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PARTIAL TOPOSES

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Abstract. We introduce various notions of *partial topos*, i.e. "topos without terminal object". The strongest one, called *local topos*, is motivated by the key examples of finite trees and sheaves with compact support. Local toposes satisfy all the usual exactness properties of toposes but are neither cartesian closed nor have a subobject classifier. Examples for the weaker notions are local homeomorphisms and discrete fibrations. Finally, for partial toposes with supports we show how they can be completed to toposes via an inverse limit construction.

1. Partial Toposes

For a category $\mathbf{B}$ with pullbacks its *fundamental fibration* $P_{\mathbf{B}} = \partial_1 : \mathbf{B}^{\mathbf{2}} \to \mathbf{B}$ is *well–powered* [Bénabou, 1980] iff for every $a : A \to I$ in $\mathbf{B}/I$ there is an object $p(a) : P(a) \to I$ in $\mathbf{B}/I$ together with a subobject $\ni_a : S_a \rightarrowtail P(a)\times_I A$ such that for every $b : B \to I$ and subobject m of $B\times_I A$ there is a *unique* map $\chi : b \to p(a)$ with $m \cong (\chi\times_I A)^*\ni_a$ as in the diagram

$$\begin{tikzcd}
S \arrow[rr] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner" very near start] & & S_a \arrow[d, tail, "\ni_a"] & \\
B\times_I A \arrow[rr, "\chi\times_I A"] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & & P(a)\times_I A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & A \arrow[d, "a"] \\
B \arrow[rr, dashed, "\chi"'] & & P(a) \arrow[r, "p(a)"'] & I
\end{tikzcd}$$

Thus $\mathbf{B}$ has pullbacks and its fundamental fibration $P_{\mathbf{B}}$ is well–powered if and only if every slice of $\mathbf{B}$ is an elementary topos. Accordingly, one may characterise elementary toposes as categories $\mathbf{B}$ with finite limits whose fundamental fibration $P_{\mathbf{B}}$ is well–powered.

General experience with the theory of fibred categories tells us that for bases of fibrations it is fairly irrelevant whether they have terminal objects, however, it is essential that they do have pullbacks. Thus, one may be inclined to define a *partial topos* as a category $\mathbf{E}$ all of whose slices are elementary toposes. This definition, however, is too general as it encompasses *all* groupoids as every slice of a groupoid is equivalent to the

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terminal category $\mathbf{1}$ and, therefore, is a trivial topos. Thus, there arises the question which conditions are missing in order to rule out such degenerate examples.

First observe that a category $\mathbf{B}$ has pullbacks iff for every morphism $\alpha : J \to I$ in $\mathbf{B}$ the functor $\Sigma_\alpha : \mathbf{B}/J \to \mathbf{B}/I : \beta \mapsto \alpha \circ \beta$ has a right adjoint α^*, i.e., as shown in [Bénabou, 1980], that change of base for fibrations along Σ_α preserves smallness of fibrations. Thus, $\mathbf{B}$ has pullbacks iff for all morphisms α in $\mathbf{B}$ change of base along Σ_α preserves[1] *all good properties of fibrations* as the Σ_α do preserve pullbacks anyway. From this point of view it appears as most natural to further require that for all objects $I \in \mathbf{B}$ the change of base along $\Sigma_I = \partial_0 : \mathbf{B}/I \to \mathbf{B}$ (i.e. localisation to I) preserves all good properties of fibrations. As the Σ_I do preserve pullbacks anyway this requirement boils down to the condition that all Σ_I have a right adjoint $I^* : \mathbf{B} \to \mathbf{B}/I$, i.e. that $\mathbf{B}$ has binary products[2].

These considerations should motivate the relevance of the following notion.

1.1. Definition. *A* partial cartesian category *is a category with pullbacks and binary products, i.e. a category with finite* non–empty *limits.*

All partial cartesian groupoids are trivial for the following reason.

