Theory and Applications of Categories, Vol. 10, No. 7, 2002, pp. 148–161.

MORE ON INJECTIVITY IN LOCALLY PRESENTABLE CATEGORIES

Dedicated to Horst Herrlich on the occasion of his 60th birthday

J. ROSICKÝ[1) 2)], J. ADÁMEK[1)] AND F. BORCEUX

ABSTRACT. Injectivity with respect to morphisms having λ-presentable domains and codomains is characterized: such injectivity classes are precisely those closed under products, λ-directed colimits, and λ-pure subobjects. This sharpens the result of the first two authors (Trans. Amer. Math. Soc. 336 (1993), 785-804). In contrast, for geometric logic an example is found of a class closed under directed colimits and pure subobjects, but not axiomatizable by a geometric theory. A more technical characterization of axiomatizable classes in geometric logic is presented.

1. Introduction

In [2], classes of objects injective with respect to a set $\mathcal{M}$ of morphisms of a locally presentable category $\mathcal{K}$ were characterized: they are precisely the classes closed under products, λ-directed colimits and λ-pure subobjects for some cardinal λ (see Part 2 below for the concept of λ-pure subobject). In fact, the formulation in [2] did not use λ-pure subobjects, but accessibility of the class in question. However, a full subcategory of $\mathcal{K}$, closed under λ-directed colimits, is accessible iff it is closed under λ'-pure subobjects for some λ' (see [3], Corollary 2.36). The main result of our paper is a "sharpening" of the previous result to a given regular cardinal λ: by a λ-*injectivity class* we call a class of all objects injective with respect to $\mathcal{M}$, where $\mathcal{M}$ is a set of morphisms with λ-presentable domains and codomains. If $\mathcal{K}$ is a locally λ-presentable category, we prove that λ-injectivity classes in $\mathcal{K}$ are precisely the full subcategories closed under products, λ-directed colimits, and λ-pure subobjects.

In contrast, the generalization from injectivity with respect to morphisms to injectivity with respect to cones, which was presented by H. Hu and M. Makkai [7], does not allow a corresponding "sharpening" – which is unfortunate, since this is closely connected to geometric logic. Hu and Makkai proved that classes given by injectivity with respect to a set of cones are precisely those closed under λ-directed colimits and λ-pure subobjects for some λ (again, we use Corollary 2.36 of [3] to put the original result into our terminology). We present an example of a class of Σ-structures, for a finitary signature Σ, which is closed

[1)] Partially supported by the Grant Agency of the Czech Republic under the grant No. 201/96/0119.
[2)] Partially supported by the Université Catholique Louvain-La-Neuve.
Received by the editors 1999 November 17 and, in revised form, 2002 March 1.
Transmitted by R.J. Wood. Published on 2002 March 15.
2000 Mathematics Subject Classification: 18C35, 03C99.
Key words and phrases: locally presentable category, injectivity class, geometric logic.
© J. Rosický, J. Adámek and F. Borceux, 2002. Permission to copy for private use granted.

under directed colimits and pure subobjects, but which is *not* a cone-injectivity class for any set of cones with finitely presentable domains and codomains. In particular, the class is not axiomatizable by geometric logic in $\mathbf{Str}\Sigma$. We introduce a rather technical concept of being strongly closed under pure subobjects. This, then, is a full characterization of classes axiomatizable by geometric logic.

Let us recall that, given an arrow $h : A \to A'$ in a category $\mathcal{K}$, an object X is called *h-injective*, provided that every morphism from A to X factors through h, i.e. the map

$$\hom(h, X) : \hom(A', X) \to \hom(A, X), \quad f \mapsto fh$$

is surjective. This is an important concept in algebra, where h is usually required to be a monomorphism, and in model theory (where general h's are considered). A class $\mathcal{A}$ of objects is called an *injectivity class* provided that there is a collection $\mathcal{M}$ of morphisms such that $\mathcal{A}$ consists of precisely all objects that are h-injective for all $h \in \mathcal{M}$ (we write $\mathcal{A} = \mathcal{M}$-Inj); if $\mathcal{M}$ is small, we speak about a *small-injectivity class*, and if all domains and all codomains of $\mathcal{M}$-maps are λ-presentable objects of $\mathcal{K}$, we speak about a λ-*injectivity class*. In case that the base category $\mathcal{K}$ is locally presentable (in the sense of Gabriel and Ulmer [5]), then every small-injectivity class is a λ-injectivity class for some λ, and conversely, every λ-injectivity class is a small-injectivity class. But small-injectivity classes are interesting also in such categories as **Top** - see e.g. [6], where they are called implicational subcategories.

A full characterization of small-injectivity classes in a locally presentable category $\mathcal{K}$ has been presented in [2]: they are precisely the classes $\mathcal{A}$ of objects, for which there exists a regular cardinal λ such that $\mathcal{A}$ is closed under

(i) products,

(ii) λ-directed colimits,

and

(iii) λ-pure subobjects.

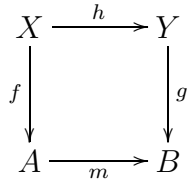
(Let us recall that λ-*pure monomorphisms* are precisely the λ-directed colimits of retractions (as objects of $\mathcal{K}^{\to}$), and if $\mathcal{K}$ is the category of all Σ-structures for some λ-ary signature Σ, then this categorical concept coincides with that of a λ-pure submodel, used in model theory.) In the present paper, we prove that for each locally λ-presentable category $\mathcal{K}$, the conditions (i) – (iii) above precisely characterize λ-injectivity classes in $\mathcal{K}$. The proof is substantially more difficult than that in [2].

