Theory and Applications of Categories, Vol. 10, No. 4, 2002, pp. 113–126.

HOMOLOGY OF LIE ALGEBRAS WITH $\Lambda/q\Lambda$ COEFFICIENTS AND EXACT SEQUENCES

EMZAR KHMALADZE

Abstract. Using the long exact sequence of nonabelian derived functors, an eight term exact sequence of Lie algebra homology with $\Lambda/q\Lambda$ coefficients is obtained, where Λ is a ground ring and q is a nonnegative integer. Hopf formulas for the second and third homology of a Lie algebra are proved. The condition for the existence and the description of the universal q-central relative extension of a Lie epimorphism in terms of relative homologies are given.

1. Introduction

Using results of [BaRo], Ellis and Rodriguez-Fernandez in [ElRo] have generalized Brown and Loday's eight term exact sequence in integral group homology [BrLo] to an eight term exact sequence in group homology with $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ coefficients, where q is a nonnegative integer. For any group G and its normal subgroup N, they obtained the following natural exact sequence

$$\begin{aligned} &H_3(G,\mathbb{Z}_q) \to H_3(G/N,\mathbb{Z}_q) \to \mathrm{Ker}(N \wedge^q G \to G) \to H_2(G,\mathbb{Z}_q) \\ &\qquad \to H_2(G/N,\mathbb{Z}_q) \to N/N\#_q G \to H_1(G,\mathbb{Z}_q) \to H_1(G/N,\mathbb{Z}_q) \to 0 \ , \end{aligned}$$

where $H_i(G,\mathbb{Z}_q)$ (i=1,2,3) denotes the i-th homology group of G with coefficients in the trivial G-module $\mathbb{Z}_q$, $N\#_q G$ denotes the subgroup of N generated by the commutators $[n,g]$ and the elements of the form n^q for $n \in N$, $g \in G$. Tensor versions of the exterior product $N \wedge^q G$ have subsequently been studied in [Br] and in [CoRo].

For an ideal M of a Lie algebra P over a commutative ring Λ, Ellis [El2] has obtained the exact sequence

$$\mathrm{Ker}(M \wedge P \to P) \to H_2(P) \to H_2(P/M) \to M/[M,P] \to H_1(P) \to H_1(P/M) \to 0 \ ,$$

where $H_n(P)$ denotes the n-th homology of P with coefficients in the trivial P-module Λ and $M \wedge P$ denotes the nonabelian exterior product of Lie algebras M and P [El1].

The author would like to thank N. Inassaridze for helpful comments. The work was partially supported by INTAS Georgia grant No 213 and INTAS grant No 00 566.

Received by the editors 2001 July 27 and, in revised form, 2002 January 14.

Transmitted by R. Brown. Published on 2002 January 21.

2000 Mathematics Subject Classification: 18G10, 18G50.

Key words and phrases: Lie algebra, nonabelian derived functor, exact sequence, homology group.

In [Kh] we introduced and studied the nonabelian tensor and exterior products of Lie algebras modulo q, these being mod q analogues of the tensor and exterior products in [El1].

The aim of this paper is to obtain the Lie algebra analogue of the eight term exact sequence of [ElRo], which will generalize the six term exact sequence above to the case of coefficients in $\Lambda/q\Lambda$ and will extend this sequence to the left by two terms.

As an application, Hopf formulas for the second and the third homologies of a Lie algebra with $\Lambda/q\Lambda$ coefficients are proved. The condition for the existence of the universal q-central relative extension of a Lie epimorphism [Kh] and the description of the kernel of such extension in terms of relative homologies are given.

Notations. Throughout the paper q denotes a nonnegative integer and Λ a commutative ring with identity. We write Λ_q instead of $\Lambda/q\Lambda$. All Lie algebras are Λ-Lie algebras and [,] denotes the Lie bracket.

2. Nonabelian derived functors of the exterior square modulo q

In this section we investigate derived functors of the nonabelian exterior square modulo q, establishing their relationship with the homology groups of a Lie algebra with coefficients in Λ_q.

First we give the definition of the nonabelian derived functors to the category of Lie algebras, denoted by $\mathcal{LIE}$ (see also [El1]).

Let $\mathcal{G} = (G, \epsilon, \delta)$ be a cotriple on a category $\mathcal{A}$ and $T : \mathcal{A} \to \mathcal{LIE}$ be a functor. For an object A of $\mathcal{A}$ let us consider the $\mathcal{G}$ cotriple resolution of A [BaBe2, Ke1]

$$\mathcal{G}(A)_* \equiv \cdots \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} G^2(A) \ \underset{d_1^1}{\overset{d_0^1}{\rightrightarrows}} G^1(A) \xrightarrow{d_0^0} A \ ,$$

where $G^n(A) = G(G^{n-1}(A))$, $d_i^n = G^i \epsilon G^{n-i}$, $s_i^n = G^i \delta G^{n-i}$. Applying T dimension-wise to $\mathcal{G}(A)_*$ yields a simplicial Lie algebra

$$T\mathcal{G}(A)_* \equiv \cdots \begin{smallmatrix}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{smallmatrix} TG^2(A) \rightrightarrows TG^1(A) \ .$$

