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RING EPIMORPHISMS AND $C(X)$

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ABSTRACT. This paper studies the homomorphism of rings of continuous functions $\rho: C(X) \to C(Y)$, Y a subspace of a Tychonoff space X, induced by restriction. We ask when ρ is an epimorphism in the categorical sense. There are several appropriate categories: we look at **CR**, all commutative rings, and **R/N**, all reduced commutative rings. When X is first countable and perfectly normal (e.g., a metric space), ρ is a **CR**-epimorphism if and only if it is a **R/N**-epimorphism if and only if Y is locally closed in X. It is also shown that the restriction of ρ to $C^*(X) \to C^*(Y)$, when X is normal, is a **CR**-epimorphism if and only if it is a surjection.

In general spaces the picture is more complicated, as is shown by various examples. Information about Spec ρ and Spec ρ restricted to the proconstructible set of prime z-ideals is given.

1. Introduction

During the 1960s considerable effort was expended in order to understand the meaning of "epimorphism" ([Mac Lane (1998), page 19]) in various concrete categories. These questions got particular attention in categories of rings. (See, for example, [Lazard (1968)], [Mazet (1968)], [Olivier (1968)] and [Storrer (1968)].) Much later the subject again received attention in categories of ordered rings (see [Schwartz & Madden (1999)] and its bibliography.) Taking the notation in [Schwartz & Madden (1999)] as a model, we will use the following symbols for the categories of interest to us (throughout, all rings referred to will be commutative and with 1); in each case, morphisms are assumed to preserve the relevant structures:

POR/N, the category of reduced partially ordered rings;

R/N, the category of reduced rings;

CR, the category of all (commutative) rings.

The ring of continuous real-valued functions, $C(X)$, on a topological space X is an object in all three categories (such rings also live in several other important categories but our attention will be restricted to the above three). In each of these categories, it is easy to find examples of morphisms $f: A \to B$ which are epimorphisms without being surjective. However, if f is an epimorphism there must, nevertheless, be some close connections between the two rings.

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It was observed in [Schwartz (2000)] that when $f: A \to B$ is a ring homomorphism between real closed rings then it is automatically a morphism in **POR/N** and **R/N**. (See [Schwartz & Madden (1999), §12] for basic properties of real closed rings and an entry into the literature.) Moreover, the criteria (quoted below) for being an epimorphism in the two categories are simultaneously satisfied in this situation. An epimorphism in **CR** is automatically an epimorphism in **R/N**, but not conversely. However, the question whether an **R/N**-epimorphism between *real closed rings* is also a **CR**-epimorphism remains open and will be discussed later in this article.

Our main topic will be the study of the morphism (in the categories) ρ induced by a subspace Y of a topological space X via restriction. Then $\rho: C(X) \to C(Y)$ is a ring homomorphism between two real closed rings ([Schwartz (1997), Theorem 1.2]). There are various ways in which a subspace can be related to the ambient space: closed, open, locally closed, zero-set, cozero-set, C-embedded, C^*-embedded, z-embedded and (see below) G-embedded, among others. We will study when ρ is an epimorphism in the three categories. This, as we will show, has connections with the other sorts of embedding of Y in X. As one would expect, the results are more complete when the ambient space, X, is "nice" in topological terms.

The statement that ρ is an epimorphism in any of the categories connects not only $\operatorname{Spec} C(Y)$ with $\operatorname{Spec} C(X)$ but also the residue fields of the two rings as well. The question is also closely related to continuous extensions of elements of $C(Y)$ to larger subsets of X.

The criteria for an epimorphism in **POR/N** and **R/N** are quoted in [Schwartz (2000), page 351]. When $f: A \to B$ is a morphism between real closed rings (as is the case for ρ) f is automatically a **POR/N** homomorphism because the positive cones are the sets of squares and the real spectrum ([Schwartz & Madden (1999), Proposition 12.4]) and the ordinary spectrum coincide. In the case of a homomorphism between real closed rings, it is easy to see that the criteria for an epimorphism in **R/N** and in **POR/N** coincide. This fact is summarized in Proposition 1.1. The symbol $\mathrm{Q_{cl}}(A)$ denotes the classical ring of quotients (or ring of fractions) of a ring A.

1.1. PROPOSITION. *Let $f: A \to B$ be a homomorphism between real closed rings. Then f is an epimorphism in **POR/N** and in **R/N** if and only if (i) $\operatorname{Spec}\rho: \operatorname{Spec} B \to \operatorname{Spec} A$ is an injection, and (ii) for each $\mathfrak{p} \in \operatorname{Spec} B$, f induces an isomorphism $\mathrm{Q_{cl}}(A/f^{-1}(\mathfrak{p})) \to \mathrm{Q_{cl}}(B/\mathfrak{p})$.*

There is an analogous characterization of epimorphisms in **CR** due to Ferrand, quoted in [Lazard (1968), Proposition IV 1.5]. However, a more down-to-earth criterion will be more convenient for us. The following is found in [Mazet (1968)] and in [Storrer (1968), page 73].

1.2. PROPOSITION. *A homomorphism $f: A \to B$ in **CR** is an epimorphism if and only if for each $b \in B$ there exist matrices C, D, E of sizes $1 \times n$, $n \times n$ and $n \times 1$, respectively, where (i) C and E have entries in B, (ii) D has entries in $f(A)$, (iii) the entries of CD*

and of DE are elements of $f(A)$ and (iv) $b = CDE$. (Such an expression is called an $n \times n$ zig-zag for b over A.*)*

Section 2 deals with the category **CR**. Some easy cases where $\rho: C(X) \to C(Y)$ is a **CR**-epimorphism are dealt with first. For example, it suffices that Y be C^*-embedded in X. The next steps are to show that the existence of a zig-zag for $f \in C(Y)$ over $C(X)$ says something about extending f to a larger domain; explicitly, f can be extended from a dense set to an open set (Proposition 2.6). The most complete result is for X first countable and perfectly normal (e.g., a metric space), in which case ρ is an epimorphism if and only if Y is locally closed (i.e., the intersection of an open and a closed, [Engelking (1968), page 96]) (Corollary 2.12).

Along the way, it is also shown that, when X is normal, the restriction $\rho^*: C^*(X) \to C^*(Y)$ of ρ is an epimorphism if and only if it is surjective (Theorem 2.4); i.e., when Y is C^*-embedded.

Section 3 looks at the categories **R/N** and **POR/N**. Since ρ is an epimorphism in one category if and only if it is an epimorphism in the other one, we will always talk about **R/N**-epimorphisms. Much of the section will deal with a necessary condition for ρ to be an **R/N**-epimorphism, namely that Y is G-embedded in X. This condition is strictly weaker than saying that ρ is an **R/N**-epimorphism; however, if the criteria (i) and (ii) of Proposition 1.1 are restricted to the z-spectra we get a characterization of when Y is G-embedded in X (Theorem 3.5). It is also related to a topic in N. Schwartz's paper on $C(X)$, [Schwartz (1997)], where the image of $\operatorname{Spec}\rho$ is studied. For X first countable and perfectly normal we have the attractive analogy to the result on (**CR**)-epimorphisms in such spaces: Y is G-embedded in X if and only if Y is a *finite union* of locally closed sets (Corollary 3.18). For these spaces, it is also shown that ρ is an **R/N**-epimorphism if and only if it is a **CR**-epimorphism.

Along the way, we look at some regular rings naturally related to $C(X)$, namely $T(X)$, $Q(X)$, $H(X)$ and $G(X)$ and observe that they are all real closed, and, hence, epi-final in **R/N** and in **POR/N** (Proposition 3.1). This fact for $G(X)$ is important in the rest of the section.

Section 4 has miscellaneous results and poses some vexing questions which remain unanswered.

Conventions of notation and terminology.

Throughout a topological space will mean a Tychonoff space. If Y is a subspace of a topological space X the restriction homomorphism $\rho: C(X) \to C(Y)$ will be the object of our study. "Regular ring" will always mean "von Neumann regular ring".

Recall some of the basic terminology: a subset Z of X is a *zero-set* if there is $f \in C(X)$ with $Z = z(f) = \{x \in X \mid f(x) = 0\}$; its complement, $\operatorname{coz} f$, is a *cozero-set.* An ideal I of $C(X)$ is a *z-ideal* if $g \in I$ whenever $f \in I$ and $z(g) = z(f)$. A subset Y of X is *locally closed* if it is the intersection of an open and a closed subset. The general references for topology are [Engelking (1968)], [Gillman & Jerison (1960)] and [Willard (1970)], while that for rings of continuous functions is, of course, [Gillman & Jerison (1960)]. Some categorical material and properties of real closed rings are taken from [Schwartz & Madden (1999)] and [Schwartz (1997)].

2. Subspaces inducing epimorphisms in **CR**.

Recall ([Engelking (1968), page 60]) that a space is *perfectly normal* if every closed set is a zero-set. (A perfectly normal space is normal. If A and B are disjoint zero-sets of f and g, resp., then the function $|f|/(|f|+|g|)$ is 0 on A and 1 on B.) One striking result in this section is that in a first countable perfectly normal space X (e.g., a metric space) ρ is an epimorphism in **CR** if and only if Y is locally closed. However, the picture is quite different when X does not have these strong conditions, as we shall see.

Let us first note that there are non-trivial cases where ρ is an epimorphism in **CR**, and, hence, in **POR/N** and **R/N** as well. In fact, to say that ρ is surjective is the same as saying that Y is C-embedded. See [Gillman & Jerison (1960), 1.18, 1.19, 1F] for examples.

We will see that ρ is an epimorphism if Y is only C^*-embedded in X. This is part of the next observation which discusses the most elementary examples of subspaces where ρ is an epimorphism in **CR** but need not be surjective. It is only a start on the topic.

2.1. Proposition. *Let Y be a subspace of a topological space X and $\rho: C(X) \to C(Y)$ the restriction homomorphism.*

*(i) If Y is C^*-embedded in X, then ρ is an epimorphism in **CR**.*

*(ii) If each $f \in C^*(Y)$ extends to a continuous function on a cozero-set of X, then ρ is an epimorphism in **CR**. This occurs, in particular, if Y is a cozero-set in X.*

(iii) If for each $f \in C^*(Y)$ there are $g, h \in C(X)$ with $Y \subseteq \operatorname{coz} g$ and $f\rho(g) = \rho(h)$, then ρ is an epimorphism.

Proof. (i) Given $f \in C(Y)$ we wish to construct a zig-zag for f over $C(X)$. Consider the two bounded functions $f_1 = f/(1+f^2)$ and $f_2 = 1/(1+f^2)$ and write $f_i = \rho(g_i)$, for $g_i \in C(X)$, $i = 1, 2$. Then, on Y, $f_2^{-1} g_1 g_2 f_2^{-1} = f$ is the desired zig-zag (here $n = 1$).

(ii) Suppose first that $f \in C^*(Y)$ and that f extends to a cozero-set $\operatorname{coz} g$, $g \in C^*(X)$ (we keep the symbol f). We get a zig-zag since fg extends to some $h \in C^*(X)$, and so, on Y, $f = g^{-1} g h g^{-1}$. Now, as above, we put $f_1 = f/(1+f^2)$ and $f_2 = 1/(1+f^2)$ and we have zig-zags with $f_i = g_i^{-1} g_i h_i g_i^{-1}$ on Y, where h_2 is non-zero on Y. These combine to give $f = (g_1^{-1} g_1 h_1 g_1^{-1})(g_2^{-1} g_2 h_2 g_2^{-1})^{-1} = (g_1^{-1} h_2^{-1})(g_1 g_2 h_1 h_2)(g_1^{-1} h_2^{-1})$ on Y.