More generally, given a cone $H = (h_i : A \to A'_i)_{i \in I}$, an object X is H-injective iff every map from A to X factors through h_i for some $i \in I$. A class of objects is called a *(small-)cone-injectivity class* if it can be axiomatized by injectivity with respect to a class (or set) of cones. If all domains and codomains of all those cones are λ-presentable, we speak about λ-*cone-injectivity* classes.

Injectivity in locally presentable categories is treated in [3] where the reader also finds standard categorical concepts used in our paper. We use the notation $\mathbf{Str}\,\Sigma$ for the category of Σ-structures and homomorphisms, where Σ is a finitary many-sorted signature. This is a locally finitely presentable category.

2. Characterization of Injectivity Classes

2.1. Remark. Recall that a morphism $m : A \to B$ of a locally λ-presentable category $\mathcal{K}$ is called λ-*pure* provided that in each commutative square



with X and Y λ-presentable the morphism f factors through h. As an object of $A \downarrow \mathcal{K}$, m is λ-pure iff m is a λ-directed colimit of retractions, see [3]. Every λ-pure morphism is a regular monomorphism.

2.2. Theorem. *Let $\mathcal{K}$ be a locally λ-presentable category. A full subcategory of $\mathcal{K}$ is a λ-injectivity class iff it is closed under*

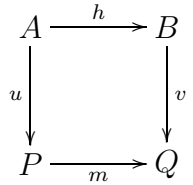
(i) *products*

(ii) *λ-directed colimits*

and

(iii) *λ-pure subobjects.*

Proof. The necessity is easy: closedness under products and λ-directed colimits is trivial. To prove closedness under λ-pure subobjects $m : P \to Q$, we are to show that if Q is h-injective then P is h-injective for all maps $h : A \to B$ with A, B both λ-presentable. Let $u : A \to P$ be a morphism. There exists $v : B \to Q$ making the following square



commutative (h-injectivity of Q) and since u factors through h (λ-purity of m), the h-injectivity of P follows.

For the sufficiency, we can assume that $\mathcal{K}$ is the category $\mathbf{Str}\,\Sigma$ of all Σ-structures for some finitary relational signature Σ. (The general case follows from the fact that $\mathcal{K}$ can be considered as a full reflective subcategory of $\mathbf{Str}\,\Sigma$ closed under λ-directed colimits – see 1.47 of [3]. This implies that $\mathcal{K}$ is closed under (i) – (iii), see Remark 2.31 in [3]. Consequently, given a full subcategory $\mathcal{L}$ of $\mathcal{K}$ closed under (i) – (iii) in $\mathcal{K}$, then $\mathcal{L} = \mathcal{M}$-Inj for some set $\mathcal{M}$ of morphisms of $\mathbf{Str}\,\Sigma$ having λ-presentable domains and codomains. The reflector $F : \mathbf{Str}\,\Sigma \to \mathcal{K}$ obviously preserves λ-presentability of objects and satisfies $\mathcal{L} = F(\mathcal{M})$-Inj.)

Thus, let $\mathcal{L}$ be a full subcategory of $\mathbf{Str}\,\Sigma$, where Σ is a finitary relational signature, closed under (i) – (iii). Put

$$\mathcal{M} = \{f : A \to B \text{ in } \mathbf{Str}\,\Sigma\,;\ A \text{ and } B \text{ are } \lambda\text{-presentable} \\ \text{and every object of } \mathcal{L} \text{ is injective with respect to } f\}.$$

We will prove that $\mathcal{L} = \mathcal{M}$-Inj. We first observe that $\mathcal{L}$ is *weakly reflective*, i.e., every object K of $\mathbf{Str}\,\Sigma$ has a morphism $r_K : K \to K^*$ (weak reflection) with K^* in $\mathcal{L}$ such that every object of $\mathcal{L}$ is injective with respect to r_K. This follows from 2.36 and 4.8 in [3]. Now $\mathcal{L} \subseteq \mathcal{M}$-Inj by the choice of $\mathcal{M}$, and to prove $\mathcal{M}$-Inj $\subseteq \mathcal{L}$, we verify the following implication:

$K \in \mathcal{M}$-Inj $\Rightarrow$ any weak reflection of K in $\mathcal{L}$ is λ-pure.

It then follows from (iii) that $K \in \mathcal{L}$, and this will prove

$$\mathcal{M}\text{-Inj} = \mathcal{L}\,.$$

Thus, given $K \in \mathcal{M}$-Inj and a weak reflection $r : K \to K^*$ in $\mathcal{L}$, we are to prove that in any commutative square

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ K & \xrightarrow[r]{} & K^* \end{array}$$

with A and B λ-presentable the map u factors through h. We will work with the arrow-category $(\mathbf{Str}\,\Sigma)^{\to}$ and consider the morphism $(u, v) : h \to r$.

Claim: There is a factorization $u = u_2 \cdot u_1$ and a morphism $(u_1, v_1) : h \to \bar{r}$ for some object $\bar{r}$ of $(\mathbf{Str}\,\Sigma)^{\to}$, where $\bar{r} : \bar{K} \to \bar{K}^*$ is a weak reflection of $\bar{K}$ in $\mathcal{L}$ and $\bar{K}$ is λ-presentable in $\mathbf{Str}\,\Sigma$.

Proof of claim. Consider all morphisms $(u_1, v_1) : h \to \bar{r}$ where $\bar{r} : \bar{K} \to \bar{K}^*$ is a weak reflection of $\bar{K}$ in $\mathcal{L}$ and $u = u_2 \cdot u_1$ for some u_2. Since $(u, v) : h \to r$ is such a morphism, we can take the smallest α such that $\bar{K}$ is α-presentable in $\mathbf{Str}\,\Sigma$. We are to prove $\alpha \leq \lambda$. Assuming $\alpha > \lambda$ we derive a contradiction. Let us first remark that an object A of $\mathbf{Str}\,\Sigma$ is α-presentable iff it has

(a) less than α elements (i.e., the union of all underlying sets of all sorts has cardinality $< \alpha$)

and

(b) less than α relational symbols $\sigma \in \Sigma$ with σ_A nonempty.