The n-th homotopy group of $T\mathcal{G}(-)_*$ is called the n-th nonabelian derived functor of T with respect to the cotriple $\mathcal{G} = (G, \epsilon, \delta)$ and it is denoted by $\mathcal{L}_n^{\mathcal{G}} T(-)$. Recall from [Cu] that the homotopy groups of $T\mathcal{G}(A)_*$ are the homology groups of the associated Moore complex

$$M_* \equiv \cdots M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} M_0 \xrightarrow{d_0} 0 \ ,$$

where $M_0 = TG(A)$, $M_n = \bigcap\limits_{i=1}^{n-1} \mathrm{Ker} T(d_i^n)$ and d_n is the restriction of $T(d_n^n)$. Hence

$$\mathcal{L}_n^{\mathcal{G}} T(A) = \mathrm{Ker} d_n / \mathrm{Im} d_{n+1} \ , \quad n \geq 0 \ .$$

Let $\mathcal{F} = (F, \epsilon, \delta)$ be the cotriple on $\mathcal{LIE}$ generated by the adjoint pair [BaBe2, Ke1]

$$\begin{tikzcd} \mathcal{LIE} \arrow[r, "F"] \arrow[dr, "U"'] & \mathcal{LIE} \\ & \mathcal{SET} \arrow[u, "F'"'] \end{tikzcd} ,$$

where U is the forgetful functor sending a Lie algebra to its underlying set, F' is the functor sending a set to the free Lie algebra generated by this set.

Let M and N be two ideals of a Lie algebra P. We denote by $M\#_q N$ the submodule of $M \cap N$ generated by the elements $[m, n]$ and qk for $m \in M$, $n \in N$, $k \in M \cap N$. Then $M\#_q N$ is an ideal of $M \cap N$. In particular, $P\#_q P$ is an ideal of P. Let us consider the endofunctors V, $\mathcal{V} : \mathcal{LIE} \to \mathcal{LIE}$ defined by

$$V(P) = P\#_q P \ \text{ and } \ \mathcal{V}(P) = P/V(P) \ .$$

2.1. Lemma. *There is a natural isomorphism*

$$\mathcal{L}_n^{\mathcal{F}} \mathcal{V}(P) \approx H_{n+1}(P, \Lambda_q) \ \ (n \geq 0),$$

where $H_n(P, \Lambda_q)$ denotes the n-th homology of a Lie algebra P with coefficients in the trivial P-module Λ_q.

Proof. As pointed out in [Qu, Chapter II, Section 5], the cotriple description of group cohomology [BaBe1] carries over to the case of Lie algebra cohomology. Hence the cotriple description of group homology [BaBe2] carries over to the description of Lie algebra homology. Now if U_P and IP denote respectively the universal enveloping algebra and the augmentation ideal of a Lie algebra P, then the isomorphism $\Lambda_q \otimes_{U_P} IP \approx P/P\#_q P$ completes the proof. ∎

Let P be a Lie algebra with an ideal M. The exterior product modulo q of M and P [Kh] is the Lie algebra $M \wedge^q P$ generated by the symbols $m \wedge p$ and $\{m\}$ with $m \in M$, $p \in P$ subject to the relations

$$\lambda(m \wedge p) = \lambda m \wedge p = m \wedge \lambda p, \tag{1}$$

$$\begin{aligned} (m + m') \wedge p &= m \wedge p + m' \wedge p, \\ m \wedge (p + p') &= m \wedge p + m \wedge p', \end{aligned} \tag{2}$$

$$\begin{aligned} [m, m'] \wedge p &= m \wedge [m', p] - m' \wedge [m, p], \\ m \wedge [p, p'] &= [p', m] \wedge p - [p, m] \wedge p', \end{aligned} \tag{3}$$

$$[m \wedge p, m' \wedge p'] = [m, p] \wedge [m', p'], \tag{4}$$

$$[\{m'\}, m \wedge p] = [qm', m] \wedge p + m \wedge [qm', p], \tag{5}$$

$$\{\lambda m + \lambda' m'\} = \lambda\{m\} + \lambda'\{m'\}, \tag{6}$$

$$[\{m\}, \{m'\}] = qm \wedge qm', \tag{7}$$

$$\{[m, p]\} = q(m \wedge p), \tag{8}$$

$$m \wedge m = 1 \tag{9}$$

for all $m, m' \in M, \quad p, p' \in P, \quad \lambda, \lambda' \in \Lambda$.

2.2. Lemma. *If M is an ideal of a Lie algebra P then there is an exact sequence of Lie algebras*

$$(M \wedge^q P) \rtimes (M \wedge^q P) \xrightarrow{\alpha} P \wedge^q P \xrightarrow{\beta} P/M \wedge^q P/M \longrightarrow 0 \ ,$$

where $\rtimes$ denotes the semidirect product and the action of $M \wedge^q P$ on itself is given by Lie multiplication.

Proof. β is the functorial homomorphism induced by the projection $P \to P/M$ and it is surjective [Kh, Proposition 1.8]. Let $\alpha' : M \wedge^q P \to P \wedge^q P$ be the functorial homomorphism induced by the inclusion $M \to P$ and by the identity map $P \to P$. We set $\alpha(x, y) = \alpha'(x) + \alpha'(y)$ for $x, y \in M \wedge^q P$. It is easy to check that α is a Lie homomorphism. The image of α is generated by the elements $m \wedge p$ and $\{m\}$ for $m \in M$, $p \in P$. Clearly $\beta\alpha$ is the trivial homomorphism. By the formulas (4), (5) $\mathrm{Im}(\alpha)$ is an ideal of $P \wedge^q P$. Let us define a homomorphism $\beta' : P/M \wedge^q P/M \longrightarrow (P \wedge^q P)/\mathrm{Im}(\alpha)$ as follows: $\beta'(\overline{p}_1 \wedge \overline{p}_2) = \overline{p_1 \wedge p_2}$, $\beta(\{\overline{p}\}) = \overline{\{p\}}$, $p, p_1, p_2 \in P$. It is easy to see that β' is correctly defined and there is an inverse homomorphism of β' induced by β. ∎

Note that there is a Lie homomorphism $\partial : M \wedge^q P \to P$ defined by $\partial(m \wedge p) = [m, p]$, $\partial(\{m\}) = qm$ [Kh, Proposition 1.3] and the image of ∂ is $M \#_q P$.

