$, etc. are such that both sides of

$$(\bar{u} +_2 \bar{w}) +_1 (\bar{v} +_2 \bar{z}) = (\bar{u} +_1 \bar{v}) +_2 (\bar{w} +_1 \bar{z}) \tag{4.12}$$

are defined. Then there are thin relative homotopies h, k, h', k' such that the following composite is defined.

$$\begin{bmatrix} u & \overset{\leftrightarrow}{h} & w \\ h' & \odot & k' \\ v & k & z \end{bmatrix}$$

Evaluating this in two ways gives the interchange law (3.5). ∎

This completes the construction of $\boldsymbol{\rho}_2^{\square}(X)$ as a double groupoid.

5. Connections and thin structure

A *connection pair* on a double groupoid, D, is given by a pair of maps

$$\Gamma^-, \Gamma^+ : D_1 \longrightarrow D_2.$$

The edges of $\Gamma^-(a)$, $\Gamma^+(a)$ for $a \in D_1$ are described by the formulae (1.6) suitably interpreted in the framework of a double groupoid.

Furthermore, Γ^- and Γ^+ are required to satisfy the usual *transport laws* (cf. 1.7 of [BS], (28) of [B 4], p. 167 of [BM]) describing the connection of a composite of edges, e.g.

$$\Gamma^-(a+b) = \begin{bmatrix} \Gamma^-(a) & \varepsilon_1(b) \\ \varepsilon_2(b) & \Gamma^-(b) \end{bmatrix}.$$

In order to construct a connection pair on $\boldsymbol{\rho}_2^{\square}(X)$ we make use of the connections

$$\Gamma^-, \Gamma^+ : R_1^{\square}(X) \longrightarrow R_2^{\square}(X)$$

(cf. (1.5)).

Let $a, a' \in R_1^{\square}(X)$ be paths, $a, a' : x \simeq y$. We claim that $a \sim_T a'$ implies $\Gamma^-(a) \equiv_T^{\square} \Gamma^-(a')$ (similarly, $\Gamma^+(a) \equiv_T^{\square} \Gamma^+(a')$), where $\sim_T$ is the relation of Definition 4.1. Let $h : a \simeq a' : x \simeq y$ be a thin relative homotopy. Then shifting $\Gamma^-(a)$ to $\Gamma^-(a')$ along $\Gamma^-(h_r), 0 \leqslant r \leqslant 1$, where $h_r : h_{r,0} \simeq h_{r,1} : x \simeq y$ we obtain a cubically thin homotopy $\Gamma^-(a) \equiv_T^{\square} \Gamma^-(a')$.

It follows that we have two induced functions

$$\Gamma^-, \Gamma^+ : \mathbf{G}(X) \longrightarrow \rho_2^{\square}(X)$$

together with the obvious relations induced by (1.6). In order to verify the transport law for Γ^-, let $a, b \in R_1^{\square}(X)$ be composable paths, $a : x \simeq y,\ b : y \simeq z$. Then we have to compare $\Gamma^-(a+b)$ and the composite square

$$u = \begin{bmatrix} \Gamma^-(a) & \varepsilon_1(b) \\ \varepsilon_2(b) & \Gamma^-(b) \end{bmatrix}.$$

We observe that we have factorisations

$$\Gamma^-(a+b) = (I^2 \xrightarrow{\phi} I \xrightarrow{a+b} X),$$
$$u = (I^2 \xrightarrow{\psi} I \xrightarrow{a+b} X)$$

of $\Gamma^-(a+b)$ resp. u through the unit interval such that ϕ and ψ coincide on the boundary $\dot{I}^2$ of I^2. Since I is a convex set it follows that ϕ and ψ are homotopic rel. $\dot{I}^2$. Hence there is a homotopy rel. $\dot{I}^2$ between $\Gamma^-(a+b)$ and u. Since a homotopy rel. $\dot{I}^2$ is a cubically thin homotopy, it follows that $\Gamma^-(a+b) \equiv^{\square}_T u$ which proves the transport law for Γ^-. The transport law for Γ^+ is proved in a similar way.