(See 1.14 (2) in [3].) Consequently, for every regular cardinal $\alpha \geq \aleph_0$, any α-presentable object in $\mathbf{Str}\,\Sigma$ is a colimit of a *smooth α-chain* (i.e., a chain whose limit steps form a colimit of the preceding part) of objects of presentability smaller than α.

Let us express the above object $\bar{K}$ as a colimit of a smooth α-chain $k_{ij} : K_i \to K_j$ $(i \leq j < \alpha)$ of objects K_i of presentability less than α, let $k_{i\alpha} : K_i \to \bar{K}$ $(i < \alpha)$ denote the colimit cocone. We define an α-chain $k^*_{ij} : K^*_i \to K^*_j$ $(i \leq j < \alpha)$ in $\mathcal{L}$ and a natural transformation

$$r_i : K_i \to K^*_i \qquad (i < \alpha)$$

by transfinite induction. (This idea has been used in [2] already, see the proof of IV. 3.)

(i) $r_0 : K_0 \to K^*_0$ is a weak reflection of K_0 in $\mathcal{L}$.

(ii) $i \mapsto i + 1$: Form a pushout of r_i and $k_{i,i+1}$

$$\begin{array}{ccc} K_i & \xrightarrow{r_i} & K^*_i \\ {\scriptstyle k_{i,i+1}}\downarrow & & \downarrow{\scriptstyle \widehat{k}_{i,i+1}} \\ K_{i+1} & \xrightarrow[\widehat{r}_i]{} & \widehat{K}_{i+1} \end{array}$$

Choose a weak reflection $r^*_{i+1} : \widehat{K}_{i+1} \to K^*_{i+1}$ in $\mathcal{L}$ and put $r_{i+1} = r^*_{i+1}\widehat{r}_i$ and $k^*_{i,i+1} = r^*_{i+1} \cdot \widehat{k}_{i,i+1}$.

(iii) Given a limit ordinal j, form a colimit

$$\widehat{K}_j = \operatorname*{colim}_{i<j} K^*_i$$

with colimit maps $\widehat{k}_{ij} : K^*_i \to \widehat{K}_j$ $(i < j)$ and choose a weak reflection $r^*_j : \widehat{K}_j \to K^*_j$ in $\mathcal{L}$, then $r_j = r^*_j \cdot (\operatorname*{colim}_{i<j} r_i)$ and $k^*_{ij} = r^*_j \widehat{k}_{ij}$.

Observe that every object L of $\mathcal{L}$ is r_i-injective for all $i < \alpha$ (i.e., each $r_i : K_i \to K^*_i$ is a weak reflection of K_i).

Proof. This is obvious for $i = 0$. If this holds for all smaller ordinals than i (> 0), then it holds for i as well: given $f : K_i \to L$, we can define a compatible cocone $f^*_{ij} : K^*_j \to L$ $(j \leq i)$ with $f^*_j r_j = f\, k_{ji}$ by transfinite induction as follows:

(i) f_0^* is any map with $f_0^* r_0 = f\, k_{0i}$ (this exists by the choice of r_0);

(ii) given f_j^*, the pushout property yields a unique $\widehat{f}_{j+1} : \widehat{K}_{j+1} \to L$ and we choose any map $f_{j+1}^* : K_{j+1}^* \to L$ with $\widehat{f}_{j+1} = f_{j+1}^* r_{j+1}^*$;

(iii) given a limit ordinal j and f_k^* for $k < j$, the j-chain-colimit property yields a unique $\widehat{f}_j : \widehat{K}_j \to L$ and we choose any map $f_j^* : K_j^* \to L$ with $\widehat{f}_j = f_j^* \cdot r_j^*$.

The object $K_\alpha^* = \operatorname{colim}_{i<\alpha} K_i^*$ lies in $\mathcal{L}$ because α-chains are λ-directed (recall that $\alpha > \lambda$ is a regular cardinal). Thus, for the map

$$r_\alpha = \operatorname*{colim}_{i<\alpha} r_i : \bar{K} \to K_\alpha^*$$

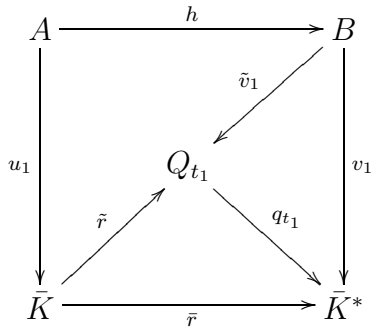
there exists $s : \bar{K}^* \to K_\alpha^*$ with $r_\alpha = s\bar{r}$. (Recall that $\bar{r}$ is a weak reflection of $\bar{K}$.)

In $(\mathbf{Str}\,\Sigma)^{\to}$ we thus obtain a morphism

$$(u_1, sv_1) : h \to r_\alpha\,.$$

Now h is, obviously, λ-presentable in $(\mathbf{Str}\,\Sigma)^{\to}$, thus that last map factors through some of the objects r_i, $i < \alpha$. This is the desired contradiction with the minimality of α: each $r_i : K_i \to K_i^*$ is a weak reflection of K_i, and K_i has smaller presentation rank than α. This proves the claim.