2.3. Lemma. *If $q \geq 1$, Λ is a q-torsion-free ground ring and F is a free Lie algebra, then the homomorphism $\partial : F \wedge^q F \to F$ induces an isomorphism $F \wedge^q F \approx F \#_q F$.*

Proof. Let $F \wedge F$ be the nonabelian exterior square (for the definition see [El1]). By [Kh, Proposition 1.6] one has the following commutative diagram of Lie algebras with exact rows

$$\begin{tikzcd}
0 \arrow[r] & F \wedge F \arrow[r, "\varphi"] \arrow[d, "\partial'"] & F \wedge^q F \arrow[r] \arrow[d, "\partial"] & F^{ab} \arrow[r] \arrow[d, "\partial''"] & 0 \\
0 \arrow[r] & {[F, F]} \arrow[r] & F \#_q F \arrow[r] & qF^{ab} \arrow[r] & 0
\end{tikzcd} \ ,$$

where $F^{ab} = F/[F, F]$, ∂' is an isomorphism [El2, Proposition 1.2] and hence φ is injective. ∂'' is induced by ∂ and clearly it is surjective. F^{ab} is a free Λ-module. Since Λ is a q-torsion-free, ∂'' is an isomorphism and so is ∂. ∎

Consider the endofunctor $\wedge^q : \mathcal{LIE} \to \mathcal{LIE}$, which we call nonabelian exterior square modulo q, defined by

$$\wedge^q(P) = P \wedge^q P \ .$$

One has the following

2.4. Proposition. *There is a natural isomorphism*

$$\mathcal{L}_0^{\mathcal{F}} \wedge^q (P) \approx P \wedge^q P \ .$$

Moreover, if $q \geq 1$ and Λ is a q-torsion-free ring, then there is a natural isomorphism

$$\mathcal{L}_n^{\mathcal{F}} \wedge^q (P) \approx H_{n+2}(P, \Lambda_q)$$

for every $n \geq 1$.

PROOF. Consider the diagram of Lie algebras

$$F^2(P)\wedge^q F^2(P) \underset{d_1^1\wedge d_1^1}{\overset{d_0^1\wedge d_0^1}{\rightrightarrows}} F(P)\wedge^q F(P) \xrightarrow{d_0^0\wedge d_0^0} P\wedge^q P \longrightarrow 0 \ .$$

We have to show $(d_1^1\wedge d_1^1)(\mathrm{Ker}(d_0^0\wedge d_0^0)) = \mathrm{Ker}(d_0^0\wedge d_0^0)$. By Lemma 2.2 we get that $\mathrm{Ker}(d_0^1\wedge d_0^1)$ is generated by the elements $x\wedge k$ and $\{k\}$ with $x\in F^2(P)$, $k\in \mathrm{Ker}d_0^1$. Thus $(d_1^1\wedge d_1^1)(\mathrm{Ker}(d_0^0\wedge d_0^0))$ is generated by the elements $x'\wedge k'$ and $\{k'\}$ with $x'\in F(P)$, $k'\in d_1^1(\mathrm{Ker}d_0^1)$. On the other hand it follows from Lemma 2.2 that $\mathrm{Ker}(d_0^0\wedge d_0^0)$ is generated by the elements $x''\wedge k''$ and $\{k''\}$ with $x''\in F(P)$, $k''\in \mathrm{Ker}d_0^0$. Then the identity $d_1^1(\mathrm{Ker}d_0^1) = \mathrm{Ker}d_0^0$ proves the first isomorphism.

Consider the $\mathcal{F}$ cotriple resolution of P

$$\mathcal{F}(P)_* \equiv \cdots \overrightarrow{\underrightarrow{\longrightarrow}}\ F^2(P) \underset{d_1^1}{\overset{d_0^1}{\rightrightarrows}} F^1(P) \xrightarrow{d_0^0} P \ .$$

By Lemma 2.3 there is a simplicial isomorphism

$$\mathcal{F}(P)_*\#_q\mathcal{F}(P)_* \approx \wedge^q\mathcal{F}(P)_* \ .$$

Thus one has the following short exact sequence of simplicial Lie algebras

$$0\to \wedge^q\mathcal{F}(P)_* \to \mathcal{F}(P)_* \to \mathcal{V}\mathcal{F}(P)_* \to 0 \ .$$

Then by Lemma 2.1 the respective long exact homotopy sequence is of the form

$$\begin{aligned}\cdots\to 0\to H_{n+2}(P,\Lambda_q)\to \mathcal{L}_n^{\mathcal{F}}\wedge^q(P)\to 0 \\ \to H_{n+1}(P,\Lambda_q)\to\cdots\to \mathcal{L}_0^{\mathcal{F}}\wedge^q(P)\to P\to P/P\#_qP \ ,\end{aligned}$$

which gives the second isomorphism. ∎

3. Eight term exact sequence of Lie algebra homology with Λ_q coefficients

Let $\mathcal{LIE}_1$ denotes the category whose objects are surjective morphisms of $\mathcal{LIE}$ and a morphism from $P\xrightarrow{\alpha} Q$ to $P'\xrightarrow{\alpha'} Q'$ is a commutative square in $\mathcal{LIE}$