1.2. Lemma. *Every groupoid* $\mathbf{G}$ *with binary products is trivial.*

Proof. First of all notice that a groupoid $\mathbf{G}$ with binary products is strongly connected in the sense that for all $X, Y \in \mathbf{G}$ there is a morphism from X to Y as we have $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ and, therefore, $\pi_2 \circ \pi_1^{-1} : X \to Y$. Moreover, $\mathbf{G}$ is posetal as for morphisms $f, g : X \to Y$ we have $\langle f, g\rangle = \delta_Y \circ h = \langle h, h\rangle$ for some $h : X \to Y$ because $\delta_Y = \langle \mathrm{id}_Y, \mathrm{id}_Y\rangle$ is an isomorphism. ∎

Notice that we could have defined the notion of "partial cartesian category" equivalently as a category $\mathbf{B}$ with binary products all of whose slices are cartesian as the latter requirement is equivalent to the existence of all pullbacks. Notice that for all $\alpha : J \to I$ in $\mathbf{B}$ the pullback functor $\alpha^* : \mathbf{B}/I \to \mathbf{B}/J$ preserves finite limits. Moreover, if $\mathbf{B}/I$ and $\mathbf{B}/J$ are cartesian closed or toposes then $\alpha^* : \mathbf{B}/I \to \mathbf{B}/J$ preserves exponentials and subobject classifiers, i.e. all structure under consideration.

This suggests the following notions of "partial locally cartesian closed category" and "partial topos".

1.3. Definition. *A* partial locally cartesian closed category *is a category with binary products all of whose slices are cartesian closed. A* partial topos *is a category with binary products all of whose slices are toposes.*

[1] It has been shown in [Bénabou, 1980] that change of base along a functor F between categories with pullbacks preserves *all good properties of fibrations* if and only if F preserves pullbacks and has a right adjoint.

[2] For objects I and J in $\mathbf{B}$ their product cone is given by $I^*J : P \to I$ and the counit $\varepsilon_J : P \to J$ of $\Sigma_I \dashv I^*$ at J.

It is a straightforward exercise to show that $\mathbf{B}$ is partial cartesian closed if and only if $\mathbf{B}$ has binary products and $P_{\mathbf{B}} = \partial_1 : \mathbf{B^2} \to \mathbf{B}$ is *locally small.*

Partial toposes have conditional colimits in the following sense.

1.4. Lemma. *In a partial topos every finite diagram has a colimit whenever it has a cocone.*

Proof. Let $\mathbf{E}$ be a partial topos. A cocone for a finite diagram $D : \mathbb{D} \to \mathbf{E}$ corresponds uniquely to a diagram $D_A : \mathbb{D} \to \mathbf{E}/A$ which has a colimit in the topos $\mathbf{E}/A$. Moreover, the functor $\Sigma_A = \partial_0 : \mathbf{E}/A \to \mathbf{E}$ preserves all colimits as it has a right adjoint. Thus, applying Σ_A to a colimiting cocone for D_A gives rise to a colimiting cocone for D as required. ∎

Thus, in particular, for partial toposes the situation is as follows.

1.5. Corollary. *For a partial topos $\mathbf{E}$ we have that*

(1) *$\mathbf{E}$ has an initial object if and only if $\mathbf{E}$ is nonempty.*

(2) *The sum $X + Y$ exists in $\mathbf{E}$ if and only if there are morphism $X \to Z$ and $Y \to Z$ in $\mathbf{E}$ for some object Z in $\mathbf{E}$.*

(3) *A parallel pair of maps $f, g : X \to Y$ in $\mathbf{E}$ has a coequaliser if and only if $h \circ f = h \circ g$ for some $h : Y \to Z$ in $\mathbf{E}$.*

Proof. Obvious from Lemma 1.4. ∎

Next we discuss a few examples of properly partial toposes, i.e. partial toposes without a terminal object[3].

1.6. Example. For a topos $\mathbf{E}$ and a set $\mathcal{I}$ of subterminals, i.e. subobjects of $1_{\mathbf{E}}$ let $\mathbf{E}_{/\mathcal{I}}$ be the full subcategory of $\mathbf{E}$ on those objects X for which there is a morphism from X to some $U \in \mathcal{I}$. One easily checks that $\mathbf{E}_{/\mathcal{I}}$ is always a partial topos and that it is a topos iff $\mathcal{I}$ has a greatest element. The next three examples are instances of this quite general scheme.

1.7. Example. Let FinSet be the category of finite sets and arbitrary maps. The category of *finitely branching trees* is given by $\mathsf{FinSet}^{\omega^{\mathrm{op}}}$, i.e. presheaves on ω with values in FinSet. It is a topos as ω/n is finite for all $n \in \omega$. Let $\mathcal{I}$ be the set of all proper subobjects of the terminal object, i.e. all presheaves isomorphic to $\partial_0 : \omega/n \to \omega$ for some $n \in \omega$. Then $\mathbf{T} = \mathsf{FinSet}^{\omega^{\mathrm{op}}}_{/\mathcal{I}}$, the category of *finite trees*, is a properly partial topos as $\mathcal{I}$ does not have a greatest element.