Finally, a similar factorisation argument together with a dilation type argument shows that

$$[\Gamma^+(a)\ \ \Gamma^-(a)] \equiv^{\square}_T \varepsilon_1(a), \text{ resp. } \begin{bmatrix}\Gamma^+(a)\\ \Gamma^-(a)\end{bmatrix} \equiv^{\square}_T \varepsilon_2(a).$$

This implies that $\Gamma^-, \Gamma^+ : \mathbf{G}(X) \longrightarrow \rho^{\square}_2(X)$ are "inverse" to each other in both directions.

This completes the construction of the homotopy double groupoid $\boldsymbol{\rho}^{\square}_2(X)$ of a Hausdorff space X.

Brown and Mosa have shown in section 4 of [BM] (following [BH 1]) how a connection on a double groupoid gives rise to a thin structure. We note that connections and thin structures have been exploited in [Spe] and [SW] in the theory of homotopy pullbacks and pushouts and the theory of homotopy commutative cubes.

For the definition of a thin structure we need the following general construction associating with an arbitrary groupoid C the *double groupoid* $\square\mathsf{C}$ *of commuting squares* in C, i.e. the squares in $\square\mathsf{C}$ are quadruples $\left(\begin{smallmatrix} & a & \\ c & & b\\ & d & \end{smallmatrix}\right)$ of arrows a, b, c, d in C such that ab, cd are defined and are equal. Then, by definition, a *thin structure* on a double groupoid D is a morphism of double groupoids

$$\Theta : \square D_1 \longrightarrow \mathsf{D}$$

which is the identity on D_1 and D_0. The elements of D_2 lying in $\Theta(\square D_1)$ are called *thin*.

If D is a double groupoid with connection pair (Γ^-, Γ^+), then by Theorem 4.3 of [BM] D inherits a thin structure Θ such that the thin elements $\Theta\left(\begin{smallmatrix} & a & \\ c & & b\\ & d & \end{smallmatrix}\right)$ can be described explicitly by the formulae

$$\begin{aligned}\Theta\left(\begin{smallmatrix} & a & \\ c & & b\\ & d & \end{smallmatrix}\right) &= (\varepsilon_1(a) +_2 \Gamma^+(b)) +_1 (\Gamma^-(c) +_2 \varepsilon_1(d))\\ &= (\varepsilon_2(c) +_1 \Gamma^+(d)) +_2 (\Gamma^-(a) +_2 \varepsilon_2(b)).\end{aligned}$$

The next aim is to characterise the thin structure on the homotopy double groupoid $\boldsymbol{\rho}^{\square}_2(X)$ inherited by the connection pair described above.

5.1. Definition. *An element* $\{u\}_T \in \rho^{\square}_2(X)$ *is called*

(1) algebraically thin *if it is thin in the sense of Brown-Mosa [BM],*

(2) geometrically thin if it has a thin representative in the sense of Definition 3.1 (1).

Our main result on this is:

5.2. Theorem. *An element* $\{u\}_T \in \rho^{\square}_2(X)$ *is algebraically thin if and only if it is geometrically thin.*

Proof. The forward implication follows from the Brown-Mosa formulae, since any composition of thin squares is thin (Proposition 3.5) and the elements $\Gamma^{\pm}(a)$, $\varepsilon_i(a)$ for $a \in R^{\square}_1(X)$ are all thin (3.3). ∎

In order to show the converse implication we first observe

5.3. PROPOSITION. *Let $u, v : I^2 \longrightarrow X$ be thin squares such that $u \mid \dot{I}^2 = v \mid \dot{I}^2$. Then*

$$u \equiv_T^{\square} v.$$

PROOF. This follows from Lemma 3.8. ■

We can generalise Proposition 5.3 to:

5.4. PROPOSITION. *Let $u, v : I^2 \longrightarrow X$ be thin squares such that $\langle \partial_i^{\alpha} u\rangle = \langle \partial_i^{\alpha} v\rangle$* for *$\alpha = \pm 1,\ i = 1, 2$. Then*

$$u \equiv_T^{\square} v.$$

PROOF. We can use the given thin homotopies and the usual retraction argument to define a cubically thin homotopy $u \equiv_T^{\square} u'$ where $u' \mid \dot{I}^2 = v \mid \dot{I}^2$ and u' is also thin. Now apply Proposition 5.3. ■