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "h_0"'] & Q \arrow[d, "h_1"] \\ P' \arrow[r, "\alpha'"'] & Q' \end{tikzcd} \ .$$

The cotriple $\mathcal{F} = (F, \epsilon, \delta)$ on $\mathcal{LIE}$ extends to a cotriple $\mathcal{F}_1 = (F_1, \epsilon_1, \delta_1)$ on $\mathcal{LIE}_1$ which is generated by the adjoint pair

$$\begin{tikzcd} \mathcal{LIE}_1 \arrow[r, "F_1"] \arrow[dr, "U_1"'] & \mathcal{LIE}_1 \\ & \mathcal{SET}_1 \arrow[u, "F'_1"'] \end{tikzcd} ,$$

where $\mathcal{SET}_1$ is the category whose objects are surjective maps of sets and whose morphisms are commutative squares in $\mathcal{SET}$, U_1 and F'_1 are induced respectively by U and F'.

We say that $(h_0, h_1) : \alpha \to \alpha'$ is a surjective morphism of $\mathcal{LIE}_1$ if $U_1(h_0, h_1)$ has a splitting in $\mathcal{SET}_1$.

Inductively we define a category $\mathcal{LIE}_m$, a cotriple $\mathcal{F}_m = (F_m, \epsilon_m, \delta_m)$ on $\mathcal{LIE}_m$ and surjective morphisms of $\mathcal{LIE}_m$ for $m \geq 0$:

$$\mathcal{LIE}_{m+1} = (\mathcal{LIE}_m)_1 \ , \ \ \mathcal{LIE}_0 = \mathcal{LIE}$$

and

$$\mathcal{F}_{m+1} = (\mathcal{F}_m)_1 \ , \ \ \mathcal{F}_0 = \mathcal{F} \ .$$

Moreover, if $T : \mathcal{LIE} \to \mathcal{LIE}$ is an endofunctor, we define $T_m : \mathcal{LIE}_m \to \mathcal{LIE}$, $m \geq 0$, as follows: if α, α' are objects of $\mathcal{LIE}_1$ and $(h_0, h_1) : \alpha \to \alpha'$ is a morphism of $\mathcal{LIE}_1$ then

$$T_1(\alpha) = \mathrm{Ker} T(\alpha) \ , \ \ T_1(h_0, h_1) = T(h_0)|_{T_1(\alpha)};$$

and

$$T_{m+1} = (T_m)_1 \ , \ \ T_0 = T \ .$$

It is easy to see that a surjective morphism $f : X \to Y$ of $\mathcal{LIE}_m$ induces a surjection of simplicial Lie algebras $f_* : T_m\mathcal{F}_m(X)_* \to T_m\mathcal{F}_m(Y)_*$, which yields a long exact sequence of homotopy groups. Thus we have immediately

3.1. Proposition. *A surjective morphism $f : X \to Y$ of $\mathcal{LIE}_m$ $(m \geq 0)$ yields a natural long exact sequence*

$$\cdots \to \mathcal{L}_n^{\mathcal{F}_{m+1}} T_{m+1}(f) \to \mathcal{L}_n^{\mathcal{F}_m} T_m(X) \to \mathcal{L}_n^{\mathcal{F}_m} T_m(Y) \to \cdots \to \mathcal{L}_0^{\mathcal{F}_m} T_m(Y) \to 0 \ .$$

Further for a functor $T : \mathcal{LIE} \to \mathcal{LIE}$ we shall write $\mathcal{L}_n T_m(-)$ to mean the n-th derived functor with respect to the cotriple $\mathcal{F}_m$.

Let V, $\mathcal{V} : \mathcal{LIE} \to \mathcal{LIE}$ be the endofunctors defined in the previous section. Then one has the following

3.2. Proposition. *Let M and N be two ideals of a Lie algebra P such that $M+N=P$. Consider the following object (α,γ) in the category $\mathcal{LIE}_2$*

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d] & N/M\cap N \arrow[d] \\ N/M\cap N \arrow[r, "\gamma"'] & 0 \end{tikzcd} .$$

Then there is a natural long exact sequence

$$\begin{aligned} \cdots \to H_{n+1}(P,\Lambda_q) \to H_{n+1}(M/M\cap N,\Lambda_q) \oplus H_{n+1}(N/M\cap N,\Lambda_q) \\ \to \mathcal{L}_{n-1}\mathcal{V}_2(\alpha,\gamma) \to \cdots \to H_2(P,\Lambda_q) \\ \to H_2(M/M\cap N,\Lambda_q) \oplus H_2(N/M\cap N,\Lambda_q) \to \mathcal{L}_0\mathcal{V}_2(\alpha,\gamma) \\ \to H_1(P,\Lambda_q) \to H_1(M/M\cap N,\Lambda_q) \oplus H_1(N/M\cap N,\Lambda_q) \to 0 \ . \end{aligned}$$