[3]Obviously, a partial topos is a topos iff it has a terminal object.

1.8. Example. The category of **Set**–valued presheaves over the large poset Ord of ordinals is a topos as Ord/α is a set for all $\alpha \in \mathsf{Ord}$. Let Wrk be the full subcategory on those presheaves A whose support is a set, i.e. for which there is an $\alpha \in \mathsf{Ord}$ such that $A(\beta) = \emptyset$ for all $\beta > \alpha$.

1.9. Example. Let X be a locally compact Hausdorff space and $\mathcal{I}$ the set of subterminals in $\mathsf{Sh}(X)$ consisting of the relatively compact open subsets of X, i.e. those open $U \subseteq X$ whose closure $\overline{U}$ is compact in X. Then $\mathsf{Sh}_c(X) = \mathsf{Sh}(X)_{/\mathcal{I}}$, the category of sheaves on X "with compact support", is a partial topos which is properly partial iff X is not compact.

1.10. Example. Let **LH** be the category of spaces and local homeomorphism. It can be shown that **LH** has binary products (see [Selinger, 1994]). Thus, **LH** is a partial topos as for every space X the slice of **LH** over X is equivalent to the topos $\mathsf{Sh}(X)$. It follows from Lemma 2.2 below that **LH** does not have a terminal object.

1.11. Example. Let **dF** be the category of small categories and discrete fibrations as morphisms. It can be shown that **dF** has binary products (the proof is a variation of Selinger's proof in [Selinger, 1994] that **LH** has binary products). Thus, **dF** is a partial topos as for every small category $\mathbb{C}$ the slice of **dF** over $\mathbb{C}$ is equivalent to the presheaf topos $\widehat{\mathbb{C}} = \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$. It follows from Lemma 2.2 below that **dF** does not have a terminal object.

We conclude this section by showing that partial cartesian categories cannot have exponentials or a subobject classifier unless they have already a terminal object, i.e. are properly cartesian.

1.12. Lemma. *Let* $\mathbf{B}$ *be a partial cartesian category. If for some object* A *in* $\mathbf{B}$ *the exponential* A^A *exists in* $\mathbf{B}$ *then* $\mathbf{B}$ *has a terminal object, namely the equalizer of the identity on* A^A *and the morphism* $r : A^A \to A^A$ *defined as the exponential transpose of* $\pi_2 : A^A \times A \to A$.

Proof. Let $e : T \rightarrowtail A^A$ be an equalizer of id_{A^A} and r. Observe that a map $f : B \to A^A$ equalizes id_{A^A} and r iff $\mathsf{ev} \circ (f \times A) = \mathsf{ev} \circ (r \times A) \circ (f \times A)$, i.e. iff $\mathsf{ev} \circ (f \times A) = \pi_2 \circ (f \times A) = \pi_2$. Thus, for every object B in $\mathbf{B}$ there exists precisely one map from B to A^A equalizing id_{A^A} and r. Accordingly, for every object B in $\mathbf{B}$ there exists precisely one map from B to T, i.e. T is terminal in $\mathbf{B}$ as claimed. ∎

1.13. Lemma. *Let* $\mathbf{B}$ *be a partial cartesian category with a subobject classifier* $\mathsf{t} : T \rightarrowtail \Omega$. *Then* T *is terminal in* $\mathbf{B}$.

PROOF. For every object A of $\mathbf{B}$ let $\top_A : A \to \Omega$ be the classifier of id_A. Notice that for $f : B \to A$ we have $\top_A \circ f = \top_B$. Let $p : \Omega \to T$ be the unique map with $\mathsf{t} \circ p = \top_\Omega$. We have $\mathsf{t} \circ p \circ \mathsf{t} = \top_\Omega \circ \mathsf{t} = \top_T = \mathsf{t}$ from which it follows that $p \circ \mathsf{t} = \mathrm{id}_T$. Thus, the monomorphism t is an equalizer for id_Ω and $\top_\Omega$. Obviously, a map $f : A \to \Omega$ equalizes id_Ω and $\top_\Omega$ iff $f = \top_\Omega \circ f = \top_A$. Thus, for every object A of $\mathbf{B}$ there exists precisely one map from A to T, i.e. T is terminal in $\mathbf{B}$ as claimed. ∎

2. Partial Toposes with Supports

Now we will discuss some further property of partial toposes satisfied by all the examples above with the exception of Examples 1.10 and 1.11.