(iii) Here $\rho(g)$ is invertible and so $f = \rho(g)^{-1}\rho(g)\rho(h)\rho(g)^{-1}$ is a (1×1) zig-zag. ∎

The ideas in Proposition 2.1 will be pushed quite a bit further. Part (iii) will show up in Proposition 2.15, below. Notice that in each of the parts of the proposition, only a 1×1 zig-zag appears. This phenomenon will be discussed in Section 4.

2.2. Proposition. [Fine, Gillman, & Lambek (1965), page 24] *Suppose that Y is a dense subset of X. Then for any $f \in C(Y)$, there is a largest subset of X to which f can be continuously extended.*

For any $u \in C(Y)$, let us write $\operatorname{dom}(u)$ for the largest subspace of X to which u can be extended continuously.

2.3. Lemma. *Suppose that X is a topological space and Y a dense subspace. Let $f \in C(Y)$ and $x \in X - Y$. Assume that $f = GAH$ is a zig-zag for f over $C(Y \cup \{x\})$ (as in Proposition 1.2). Suppose, moreover, that for some neighbourhood U of x the entries of G and H are bounded in $U \cap Y$. Then f can be extended continuously to x.*

Proof. Let $\widehat{A}$ denote the constant matrix $A(x)$ and $\widehat{f} = G\widehat{A}H$. Let U be a neighbourhood of x as in the statement. Suppose that M is a bound on both $\|G\|$ and $\|H\|$ in $U \cap Y$. Then we have that in U,

$$|f - \widehat{f}| = \|GAH - G\widehat{A}H\| \le \|G\|\,\|A - \widehat{A}\|\,\|H\| \le M^2\|A - \widehat{A}\|$$

and then $\lim_{y\to x} A(y) = \widehat{A}$ implies that

$$\lim_{y\to x} |f(y) - \widehat{f}(y)| = 0 \tag{$*$}$$

We also have that in $U \cap Y$,

$$\|GA - G\widehat{A}\| \le \|G\|\|A - \widehat{A}\| \le M\|A - \widehat{A}\|$$

from which we infer that $\lim_{y\to x}(G(y)A(y) - G(y)\widehat{A}) = 0$ and hence that $\lim_{y\to x} G(y)\widehat{A}$ exists and equals $\lim_{y\to x} G(y)A(y)$. Similarly, $\lim_{y\to x} \widehat{A}H(y) = \lim_{y\to x} A(y)H(y)$ exists. Since $\widehat{A}$ is constant, there is a matrix B such that $\widehat{A}B\widehat{A} = \widehat{A}$ (the ring of matrices over $\mathbf{R}$ is regular) and then $\widehat{f} = G\widehat{A}H = G\widehat{A}B\widehat{A}H$ is defined at x, and then so is f. ∎

Lemma 2.3 gives us information about when $\rho^*\colon C^*(X) \to C^*(Y)$, the restriction of ρ to $C^*(X)$, is an epimorphism. This behaviour is quite different from that of ρ itself.

2.4. Theorem. *Suppose that Y is a proper dense subset of X. Then $\rho^*\colon C^*(X) \to C^*(Y)$ is an epimorphism in **CR** if and only if it is surjective, that is, if and only if Y is C^*-embedded in X. If, moreover, X is normal then the conclusion is valid even when Y is not dense.*

Proof. If it is an epimorphism and $f \in C^*(Y)$, then there is a zig-zag $f = GAH$ over $C^*(X)$, as in the lemma. Since G and H have entries in $C^*(Y)$ they are necessarily bounded and hence f can be extended to any point of $X - Y$. By bounding the extension with the bounds of f we may assume that the extension is in $C^*(X)$.

When X is normal and $C^*(X) \to C^*(Y)$ is an epimorphism then for $f \in C^*(Y)$ we can restrict a zig-zag for f over $C^*(X)$ to get one over $C^*(\operatorname{cl}(Y))$. This allows us to extend f to $\tilde{f} \in C^*(\operatorname{cl}(Y))$. However, $\operatorname{cl}(Y)$ is C^*-embedded in X, completing the proof. ∎

When $C^*(Y) = C(Y)$, that is, when Y is pseudocompact, the same reasoning as in Theorem 2.4 gives us the following corollary.

2.5. Corollary. *Suppose that Y is a pseudocompact proper dense subset of X. Then $\rho^*: C(X) \to C(Y)$ is an epimorphism in* ***CR*** *if and only if it is surjective, that is, if and only if Y is C-embedded. If, moreover, X is normal then the conclusion is valid for pseudocompact Y even when not dense.*

In light of the discussion in Section 3B, note that if Y is C^*-embedded then it is z-embedded. An almost compact space ([Gillman & Jerison (1960), 6J]) is C-embedded in any ambient space. If Y is pseudocompact but not almost compact then there is a compactification of Y in which Y is not z-embedded ([Blair & Hager (1974), Theorem 4.1]).

The next corollary is a partial converse to Proposition 2.1(ii).

2.6. Proposition. *Suppose that X is a topological space and Y a dense subspace such that $\rho: C(X) \to C(Y)$ is an epimorphism in* ***CR****. Then for $f \in C(Y)$,* $\mathrm{dom}(f)$ *contains an open set containing Y.*

Proof. We know that there must exist G, A, and H as in Lemma 2.3 for which $f = GAH$. Suppose that $y \in Y$. Since G and H are defined and continuous on Y, there is an open neighbourhood U of y on which both G and H are bounded. For any $x \in U$, U is also a neighbourhood of x and it follows from 2.3 that f can be extended to x. ∎

2.7. Proposition. [Gillman & Jerison (1960), 3.11(b)] *In a space X, every first countable point is a zero-set.*

2.8. Lemma. *Suppose that $t_1 > t_2 > t_3 > \cdots$ is a sequence of numbers in the open unit interval that converges to 0. Then there is a continuous function $h : (0, \infty) \to [0, 1]$ such that $h(t) = 0$ for $t \geq t_1$, $h(t_n) = 0$ when n is odd and $h(t) = 1$ when n is even.*

Proof. Just interpolate linearly between those values. ∎

2.9. Proposition. *The complement of a first countable, non-isolated point is not C^*-embedded.*

Proof. Let $x \in X$ be a first countable non-isolated point. If $Y = X - \{x\}$ were C^*-embedded in X then $\beta Y = \beta X$. Then by [Gillman & Jerison (1960), 9.6 and the remarks of 9.7], x is not a G_δ-set in βY but is first countable in βY. This is impossible. ∎

2.10. Theorem. *If a dense subset Y of a first countable space X induces an epimorphism in $\boldsymbol{CR}$ then Y is open in X.*

Proof. If $Y \subseteq X$ is not open, then since X is first countable, there must be a sequence of points $x_1, x_2, \dots$ of $X - Y$ which converges to some $y \in Y$. By using the device of Proposition 2.7, it can be assumed that the points are distinct and form a discrete subspace of X.

Choose, for each n, a function $f_n : X - \{x_n\} \to [0,1]$ that cannot be extended to x_n. Then $f_n \in C(Y)$ and so is $f = \sum 2^{-n} f_n$, since the convergence is uniform. For each n there is a neighbourhood V of x_n excluding the other points of the sequence. Then $f - 2^{-n} f_n$ behaves like $\sum_{m \neq n} 2^{-m} f_m$ on V and so extends continuously to x_n. This implies that f cannot be extended continuously to x_n. Thus $f \in C(Y)$ cannot be extended to any open set containing Y and the conclusion follows. ∎

2.11. Corollary. *Let Y be a subspace of a first countable space X. If Y induces an epimorphism in $\boldsymbol{CR}$ then Y is locally closed in X*

Proof. Since $C(\mathrm{cl}(Y)) \to C(Y)$ is an epimorphism (the second factor of an epimorphism), Theorem 2.10 applies. ∎

2.12. Corollary. *A subspace Y of a perfectly normal first countable space X induces an epimorphism in $\boldsymbol{CR}$ if and only if it is locally closed.*

Proof. We already have one direction in Corollary 2.11. In the other direction, if $Y = U \cap V$, where V is closed and U open, U is a cozero-set in X, and, hence, also in V. Then $C(V) \to C(Y)$ is an epimorphism. Moreover, V is C^*-embedded in X; thus, $C(X) \to C(V)$ is also an epimorphism. ∎

We will see in the next section, that there are dense open subsets of normal spaces which do not induce epimorphisms and a first countable space containing a zero-set which does not induce an epimorphism, even in $\mathbf{R}/\mathbf{N}$. There is a compact space with a dense subset which induces an epimorphism but is not even a finite union of locally closed subsets. See Examples 3.11, below. These show that there are limitations to generalizations of the last two corollaries. The following space shows that the assumption "first countable" cannot be removed from Corollary 2.12.

2.13. Example. There is a countable (hence, perfectly normal) space X such that every subspace induces an epimorphism in $\mathbf{CR}$ and not all subspaces are locally closed.

Proof. A space is called *resolvable* if it is the union of two disjoint dense subsets. Now suppose that X is a countable resolvable F-space ([Gillman & Jerison (1960), Theorem 14.25]). Since X is a countable F-space, every subset is C^*-embedded ([*ibid.*, 14 N5]).

Since it is resolvable, it has a dense subset which is not open. Since X is countable, it is perfectly normal ([Gillman & Jerison (1960), 3B(1)]). However, X has a dense non-open subset that induces an epimorphism.

We must now produce an example of such a space. We are grateful to R.G. Woods for pointing out the following. For the notion of "absolutes" and "perfectly irreducible surjections", see [Porter & Woods (1988), Chapter 6]. We will need [*ibid.*, 6.5(b)(4)]: if $f\colon X \to Y$ is irreducible and S is dense in Y, $f^{-1}(S)$ is dense in X and $f|_{f^{-1}(S)}$ is irreducible from $f^{-1}(S)$ onto S.

Let E denote the absolute of the rationals $\mathbf{Q}$ and let A and B be complementary dense subsets of $\mathbf{Q}$. Suppose that $k\colon E \to \mathbf{Q}$ is the canonical perfect irreducible surjection and $C = k^{-1}(A)$ and $D = k^{-1}(B)$. Then C and D are dense in E. Since A and B are separable so are C and D. Now let R and S be countable dense subsets of C and D, respectively. Define $X = R \cup S$; it is countable and resolvable. Since it is dense in E, it is extremally disconnected by [Gillman & Jerison (1960), 1H(4)] since E is. The space X has no isolated points because $\mathbf{Q}$, and, hence, E does not have isolated points. ∎

Proposition 2.9 is not the last word on complements of points.

2.14. Example. There are easy examples of non-isolated points in normal (even compact) spaces whose complements are C^*-embedded. If $x \in X$ is such an example, then $\rho\colon C(X) \to C(X - \{x\})$ induces an epimorphism in **CR** but $X - \{x\}$ is not a cozero-set.