Proof. First note that $\mathcal{L}_n\mathcal{V}_2(\alpha,\gamma) = \mathcal{L}_n\mathcal{V}_2(h_0,h_1)$, $n \geq 0$. Then using Proposition 3.1 it is easy to get the following natural long exact sequence (compare [El1, Lemma 31])

$$\begin{aligned} \cdots \to \mathcal{L}_n\mathcal{V}(P) \to \mathcal{L}_n\mathcal{V}(M/M\cap N) \oplus \mathcal{L}_n\mathcal{V}(N/M\cap N) \\ \to \mathcal{L}_{n-1}\mathcal{V}_2(\alpha,\gamma) \to \cdots \to \mathcal{L}_0\mathcal{V}_2(\alpha,\gamma) \to \mathcal{L}_0\mathcal{V}(P) \\ \to \mathcal{L}_0\mathcal{V}(M/M\cap N) \oplus \mathcal{L}_0\mathcal{V}(N/M\cap N) \to 0 \ . \end{aligned}$$

Then the isomorphism of Lemma 2.1 gives the result. ■

3.3. Corollary. *Let M be an ideal of a Lie algebra P and $\alpha : P \to P/M$ the natural epimorphism. One has the following exact sequence*

$$\begin{aligned} \cdots \to H_{n+1}(P,\Lambda_q) \to H_{n+1}(P/M,\Lambda_q) \to \mathcal{L}_{n-1}\mathcal{V}_1(\alpha) \\ \to \cdots \to H_3(P,\Lambda_q) \to H_3(P/M,\Lambda_q) \to \mathcal{L}_1\mathcal{V}_1(\alpha) \to H_2(P,\Lambda_q) \\ \to H_2(P/M,\Lambda_q) \to \mathcal{L}_0\mathcal{V}_1(\alpha) \to H_1(P,\Lambda_q) \to H_1(P/M,\Lambda_q) \to 0 \ . \end{aligned}$$

Proof. The result follows from the previous proposition by considering $N=P$ and the object (α,γ) in the category $\mathcal{LIE}_2$

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d] & P/M \arrow[d] \\ 0 \arrow[r, "\gamma"'] & 0 \end{tikzcd} ,$$

for which we have $\mathcal{L}_n\mathcal{V}_2(\alpha,\gamma) = \mathcal{L}_n\mathcal{V}_1(\alpha)$, $n \geq 0$. ■

Now we compute $\mathcal{L}_0\mathcal{V}_1(-)$ and $\mathcal{L}_1\mathcal{V}_1(-)$ to give an interpretation of the last eight term of the long exact sequence of Corollary 3.3.

First we recall the following fact from [El1]. (See [Ke2] for the group case). For any ideal M of a Lie algebra P let us consider the diagram

$$(M \oplus M) \rtimes P \underset{l_3}{\overset{l_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xrightarrow{l_2}}}}} M \rtimes P \quad \underset{p_2}{\overset{p_1}{\rightrightarrows}} P \ ,$$

where $\rtimes$ denotes a semi-direct product; the action of P on M is given by Lie multiplication; the action of P on $M \oplus M$ is ${}^p(m, m') = ([p, m], [p, m'])$; the homomorphisms are defined by

$$p_1(m, p) = m + p, \quad p_2(m, p) = p \ ;$$

$$l_1(m', m, p) = (m' - m, m + p), \quad l_2(m', m, p) = (m', p), \quad l_3(m', m, p) = (m, p) \ .$$

If $T : \mathcal{LIE} \to \mathcal{LIE}$ is any endofunctor, on applying $\mathcal{L}_0T$ to the above diagram we obtain a diagram

$$\mathcal{L}_0T((M \oplus M) \rtimes P) \underset{l'_3}{\overset{l'_1}{\underset{\longrightarrow}{\overset{\longrightarrow}{\xrightarrow{l'_2}}}}} \mathcal{L}_0T(M \rtimes P) \quad \underset{p'_2}{\overset{p'_1}{\rightrightarrows}} \mathcal{L}_0T(P) \ ,$$

where we write p'_i and l'_i instead of $\mathcal{L}_0T(p_i)$ and $\mathcal{L}_0T(l_i)$. Suppose $\alpha : P \to P/M$ is the natural epimorphism. Then we have

3.4. Lemma. [E1] *There is an isomorphism*

$$\mathcal{L}_0T_1(\alpha) = \{\mathrm{Ker}p'_2\}/\{l'_1(\mathrm{Ker}l'_2 \cap \mathrm{Ker}l'_3)\} \ .$$

3.5. Proposition. *Let $0 \to M \to P \to P/M \to 0$ be a short exact sequence of Lie algebras, then*

(i) $\mathcal{L}_0\mathcal{V}_1(\alpha) \approx M/M\#_qP$,

(ii) $\mathcal{L}_1\mathcal{V}_1(\alpha) \approx \mathrm{Ker}(\partial : M \wedge^q P \to P)$, *if $q \geq 1$ and Λ is a q-torsion-free ring.*

Proof. (i) Consider $l'_i : \mathcal{V}((M \oplus M) \rtimes P) \to \mathcal{V}(M \rtimes P)$. It is easy to check that $\mathrm{Ker}l'_2 \cap \mathrm{Ker}l'_3 = 0$, then by Lemma 3.4 one has

$$\mathcal{L}_0\mathcal{V}_1(\alpha) \approx \mathrm{Ker}\{\mathcal{V}(M \rtimes P) \xrightarrow{p'_2} \mathcal{V}(P)\} \approx M/M\#_qP \ .$$