2.1. DEFINITION. *Let* $\mathbf{B}$ *be a category. An object* $U \in \mathbf{B}$ *is called* subterminal *if for all objects* $X \in \mathbf{B}$ *there exists at most one morphism in* $\mathbf{B}$ *from* X *to* U*. We write* $\mathsf{st}(\mathbf{B})$ *for the full subcategory of* $\mathbf{B}$ *on subterminal objects of* $\mathbf{B}$ *and say that* $\mathbf{B}$ *is a* category with supports *or* has supports *iff the inclusion* $i : \mathsf{st}(\mathbf{B}) \hookrightarrow \mathbf{B}$ *has a left adjoint* supp*.*

Obviously, the category $\mathsf{st}(\mathbf{B})$ is posetal as for subterminals U and V there is at most one arrow from U to V. Accordingly, we write $U \leq V$ iff there is an arrow from U to V. Notice, moreover, that in a category with a terminal object the subterminals in the sense of the previous definition coincide with the subobjects of the terminal object.

Obviously, for $U \in \mathsf{st}(\mathbf{B})$ it leads to no confusion when identifying $\mathbf{B}/U$ with the full subcategory of $\mathbf{B}$ on those objects X for which there is a morphism $X \to U$ in $\mathbf{B}$. Hence, in the sequel we employ this harmless identification whenever convenient.

For partial cartesian categories $\mathbf{B}$ with supports Lemmas 1.12 and 1.13 can be shown even more easily as follows. If A^A exists in $\mathbf{B}$ then for all objects B in $\mathbf{B}$ there is a map from B to A^A (the exponential transpose of $\pi_2 : B \times A \to A$) and accordingly $\mathsf{supp}(B) \leq \mathsf{supp}(A^A)$ from which it follows that $\mathsf{supp}(A^A)$ is terminal in $\mathbf{B}$. If there is a subobject classifier $\mathsf{t} : T \rightarrowtail \Omega$ in $\mathbf{B}$ then for all objects B in $\mathbf{B}$ there is a map from B to Ω (the classifier of id_B) and accordingly $\mathsf{supp}(B) \leq \mathsf{supp}(\Omega)$ from which it follows that $\mathsf{supp}(\Omega)$ is terminal in $\mathbf{B}$.

The next lemma shows that not every partial topos has supports.

2.2. LEMMA. *Both in* $\mathbf{LH}$ *and* $\mathbf{dF}$ *there are objects* X *such that there is no map from* X *to a subterminal object* U*.*

PROOF. First notice that up to isomorphism the monos in $\mathbf{LH}$ are the open subspace inclusions. Let $\nabla(2)$ be the codiscrete space with 2 elements. The space $\nabla(2)$ cannot be subterminal as it admits endomaps different from $\mathrm{id}_{\nabla(2)}$. As every local homeomorphism $h : \nabla(2) \to X$ must have discrete fibres it follows that h is one–to–one, i.e. monic and, therefore, isomorphic to a subspace inclusion. Thus, if there were a map $h : \nabla(2) \to U$

with U subterminal then $\nabla(2)$ were subterminal, too. Thus, there is no map from $\nabla(2)$ to a subterminal object.

First notice that up to isomorphism the monos in **dF** are inclusions of sieves. Let $\nabla(2)$ be the codiscrete category with 2 elements. The category $\nabla(2)$ cannot be subterminal as it admits endomaps different from $\mathrm{id}_{\nabla(2)}$. As every discrete fibration $P : \nabla(2) \to X$ must have discrete fibres it follows that P is isomorphic to a sieve inclusion and, therefore, a monomorphism. Now if $P : \nabla(2) \to \mathbb{B}$ were a discrete fibration with $\mathbb{B}$ subterminal then $\nabla(2)$ were subterminal, too. Thus, there is no map from $\nabla(2)$ to a subterminal object. ∎

Obviously, from Lemma 2.2 it follows that **LH** and **dF** have neither supports nor a terminal object.