Proof. Let Z be a non-compact space and $a \in \beta Z - Z$, $X = \beta Z$ and $Y = \beta Z - \{a\}$. Since a is not isolated, Y is dense open in X. By [Gillman & Jerison (1960), 1H(6))], Y is C^*-embedded in X and therefore Y induces an epimorphism. However, Y is not a cozero-set by [Gillman & Jerison (1960), 9.5]. Thus open sets can induce epimorphisms without being cozero-sets. ∎

We finish this section with some special families of examples. They will appear again in Section 3. In particular, we will see that if $Y \subseteq X$ and Y is homeomorphic to $\mathbf{N}$, then ρ is an epimorphism.

2.15. Proposition. *(i) Let Y be a subspace of a normal space X which has the following form: $Y = \bigcup_{n\in\mathbf{N}} B_n$ where the B_n are closed sets of X and there is a family of pairwise disjoint open sets U_n from X with $B_n \subseteq U_n$, for each $n \in \mathbf{N}$. Then Y induces an epimorphism in **CR**.*

(ii) If, under the hypotheses of (i), each B_n is compact then then the conclusion of (i) holds for any X. This applies, in particular, if Y is homeomorphic to $\mathbf{N}$.

*(iii) Let Y be a subspace of a space X which has the following form: $Y = \bigcup_{n\in\mathbf{N}} B_n$, where there is a family of pairwise disjoint cozero-sets U_n, with $B_n \subseteq U_n$, for $n \in \mathbf{N}$. Suppose, moreover, that each B_n is C^*-embedded in U_n. Then $Y \subseteq X$ induces an epimorphism in **CR**.*

Proof. (i) Let $f \in C^*(Y)$. Set $M = \sup_Y |f(y)|$. We use the sequence $\{U_n\}$. Now find $h_n \in C(X)$ with h_n constantly 1 on B_n and zero on $X - U_n$. We may assume that values of h_n are in $[0,1]$. Define $g \in C^*(Y)$ by $g(y) = (1/n)f(y)$, for $y \in B_n$. It will be shown that g extends to X. Since X is normal and B_n closed, $g_n = g|_{B_n}$ extends to $l_n \in C(X)$ and we may assume that, for all $x \in X$, $|l_n(x)| \leq M/n$.

The next step is to define $a, b \in C(X)$, as follows. Put $a(x) = h_n(x)l_n(x)$ if $x \in U_n$, and $a(x) = 0$ if $x \in \bigcap_{\mathbf{N}}(X - U_n)$. Similarly, $b(x) = (1/n)h_n(x)$ if $x \in U_n$ and $b(x) = 0$ for $x \in \bigcap_{\mathbf{N}}(X - U_n)$. We check that a is continuous (the continuity of b is similar). For an interval (u, v), $a^{-1}((u,v)) = \bigcup_{\mathbf{N}}(h_n l_n)^{-1}((u,v))$, an open set, if $0 \notin (u,v)$. If $0 \in (u,v)$, the inverse image includes all but finitely many of the U_n. Its complement is a *finite* union of sets of the form $(h_n l_n)^{-1}((-\infty, u] \cup [v, \infty))$, which is closed. Finally, $f = \rho(a)\rho(b)^{-1}$. Then Proposition 2.1(iii) applies.

(ii) This is done by first embedding X in βX. There are open sets V_n, $n \in \mathbf{N}$, in βX with $U_n = X \cap V_n$; since X is dense in βX, the V_n are disjoint. The compact sets B_n are closed in βX. However, βX is normal and part (i) applies there. The functions constructed on βX, restricted to X, give the result.

(iii) Let $U_n = \operatorname{coz} \varphi_n$, where we can suppose $\varphi_n \colon X \to [0,1]$. The proof here is similar to that of (i) with the following modifications. Here, $g(y) = (1/n)f(y)\varphi_n(y)$, $y \in B_n$, and it extends to X since $f|_B$ extends to U_n. From there, $g|_{U_n}$ extends to $l_n \in C(X)$ and $a(x) = l_n(x)$, for $x \in U_n$. In addition, b is constructed via $b(x) = (1/n)\varphi_n(x)$, for $x \in U_n$. ∎

3. Epimorphisms in $\mathbf{R/N}$ and in $\mathbf{POR/N}$, and generalizations.

A. Real closed regular rings.

It was mentioned earlier that the restriction homomorphism $\rho \colon C(X) \to C(Y)$ induces an epimorphism in one of these two categories if and only if it induces one in the other. In this section we will use the phrase "an epimorphism in $\mathbf{R/N}$" or "$\mathbf{R/N}$-epimorphism" to cover both categories.

As we saw in Proposition 1.1, the homomorphism ρ is an epimorphism in $\mathbf{R/N}$ if and only if $\operatorname{Spec}\rho \colon \operatorname{Spec} C(Y) \to \operatorname{Spec} C(X)$ is injective and the residual fields are isomorphic via ρ. In many situations the prime z-ideals are easier to study. For that reason we will study a necessary condition for an epimorphism in $\mathbf{R/N}$ which only involves the prime z-ideals. The subspace of $\operatorname{Spec} C(X)$ consisting of prime z-ideals is called $\operatorname{zSpec} C(X)$; it is a proconstructible subset, i.e., it is closed in the constructible topology (see [Schwartz & Madden (1999), Section 4] for terminology and basic facts).

However, our first observations are about rings related to $C(X)$. For any commutative ring there is a universal regular ring functor, often called T, which is the left adjoint to the inclusion of the category of (commutative von Neumann) regular rings into $\mathbf{CR}$. For $f \colon A \to B$ in $\mathbf{CR}$, where A and B are rings, if f is an epimorphism in $\mathbf{CR}$ then $T(f)$ is surjective. However, we will see that for real closed rings $T(f)$ surjective implies that f is an epimorphism in $\mathbf{R/N}$.

An object A in a concrete category is called *epi-final* if each epimorphism $f\colon A \to B$ is a surjection. Note that a regular f-ring is real closed if and only if its residue fields (or, equivalently, its Pierce stalks) are all real closed (see [Lipshitz (1977), Definitions (vi)]; the conditions for a real closed regular ring go up and down from the Pierce stalks which are ordered fields).

3.1. Proposition. *(1) A regular ring with factor fields of characteristic zero is epi-final in* ***R/N****. (2) A real closed regular ring is epi-final in* ***POR/N****.*

Proof. The two proofs are similar. Part (2) is as follows. If $f\colon (A,P) \to (B,Q)$ is an epimorphism in **POR/N** and A is regular and real closed then the factor fields of A are real closed. If $\mathfrak{p}$ is the support of a prime cone in $\mathrm{Sper}(B,Q)$ then $A/f^{-1}\mathfrak{p}$ is the real closure of $\mathrm{Q_{cl}}(B/\mathfrak{p})$. However, $A/f^{-1}\mathfrak{p}$ order embeds in $B/\mathfrak{p}$ and so we have equality. Therefore, $\mathfrak{p}$ is a maximal ideal. Moreover, every minimal prime of B is the support of a prime cone in $\mathrm{Sper}\, B$ ([Schwartz & Madden (1999), Proposition 4.4]) and it follows that the minimal primes are maximal; in other words, B is regular and real closed. In that case $\mathrm{Sper}\, B$ and $\mathrm{Spec}\, B$ are homeomorphic and $\mathrm{Spec}\, f(\mathrm{Spec}\, B)$ is a closed subset of $\mathrm{Spec}\, A$. Then B is isomorphic to A/I, where I is the intersection of all the maximal ideals not in the image of $\mathrm{Spec}\, f$. ∎

There are several regular rings in the literature related to a ring of the form $C(X)$. Let us recall some of them. Firstly, there is $T(C(X))$, abbreviated to $T(X)$, secondly $F(X) = \mathbf{R}^X$ and thirdly the complete ring of quotients, $Q(X)$. The universal property of the functor T says that the inclusion of $C(X)$ into $F(X)$ factors through $T(X)$ with image denoted $G(X)$, and the inclusion of $C(X)$ into $Q(X)$ factors through $T(X)$ with image $H(X)$. The ring $G(X)$ is the one of most interest in the rest of this section but the others all have been studied. See [Fine, Gillman, & Lambek (1965)] for $Q(X)$, [Olivier (1968)] and [Wiegand (1971)] for the functor T, [Raphael & Woods (2000)] for $H(X)$ and [Henriksen, Raphael, & Woods (2001)] for $G(X)$.

3.2. Proposition. *For a topological space X, the regular rings $T(X)$, $Q(X)$, $G(X)$, and $H(X)$ are all real closed.*

Proof. The residue fields of $T(X)$ are those of $C(X)$ (i.e., the fields $\mathrm{Q_{cl}}(C(X)/\mathfrak{p})$, $\mathfrak{p} \in \mathrm{Spec}\, C(X)$). These fields are all real closed by [Schwartz (1997), Theorem 1.2]. As already remarked in [*ibid.*, page 299] , since $Q(X)$ is a direct limit of real closed rings, it is real closed. Finally, $G(X)$ and $H(X)$ are regular homomorphic images of real closed rings and are, hence, real closed (see the table [Schwartz & Madden (1999), page 255]). ∎

B. G-embeddings and $\mathrm{zSpec}\, C(X)$**.**

For a space X, the elements of $G(X)$ can be described explicitly ([Henriksen, Raphael, & Woods (2001), Theorem 1.1]) as follows. For $a \in C(X)$, $a^*\colon X \to \mathbf{R}$ is defined by: $a^*(x) = 1/a(x)$ if $a(x) \neq 0$, and is zero otherwise. Note that a^* is continuous on the dense open set $\operatorname{coz} a \cup (X - \operatorname{cl}(\operatorname{coz} a))$. A typical element of $G(X)$ has the form $\sum_{i=1}^m a_i b_i^*$ for some $m \in \mathbf{N}$, where the $a_i, b_i \in C(X)$. This shows that $G(X)$ is generated, as a regular ring, by the elements of $C(X)$. Notice that $C(X)$ and $G(X)$ have the same cardinality.

When Y is a subspace of X, there is an induced homomorphism $G(\rho)\colon G(X) \to G(Y)$: if $\sigma = \sum_{i=1}^m f_i h_i^* \in G(X)$, then $G(\rho)(\sigma) = \sum_{i=1}^m \rho(f_i)(\rho(h_i))^*$. When ρ is an epimorphism in **CR** or in **R/N** then $G(\rho)$ is a surjection. This follows since T is a left adjoint and thus preserves epimorpisms; moreover, $T(X) \to T(Y)$ is surjective (Proposition 3.1), while $T(Y) \to G(Y)$ is surjective by construction. Since the composite $T(X) \to T(Y) \to G(Y) = T(X) \to G(X) \to G(Y)$, it follows that the second factor is surjective .

The surjectivity of $G(\rho)$ is a necessary condition for an epimorphism in **R/N** and when this occurs we will say, following [Henriksen, Raphael, & Woods (2001), Proposition 2.1], that Y is *G-embedded* in X. (There are special cases where G-embedded suffices for a **CR**-epimorphism; see Proposition 4.1(ii), below.) The next observations will characterize when Y is G-embedded in X. The conditions are exactly those of Proposition 1.1, but restricted to the z-spectra of $C(X)$ and $C(Y)$.