We are ready to prove that u factors through h. Let us consider a factorization $u = u_2 \cdot u_1$ and a morphism $(u_1, v_1) : h \to \bar{r}$ as in the above claim. Let us express $\bar{K}^*$ as a λ-directed colimit of λ-presentable objects Q_t, $t \in T$, in $\mathbf{Str}\,\Sigma$ with a colimit cocone $q_t : Q_t \to \bar{K}^*$. Since both $\bar{K}$ and B are λ-presentable, the maps $\bar{r}$ and v_1 both factor through q_{t_0} for some $t_0 \in T$ and then there exists (since A is λ-presentable) $t_1 \geqslant t_0$ in T with a commutative diagram as follows:



Since all objects of $\mathcal{L}$ are $\bar{r}$-injective, they are also $\tilde{r}$-injective; moreover, $\bar{K}$ and Q_{t_1} are both λ-presentable, thus,

$$\tilde{r} \in \mathcal{M}\,.$$

This implies that K is $\tilde{r}$-injective. Choose $d : Q_{t_1} \to K$ with $u_2 = d\tilde{r}$ to obtain

$$u = u_2 u_1 = d\tilde{r}u_1 = d\tilde{v}_1 h\,,$$

the desired factorization. Thus, r is λ-pure, and $K \in \mathcal{L}$. ∎

2.3. Remark. The reader may well wonder whether the condition of closedness under λ-pure subobjects is not superfluous: after all

(i) closedness under λ-directed colimits implies closedness under split subobjects

and

(ii) every λ-pure subobject is a λ-directed colimit of split subobjects, see [3] 2.30.

However, (ii) holds in the arrow-category of $\mathcal{K}$, not in $\mathcal{K}$ itself. The following Example demonstrates that, indeed, λ-pure subobjects are essential:

2.4. Example. A class of graphs which is closed under products and directed colimits, but is not an ω-injectivity class: We denote by **Gra** the category of graphs, i.e., pairs (X, R) of sets with $R \subseteq X \times X$, and graph homomorphisms. Let $\mathcal{A}$ be the class of all graphs containing nodes $x_n \in X \quad (n \in N)$ such that $x_n R x_{n+1}$ for each $n \in N$. It is clear that $\mathcal{A}$ is closed under products and nonempty colimits in **Gra**. However, $\mathcal{A}$ is not closed under ω-pure subobjects. In fact, let A denote the graph obtained from an infinite path $x_0, x_1, x_2, \ldots$ (starting in the node x_0) by gluing a path of length n to x_0 for each $n \in N$. Then

$$A \in \mathcal{A} \quad \text{and} \quad B \notin \mathcal{A}$$

where B is the strong subgraph of A over all nodes distinct from x_k for $k \geq 1$. However, B is an ω-pure subgraph of A because given any finite (= finitely presentable) subgraph V of A there exists a morphisms $h : V \to B$ with $h(x) = x$ for all nodes x of B. In fact: let n be the largest index with $x_n \in V$ and for each $k \leq n$ denote by x'_k the k-th node on the path of length n in A, then put $h(x) = x$ if $x = B$ and $h(x_k) = x'_k$.

3. Characterization of Classes Axiomatizable by Geometric Logic

Recall that *geometric logic* is the first-order logic using formulas of the following form

$$(\forall\, x)\big(\varphi(x) \to \bigvee_{i \in I} (\exists\, y_i)\ \psi(x, y_i)\big) \tag{1}$$

where x and each y_i is a finite string of variables, and φ and each ψ_i is a finite conjunction of atomic formulas. For example, given a cone H with finitely presentable domain and codomains in $\mathbf{Str}\,\Sigma$, injectivity with respect to H can be axiomatized by a formula (1), where $\varphi(x)$ expresses the presentation of the domain of H, and $\psi(x, y_i)$ expresses the presentation of the i-th codomain, and the connecting morphism. Consequently, every ω-cone-injectivity class in $\mathbf{Str}\,\Sigma$ is axiomatizable in geometric logic. Conversely, every formula (1) can be translated to ω-cone-injectivity in $\mathbf{Str}\,\Sigma$, see [3].

Observation. Every class of structures axiomatizable by geometric logic is closed in $\mathbf{Str}\,\Sigma$ under directed colimits and pure subobjects.

The converse does not hold:

3.1. Example. A class of Σ-structures (Σ finitary) closed under directed colimits and strong subobjects, which is not an ω-cone-injectivity class: Let Σ be the (one-sorted) signature of countably many unary relation symbols σ_n $(n \in \omega)$. For every set $A \subseteq \omega$ denote by $\bar{A}$ the Σ-structure over the set $\{0\}$ with $\sigma_n \neq \emptyset$ iff $n \in A$. Further denote by I the initial (empty) object of $\mathbf{Str}\,\Sigma$.

Choose a disjoint decomposition

$$\omega = \bigcup_{n\in\omega} B_n\,, \quad B_n \text{ infinite}$$

and denote by $\mathcal{B}$ the full subcategory of $\mathbf{Str}\,\Sigma$ consisting precisely of

(i) I

and

(ii) all $\bar{B}$ where $B \subseteq \omega$ is a set such that there exists $n \in \omega$ and a set $M \subseteq \omega$ of cardinality $\leq n$ with $B = B_n \cup M$.

$\mathcal{B}$ is obviously closed under substructures (strong subobjects), since $\mathcal{B}$-objects have no non-trivial substructures. And it is closed under directed colimits: if $\bar{B}_t$ $(t \in T)$ is a directed collection in $\mathcal{B}$, then there exists $n \in \omega$ such that each $\bar{B}_t$ has the form $\bar{B}_t = B_n \cup M_t$ for M_t of at most n elements. Since T is directed, the set $M = \bigcup_{t\in T} M_t$ also has at most n elements, and $\operatorname{colim} \bar{B}_t = B_n \cup M \in \mathcal{B}$.