Now we can complete the proof of Theorem 5.2. Suppose $u : I^2 \longrightarrow X$ is thin. Let $v = \Theta(u \mid \dot{I}^2)$ be one of the thin elements given by the Brown-Mosa formulae (Theorem 4.3 of [BM]). By Proposition 5.4, $u \equiv_T^{\square} v$. But $\{v\}_T$ is thin in the sense of Brown-Mosa. ■

Next we show that $\boldsymbol{\rho}_2^{\square}(X)$ satisfies the homotopy addition lemma in dimension 2.

If $U : I^3 \longrightarrow X$ is a cube in X, then we denote the faces of U by

$$a_{\alpha} = (\partial_1^{\alpha} U)^{\leftrightarrow},\ \ b_{\alpha} = \partial_2^{\alpha} U, \quad c_{\alpha} = \partial_3^{\alpha} U \quad (\alpha = \pm 1).$$

We write $\bar{a}_{\alpha}$, etc. for the corresponding elements of $\rho_2^{\square}(X)$. We use matrix notation in the double groupoid $\boldsymbol{\rho}_2^{\square}(X)$.

5.5. PROPOSITION. [homotopy addition lemma] *If $U \in R_3^{\square}(X)$, then*

$$\bar{c}_+ = \begin{bmatrix} \Gamma^+ & -_1\bar{a}_- & -_1\Gamma^- \\ -_2\bar{b}_- & \bar{c}_- & \bar{b}_+ \\ -_2\Gamma^- & \bar{a}_+ & \Gamma^- \end{bmatrix}$$

in $\rho_2^{\square}(X)$, where each Γ^- (resp. Γ^+) stands for $\Gamma^-\langle p\rangle$ (resp. $\Gamma^+\langle p\rangle$) for an appropriate $p \in R_1^{\square}(X)$.

PROOF. We adapt the proof of Proposition 5 of [BH 1], to the present situation, see also Proposition 6.3 of [B 4].

Consider the maps $\varphi_0, \varphi_1 : I^2 \longrightarrow I^3$ defined by

$$\varphi_0 = \begin{bmatrix} \Gamma^+ & -_1 \overleftrightarrow{\eta_1^0} & -_1\Gamma^- \\ -_2\eta_2^0 & \eta_3^0 & \eta_2^1 \\ -_2\Gamma^- & \overleftrightarrow{\eta_1^1} & \Gamma^- \end{bmatrix}, \quad \varphi_1 = \begin{bmatrix} \odot & \varepsilon_1 & \odot \\ \varepsilon_2 & \eta_3^1 & \varepsilon_2 \\ \odot & \varepsilon_1 & \odot \end{bmatrix},$$

where $\eta_i^\alpha : I^2 \longrightarrow I^3$ denotes the map which inserts α at the i–th coordinate.

Then φ_0, φ_1 agree on $\dot{I}^2$ and so, since I^3 is convex, are homotopic rel. $\dot{I}^2$. It follows that $U \circ \varphi_0$ and $U \circ \varphi_1$ are homotopic rel. $\dot{I}^2$ which implies that $U \circ \varphi_0$ and $U \circ \varphi_1$ are cubically T-equivalent. But the composite matrix given in the proposition is the element in $\rho_2^\square(X)$ corresponding to $U \circ \varphi_0$ whereas by a dilation argument it can be shown that $U \circ \varphi_1$ is cubically T-equivalent to c_+. ■

6. Relation with the homotopy 2-groupoid

In [BM] Brown and Mosa present an equivalence of categories

$$\lambda : 2-\mathsf{Cat} \rightleftarrows \boldsymbol{\square}\!-\!\mathsf{Cat} : \gamma,$$

where $2-\mathsf{Cat}$ denotes the category of small 2-categories and $\boldsymbol{\square}\!-\!\mathsf{Cat}$ is the category of (edge symmetric) double categories with connection. The functor λ associates to a 2-category, C, the double category, $\lambda\mathsf{C}$, whose squares are the squares of 1-cells in C inhabited by a 2-cell of C, a construction which goes back to Ehresmann [E].