3.3. Proposition. *Let Y be a subspace of a topological space X. Then Y is G-embedded in X if and only if every $f \in C(Y)$ is in the image of $G(\rho)$. In other words, each $f \in C(Y)$ can be written $\sum_{i=1}^m \rho(a_i)\rho(b_i)^*$ for some $m \in \mathbf{N}$, with $a_i, b_i \in C(X)$.* ∎

We next describe more explicitly the relation between $T(X)$ and $G(X)$. Recall that the maximal ideals of $T(X)$ are in one-to-one correspondence with the primes of $T(X)$. For $P \in \operatorname{Spec} C(X)$, let the corresponding maximal ideal be $\tilde{P}$.

3.4. Proposition. *Let X be a topological space. (1) The kernel of the natural surjection $\nu_X\colon T(X) \to G(X)$ is $\ker \nu_X = \bigcap_{P \in \operatorname{zSpec} C(X)} \tilde{P}$. (2) $\operatorname{Spec} G(X)$ is $\operatorname{zSpec} C(X)$ as a subspace of $\operatorname{Spec} C(X)$ endowed with the constructible topology.*

Proof. We will use the subscript 'c' when using the constructible topology, for example $\operatorname{Spec}_c C(X)$. As always with regular rings, we can view them (see [Pierce (1967)] or [Burgess & Stephenson (1976)]) as rings of sections of a sheaf of fields over a boolean space. The kernel of a homomorphism from $T(X)$ will be the set of sections which vanish on an open subset U of the boolean space $\operatorname{Spec}_c C(X)$ and the image is the restriction of the sheaf to the complement of U. We will show that elements of $\ker \nu_X$ have supports in $\operatorname{Spec}_c C(X) - \operatorname{zSpec}_c C(X)$. We need the fact ([Schwartz (1997), Theorem 3.2] or [Montgomery (1973), Theorem 7.1]) that the fixed maximal ideals are constructibly dense in $\operatorname{zSpec} C(X)$.

In one direction, note what happens to elements of $T(X)$ under ν_X. For $x \in X$ denote by $\mathfrak{m}_x$ the fixed maximal ideal for x. Then for $\alpha = \sum_{i=1}^m a_i b_i'$ and $x \in X$, $\nu_X(\alpha)(x) = \sum_{i=1}^m a_i(\mathfrak{m}_x)(b_i'(\mathfrak{m}_x) = \sum_{i=1}^m a_i(x) b_i'(x)$, where $b_i'(x) = 1/b_i(x)$ or 0. Thus, if $\nu_X(\alpha) = 0$, $\alpha(\mathfrak{m}_x) = 0$ for all $x \in X$. Then, by the density of the of fixed maximal ideals, α is zero at all the prime z-ideals, since its support is both open and closed.

Next, suppose the support of α is in $\operatorname{Spec_c} C(X) - \operatorname{zSpec_c} C(X)$. Take a basic set $U = V(I) \cap D(b)$ in the support of α. For any $\mathfrak{p} \in U$, $b \notin \mathfrak{p}$ and $b \notin \mathfrak{m}$, for the unique maximal ideal containing $\mathfrak{p}$, because the maximal ideals are z-ideals. By [Gillman & Jerison (1960), 14.5], $b(\mathfrak{p})$ is an infinitesimal for each $\mathfrak{p} \in U$. If e is the idempotent in $T(X)$ whose support is U, for all $n \in \mathbf{Z}$, $nbe \le \mathbf{1}$. However, $G(X)$ is archimedean and so $\nu_X(be) = 0$. Moreover, $e = (be)(be)'$, showing that $\nu_X(e) = 0$. It follows that any element of $T(X)$ whose support does not meet $\operatorname{zSpec_c} C(X)$ is in the kernel of ν_X. ∎

We can now characterize G-embedded subspaces in terms like those of Proposition 1.1.

3.5. Theorem. *Let Y be a subspace of a topological space X. Then Y is G-embedded in X if and only if (i)* $\operatorname{zSpec}\rho\colon \operatorname{zSpec} C(Y) \to \operatorname{zSpec} C(X)$ *is injective and (ii) for every* $\mathfrak{p} \in \operatorname{zSpec} C(Y)$, *$\rho$ induces an isomorphism* $\mathrm{Q_{cl}}(C(X)/\rho^{-1}(\mathfrak{p})) \to \mathrm{Q_{cl}}(C(Y)/\mathfrak{p})$.

Proof. We need to know that, in fact, $\operatorname{Spec}\rho(\operatorname{zSpec} C(Y)) \subseteq \operatorname{zSpec} C(X)$. This is clear, for if $\mathfrak{p} \in \operatorname{zSpec} C(Y)$, $a \in \rho^{-1}\mathfrak{p}$ and $b \in C(X)$ with $z(b) = z(a)$, then $\rho(b) \in \mathfrak{p}$ as well. With this in hand, $G(\rho)$ is surjective if and only if the two conditions hold, by Proposition 1.1 applied to $G(\rho)$. ∎

When the subspace Y is z-embedded in X then it is automatically the case that $\operatorname{Spec}\rho$ is injective on $\operatorname{zSpec} C(Y)$ by [Schwartz (1997), pages 288 and 289].

3.6. Definition. *For a topological space X, the boolean algebra generated by the zero-sets of X will be denoted $\mathfrak{Z}(X)$.*

3.7. Proposition. *Let X be a normal space. Suppose that $Y \subseteq X$ and $Y \in \mathfrak{Z}(X)$. Then Y is G-embedded in X.*

Proof. We know that the result is true if Y is a cozero-set (Proposition 2.1) and if it is a zero-set, since closed sets in a normal space are C^*-embedded ([Gillman & Jerison (1960), 3D]). Recall that a closed subspace of a normal is space is normal. Moreover, every finite boolean combination of zeros sets in X is the support of an idempotent in $G(X)$ (for $a \in C(X)$, $aa^* \in G(X)$ is zero on $Z(a)$ and constantly 1 on $\operatorname{coz} a$).

Let $Y \in \mathfrak{Z}(X)$. It is a boolean combination of zero-sets and so it can be expressed in conjunctive normal form, as a union of intersections of zero-sets and cozero-sets. By combining all the cozero-sets and all the zero-sets in each term, we get an expression $Y = \bigcup_S(\operatorname{coz} g_s \cap z(h_s))$.

The proof will be by induction on $n = |S|$. When $n = 1$, $Y = \operatorname{coz} g \cap z(h)$ and ρ is an epimorphism in **CR**, a stronger property than G-embedding. Then, for $f \in C^*(Y)$ there is $\widehat{g} \in G(X)$ so that $\widehat{g}|_Y = f$.

Suppose now that we have the result for $n \ge 1$ and that $Y = \bigcup_T(\operatorname{coz} g_t \cap z(h_t))$, with $|T| = n+1$. Write $T = S \cup \{u\}$. Put $A_S = \bigcup_S(\operatorname{coz} g_s \cap z(h_s)$. For $f \in C^*(Y)$ there is, by the induction assumption, $\widehat{g}_S \in G(X)$ with $\widehat{g}_S = f|_S$. Let $e_S = e_S^2 \in G(X)$ have support

A_S. There is, also, $\widehat{g}_u \in G(X)$ with $\widehat{g}_u|_{\operatorname{coz} g_u \cap z(h_u)} = f|_{\operatorname{coz} g_u \cap z(h_u)}$. Let $e_u = e_u^2 \in G(X)$ have support $\operatorname{coz} g_u \cap z(h_u)$. Then, $f = (\widehat{g}_S e_S + \widehat{g}_u e_u(1 - e_S))|_Y$. ∎

Proposition 3.7 shows that there are G-embedded subspaces which do not induce **CR** epimorphisms because there are subspaces, even of metric spaces X, which are in $\mathfrak{Z}(X)$ but are not locally closed. As an example, we can take $Y = \bigcup_{n\in\mathbf{N}}(1/n+1, 1/n) \cup \{0\}$ in $\mathbf{R}$ because Y is not locally compact.

Recall some notation from N. Schwartz, [Schwartz (1997), page 288]). For a subspace $Y \subseteq X$: (i) $e(Y) \subseteq \operatorname{Spec} C(X)$ is the set of fixed maximal ideals of $C(X)$ attached to the elements of Y, (ii) $\tilde{Y}$ is the constructible closure of $e(Y)$ in $\operatorname{zSpec} C(X)$. Schwartz then asks ([*ibid.*, page 289]) whether the image of $\operatorname{Spec}\rho$ is the convex hull of $\tilde{Y}$, $\operatorname{cvx}\tilde{Y}$, and gives a positive answer when Y is z-embedded in X. One key step in Schwartz' proof is to show that when Y is z-embedded then $\operatorname{Spec}\rho$ is injective on $\operatorname{zSpec} C(Y)$. The conclusion does not, as we have seen, depend on Y being z-embedded. In fact the injectivity of $\operatorname{Spec}\rho$ on $\operatorname{zSpec} C(Y)$ will follow if $G(\rho)$ sends the set of idempotents of $G(X)$ onto that of $G(Y)$. When Y is z-embedded, $G(\rho)$ has this property. These remarks are summarized in the next two results.

3.8. Corollary. *Let $Y \subseteq X$ and suppose that $G(\rho)$ sends the algebra of idempotents of $G(X)$ onto that of $G(Y)$. Then* $\operatorname{Spec}\rho$ *is injective on* $\operatorname{zSpec} C(Y)$.

Proof. Since $G(X)$ and $G(Y)$ are regular rings, every idempotent in the image of $G(\rho)$ is the image of an idempotent in $G(X)$. Moreover, the spectrum of a regular ring is the same as that of its algebra of idempotents. Hence, a surjection at the level of idempotents yields an injection of spectra. ∎

3.9. Corollary. [cf. Schwartz (1997), Lemma 4.1 and Proposition 4.2] *Let Y be a z-embedded subspace of a space X. Then* $\operatorname{Spec}\rho$ *is injective on* $\operatorname{zSpec} C(Y)$.

Proof. By Corollary 3.8 it suffices to show that every idempotent of $G(Y)$ is in the image of $G(\rho)$. However, the idempotents of $G(Y)$ are boolean combinations of idempotents of the form aa^*, $a \in C(Y)$. (This is the idempotent which is the characteristic function of $\operatorname{coz} a$.) There is some $b \in C(X)$ so that $z(b) \cap Y = z(a)$. Then $aa^* = G(\rho)(bb^*)$. ∎

Note that it is easy to find examples where $G(\rho)$ restricted to the idempotents is surjective but $G(\rho)$ is not. The following remark shows, as an illustration, that the set of irrationals in $\mathbf{R}$ is such an example, by [Fine, Gillman, & Lambek (1965), Theorem 3.10(2)].

3.10. Remark. (cf., Theorem 2.10) Let Y be a dense subspace of a space X which is G-embedded and suppose there is a function on Y which cannot be extended continuously to a larger subset. Then Y contains a dense open set.

Proof. Let $f \in C(Y)$ be as in the statement. If there is $\widehat{g} \in G(X)$ with $G(\rho)(\widehat{g}) = f$ then $\widehat{g}$ is continuous on a dense open set D. As a result f can be extended to a continuous function on $Y \cup D$, showing that $D \subseteq Y$. ∎

There is a more complete statement than that in Corollary 3.9. It is easy to see that $G(\rho)$ is surjective on idempotents if and only if for every zero-set Z of Y there is $T \in \mathfrak{Z}(X)$ with $Z = T \cap Y$. We do not yet have an example of such a subset which is not also z-embedded.

It is time for some more examples: in the first three, Y is not G-embedded and hence ρ is not an epimorphism even in $\mathbf{R}/\mathbf{N}$.