However, $\mathcal{B}$ is not an ω-cone-injectivity class. In fact, $\mathcal{B}$ does not contain the terminal Σ-structure $\bar{\omega}$, although any ω-cone-injectivity class containing $\mathcal{B}$ contains $\bar{\omega}$ too. Indeed: $\bar{\omega}$ is injective with respect to all nonempty cones, and we observe that there exists no empty ω-cone H to which all $\mathcal{B}$-objects are injective. (In fact, let M be the domain of H. Since M is finitely presentable, there exists $n \in \omega$ such that $(\sigma_k)_M = \emptyset$ for all $k \geq n$. Put $B = B_n \cup \{0, 1, \ldots, n-1\}$, then the constant map is a Σ-homomorphism from M to $\bar{B}$, consequently, $\bar{B}$ is not injective with respect to the empty cone with domain H.)

3.2. Definition. *A span $A \xleftarrow{n} C \xrightarrow{m} B$ is called* λ-pure *provided that for each commutative square*

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ A \leftarrowtail C & \xrightarrow[m]{} & B \end{array}$$

with X and Y λ-presentable the morphism nf factors through h (i.e., there is $t : Y \to A$ with $nf = th$). If $\lambda = \omega$ we say just pure.

3.3. Examples. (1) For each morphism $r : B \to A$ the inverse relation of the graph of r, i.e., $A \xleftarrow{r} B \xrightarrow{1} B$, is pure. Also any subspan of the inverse of a graph is pure. These are precisely the relations $A \xleftarrow{n} C \xrightarrow{m} B$ where n factors through m; we call them *split spans*.

(2) A λ-directed colimit of split spans is λ-pure. More precisely, let $\mathbf{Sp}(\mathcal{K})$ denote the category whose objects are spans $A \xleftarrow{n} C \xrightarrow{m} B$ in $\mathcal{K}$ and whose morphisms are commutative diagrams

$$\begin{array}{ccccc} A & \xleftarrow{n} & C & \xrightarrow{m} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle w} \\ A' & \xrightarrow[n']{} & C' & \xrightarrow[m']{} & B \end{array}$$

If $\mathcal{K}$ has λ-directed colimits, then so does $\mathbf{Sp}(\mathcal{K})$, and a λ-directed colimit of split spans is λ-pure. This follows easily from the fact that (λ-directed) colimits are formed object-wise in $\mathbf{Sp}(\mathcal{K})$.

3.4. Remark. In a locally λ-presentable category the latter example of λ-pure spans is canonical: every λ-pure span is a λ-directed colimit of split spans. The proof is analogous to that of 2.30 in [3].

3.5. Definition. *A subcategory $\mathcal{L}$ of a category $\mathcal{K}$ is* strongly closed under λ-pure subobjects *provided that $\mathcal{L}$ contains every object A of $\mathcal{K}$ with the following property:*

> *given a morphism $n : C \to A$ in $\mathcal{K}$ with C λ-presentable, there exists a λ-pure span $A \xleftarrow{n} C \xrightarrow{m} B$ with $B \in \mathcal{L}$.*

3.6. Remark. (1) Given a morphism $m : C \to D$, for each cardinal λ let us denote by $H_\lambda(m)$ the set of all spans (h, f) where $h : X \to Y$ is a morphism with X and Y λ-presentable, and $f : X \to C$ is a morphism with a commutative square

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ {\scriptstyle f}\downarrow & & \vdots \\ C & \xrightarrow[m]{} & D \end{array}$$

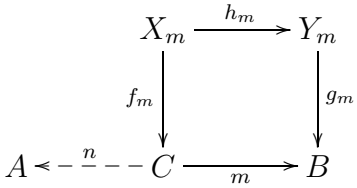
A span $A \xleftarrow{n} C \xrightarrow{m} B$ is λ-pure iff $H_\lambda(m) \subseteq H_\lambda(n)$. Thus, $\mathcal{L}$ is strongly closed under λ-pure subobjects iff $\mathcal{L}$ contains every object $A \in \mathcal{K}$ with the following property: given $n : C \to A$, C λ-presentable, there exists $m : B \to A$ with $B \in \mathcal{L}$ such that $H_\lambda(m) \subseteq H_\lambda(n)$.

(2) This is, indeed, stronger than closedness under λ-pure subobjects. For, given $L \in \mathcal{L}$ and a λ-pure subobject $i : A \to L$, we prove that $A \in \mathcal{L}$ as follows: Let $n : C \to A$ be a morphism with C λ-presentable, then $A \xleftarrow{n} C \xrightarrow{in} L$ is a λ-pure span.

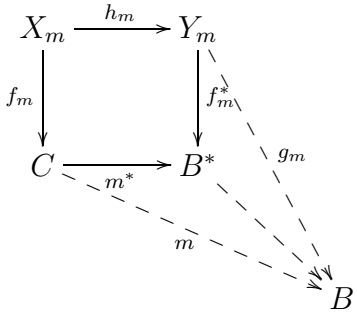
(3) Strong closedness under λ-pure subobjects implies closedness under λ-directed colimits (and thus is, indeed, much stronger than closedness under λ-pure subobjects). In fact, let A be a λ-directed colimit of $A_i \in \mathcal{L}$ with a colimit cocone $\left(A_i \xrightarrow{h_i} A\right)_{i\in I}$. For each morphism $n : C \to A$ with C λ-presentable there exists $i \in I$ such that n factors as $n = a_i n'$ for some $n' : C \to A_i$, and we have a pure span $A \xleftarrow{n} C \xrightarrow{n'} A_i$, thus, $A \in \mathcal{L}$.