The functor γ assigns to a double category, D, the 2-category, $\gamma\mathsf{D}$, it contains in the obvious way, i.e. $\gamma\mathsf{D}$ consists of those squares in D whose vertical edges are identities.

In this section we clarify the relation between the homotopy double groupoid $\boldsymbol{\rho}_2^\square(X)$ of a Hausdorff space X and the homotopy 2-groupoid $\mathbf{G}_2(X)$ of [HKK]. We show that $\mathbf{G}_2(X)$ is isomorphic to the 2-groupoid $\gamma\boldsymbol{\rho}_2^\square(X)$ contained in $\boldsymbol{\rho}_2^\square(X)$.

We recall the definition of $\mathbf{G}_2(X)$. As for the homotopy double groupoid the 1-dimensional structure of $\mathbf{G}_2(X)$ is the path groupoid $\mathbf{G}(X)$. In [HKK] the 2-cells of $\mathbf{G}_2(X)$ have been constructed by a two stage process, part geometric and part algebraic. If x, y are points of X, then $\mathbf{G}_2X(x,y)$ has been defined as the quotient groupoid $\mathbf{\Pi}_2X(x,y)/\mathbf{N}X(x,y)$ where $\mathbf{\Pi}_2X(x,y)$ denotes the groupoid of 2-tracks between paths in X from x to y and $\mathbf{N}X(x,y)$ is the normal subgroupoid of $\mathbf{\Pi}_2X(x,y)$ consisting of those 2-tracks which are thin (see section 2 of [HKK]). However, it has been noted in [HKK] that the 2-cells of $\mathbf{G}_2(X)$ can also be obtained in a one stage geometric process determined by the notion of globularly thin homotopy defined as follows.

6.1. Definition. *Let $u : a \simeq a' : x \simeq y$ and $u' : b \simeq b' : x \simeq y$ be relative homotopies.*

(1) *A* globularly thin homotopy *$U : u \equiv_T^{\bigcirc} u'$ between u and u' is a cube $U \in R_3^\square(X)$ such that*
$$\partial(U) = (u, u', \partial_2^-(U), \partial_2^+(U), \odot_x, \odot_y)$$
and
$$\partial_2^-(U) : a \simeq b : x \simeq y \text{ resp. } \partial_2^+(U) : a' \simeq b' : x \simeq y$$
are thin.

(2) *The square u is* globularly T-equivalent *to u', denoted $u \equiv_T^{\bigcirc} u'$, if there is a globularly thin homotopy between u and u'.*

Let $R_2^{\bigcirc}(X) \subset R_2^{\square}(X)$ be the subset of relative homotopies. We denote the set of equivalence classes $R_2^{\bigcirc}(X)/\equiv_T^{\bigcirc}$ by $\rho_2^{\bigcirc}(X)$.

The following result is clear.

6.2. Proposition. *The 2-cells of the homotopy 2-groupoid $\mathbf{G}_2(X)$ are in bijective correspondence with the elements of $\rho_2^{\bigcirc}(X)$.* ∎

6.3. Theorem. *The inclusion $R_2^{\bigcirc}(X) \longrightarrow R_2^{\square}(X)$ of the subset of relative homotopies induces an injection $\iota : \rho_2^{\bigcirc}(X) \longrightarrow \rho_2^{\square}(X)$.*

The proof of Theorem 6.3 will make use of a folding map assigning a relative homotopy to an arbitrary square (cf. [BM]).

6.4. Definition. *The* folding map

$$\Phi : R_2^{\square}(X) \longrightarrow R_2^{\bigcirc}(X)$$

is given by

$$\Phi(u) = [\ \ulcorner\ u\ \lrcorner\ ]$$

for $u \in R_2^{\square}(X)$, where $\ulcorner = \Gamma^+\partial_2^-(u)$ and $\lrcorner = \Gamma^-\partial_2^+(u)$.

Proof of Theorem 6.3. Let

$$u : a \simeq a' : x \simeq y \text{ and } u' : b \simeq b' : x \simeq y$$

be relative homotopies such that u is cubically T-equivalent to u'.