Notice that already for the space $\mathbb{R}$ of real numbers there cannot exist a reflection map $\eta_{\mathbb{R}}$ from $\mathbb{R}$ to a subterminal U. The reason is that for every $c \in \mathbb{R}$ the map $f(x) = c + x$ is an automorphism of $\mathbb{R}$ from which it follows that $\eta_{\mathbb{R}} : \mathbb{R} \to U$ has to be constant and, therefore, cannot be a local homeomorphism as it has a non-discrete fibre, namely $\mathbb{R}$.

There is also another argument showing that **dF** is not a category with supports. First observe that for a subterminal object $\mathbb{B}$ in **dF** it holds that objects $I, J \in \mathbb{B}$ are isomorphic whenever $\mathbb{B}/I \cong \mathbb{B}/J$. Let $\mathbb{C}$ be the category

$$A \xrightarrow{\alpha} T \xleftarrow{\beta} B$$

where the objects A, B and T are all different. We show that for every discrete fibration $P : \mathbb{C} \to \mathbb{B}$ the category $\mathbb{B}$ cannot be subterminal. The objects $P(A)$ and $P(B)$ are not isomorphic as otherwise there were a map from A to B in $\mathbb{C}$. But on the other hand we have

$$\mathbb{B}/P(A) \cong \mathbb{C}/A \cong \mathbf{1} \cong \mathbb{C}/B \cong \mathbb{B}/P(B)$$

from which it follows by the above observation that $P(A)$ and $P(B)$ are isomorphic in $\mathbb{B}$. Thus, the category $\mathbb{B}$ cannot be subterminal in **dF** and, accordingly, in **dF** there is no map from $\mathbb{C}$ to a subterminal object. From this it follows that **dF** has neither supports nor a terminal object.

Next we show how under the assumption of having supports partial cartesian categories can be characterised more simply.

2.3. LEMMA. *Let* $\mathbf{B}$ *be a category with supports and pullbacks. Then* $\mathbf{B}$ *has binary products if and only if* $\mathsf{st}(\mathbf{B})$ *has binary infima.*

PROOF. Assume that $\mathbf{B}$ has supports and pullbacks.

If $\mathbf{B}$ has binary products then $\mathsf{st}(\mathbf{B})$ has binary infima as subterminals are closed under binary products.

On the other hand if $\mathsf{st}(\mathbf{B})$ has binary infima then the binary product of A and B is constructed as follows

$$\begin{tikzcd}
A \times B \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & B' \arrow[r, tail, "m_2"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d] \\
A' \arrow[r] \arrow[d, tail, "m_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \cap V \arrow[r, tail] \arrow[d, tail] & V \\
A \arrow[r] & U &
\end{tikzcd}$$

where $U = \mathsf{supp}(A)$, $V = \mathsf{supp}(B)$ and $U \cap V$ is the infimum of U and V in $\mathsf{st}(\mathbf{B})$. The projections π_i are given by $m_i \circ p_i$ as indicated in the diagram. ∎

We have already observed that most examples of partial toposes satisfy the additional property of having supports. In the Examples 1.7–1.9 it holds moreover that the subcategory of subterminals is *filtered* in the sense that for all subterminals U and V there is a subterminal W with $U, V \leq W$.

Next we show that categories $\mathbf{B}$ with supports and pullbacks have binary products whenever $\mathsf{st}(\mathbf{B})$ is filtered.

2.4. Lemma. *Let $\mathbf{B}$ be a category with pullbacks and supports where $\mathsf{st}(\mathbf{B})$ is filtered, i.e. for all $U, V \in \mathsf{st}(\mathbf{B})$ there is a $W \in \mathsf{st}(\mathbf{B})$ with $U, V \leq W$. Then $\mathbf{B}$ has binary products.*

Proof. By Lemma 2.3 it suffices to show that $\mathsf{st}(\mathbf{B})$ has binary infima. Suppose U and V are in $\mathsf{st}(\mathbf{B})$. As $\mathsf{st}(\mathbf{B})$ is filtered there is a W in $\mathsf{st}(\mathbf{B})$ with $U, V \leq W$. Then the infimum of U and V is given by the pullback

$$\begin{tikzcd}
U \cap V \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & V \arrow[d] \\
U \arrow[r] & W
\end{tikzcd}$$

The object $U \cap V$ is subterminal as for every object X there is at most one arrow from X to $U \cap V$ (because there is at most one arrow from X to U and at most one arrow from X to V). ∎

In the light of the previously considered examples the following more restrictive notion of partial topos seems to be also the more useful one.