3.11. Examples. **1.** Let X be a one-point compactification of an uncountable discrete set D, $X = D \cup \{*\}$. Then $\{*\}$ induces an epimorphism (in $\mathbf{CR}$), while its complement, D, is not G-embedded.

2. There is a closed subset L of the space $\mathbf{W}$ of ordinals less than the first uncountable ordinal whose complement is not G-embedded.

3. There is a first countable space with a zero-set which is not G-embedded.

4. There is a space with a C^*-embedded subset which is not a finite boolean combination of closed sets.

Proof. 1. The space X is functionally countable (quoted in [Levy & Rice (1981), Proposition 3.1]). Therefore every function in $G(X)$ has countable range. If D were G-embedded, then elements of $C(D)$ would be images of elements of $G(X)$, by Proposition 3.3. That would mean that elements of $C(D)$ also had countable range, but this is false. Therefore D is not G-embedded.

2. Take L to be the subspace of limit ordinals. By [Gillman & Jerison (1960), 5M (2)], L is closed in the normal space $\mathbf{W}$ and, hence, induces an epimorphism in $\mathbf{CR}$. Its complement D is discrete ([*ibid.*; 5.11]) and uncountable. Moreover, elements of $C(\mathbf{W})$ are constant on a “tail” of $\mathbf{W}$ and, hence, so are the functions in $G(\mathbf{W})$. This is not true of $C(D)$.

3. The famous example Γ (e.g., [Gillman & Jerison (1960), 3K] and also [Blair & Hager (1974), Remarks 2.5(a)]) is a completely regular space which is not normal (the closed upper half plane with the “tangent circle” topology). There is a subspace D which is uncountable and discrete. Then $|C(D)| = 2^{\mathfrak{c}}$ whereas $|C(\Gamma)| = \mathfrak{c}$. But then, also, $|G(\Gamma)| = \mathfrak{c}$. This shows that D, while a zero-set, cannot be G-embedded in Γ.

4. We take any space X for which $\beta X - X$ is dense in βX (e.g., $X = \mathbf{Q}$, [Gillman & Jerison (1960), 6.10]). Suppose $X = \bigcup_{i=1}^{n}(U_i \cap V_i)$, where the U_i are open and the V_i are closed in βX. We will show that the U_i are pairwise disjoint and use this fact to get a contradiction.

Downward induction on n is used. Suppose first that $U = \bigcap_{i=1}^{n} U_i \neq \emptyset$. Then there is $p \in U - X$. Then $p \notin \bigcup_{i=1}^{n} V_i$. For any $x \in X$, $x \in U_i \cap V_i \subseteq U$ for some i. Hence $X \subseteq \bigcup_{i=1}^{n} V_i \cup U^c = V$. But V is a proper closed subset of βX containing X, which is impossible.

Next, suppose that for some r, $1 < r \le n$, the intersection of any r of the U_i is empty. Suppose $U = \bigcap_{i=1}^{r-1} U_i \neq \emptyset$ (re-indexed if necessary). Then there is some $p \in U - X$. Again, $p \notin \bigcup_{i=1}^{r-1} V_i$. On the other hand, if $x \in X$ then $x \in \bigcup_{i=1}^{r-1}(U_i \cap V_i) \subseteq U$ or $x \in U_j \cap V_j$, some $r \le j \le n$. But, by the induction hypothesis, $x \notin U$. Hence, $X \subseteq \bigcup_{i=1}^{r-1} V_i \cup U^c$, a proper closed subset of βX. This is impossible and we see that the U_i are pairwise disjoint.

This shows that X is a disjoint union of a finite number of closed (in X) subsets $U_i \cap V_i$, since $U_i \cap V_i = (\bigcup_{i=1}^{n} U_i) \cap V_i = X \cap V_i$. These are clopen in X and, hence, C^*-embedded. They are, thus, clopen in βX ([Gillman & Jerison (1960), 9.6(c)]), making X open in βX. This is not the case. ∎

C. Necessary conditions for G-embeddings.

We saw in Proposition 3.7 that in a normal space X, a subset which is G-embedded is in $\mathfrak{Z}(X)$. The present task is to find a converse. In Proposition 3.17, below, we will show that if X is first countable then a G-embedded subspace must be a boolean combination of closed sets. The two propositions together give a characterization of "G-embedded" in first countable, perfectly normal spaces.

We begin with a rather specialized situation. However, it will twice later turn up in two more general settings. As it stands, it give an example, for each $n \in \mathbf{N}$, of a G-embedded subset Y in a perfectly normal first countable space X for which there is $f \in C^*(Y)$ so that when $f = G(\rho)(\widehat{g})$ for some $\widehat{g} \in G(X)$, then $\widehat{g}$ has at least $n+1$ terms.

3.12. Example. Suppose in a first countable space Y we have *disjoint* subsets $A_0, A_1, \ldots, A_{2n}$ with the following properties. (A specific example will be presented below.)

(i) $A_0 = \{x_0\}$.

(ii) $A_1 = \{x_{m(1)}\}_{m(1)\in\mathbf{N}}$, a sequence of distinct points converging to x_0.

(iii) $A_2 = \{x_{m(1),m(2)}\}$, $(m(1), m(2)) \in \mathbf{N}^2$, where $\{x_{m(1),m(2)}\}_{m(2)\in\mathbf{N}}$ is a sequence of distinct points converging to $x_{m(1)}$, for each $m(1) \in \mathbf{N}$.

(iv) For all k, $1 \le k \le 2n$, $A_k = \{x_{m(1),\ldots,m(k)}\}$, $(m(1), \ldots, m(k)) \in \mathbf{N}^k$, where, for each $k-1$-tuple $m(1), \ldots, m(k-1)$, $\{x_{m(1),\ldots,m(k-1),m(k)}\}_{m(k)\in\mathbf{N}}$ is a sequence of distinct points converging to $x_{m(1),\ldots,m(k-1)}$.

(v) For each k, $1 \le k \le 2n$, $\mathrm{cl}(A_k) = \bigcup_{i=0}^{k} A_i$.

Define $Z = \bigcup_{i=0}^{n} A_{2i}$. Let $f \in C^*(Z)$ have the following properties: f is non-zero on Z and cannot be extended to any point of any A_{2i-1}, $i = 1, \ldots, n$. (As an example, let $f(x_0) = 2$ and $f(x_{m(1),\ldots,m(2k)}) = 2 + (-1)^{m(2)}/m(1) + \cdots + (-1)^{m(2k)}/m(2k-1)$.) We make two claims about Z and f.

Claim 1: $Z = \bigcup_{i=0}^{n}(F_i \cap G_i)$, where the F_i are closed and the G_i open in Y. (Hence, if Y is perfectly normal, Z is G-embedded in Y.)

Claim 2: If $f = G(\rho)(\widehat{g})$ for some $\widehat{g} \in G(Y)$, then $\widehat{g}$ has at least $n+1$ terms and at least n of these terms are zero at x_0.

PROOF. Claim 1: This is done by induction: A_0 is closed and, by property (v), $\bigcup_{i=0}^{k+1} A_{2k} = \bigcup_{i=0}^{k} A_{2i} \cup [\mathrm{cl}(A_{2k+2}) - \mathrm{cl}(A_{2k+1})]$.

Claim 2: This will be by induction on n. When $n = 1$, suppose that our function $f = ab^*|_Y$ for some $a, b \in C(Y)$. Since $f(x_0) \neq 0$, ab^* is continuous on a neighbourhood of x_0 (on $\operatorname{coz} b$, in fact) and so f can be extended to the points of $A_1 \cap \operatorname{coz} b$. But this is not possible by the nature of f. Hence, there are at least two terms, say $\widehat{g} = \sum_{i=1}^{s} a_i b_i^*$ whose restriction to Z is f. If all were non-zero at x_0, $\widehat{g}$ would be continuous on a neighbourhood of x_0 and f would extend to some elements of A_1. This is not possible.

Now suppose that $n > 1$ and that the claim is true for $n - 1$. Suppose f can be expressed by $\widehat{g}|_Y$ where $\widehat{g} = \sum_{i=1}^{n} a_i b_i^*$, $a_i, b_i \in C(Y)$. Assume that $a_i b_i^*(x_0) \neq 0$ for $i = 1, \ldots, r$, while the other terms are zero at x_0. By replacing the various sequences in the A_j by cofinite subsequences, we may assume, by intersecting Y with $\bigcap_{i=1}^{r} \operatorname{coz} a_i b_i$, that the first r terms are non-zero everywhere on Z.

Suppose that for all $x_{m(1),m(2)} \in A_2$ we have $a_j b_j^*(x_{m(1),m(2)}) = 0$, for $j = r+1, \ldots, n$. Then f and $\widehat{h} = \sum_{i=1}^{r} a_i b_i^*$ coincide on $A_0 \cup A_2$. However, $\widehat{h}$ is continuous on a neighbourhood of x_0 which must contain points of A_1. Hence, f can be extended continuously to some points of A_1, which is impossible. Hence, we may assume that $a_{r+1} b_{r+1}^*$, say, is non-zero at some $x_{u,v} \in A_2$.

Consider the subset T of Z consisting of all points whose first two indices are u, v. Then $f|_T$ and T are an example of the claim for $n - 1$. Hence, $f|_T$ requires at least n terms, of which at least $n-1$ are zero at $x_{u,v}$. Since r is at least 1, there are at least $r+1$ terms non-zero at $x_{u,v}$; this give $(n-1) + (r+1) = n + r$ terms. This is impossible.

We now know that $\widehat{g}$ has more than n terms, say s. Finally, suppose, again that r terms of $\widehat{g}$ are non-zero at x_0. If $s - r < n$, we see that, as above, we get some $x_{u,v} \in A_2$ where $\widehat{g}$ is non-zero in $r+1$ terms and zero on fewer than $s - r - 1 < n - 1$. This is a contradiction. Hence, $s - r \geq n$. ∎

3.13. EXAMPLE. For each $n \in \mathbf{N}$, there is an example of the construction of 3.12 in the interval $[0, 1] \subseteq \mathbf{R}$.

PROOF. Let $A_0 = \{0\}$, $A_1 = \{1/n\}_{n>1}$, $A_2 = \{1/n + 1/mn(n+1)\}_{n,m>1}$, $A_3 = \{(1/n + 1/mn(n+1)) + 1/km(m+1)n(n+1)\}_{m,n,k>1}$, and so on. ∎

We now look for instances of the construction in other spaces. For a space X and a subspace Y we define, inductively, a sequence of subspaces as follows: $Y_0 = Y$ and $Y_1 = \mathrm{cl}(Y) - Y$. For $k \geq 1$, once Y_k has been defined, $Y_{k+1} = \mathrm{cl}(Y_k) - Y_k$. We need the following elementary facts.

3.14. LEMMA. *Let Y be a subspace of a topological space X. Consider the sequence of subspaces $\{Y_k\}_{k\in\mathbf{N}}$. Then*

1) For each $k \geq 0$, Y_{2k+1} is a closed subspace of Y_1 and $Y_1 \supseteq Y_3 \supseteq Y_5 \supseteq \cdots$.

2) For each $k \geq 0$, Y_{2k} is a closed subspace of Y and $Y = Y_0 \supseteq Y_2 \supseteq Y_4 \supseteq \cdots$.