3.7. Theorem. *Let $\mathcal{K}$ be a locally λ-presentable category. Then λ-cone-injectivity classes of $\mathcal{K}$ are precisely the full subcategories strongly closed under λ-pure subobjects.*

Proof. I. Suppose that $\mathcal{L}$ is strongly closed in $\mathcal{K}$ under λ-pure subobjects. For every object A in $\mathcal{K} - \mathcal{L}$ we construct a sink γ_A as follows: choose a morphism $n : C \to A$, C λ-presentable, such that no λ-pure span $A \xleftarrow{n} C \xrightarrow{m} B$ with $B \in \mathcal{L}$ exists. Given any morphism $m : C \to B$ with $B \in \mathcal{L}$ we thus have a commutative square



on $\mathcal{K}$ with X_m and Y_m λ-presentable such that $n \cdot f_m$ does not factor through h_m. Form a pushout of f_m and h_m:



Since C, X_m and Y_m are λ-presentable, so is B^*. We denote by γ_A the cone of all $m^* : C \to B^*$ (indexed by all $m \in \bigcup_{B\in\mathcal{L}} \hom(C, B)$), and we first observe that since γ_A has a λ-presentable domain and all codomains, it is essentially small. We claim that

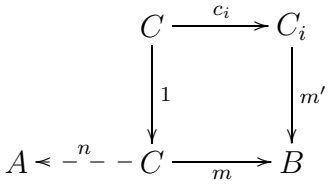
$$\mathcal{L} = \{\gamma_A\}_{A\in\mathcal{K}-\mathcal{L}}\text{-Inj}\,,$$

i.e., that an object of $\mathcal{K}$ lies in $\mathcal{L}$ iff it is cone-injective with respect to each γ_A. In fact, given $B \in \mathcal{L}$ then for every morphism $m : C \to B$ we have a factorization of m through m^*, a member of γ_A, thus, B is $\mathcal{M}$-injective.

Conversely, let A be an object of $\mathcal{K}$ injective to all of the above cones, then we prove that $A \in \mathcal{L}$. In fact, assuming $A \in \mathcal{K} - \mathcal{L}$ we have the above cone γ_A and since A is

γ_A-injective, the morphism $n : C \to A$ factors through m^* for some $m : C \to B$ with $B \in \mathcal{L}$. But then $n \cdot f_m$ factors through h_m (since if $t \cdot m^* = n$ then $t \cdot f_m^* \cdot h_m = n \cdot f_m$), in contradiction to the above choice of h_m, f_m, and g_m.

II. Suppose that $\mathcal{L}$ is a λ-cone-injectivity class in $\mathcal{K}$. Let $\mathcal{M}$ be a set of cones with λ-presentable domains and codomains such that $\mathcal{L} = \mathcal{M}$-Inj. We will prove that $\mathcal{L}$ contains any object A satisfying the condition of the above definition of strong closedness under λ-pure subobjects. Thus, given a cone $\gamma = (C \xrightarrow{c_i} C_i)_{i \in I}$ in $\mathcal{M}$, we are to show that A is γ-injective. In fact, let $n : C \to A$ be a morphism. There exists a λ-pure span $A \xleftarrow{n} C \xrightarrow{m} B$ with $B \in \mathcal{L}$. Since $B \in \mathcal{M}$-Inj, the morphism m factors through some c_i, say, $m = m' \cdot c_i$.



From the λ-purity we conclude that $n \cdot 1$ factors through c_i, thus, A is γ-injective. ∎

3.8. Corollary. *A class of Σ-structures can be axiomatized by geometric logic iff it is strongly closed under pure subobjects in* $\mathbf{Str}\,\Sigma$.

3.9. Remark. (1) We can speak, more generally, about infinitary geometric logic: given a regular cardinal λ, a formula (1) is λ-geometric provided that x and y_i are strings of less than λ variables and φ and ψ_i are conjunctions of less than λ atomic formulas. The above corollary generalizes as expected: classes axiomatized by λ-geometric theories are precisely those strongly closed under λ-pure subobjects.

(2) λ-geometric logic precisely describes λ-cone-injectivity classes in $\mathbf{Str}\,\Sigma$. Analogously, λ-injectivity classes are described by λ-*regular logic*, i.e., logic using formulas

$$(\forall\, x)\big(\varphi(x) \to \exists\, y\ \psi(x, y)\big) \tag{2}$$

where, again, x and y are strings of less than λ variables and φ and ψ are conjunctions of less than λ atomic formulas. Thus, Theorem 2.2 characterizes classes of structures axiomatizable by λ-regular logic: they are the classes closed under products, λ-directed colimits and λ-pure subobjects.

This seems to be a new result for $\lambda > \aleph_0$; let us remark that, however, for $\lambda = \aleph_0$ a more elementary proof follows from Compactness Theorem. A class closed under products and directed colimits is namely closed under ultraproducts; being closed under pure (in particular, under elementary) subobjects, the class is axiomatizable in the first-order logic, and our result then easily follows.

(3) We now turn to a special case of cone-injectivity classes for which a good characterization can be formulated. If $\mathcal{M}$ is a set of cones and each cone consists of strong epimorphisms only, we call $\mathcal{M}$-Inj a *strong cone-injectivity class*.

If the domains and codomains in all those cones are λ-generated objects (i.e., objects A such that $\hom(A,-)$ preserves λ-directed unions), then we speak about *strong λ-cone-injectivity* classes, and we prove that those are precisely the classes closed under λ-directed unions and subobjects. This depends on (strong epi, mono)-factorizations of morphisms. We work, more generally, with an abstract factorization system. Following [1] we say that $\mathcal{K}$ is an $(\mathcal{E},\mathcal{M})$-*structured category* provided that $\mathcal{E}$ is a class of epimorphisms, $\mathcal{M}$ is a class of monomorphisms, both closed under composition, with $\mathcal{E}\cap\mathcal{M}=\mathrm{Iso}$ and every morphism has an essentially unique $(\mathcal{E},\mathcal{M})$-factorization.