(ii) Let $I : \mathcal{LIE} \to \mathcal{LIE}$ be the identity functor. Then

$$0 \to V_1 \to I_1 \to \mathcal{V}_1 \to 0$$

is an exact sequence of functors from $\mathcal{LIE}_1$ to $\mathcal{LIE}$. Since $\mathcal{L}_0I_1 \approx I_1$ and $\mathcal{L}_nI_1 = 0$ for $n \geq 1$, the resulting long exact homotopy sequence provides an isomorphism

$$\mathcal{L}_1\mathcal{V}_1(\alpha) \approx \mathrm{Ker}(\mathcal{L}_0V_1(\alpha) \to I_1(\alpha)) \ .$$

Since $I_1(\alpha) = M$, by Lemma 2.3 we get an isomorphism

$$\mathcal{L}_1\mathcal{V}_1(\alpha) \approx \mathrm{Ker}(\mathcal{L}_0 \wedge_1^q (\alpha) \to M) \ .$$

Thus to prove the isomorphism (ii) we need to show that there is an isomorphism $M \wedge^q P \xrightarrow{\approx} \mathcal{L}_0 \wedge_1^q (\alpha)$ such that the diagram

$$\begin{tikzcd} M \wedge^q P \arrow[r, "\partial"] \arrow[d, "\approx"'] & M \arrow[d, equal] \\ \mathcal{L}_0 \wedge_1^q (\alpha) \arrow[r] & M \end{tikzcd}$$

commutes. Consider the diagram

$$((M \oplus M) \rtimes P) \wedge^q ((M \oplus M) \rtimes P) \overset{l'_1}{\underset{l'_3}{\overset{\longrightarrow}{\underset{\longrightarrow}{\xrightarrow{l'_2}}}}} (M \rtimes P) \wedge^q (M \rtimes P) \ \overset{p'_1}{\underset{p'_2}{\rightrightarrows}} P \wedge^q P \ ,$$

where l'_i, p'_i are the homomorphisms of Lemma 3.4. By Lemma 2.2 $\mathrm{Ker}l'_2$ is generated by the elements $(0, m, 0) \wedge (m_1, m_2, p)$ and $\{(0, m, 0)\}$, $\mathrm{Ker}l'_3$ is generated by the elements $(m, 0, 0) \wedge (m_1, m_2, p)$ and $\{(m, 0, 0)\}$. Thus $\mathrm{Ker}l'_2 \cap \mathrm{Ker}l'_3$ is generated by the elements $(m, 0, 0) \wedge (0, m', 0)$ and then $l'_1(\mathrm{Ker}l'_2 \cap \mathrm{Ker}l'_3)$ is generated by elements of the form $(m, 0) \wedge (-m', m')$. It is easy to check that $\mathrm{Ker}p'_2 = (M \rtimes 0) \wedge^q (M \rtimes P)$. Then by Lemma 3.4 one has

$$\mathcal{L}_0 \wedge_1^q (\alpha) \approx (M \rtimes 0) \wedge^q (M \rtimes P)/l'_1(\mathrm{Ker}l'_2 \cap \mathrm{Ker}l'_3) \approx M \wedge^q P \ ,$$

where the last isomorphism is defined by $\overline{(m, 0) \wedge (m', p)} \mapsto m \wedge (m' + p)$, $\{\overline{(m, 0)}\} \mapsto \{m\}$. It is readily seen that the above diagram commutes. ■

The previous results give immediately the following

3.6. Theorem. *Let $q \geq 1$, Λ is a q-torsion-free ground ring and P be a Lie algebra with an ideal M. There is a natural exact sequence*

$$\begin{aligned} H_3(P, \Lambda_q) &\to H_3(P/M, \Lambda_q) \to \mathrm{Ker}(M \wedge^q P \xrightarrow{\partial} P) \to H_2(P, \Lambda_q) \\ &\to H_2(P/M, \Lambda_q) \to M/M\#_qP \to H_1(P, \Lambda_q) \to H_1(P/M, \Lambda_q) \to 0 \ . \end{aligned}$$

Observe that the exact sequence of Theorem 3.6 generalizes the six term exact sequence in [El2] to eight term and to the case of coefficients in Λ_q. The group theoretic version of this sequence is obtained in [ElRo].

3.7. Corollary. *Let $q \geq 1$ and P be a Lie algebra over a q-torsion-free ground ring Λ. There is an isomorphism*

$$H_2(P, \Lambda_q) \approx \mathrm{Ker}(P \wedge^q P \xrightarrow{\partial} P).$$

Furthermore, for any free presentation

$$0 \to R \to F \to P \to 0$$

of P, there is an isomorphism

$$H_3(P, \Lambda_q) \approx \mathrm{Ker}(R \wedge^q F \xrightarrow{\partial} F).$$

In the rest of this section, as an application of the previous results, we prove Hopf formulas for the second and the third homology groups of a Lie algebra with Λ_q coefficients. Also we give the condition for the existence and the description of the universal q-central relative extension [Kh] in terms of relative homologies.