3) If, for some $k \geq 0$, Y_k is closed, then Y is a finite boolean combination of closed sets. In fact, if, for some $k > 0$, Y_{2k} is closed, then Y is a union of k locally closed sets, one of which is closed. If, for some $k \geq 0$, Y_{2k+1} is closed, then Y is a union of k locally closed sets.

PROOF. 1) This is clear for $k = 0$. Suppose for some $k \geq 0$ that $Y_1 \supseteq \cdots \supseteq Y_{2k+1}$ and each Y_{2l+1},. $0 \leq l \leq k$, is a closed subset of Y_1. Then

$$\begin{aligned} Y_{2k+3} &= \mathrm{cl}(Y_{2k+2}) - Y_{2k+2} = \mathrm{cl}(Y_{2k+2}) \cap Y_{2k+2}^c \\ &= \mathrm{cl}(\mathrm{cl}(Y_{2k+1}) - Y_{2k+1}) \cap (\mathrm{cl}(Y_{2k+1}) - Y_{2k+1})^c \\ &= \mathrm{cl}(\mathrm{cl}(Y_{2k+1}) \cap Y_{2k+1}^c) \cap (\mathrm{cl}(Y_{2k+1})^c \cup Y_{2k+1}) \,. \end{aligned}$$

The first term is in $\mathrm{cl}(Y_{2k+1})$ and so $Y_{2k+3} \subseteq Y_{2k+1}$, and it is closed in Y_{2k+1} and, hence, also in Y_1.

2) This is similar.

3) We need only apply the identity $A = (\mathrm{cl}(A) - A)^c \cap \mathrm{cl}(A)$ several times. Then

$$\begin{aligned} Y &= Y_1^c \cap \mathrm{cl}(Y) = [Y_2^c \cap \mathrm{cl}(Y_1)]^c \cap \mathrm{cl}(Y) \\ &= [Y_2 \cup \mathrm{cl}(Y_1)^c] \cap \mathrm{cl}(Y) = [Y_2 \cap \mathrm{cl}(Y)] \cup [\mathrm{cl}(Y_1)^c \cap \mathrm{cl}(Y)] \\ &= [(Y_3^c \cap \mathrm{cl}(Y_2)) \cap \mathrm{cl}(Y)] \cup [\mathrm{cl}(Y_1)^c \cap \mathrm{cl}(Y_0)] \\ &= [Y_3^c \cap \mathrm{cl}(Y_2)] \cup [\mathrm{cl}(Y_1)^c \cap \mathrm{cl}(Y)] \text{ , etc.} \end{aligned}$$

∎

We need another technical lemma.

3.15. LEMMA. *Let X be a first countable space and F a countable compact subset. Then F is a zero-set in X.*

PROOF. For each $y \in F$ we take a countable neighbourhood base for y and let $\mathcal{U}$ be the union of all of these. A subset of X is called *good* if it contains F and is the union of finitely many elements of $\mathcal{U}$. For any $x \in X - F$ there is a good subset excluding x; this is because $X - \{x\}$ is a neighbourhood of each of the points of F. Compactness is then used to get a good subset. The good subsets can be indexed by $\mathbf{N}$, say $\{V_n\}_{n\in\mathbf{N}}$, and for each V_n there is, by the compactness of F ([Gillman & Jerison (1960), 3.11(a)]), a non-negative function f_n bounded by $\mathbf{1}$ which is zero on F and 1 on $X - V_n$. Now take $f = \sum_{n\in\mathbf{N}} 2^{-n} f_n$ whose zero-set is F. ∎

3.16. Proposition. *Let Y be a subspace of a first countable space X which is G-embedded. If, for some $k > 0$, Y_{2k} is non-empty, then there is $f \in C^*(Y)$, so that if $f = G(\rho)(\widehat{g})$ then $\widehat{g}$ requires at least $k+1$ terms.*

Proof. We will build a closed subset F of Y which has the properties of the subset Z in Example 3.12. Suppose f is a function as in 3.12. If $f = \widehat{g}|_F$, for some $\widehat{g} \in G(X)$, $\widehat{g}$ will require at least $k+1$ terms because of Claim 2 in 3.12. However, as we shall see, F is C-embedded in Y and so an extension $\tilde{f} \in C^*(Y)$ of f will also require $k+1$ terms. The rest of the proof will be a construction of such a closed subset F and a verification that it is C-embedded in Y.

I. Since Y_{2k} is non-empty, there is $x_0 \in Y_{2k}$. It is in $\mathrm{cl}(Y_{2k-1}) - Y_{2k-1}$ so there is a sequence $\{x_{m(1)}\}_{m(1)\in\mathbf{N}}$ from Y_{2k-1} converging to x_0. By, for example, [Gillman & Jerison (1960), 0.13] we may assume that $\{x_{m(1)}\}_{m(1)\in\mathbf{N}}$, as a subset of X, is discrete. Pick a basis for the neighbourhoods of x_0 which consists of a strictly decreasing sequence of open sets W_n, $n \in \mathbf{N}$. By deleting some of the $x_{m(1)}$ and some of the W_n, we may assume that each $x_{m(1)} \in W_{m(1)} - W_{m(1)+1}$.

This means that there are pairwise disjoint open sets $U_{m(1)}$ with $x_{m(1)} \in U_{m(1)}$. We will choose each of the open sets $U_{m(1)}$ so that $U_{m(1)} \subseteq W_{m(1)}$.

As a next step we note that each $x_{m(1)}$, $m(1) \in \mathbf{N}$, is in $\mathrm{cl}(Y_{2k-2}) - Y_{2k-2}$; thus there is a sequence $\{x_{m(1),m(2)}\}_{m(2)\in\mathbf{N}}$ from Y_{2k-2} converging to $x_{m(1)}$. We make sure that the terms are in $U_{m(1)}$ and follow the same rules as for the construction of the first sequence.

Suppose, more generally, that we have constructed sequences as far as $\{x_{\mathbf{a}}\}_{\mathbf{a}\in\mathbf{N}^l}$, $1 \le l < 2k$, so that $x_{\mathbf{a}} \in Y_{2k-l}$ and that $\{x_{m(1),\dots,m(l)}\}_{m(l)\in\mathbf{N}}$ converges to $x_{m(1),\dots,m(l-1)}$. Moreover, we make the same provisos about the nature of these sequences; i.e., (i) for $\mathbf{b} = m(1),\dots,m(l-1)$, the sequence $\{x_{\mathbf{b},m(l)}\}_{m(l)\in\mathbf{N}}$, is discrete (as a set), (ii) there is a basic set of neighbourhoods of $x_{\mathbf{b}}$, $W_{\mathbf{b}n}$ which is strictly decreasing by inclusion and $x_{\mathbf{b},m(l)} \in W_{\mathbf{b}m(l)} - W_{\mathbf{b}\,m(l)+1}$ and (iii) there are pairwise disjoint open sets $U_{\mathbf{b},m(l)} \subseteq W_{\mathbf{b}m(l)}$ with $x_{\mathbf{b},m(l)} \in U_{\mathbf{b},m(l)}$.

The process stops when $l = 2k$. Let $F = \{x_0\} \cup \{x_{\mathbf{a}} \mid \mathbf{a} \in \mathbf{N}^{2l}, 1 \le l \le k\}$. By Lemma 3.14(2), $F \subseteq Y$ and we must show that F is closed in Y.

The following observation will be needed.

$(*)$ For any $\mathbf{a} \in \mathbf{N}^r$, $1 \le r < 2k$, a sequence $\{v_i\}_{i\in\mathbf{N}}$ with $v_i \in U_{\mathbf{a},i}$ will converge to $x_{\mathbf{a}}$. This is because each $U_{\mathbf{a},i} \subseteq W_{\mathbf{a}i}$.

Now suppose we have a sequence of distinct terms $S = \{x_{\mathbf{a}_i}\}$ from F converging to x. We will show that either $x \notin Y$ or $x \in F$.

We will work by induction on l, $1 \le l < 2k$, by looking at terms of S which lie in the various $U_{\mathbf{a}}$, $\mathbf{a} \in \mathbf{N}^{2k-l}$.

Case $l = 1$: if there are infinitely many terms of S in some $U_{\mathbf{a}}$, $\mathbf{a} \in \mathbf{N}^{2k-1}$, then we have $x = x_{\mathbf{a}}$ since S has infinitely many terms of the sequence $\{x_{\mathbf{a},j}\}$, which, in turn, converges to $x_{\mathbf{a}}$.

Induction Step: Now suppose that for some l, $1 \le l < 2k-1$, that if for some $\mathbf{a} \in \mathbf{N}^r$, with $2k - l \le r < 2k$, there are infinitely many terms of S in $U_{\mathbf{a}}$ then $x = x_{\mathbf{b}}$, where

$\mathbf{b} = \mathbf{a}$ or $\mathbf{b} = \mathbf{a}, \mathbf{c}$, some $\mathbf{c}$.

Take $\mathbf{a} \in \mathbf{N}^{2k-l-1}$ and suppose that infinitely many terms of S lie in $U_{\mathbf{a}}$. There are two situations to explore. Firstly, if for some $\mathbf{c} \in \mathbf{N}^s$, $s \geq 1$, infinitely many terms of S lie in $U_{\mathbf{a},\mathbf{c}}$ then, by the induction hypothesis, $x = x_{\mathbf{b}}$ where $\mathbf{b} = \mathbf{a}, \mathbf{c}$ or $\mathbf{b} = \mathbf{a}, \mathbf{c}, \mathbf{d}$. Secondly, the first case does not occur and then there are only finitely many terms of S in each $U_{\mathbf{a},j}$, $j \in \mathbf{N}$. This means that there are terms of S in infinitely many of the sets $U_{\mathbf{a},j}$ and $(*)$ tells us that S converges to $x_{\mathbf{a}}$.

There is only one remaining situation to explore: no U_j, $j \in \mathbf{N}$, has infinitely many terms of S. Here, the above induction does not apply but we then know that there are infinitely many U_j, $j \in \mathbf{N}$, containing terms of S. Then $(*)$ tells us that S converges to x_0.

Hence, in all cases S converges to some $x_{\mathbf{a}}$ or to x_0. Thus either $x \in F$ or $x \notin Y$, as required.

II. The next step will be to show that $\mathrm{cl}(F)$ is, in fact, compact. Let the space in Example 3.13 be S with closure T found in the interval $[0, 1]$. Then T is a compact space which is in one-to-one correspondence with $\mathrm{cl}(F)$ and the obvious function $T \to \mathrm{cl}(F)$ preserves sequential convergence. It is then continuous ([Willard (1970), Corollary 10.5(c)]), showing that $\mathrm{cl}(F)$ is compact.

III. Since $\mathrm{cl}(F)$ is countable and compact, it is a zero-set in X by Lemma 3.15. Then $F = \mathrm{cl}(F) \cap Y$ is a zero-set in Y. Since it is Lindelöf, it is z-embedded ([Blair & Hager (1974), Theorem 4.1]) and then [*ibid.*, Proposition 4.4] says that F is C-embedded in Y. ∎

The next theorem is a converse to Proposition 3.7 but "normal" can be dropped while "first countable" is added.