3.10. Definition. *Let $\mathcal{K}$ be an $(\mathcal{E},\mathcal{M})$-structured category. A λ-directed colimit whose colimit cocone is formed by $\mathcal{M}$-morphisms is called a* λ-directed union. *An object K is called* λ-generated *provided that* $\hom(K,-)$ *preserves λ-directed unions.*

3.11. Lemma. *In a locally λ-presentable $(\mathcal{E},\mathcal{M})$-structured category every object is a λ-directed union of λ-generated objects.*

Proof. Every object K is a λ-directed colimit of λ-presentable objects K_i $(i\in I)$. If $k_i: K_i\to K$ $(i\in I)$ denotes the colimit cocone and $k_i=m_ie_i$ is an $(\mathcal{E},\mathcal{M})$-factorization of k_i, with $m_i: K_i'\to K$, then the objects K_i' form a λ-directed diagram with a colimit cocone $m_i: K_i'\to K$ $(i\in I)$, due to the diagonal fill-in. Since K_i is λ-presentable and $e_i: K_i\to K_i'$ lies in $\mathcal{E}$, it follows easily from the diagonal fill-in that K_i' is λ-generated. ∎

3.12. Remark. In a locally λ-presentable $(\mathcal{E},\mathcal{M})$-structured category, λ-generated objects are precisely the $\mathcal{E}$-quotients of all λ-presentable objects – this has been proved for $\mathcal{E}=$ strong epis in [3], 1.61, and the general case is analogous.

3.13. Definition. *A class of objects is called a* strong λ-cone-injectivity *class if it has the form $\mathcal{M}$-Inj for a set of cones formed by $\mathcal{E}$-morphisms whose domains and codomains are λ-generated.*

3.14. Theorem. *Let $\mathcal{K}$ be a locally λ-presentable $(\mathcal{E},\mathcal{M})$-structured category. A full subcategory of $\mathcal{K}$ is a strong λ-cone-injectivity class iff it is closed in $\mathcal{K}$ under λ-directed unions and $\mathcal{M}$-subobjects.*

Proof. Necessity is clear. For the sufficiency, let $\mathcal{L}$ be a full subcategory of $\mathcal{K}$ closed under λ-directed unions and $\mathcal{M}$-subobjects.

(a) $\mathcal{L}$ is an accessible category. In fact, $\mathcal{L}$ is closed under λ-pure subobjects in $\mathcal{K}$, since by 2.31 in [3],

$$\lambda\text{-pure} \;\Rightarrow\; \text{regular mono} \;\Rightarrow\; \text{member of } \mathcal{M}\,.$$

By 2.36 of [3] it is sufficient to prove that $\mathcal{L}$ is closed under μ-directed colimits for some regular cardinal μ. By the above remark, $\mathcal{K}$ has only a set of λ-generated objects, thus, there exists a regular cardinal μ such that

$$\lambda\text{-generated} \;\Rightarrow \mu\text{-presentable}\,.$$

Let L_i $(I\in I)$ form a μ-directed diagram in $\mathcal{L}$ with a colimit $l_i: L_i\to K$ $(i\in I)$ in $\mathcal{K}$. We are to prove that $K\in\mathcal{L}$. By the above Lemma, K is a λ-directed union of λ-generated

objects, say, $k_j : K_j \to K \quad (j \in J)$. Since K_j is μ-presentable, k_j factors through some l_i, i.e., there exists $i \in I$ and $d : K_j \to L_i$ with $k_j = l_i \cdot d$. Now $k_j \in \mathcal{M}$ implies $d \in \mathcal{M}$, thus, $K_j \in \mathcal{L}$ (since $\mathcal{L}$ is closed under $\mathcal{M}$-subobjects). Thus, K is a λ-directed union of $\mathcal{L}$-objects, which proves $K \in \mathcal{L}$.

(b) $\mathcal{L}$ is a cone-reflective subcategory of $\mathcal{K}$, i.e., for every object $K \in \mathcal{K}$ there exists a cone with domain K to which all $\mathcal{L}$-objects are injective. This is proved in 2.53 of [3]. Moreover, every object $K \in \mathcal{K}$ has a cone-reflection in $\mathcal{L}$ all members of which are $\mathcal{E}$-epis: this follows immediately from the closedness of $\mathcal{K}$ under $\mathcal{M}$-subobjects.

(c) Let $\mathcal{H}$ be the following collection of cones in $\mathcal{K}$: for each λ-generated object K of $\mathcal{K}$ we choose a cone-reflection $(r_i^K : K \to K_i)_{i \in I(K)}$ of K in $\mathcal{L}$ with $r_i^K \in \mathcal{E}$ for each $i \in I$ and let $\mathcal{H}$ be the set of all these cones. It is sufficient to prove that an object $K \in \mathcal{K}$ lies in $\mathcal{L}$ iff K is injective with respect to all cones of $\mathcal{H}$.

Thus, let K be injective with respect to $\mathcal{H}$-cones. By the above Lemma, we can express K as a λ-directed union of λ-generated objects $k_t : K_t \to \mathcal{K} \quad (t \in T)$. Since K is initial with respect to a cone-reflection $(r_i^{K_t})$, k_t factors through some $r_i^{K_t}$ which, since $k_t \in \mathcal{M}$ and $r_i^{K_t} \in \mathcal{E}$, implies that $r_i^{K_t}$ is an isomorphism. In other words, $K_t \in \mathcal{L}$ (for each $t \in T$). Since $\mathcal{L}$ is closed under λ-directed unions, this implies $K \in \mathcal{L}$. ∎

3.15. Remark. The factorization of Σ-homomorphisms to surjective Σ-homomorphisms followed by inclusions of substructures yields $\mathbf{Str}\,\Sigma$ as an (epi, regular mono)-structured category. Then λ-generated Σ-structures are Σ-structures generated by less than λ elements in a usual sense. If a cone $(h_i : A \to B_i)_{i \in I}$ has all h_i surjective and all A and B_i λ-generated then (2) is (equivalent to) a universal sentence of the logic $L_{\infty,\lambda}$. We obtain, from Theorem 3.7,

3.16. Corollary. *A class of Σ-structures is axiomatizable by a universal theory of the logic $L_{\infty,\lambda}$ iff it is closed under submodels and λ-directed unions.*

3.17. Remark. Corollary 3.16 has an easy direct proof: any class $\mathcal{L}$ of Σ-structures which is closed under substructures and λ-directed unions can be axiomatized by universal sentences of $L_{\infty,\lambda}$

$$(\forall \vec{x}) \bigvee_{i \in I} \pi_{A_i}(\vec{x})$$

where $\{A_i\}_{i \in I}$ is a representative set of Σ-structures $A_i \in \mathcal{L}$ generated by less then λ-elements and π_{A_i} is the diagram of A_i reduced to generators.