Let $U \in R_3^{\square}(X)$, $U : u \equiv_T^{\square} u'$, be a cubically thin homotopy between u and u', i.e. U is a homotopy rel. vertices of I^2 between u and u' such that

$$\tau_1 = \partial_2^-(U),\ \tau_2 = \partial_2^+(U),\ \tau_3 = \partial_3^-(U),\ \tau_4 = \partial_3^+(U)$$

are thin.

We shall construct a 3-cube $V \in R_3^{\square}(X)$ with the following properties:

(1) $v = \partial_1^-(V)$ and $v' = \partial_1^+(V)$ are relative homotopies such that

$$v \equiv_T^{\bigcirc} u \text{ and } v' \equiv_T^{\bigcirc} u'\ ,$$

(2) $\partial_3^-(V) = \odot_x$ and $\partial_3^+(V) = \odot_y$,

(3) the faces $\partial_2^-(V)$ and $\partial_2^+(V)$ are thin.

In particular, V is a globularly thin homotopy, $V : v \equiv_T^{\bigcirc} v'$. Then it follows that

$$u \equiv_T^{\bigcirc} v \equiv_T^{\bigcirc} v' \equiv_T^{\bigcirc} u',$$

hence u is globularly T-equivalent to u', as required.

The construction of V proceeds as follows. For $0 \leqslant r \leqslant 1$ let $U_r : I^2 \longrightarrow X$ be given by $U_r(s,t) = U(r,s,t)$. We define $V_r : I^2 \longrightarrow X$ by

$$V_r = \Phi(U_r)$$

where Φ is the folding map of 6.4 and obtain a continuous map

$$V : I^3 \longrightarrow X,\ V(r,s,t) = V_r(s,t).$$

The properties of the connections Γ^-, Γ^+ used in the definition of the folding map Φ allow us to read off the boundary of V:

$$v = [\,\odot_x\ u\ \odot_y\,],\ v' = [\,\odot_x\ u'\ \odot_y\,],$$
$$\partial_3^-(V) = \odot_x,\ \partial_3^+(V) = \odot_y,$$
$$\partial_2^-(V) = [\,\odot_x\ \tau_1\ \tau_4\,],\ \partial_2^+(V) = [\,\tau_3\ \tau_2\ \odot_y\,].$$

Hence, (2) is fulfilled. By a dilation type argument v, resp. v', are globularly T-equivalent to u resp. u'. Thus, (1) is true. Finally, (3) holds since a vertical composite of thin homotopies is thin by Proposition 3.5. ∎

6.5. Corollary. *Under the equivalence of categories*

$$\lambda : 2 - \mathsf{Cat} \rightleftarrows \square\!-\!\mathsf{Cat} : \gamma$$

constructed in Section 5 of [BM] the homotopy 2-groupoid $\mathbf{G}_2(X)$ *and the homotopy double groupoid* $\boldsymbol{\rho}_2^{\square}(X)$ *correspond to each other.*

Proof. Let $\iota : \mathbf{G}_2(X) \longrightarrow \gamma\boldsymbol{\rho}_2^{\square}(X)$ denote the obvious functor which maps 0–cells and 1–cells identically. By Theorem 6.3 ι is an injection on 2–cells. Using a simple filling argument it is not hard to show that ι is also surjective on 2–cells. Hence ι is an isomorphism. ∎

Remark. Since $\mathbf{G}_2(X)$ contains the information on all $\pi_2(X,x)$, $x \in X$, within it, clearly this information is also encoded in $\boldsymbol{\rho}_2^{\square}(X)$. This can be extracted directly without recourse to the above result: If $x \in X$, $\pi_2(X,x)$ is precisely the set of $\{u\}_T$ for u with all its faces at x. There is an obvious action of $\mathbf{G}(X)$ on $\{\pi_2(X,x)\}_{x\in X}$ which induces the usual action of the fundamental groupoid, $\Pi_1(X)$, of X on this family of groups. We leave the details for the reader. ∎

This gives: From $\boldsymbol{\rho}_2^{\square}(X)$ one can extract $\Pi_1(X)$ and its action on the family $\{\pi_2(X,x)\}_{x\in X}$ of second homotopy groups of X based at points of X.

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