3.17. Theorem. *Let X be a first countable space. If a subspace Y is G-embedded then it is a finite boolean combination of closed sets.*

Proof. Suppose, in a general first countable space, that Y is G-embedded. We will show that for some k, $Y_k = \emptyset$ (with sets Y_k as in Lemma 3.14) and, hence, that Y is a finite boolean combination of closed sets. One possibility is that all the Y_k are non-empty; then the subsets $Y = Y_0 \supset Y_2 \supset Y_4 \supset \cdots$ form an infinite chain of proper inclusions. Choose a sequence of points $T = \{x_i\}$ with $x_i \in Y_{2i} - Y_{2i-2}$. The set T contains a countable subset which is discrete ([Gillman & Jerison (1960), 0.13]), say $S = \{x_{i_j}\}$, which may be ordered in the same way as T. There is, then, a family of pairwise disjoint open sets U_{i_j} with $x_{i_j} \in U_{i_j}$.

We proceed as in Proposition 3.16 to construct sets $F_{i_j} \subseteq U_{i_j} \cap Y$ which are C-embedded in $U_{i_j} \cap Y$ and which cannot be expressed as the restriction of any $\widehat{g} \in G(X)$ with fewer than $i_j + 1$ terms. Then we can use Proposition 2.15 (iii) within Y, as follows. The role of the B_n is taken by the F_{i_j}, which are closed in Y, and that of the U_n by $U_{i_j} \cap Y$. Put $W = \bigcup_{j \in \mathbf{N}} F_{i_j}$ and define $f \in C^*(W)$ by having its restriction to F_{i_j} be f_{i_j}. Then Proposition 2.15 says that $f = ab^*|_W$ for some $a, b \in C(Y)$. In turn $a = \widehat{g}|_Y$ and

$b = \widehat{h}|_Y$ for some $\widehat{g}, \widehat{h} \in G(X)$, because $Y \subseteq X$ induces an epi. This means that f is the restriction of some element of $G(X)$ to W. This is impossible.

The other possibility is that for some k, all the subsets Y_{2l}, $l \geq k$, are equal. Since the sequence does not stop, Y_{2k} is infinite. We then pick our sequence from Y_{2k} and proceed as above; since the elements of Y_{2k} are in all the later ones, any such can be used for the construction with as many steps as we like. ∎

3.18. Corollary. *Let Y be a subspace of a first countable perfectly normal space X. Then Y is G-embedded if and only if $Y \in \mathfrak{Z}(X)$.*

Proof. We combine Theorem 3.17 and Proposition 3.7. ∎

D. Comparing epimorphisms in CR and in R/N.

We are now in a position to compare **CR** and **R/N** epimorphisms in some cases. We will show that for a first countable perfectly normal space X, ρ is an **R/N**-epimorphism if and only if it is a **CR**-epimorphism, if and only if Y is locally closed in X.

Consider the space in Example 3.12 with $n = 1$, namely $X = \{x_0\} \cup \{x_i\}_{i\in\mathbf{N}} \cup \{x_{ij}\}_{ij\in\mathbf{N}^2}$ and $Y = \{x_0\} \cup \{x_{ij}\}_{ij\in\mathbf{N}^2}$. We will show that $\rho\colon C(X) \to C(Y)$ is not an **R/N**-epimorphism. To do this we will exhibit a semiprime ring A and ring homomorphisms $\alpha, \beta\colon C(Y) \to A$ with $\alpha\rho = \beta\rho$ but $\alpha \neq \beta$.

Let $B \subseteq \mathbf{R}^{\mathbf{N}^2}$ where $(b_{ij}) \in B$ if there is a neighbourhood U of $x_0 \in X$ and $M > 0$ such that for all $x_{ij} \in U$, $|b_{ij}| < M$. Then B is a subring of the product. Define $I \subseteq B$ as follows: $(b_{ij}) \in I$ if and only if there is a neighbourhood U of x_0 in X so that for all $i \in \mathbf{N}$ with $x_i \in U$, the sequence $\{b_{ij}\}_{j\in\mathbf{N}}$ converges to 0.

3.19. Lemma. *The subset I of B is a semiprime ideal.*

Proof. The subset I is clearly closed under subtraction. Let $(b_{ij}) \in B$ and $(a_{ij}) \in I$. Then there is a neighbourhood U of x_0 in X and $M > 0$ so that for $x_{ij} \in U$, $|b_{ij}| < M$ *and* for all $x_i \in U$ each sequence $\{a_{ij}\}$ converges to 0. Then for all $x_i \in U$, the sequence $\{a_{ij}b_{ij}\}$ also converges to 0. Moreover, if $(a_{ij})^2 \in I$, so is (a_{ij}). ∎

Define $A = B/I$. The homomorphisms $\alpha, \beta\colon C(Y) \to A$ are defined as follows. For $f \in C(Y)$, $\alpha(f) = (u_{ij})+I$ where $u_{ij} = f(x_{i\,2j})$; and $\beta(f) = (v_{ij})+I$ where $v_{ij} = f(x_{i\,2j+1})$.

3.20. Proposition. *With the above notation: (i) α and β are homomorphisms; (ii) $\alpha\rho = \beta\rho$; and (iii) $\alpha \neq \beta$. Hence, $C(X) \to C(Y)$ is not an **R/N**-epimorphism.*

Proof. (i) We need only check that (u_{ij}) and (v_{ij}) are elements of B. However, by the continuity of f at $x_0 \in Y$ there is a neighbourhood U of x_0 in X such that f is bounded on $U \cap Y$. (ii) When $f = \rho(g)$, the sequences $\{f(x_{i\,2j})\}$ and $\{f(x_{i\,2j+1})\}$ both converge to $g(x_i)$. Then $\{u_{ij} - v_{ij}\}_{j\in\mathbf{N}}$ converges to 0, for each i. This shows $(u_{ij}) - (v_{ij}) \in I$ and $\alpha(f) = \beta(f)$. (iii) We need only exhibit $f \in C(Y)$ so that $\alpha(f) \neq \beta(f)$. Take f such that

$f(x_{ij}) = 1/i$, if j is even and $2/i$ if j is odd, and $f(x_0) = 0$. Then $(u_{ij})-(v_{ij}) = (-1/i) \notin I$. ∎

It would be interesting to know how the criteria of Proposition 1.1 fail in this case. Proposition 3.20 will enable us to look at more general situations.

3.21. Theorem. *Let X be a first countable space and Y a subspace of X.*

(i) (Cf. Corollary 2.11) If $\rho: C(X) \to C(Y)$ is an ***R/N****-epimorphism, then Y is locally closed in X.*

(ii) Suppose, in addition, that X is perfectly normal. Then ρ is an ***R/N****-epimorphism if and only if it is a* ***CR****-epimorphism.*

Proof. (i) Assume that Y is not locally closed. Then $\mathrm{cl}(Y) - Y$ is not closed and so there exists $x_0 \in Y$ and a sequence $\{x_i\}$ of distinct points in $\mathrm{cl}(Y) - Y$ converging to x_0. Hence, there is a subset $Z = \{x_0\} \cup \{x_{ij}\}_{ij \in \mathbf{N}^2}$ of Y, constructed as in Example 3.12 (with $n = 1$), whose closure is $\mathrm{cl}(Z) = Z \cup \{x_i\}_{i\in\mathbf{N}}$. Now consider the diagram

$$\begin{array}{ccc} C(X) & \to & C(Y) \\ \downarrow & & \downarrow \\ C(\mathrm{cl}(Z)) & \to & C(Z) \end{array}.$$

As shown in part III of the proof of Proposition 3.16, Z is C-embedded in Y so that $C(Y) \to C(Z)$ is a surjection. We have seen (Proposition 3.20) that $C(\mathrm{cl}(Z)) \to C(Z)$ is not an **R**/**N**-epimorphism. Hence, $C(X) \to C(Z)$ cannot be an **R**/**N**-epimorphism. This shows that $C(X) \to C(Y)$ is not an **R**/**N**-epimorphism.

(ii) If ρ is a **CR**-epimorphism, it is automatically an **R**/**N**-epimorphism. If it is an **R**/**N**-epimorphism, the first part of this theorem shows that Y is locally closed. Then ρ is a **CR**-epimorphism by Corollary 2.12. ∎

4. Remarks and questions.

A. On $\mathbf{1 \times 1}$ zig-zags and P-spaces.

In all the examples above where ρ is a **CR**-epimorphism, the existence of an epimorphism can be demonstrated using a 1×1 zig-zag. In particular, this is true for the subspaces found in Propositions 2.1 and 2.15, and where Corollary 2.12 applies. The question naturally arises whether or not there can be a case where ρ is a **CR**-epimorphism but there is an $f \in C(Y)$ not expressible as a 1×1 zig-zag over $C(X)$. The following result gives information about the case where Y is a P-space and suggests where one could look for examples where bigger zig-zags are needed and where they are not to be found. However, the question has not been answered.

4.1. Proposition. *Let Y be a subspace of a space X, where Y is a P-space.*

(i) If every element of $C(Y)$ satisfies a 1×1 zig-zag over $C(X)$, then Y is z-embedded in X.

(ii) If Y is z-embedded and G-embedded in X then every element of $C(Y)$ can be expressed as a 1×1 zig-zag over $C(X)$; and, hence, ρ is a ***CR****-epimorphism.*

(iii) If Y z-embedded in X, and, moreover, Y is functionally countable (e.g., Y is Lindelöf), then each element of $C(Y)$ satisfies a 1×1 zig-zag over $C(X)$.

Proof. (i) If $f \in C(Y)$ and there are $a, c \in C(Y)$ with $b \in C(X)$ so that $f = a\rho(b)c$ is a 1×1 zig-zag then, since $C(Y)$ is a regular ring, $f = a\rho(b)\rho(b)^*\rho(b)c = \rho(b)^*a\rho(b)\rho(b)c$. Hence, $f\rho(b)^*\rho(d)$, where $d \in C(X)$ is an extension of $a\rho(b)\,\rho(b)c$. It follows that $z(f) = z(bd) \cap Y$.

(ii) It would suffice to have the conclusion for Y in βX, since a 1×1 zig-zag over $C(\beta X)$ restricts to one over $C(X)$. However, X is C^*-embedded in βX; hence, it is both z-embedded and G-embedded. By replacing X by βX, we may assume that X is normal. Then, $\mathrm{cl}(Y)$ is C-embedded in X and so we may assume, in addition to the other hypotheses, that Y is dense in X.

Since $G(\rho)$ is surjective, every $f \in C(Y)$ can be put in the form $f = \sum_{i=1}^{m} \rho(a_i)\rho(b_i)^*$ for some $a_i, b_i \in C(X)$. By Proposition 2.1(ii), it suffices to show that any $f \in C^*(Y)$ can be extended continuously to a cozero-set containing Y. We will show that each $\rho(b_i)^*$ can be so extended, which will give the result. Consider $\rho(b)^*$ for some $b \in C(X)$. Its zero-set and cozero-set are both clopen in Y and so are both zero-sets. Clearly $\mathrm{coz}(b) \cap Y = \mathrm{coz}(\rho(b)^*)$ and there is some $c \in C(X)$ so that $\mathrm{coz}(c) \cap Y = z(\rho(b)^*)$. By density of Y, $\mathrm{coz}(b) \cap \mathrm{coz}(c) = \emptyset$. Then $\rho(b)^*$ can be extended to g on $\mathrm{coz}(b) \cup \mathrm{coz}(c)$ by $g(x) = 1/b(x)$ for $x \in \mathrm{coz}(b)$ and $g(x) = 0$ for $x \in \mathrm{coz}(c)$.