Corollary 3.16 documents the well-known fact that a class of Σ-structures with at most countably many elements, which is axiomatizable in L_{ω_1,ω_1}, cannot be axiomatized by a universal theory in $L_{\omega_1\omega_1}$ (see [4] 2.4.11). An analogous example cannot be found in $L_{\omega,\omega}$: any class of Σ-structures axiomatizable in $L_{\omega,\omega}$ and closed under substructures is axiomatizable by a universal theory in $L_{\omega,\omega}$.

References

[1] J. Adámek, H. Herrlich and G. Strecker, *Abstract and Concrete Categories*, John Wiley, New York 1990.

[2] J. Adámek and J. Rosický, *On injectivity in locally presentable categories*, Trans. Amer. Math. Soc. **336** (1993), 785–804.

[3] J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, Cambridge Univ. Press 1994.

[4] M. A. Dickmann, *Large Infinitary Languages*, North-Holland, Amsterdam 1975.

[5] P. Gabriel and F. Ulmer, *Lokal Präsentierbare Kategorien*, Lect. Notes in Math. **221**, Springer-Verlag, Berlin 1971.

[6] H. Herrlich, *Almost reflective subcategories of Top*, Topology and Appl. **49** (1993), 251–261.

[7] H. Hu and M. Makkai, *Accessible embeddings and the solution set conditions*, Cah. Top. Géom. Diff. Cat. **XXXV** (1994), 99–108.

[8] M. Makkai and R. Paré, *Accessible categories: the foundations of categorical model theory*, Contemporary Math. 104, Amer. Math. Soc., Providence 1989.

Department of Algebra and Geometry
Masaryk University
*Janáčkovo nám. 2*a
662 95 Brno, Czech Republic

Institut für Theoretische Informatik
Technical University Braunschweig
Postfach 3329
38023 Braunschweig, Germany

Departement des Mathématiques
Université Catholique Louvain-La-Neuve
2, Chemin du Cyclotron
1348 Louvain-La-Neuve, Belgium

Email: rosicky@math.muni.cz
adamek@iti.cs.tu-bs.de
borceux@agel.ucl.ac.be

This article may be accessed via WWW at http://www.tac.mta.ca/tac/ or by anonymous ftp at ftp://ftp.tac.mta.ca/pub/tac/html/volumes/10/7/10-07.{dvi,ps}

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

The method of distribution of the journal is via the Internet tools `WWW/ftp`. The journal is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION. Individual subscribers receive (by e-mail) abstracts of articles as they are published. Full text of published articles is available in .dvi, Postscript and PDF. Details will be e-mailed to new subscribers. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

INFORMATION FOR AUTHORS. The typesetting language of the journal is TEX, and LATEX is the preferred flavour. TEX source of articles for publication should be submitted by e-mail directly to an appropriate Editor. They are listed below. Please obtain detailed information on submission format and style files from the journal's WWW server at `http://www.tac.mta.ca/tac/`. You may also write to `tac@mta.ca` to receive details by e-mail.

EDITORIAL BOARD.
John Baez, University of California, Riverside: `baez@math.ucr.edu`
Michael Barr, McGill University: `barr@barrs.org`, *Associate Managing Editor*
Lawrence Breen, Université Paris 13: `breen@math.univ-paris13.fr`
Ronald Brown, University of North Wales: `r.brown@bangor.ac.uk`
Jean-Luc Brylinski, Pennsylvania State University: `jlb@math.psu.edu`
Aurelio Carboni, Università dell Insubria: `aurelio.carboni@uninsubria.it`
P. T. Johnstone, University of Cambridge: `ptj@dpmms.cam.ac.uk`
G. Max Kelly, University of Sydney: `maxk@maths.usyd.edu.au`
Anders Kock, University of Aarhus: `kock@imf.au.dk`
F. William Lawvere, State University of New York at Buffalo: `wlawvere@acsu.buffalo.edu`
Jean-Louis Loday, Université de Strasbourg: `loday@math.u-strasbg.fr`
Ieke Moerdijk, University of Utrecht: `moerdijk@math.uu.nl`
Susan Niefield, Union College: `niefiels@union.edu`
Robert Paré, Dalhousie University: `pare@mathstat.dal.ca`
Andrew Pitts, University of Cambridge: `Andrew.Pitts@cl.cam.ac.uk`
Robert Rosebrugh, Mount Allison University: `rrosebrugh@mta.ca`, *Managing Editor*
Jiri Rosicky, Masaryk University: `rosicky@math.muni.cz`
James Stasheff, University of North Carolina: `jds@math.unc.edu`
Ross Street, Macquarie University: `street@math.mq.edu.au`
Walter Tholen, York University: `tholen@mathstat.yorku.ca`
Myles Tierney, Rutgers University: `tierney@math.rutgers.edu`
Robert F. C. Walters, University of Insubria: `robert.walters@uninsubria.it`
R. J. Wood, Dalhousie University: `rjwood@mathstat.dal.ca`