3.8. Theorem. *Let P be a Lie algebra and*

$$0 \to R \to F \xrightarrow{\alpha} P \to 0$$

be a free presentation of P. Then there is an isomorphism

$$H_2(P, \Lambda_q) \approx (R \cap (F \#_q F))/(R \#_q F) \ .$$

Proof. Since $H_2(F, \Lambda_q) = 0$, by Corollary 3.3 and Proposition 3.5(i) we get

$$\begin{aligned} H_2(P, \Lambda_q) &\approx \mathrm{Ker}(R/R\#_q F \to H_1(F, \Lambda_q)) \\ &\approx \mathrm{Ker}(R/R\#_q F \to F/F\#_q F) \approx (R \cap (F\#_q F))/(R\#_q F) \ . \end{aligned}$$

■

Note that the isomorphism of Theorem 3.8 is the mod q version of the well known Hopf formula for the second homology of a Lie algebra (see for example [HiSt]). Now we prove the mod q version of the Hopf formula for the third homology (see [El1]). In order to do this we need the following lemma which can be proved in a similar way as Theorem 35(ii) of [El1].

3.9. Lemma. *For the following object (α, γ) in the category $\mathcal{LIE}_2$*

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d] & P/M \arrow[d] \\ P/N \arrow[r, "\gamma"'] & P/(M+N) \end{tikzcd} ,$$

where M and N are two ideals of a Lie algebra P, there is an isomorphism

$$\mathcal{L}_0\mathcal{V}_2(\alpha, \gamma) = (M \cap N)/(P\#_q(M \cap N) + M\#_q N) \ .$$

3.10. THEOREM. *Let F be a Lie algebra and $H_2(F,\Lambda_q)=0$ (for example, F is a free Lie algebra). Let R and S be two ideals of F such that $H_i(F/R,\Lambda_q)=H_i(F/S,\Lambda_q)=0$ for $i=2,3$ (for example, the Lie algebras F/R and F/S are free). Then there is an isomorphism*

$$H_3(F/(R+S),\Lambda_q)\approx (R\cap S\cap F\#_qF)/((R\cap S)\#_qF+R\#_qS)\ .$$

PROOF. Consider the object (α,γ) in $\mathcal{LIE}_2$

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[d, "h_0"'] & F/R \arrow[d, "h_1"] \\ F/S \arrow[r, "\gamma"'] & F/(R+S) \end{tikzcd}\ .$$

By Proposition 3.1 and by Lemma 2.1 there are the following three long exact sequences

$$\cdots\to\mathcal{L}_1\mathcal{V}_2(\alpha,\gamma)\to\mathcal{L}_1\mathcal{V}_1(h_0)\to\mathcal{L}_1\mathcal{V}_1(h_1)$$
$$\to\mathcal{L}_0\mathcal{V}_2(\alpha,\gamma)\to\mathcal{L}_0\mathcal{V}_1(h_0)\to\mathcal{L}_0\mathcal{V}_1(h_1)\to 0\ ;\quad (*)$$

$$\cdots\to H_3(F,\Lambda_q)\to H_3(F/S,\Lambda_q)\to\mathcal{L}_1\mathcal{V}_1(h_0)\to H_2(F,\Lambda_q)$$
$$\to H_2(F/S,\Lambda_q)\to\mathcal{L}_0\mathcal{V}_1(h_0)\to H_1(F,\Lambda_q)\to H_1(F/S,\Lambda_q)\to 0\ ;\quad (**)$$

$$\cdots\to H_3(F/R,\Lambda_q)\to H_3(F/(R+S),\Lambda_q)\to\mathcal{L}_1\mathcal{V}_1(h_1)$$
$$\to H_2(F/R,\Lambda_q)\to H_2(F/(R+S),\Lambda_q)\to\mathcal{L}_0\mathcal{V}_1(h_1)$$
$$\to H_1(F/R,\Lambda_q)\to H_1(F/(R+S),\Lambda_q)\to 0\ .\quad (***)$$

$(***)$ gives us an isomorphism $H_3(F/(R+S),\Lambda_q)\approx\mathcal{L}_1\mathcal{V}_1(h_1)$ since $H_i(F/R,\Lambda_q)=0$ for $i=2,3$. From $(**)$ we have $\mathcal{L}_1\mathcal{V}_1(h_0)=0$ since $H_2(F,\Lambda_q)=0$ and $H_i(F/S,\Lambda_q)=0$ for $i=2,3$. Thus from $(*)$ we get

$$H_3(F/(R+S),\Lambda_q)\approx\operatorname{Ker}(\mathcal{L}_0\mathcal{V}_2(\alpha,\gamma)\to\mathcal{L}_0\mathcal{V}_1(h_0)).$$

By Theorem 2.8 $S\#_qF=S\cap(F\#_qF)$ since $0\to S\to F\to F/S\to 0$ is a free presentation of F/S. Then by Lemma 3.9 and Proposition 3.5(i) we have

$$H_3(F/(R+S),\Lambda_q)\approx\operatorname{Ker}((R\cap S)/((R\cap S)\#_qF+R\#_qS)\to S/S\#_qF)$$
$$\approx (R\cap S\cap F\#_qF)/((R\cap S)\#_qF+R\#_qS)\ .$$

∎

Let $\alpha : P \to Q$ be a Lie epimorphism and A be a Q-module. Recall from [KaLo] that a relative extension of α by A is an exact sequence of Lie algebras

$$0 \to A \to E \xrightarrow{\mu} P \xrightarrow{\alpha} Q \to 0 \ ,$$

where μ is a crossed module. Such extension is called a q-central relative extension [Kh] if Q acts trivially on A and $qa = 0$ for any $a \in A$. q-central relative extension of α is called universal if there exists a unique morphism of relative extensions [KaLo] from it to any q-central relative extension of α.