(iii) As in (ii), it suffices to assume that Y is dense in X. If $f \in C(Y)$ then its image is countable, say $\{r_i\}_{i \in \mathbf{N}}$. Then, as in (ii), each $f^{-1}(r_i) = B_i$ is a cozero-set in Y. Write it as $\mathrm{coz}(a_i) \cap Y = B_i$, for some $a_i \in C(X)$. By the density, the $\mathrm{coz}(a_i)$ are disjoint and $\bigcup_{i \in \mathbf{N}} \mathrm{coz}(a_i)$ is a dense cozero-set to which f can be extended (by letting the new function be constantly r_i on $\mathrm{coz}(a_i)$). ■

For an example of a P-subspace Y of X which is functionally countable, but not Lindelöf, see [Levy & Rice (1981), Example 4]. It induces an epimorphism via 1×1 zig-zags.

It would be interesting to find an example with the following properties: A space X and subspace Y such that

(i) Y is an uncountable discrete space not z-embedded in X;

(ii) Y is the intersection of the dense cozero sets of X;

(iii) $X - Y$ is strongly discrete (see [Henriksen, Raphael, & Woods (2001), Proposition 2.9]). In such a situation ρ would be an epimorphism but there would be an element of $C(Y)$ which is not expressible as a 1×1 zig-zag over $C(X)$, and Y would be G-embedded but not z-embedded.

B. On nilpotents in $\mathbf{C(Y)} \otimes_{\mathbf{C(X)}} \mathbf{C(Y)}$.

It is clear from the proof of the spectrum characterization of epimorphisms in **CR** ([Lazard (1968), Proposition IV 1.5]) that if ρ is an $\mathbf{R/N}$-epimorphism then it would be a **CR**-epimorphism, if we knew that $C(Y) \otimes_{C(X)} C(Y)$ had no non-zero nilpotent elements. On the one hand, the fact that all the residue fields here are of characteristic zero and the corresponding extensions are trivial or purely transcendental would suggest that there are no nilpotents in the tensor product. On the other hand, it is easy to find G-embedded subspaces Y of X, where there are elements $f \in C(Y)$ not extendible to X whose squares do extend (for example, in the space of Example 3.12, with $n = 1$). This makes the construction of a nilpotent element seem plausible.

We know that for first countable, perfectly normal spaces X, the two kinds of epimorphisms coincide. The question is still open for more general kinds of spaces. If it turns out that the two kinds of epimorphisms differ for ρ, the relationship between "ρ an $\mathbf{R/N}$-epimorphism" and "Y is G-embedded" still remains to be clarified.

C. On the embedding of $\operatorname{Spec} C(Y)$ **in** $\operatorname{Spec} C(X)$**.**

In [Schwartz (1997)] it is shown that when Y is z-embedded in X, the image of $\operatorname{Spec}\rho$ can be described as $\operatorname{cvx}\tilde{Y}$. There are two ingredients: (1) the injectivity of $\operatorname{Spec}\rho$ on $\operatorname{zSpec} C(Y)$ and (2) that $\operatorname{Spec}\rho$ reflects inclusions in $\operatorname{Spec}\rho\,(\operatorname{zSpec} C(Y))$. The first part has been discussed in detail in Section 3. Is there a version of [Schwartz (1997), Lemma 4.1] if, instead of assuming that Y is z-embedded in X, we only say that $G(\rho)$ is surjective on the set of idempotents of $G(Y)$?

D. Remarks on $H(X)$ and $\mathrm{Q_{cl}}(X)$.

The ring $H(X)$, the epimorphic hull of $C(X)$ in **CR**, was mentioned briefly in Section 3. It is the smallest regular ring between $C(X)$ and the complete ring of quotients, $Q(X)$, and satisfies the universal property of being the largest essential extension which is an epimorphism in **CR** ([Storrer (1968), Definition 8.3]). Schwartz has shown ([Schwartz (2000), Theorem 1.2]) that if a real closed ring A has an epimorphic hull in $\mathbf{POR/N}$ then it coincides with that in $\mathbf{R/N}$ and in **CR**. Moreover, [*ibid.*, Theorem 4.4] shows that $C(X)$ does have an epimorphic hull in $\mathbf{POR/N}$; it is, hence, $H(X)$. We always have $\mathrm{Q_{cl}}(X) \subseteq H(X)$. In [Raphael & Woods (2000), Open question 8.1] it was asked when $\mathrm{Q_{cl}}(X)$ and $H(X)$ are, themselves, of the form $C(Z)$, for a space Z. In Theorem 4.2, we will answer this question for some spaces. These results are closely related to our theme because in each case discussed the subspace gX of X induces an epimorphism of rings. Recall (e.g., [Raphael & Woods (2000)]) that gX denotes the intersection of the dense cozero sets of X.

A space is called a *quasi-F space* if each dense cozero-set is C^*-embedded. Much is known about these spaces (for example, [Dashiell, Hager & Henriksen (1980), Theorem 5.1] and [Schwartz, 1989b, Theorem 6.2]). By [Gillman & Jerison (1960), 14N], basically disconnected spaces are F-spaces and therefore quasi-F spaces. In particular, extremally disconnected spaces are quasi-F spaces.

4.2. Theorem. *(i) Let X be a quasi-F space. Then* $\mathrm{Q_{cl}}(X)$ *is a ring of continuous functions if and only if gX is dense and C^*-embedded in X.*

(ii) Let X be basically disconnected. Then the epimorphic hull, $H(X)$, of $C(X)$ is a ring of continuous functions if and only if its set of P-points is dense and C^*-embedded in X.

(iii) Let X be an extremally disconnected space in which gX is of non-measurable cardinality. Then $H(X)$ is a ring of continuous functions if and only if the set of isolated points of X is dense and C^*-embedded.

Proof. i) Suppose $\mathrm{Q_{cl}}(X)$ is a ring of continuous functions. By Theorem 7.2 of [Raphael & Woods (2000)], gX is dense in X, $\mathrm{Q_{cl}}(X) = C(gX)$ and every function in $C(gX)$ extends to a dense cozero set of X. Let $f \in C^*(gX)$. It extends to a bounded function on a dense cozero set of X and from it to a bounded function on X. Thus $C^*(X) \to C^*(gX)$ is surjective and gX is C^*-embedded in X.

Conversely let gX be dense and C^*-embedded in X. By [*ibid.*, Theorem 7.2] again, it suffices to show that each $f \in C(gX)$ extends to a dense cozero set of X. As usual, let $f_1 = f/(f^2+1)$ and $f_2 = 1/(f^2+1)$. Let them extend to F_1 and F_2 in $C(X)$, respectively. Then f extends to F_1/F_2 on $\mathrm{coz}(F_2)$.

(ii) This becomes an instance of (i) if one notes that when X is basically disconnected gX is a P-space ([Raphael & Woods (2000), Corollary 4.12]), and that $\mathrm{Q_{cl}}(X)$ and $H(X)$ coincide because in a basically disconnected space the annihilator on an element is a principal ideal. This says ([Raphael (1972), Corollary 1.6]) that $\mathrm{Q_{cl}}(X)$ is regular and, hence, equals $H(X)$).

(iii) This is also an instance of (i), but one needs [Gillman & Jerison (1960), 12H]. When X is extremally disconnected then gX is, as well, if it is dense in X. It is a P-space because X is basically disconnected. Thus it is the set of isolated points of X, if gX is of non-measurable cardinality. ∎

Since almost-P spaces are quasi-F spaces [Dashiell, Hager & Henriksen (1980), p. 681], there are instances where case (i) holds but case (ii) fails. An example is the one point compactification of an uncountable discrete space.

References

R.L. Blair and A.W. Hager (1974), *Extensions of zero-sets and real-valued functions*, Math. Zeit. **136**, 41-52.

W.D. Burgess and W. Stephenson (1976), *Pierce sheaves of non-commutative rings*, Comm. Algebra, **4**, 51-75.

W.W. Comfort and S. Garcia-Ferreira (1996), *Resolvability: a selective survey and some new results*, Topology Appl., **74**, 149-162.

F. Dashiell, A. Hager and M. Henriksen (1980), *Order-Cauchy completions of ring and vector lattices of continuous functions.* Canad. J. Math. **32**, 657-685.

R. Engelking (1968), *Outline of General Topology.* North-Holland, Amsterdam.

N. Fine, L. Gillman and J. Lambek (1965), *Rings of Quotients of Rings of Functions.* McGill University Press, Montreal.

L. Gillman and M. Jerison (1960), *Rings of Continuous Functions.* D. Van Nostrand, Princeton.

M. Henriksen, R. Raphael and R.G. Woods (2002), *A minimal regular ring extension of* $C(X)$, Fund. Math. **172**, 1-17.

D. Lazard (1968), *Autour de la platitude*, Bull. Soc. Math. France **97**, 6-127.

R. Levy and M.D. Rice (1981), *Normal P-spaces and the G_δ-topology*, Coll. Math. **44**, 227-240.

L. Lipshitz (1977), *The real closure of a commutative regular ring*, Fund. Math. **94**, 173-176.

S. Mac Lane (1998), *Categories for the Working Mathematician.* Second Edition, Springer-Verlag, New York, Heidelberg, Berlin.

P. Mazet (1968), *Générateurs, relations et épimorphismes d'anneaux*, C. R. Acad. Sc. Paris, **t. 266**, 309-311.

R. Montgomery (1973), *The mapping of prime z-ideals*, Symp. Math. **17**, 113-124.

J.-P. Olivier (1968), *Anneaux absolument plats universels et épimorphismes*, C.R. Acad. Sc. Paris, **t. 266**, 317-318.

R.S. Pierce, (1967). *Modules over Commutative Regular Rings.* Memoirs Amer. Math. Soc., vol. 70.

J.R. Porter and R.G. Woods (1988), *Extensions and Absolutes of Hausdorff Spaces.* Springer-Verlag, New York.

R. Raphael (1972), *Injective rings*, Comm. Algebra **1**, 403-414.

R. Raphael and R.G. Woods (2000), *The epimorphic hull of* $C(X)$, Topology Appl. **105**, 65-88.

N. Schwartz, (1989a), *The basic theory of real closed spaces*, Memoirs of the Amer. Math. Soc., **397**.

N. Schwartz (1989b), *Epimorphisms of f-rings*, in: Ordered Algebraic Structures (Ed. J. Martinez), Kluwer, Dordrecht, 187-195.

N. Schwartz (1997), *Rings of continuous functions as real closed rings*, in: Ordered Algebraic Structures (eds. W.C. Holland and J. Martinez), Kluwer, Dordecht, 277-313.

N. Schwartz (2000), *Epimorphic extensions and Prüfer extensions of partially ordered rings*, Manuscripta Math. **102**, 347-381.

N. Schwartz and J. Madden (1999) *Semi-algebraic Function Rings and Reflectors of Partially Ordered Rings.* Lecture Notes in Mathematics, vol. 1712, Springer-Verlag, Berlin.

H. Storrer (1968), *Epimorphismen von kommutativen Ringen*, Comm. Math. Helv. **43**, 378-401.

R. Wiegand (1971), *Modules over universal regular rings*, Pacific J. Math. **39**, 807-819.

S. Willard (1970), *General Topology.* Addison-Wesley, Reading, Mass.

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