Let

$$0 \to M \to P \xrightarrow{\alpha} Q \to 0$$

be a short exact sequence of Lie algebras. The Lie epimorphism α has a universal q-central relative extension if and only if $M = M\#_q P$ and such extension is given by the following exact sequence [Kh, Theorem 2.8]

$$0 \to \mathrm{Ker}\partial \to M \wedge^q P \xrightarrow{\partial} P \xrightarrow{\alpha} Q \to 0 \ .$$

Using notations of [KaLo] let us denote by $H_n(\alpha, \Lambda_q)$, $n \geq 0$, the n-th relative homology group of a Lie epimorphism $\alpha : P \to Q$ with coefficients in the trivial Q-module Λ_q. Clearly $H_{n+2}(\alpha, \Lambda_q) \approx \mathcal{L}_n \mathcal{V}_1(\alpha)$, $n \geq 0$. Then from Proposition 3.5 we get

$$H_2(\alpha, \Lambda_q) \approx M/M\#_q P$$

and if $q \geq 1$ and Λ is a q-torsion-free ring then

$$H_3(\alpha, \Lambda_q) \approx \mathrm{Ker}(M \wedge^q P \xrightarrow{\partial} P) \ .$$

So the description of the universal q-central relative extension can be expressed in terms of relative homologies as follows:

3.11. Theorem. *The Lie epimorphism α has a universal q-central relative extension if and only if $H_2(\alpha, \Lambda_q) = 0$. Moreover, if $q \geq 1$ and Λ is a q-torsion-free ring, then the sequence*

$$0 \to H_3(\alpha, \Lambda_q) \to M \wedge^q P \xrightarrow{\partial} P \xrightarrow{\alpha} Q \to 0 \ ,$$

is the universal q-central relative extension of α.

This result is mod q version of [KaLo, Theorem A.4], or alternatively, it is the Lie algebra version of [CoRo, Corollary 2.16].

References

[BaBe1] M.Barr and J.Beck, *Acyclic models and triples*, in S.Eilenberg, D.K.Harrison, S.MacLane, H.Rohrl, ed., Proceedings of the conference on categorical algebra La Jolla 1965 (Springer, 1966), 344-354.

[BaBe2] M.Barr and J.Beck, *Homology and Standard Constructions*, Lecture Notes in Mathematics, Vol.**80**, Springer-Verlag, Berlin/New York (1969), 245-335.

[BaRo] J.Barja and C.Rodriguez, *Homology groups $H_n^q(-)$ and eight-term exact sequences*, Cahiers Topologie Géom. Différentielle Catégoriques **XXXI** (1990), 91-120.

[Br] R.Brown, *q-perfect groups and universal q-central extensions*, Publicacions Mat. **34** (1990), 291-297.

[BrLo] R.Brown and J.-L.Loday, *Van Kampen theorems for diagrams of spaces*, Topology **26** (1987), 311-335.

[CoRo] D.Conduché and C.Rodriguez-Fernandez, *Non-abelian tensor and exterior products modulo q and universal q-central relative extension*, J. Pure Appl. Algebra, **78** (1992), 139-160.

[Cu] E.B.Curtis, *Simplicial homotopy theory*, Adv. Math. **6** (1971), 107-204.

[El1] G.J.Ellis, *A non-abelian tensor product of Lie algebras*, Glasgow Math. J **33** (1991), 101-120.

[El2] G.J.Ellis, *Non-abelian exterior product of Lie algebras and an exact sequence in the homology of Lie algebras*, J. Pure Appl. Algebra **46** (1987), 111-115.

[ElRo] G.J.Ellis and C.Rodriguez, *An exterior product for the homology of groups with integral coefficients modulo p*, Cahiers Topologie Geom. Differentielle Categoriques **XXX** (1989), 339-344.

[HiSt] P.J.Hilton and U.Stammbach, *A course in homological algebra*, Graduate Texts in Mathematics 4 (Springer, 1971).

[KaLo] C.Kassel and J.-L.Loday, *Extensions centrales d'algebres de Lie*, Ann. Inst. Fourier (Grenoble) **33** (1982), 119-142.

[Ke1] F.Keune, *Homotopical algebra and algebraic K-theory*, Ph.D.Thesis, University of Amsterdam (1972).

[Ke2] F.Keune, *The relativization of K_2*, J. Algebra **54** (1978), 159-177.

[Kh] E.Khmaladze, *Non-abelian tensor and exterior products modulo q and universal q-central relative extension of Lie algebras*, Homology, Homotopy and Applications **1(9)** (1999), 187-204.

[Qu] D.Quillen, *Homotopical Algebra*, Lecture Notes in Mathematics **43** (Springer, 1967).

A.Razmadze Mathematical Institute,
Georgian Academy of Sciences,
M.Alexidze St. 1,
Tbilisi 380093. Georgia

Email: `khmal@@rmi.acnet.ge`

This article may be accessed via WWW at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/10/4/10-04.{dvi,ps}`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

The method of distribution of the journal is via the Internet tools WWW/ftp. The journal is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION. Individual subscribers receive (by e-mail) abstracts of articles as they are published. Full text of published articles is available in .dvi, Postscript and PDF. Details will be e-mailed to new subscribers. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, rrosebrugh@mta.ca.

INFORMATION FOR AUTHORS. The typesetting language of the journal is TEX, and LATEX is the preferred flavour. TEX source of articles for publication should be submitted by e-mail directly to an appropriate Editor. They are listed below. Please obtain detailed information on submission format and style files from the journal's WWW server at http://www.tac.mta.ca/tac/. You may also write to tac@mta.ca to receive details by e-mail.