2.5. Definition. *A category $\mathbf{E}$ is called a* local topos *iff the following three conditions hold*

(1) *every slice of $\mathbf{E}$ is a topos*

(2) *the inclusion* $i : \mathsf{st}(\mathbf{E}) \to \mathbf{E}$ *has a left adjoint* supp

(3) *the category* $\mathsf{st}(\mathbf{E})$ *is* filtered*, i.e. for all* $U, V \in \mathsf{st}(\mathbf{B})$ *there is a* $W \in \mathsf{st}(\mathbf{B})$ *with* $U, V \leq W$.

Notice that for partial toposes $\mathbf{E}_{/\mathcal{I}}$ as in Example 1.6 their categories of subterminals are filtered if and only if $\mathcal{I}$ is filtered. This, however, need not be the case as e.g. if $\mathbf{E} = \mathbf{Set}^I$ for some infinite set I and $\mathcal{I}$ is given by all subsets of I having at most n elements (for a finite cardinal n fixed in advance).

We conclude this section by observing that most of the exactness properties of toposes extend to local toposes.

From Lemma 1.4 it follows that finite colimits exists (as filtered supports guarantee that every finite diagram has a cocone). Moreover, finite colimits are preserved by pullbacks as every slice is a topos and in toposes colimits are preserved by pullbacks. For the same reason a morphism is epic iff it is a coequaliser (of its kernel pair) and monos are preserved by pushouts along arbitrary morphism and such pushout squares are also pullback squares. Furthermore, all local toposes are extensive categories with pullbacks because they have finite sums and all its slices are extensive as they are toposes. Moreover, all properties of internal categories, functors and distributors (as studied in [Johnstone, 1977]) are valid also for local toposes. Finally, notice that all partial toposes are locally cartesian closed, i.e. all pullback functors f^* have right adjoints Π_f satisfying the (Beck-) Chevalley condition. Thus, fibred over itself every partial topos has internal products (see [Bénabou, 1980, Streicher, 2003]).

3. Topos Completion of Partial Toposes with Supports

Now we will show how to complete a partial topos $\mathbf{E}$ with supports to a topos $\mathbf{E}_\infty$. For the rest of this section all partial toposes are assumed to have supports.

Notice that for every map $i : U \to V$ between subterminals for the induced essential geometric morphism

$$\Sigma_i \dashv i^* \dashv \Pi_i : \mathbf{E}/U \to \mathbf{E}/V$$

it holds that i^* is logical and its adjoints are full and faithful.

We write $\mathbf{E}/(-)$ for the contravariant (pseudo-)functor from $\mathsf{st}(\mathbf{E})$ to the (2-)category $\mathbf{Log}$ of toposes and logical morphisms sending U to $\mathbf{E}/U$ and $i : U \to V$ to the logical functor $i^* : \mathbf{E}/V \to \mathbf{E}/U$.

3.1. Theorem. *Let* $\mathbf{E}$ *be a partial topos with supports and*

$$\mathbf{E}_\infty = \mathsf{Lim}_{U \in \mathsf{st}(\mathbf{E})}\, \mathbf{E}/U$$

the inverse limit of $\mathbf{E}/(-) : \mathsf{st}(\mathbf{E})^{\mathrm{op}} \to \mathbf{Log}$ *called* topos completion *of* $\mathbf{E}$.

Then for every $U \in \mathsf{st}(\mathbf{E})$ *the projection* $\pi_U : \mathbf{E}_\infty \to \mathbf{E}/U$ *is equivalent to* $U^* : \mathbf{E}_\infty \to \mathbf{E}/U$ *and has full and faithful left and right adjoints* Σ_U *and* Π_U*, respectively.*

Moreover, $\mathsf{st}(\mathbf{E})$ *can be embedded into* $\mathsf{st}(\mathbf{E}_\infty)$ *by sending* $U \in \mathsf{st}(\mathbf{E})$ *to* $\iota(U) = (U \cap V)_{V \in \mathsf{st}(\mathbf{E})}$. *Then* $\mathbf{E}$ *can be recovered up to equivalence from* $\mathbf{E}_\infty$ *as* $\mathbf{E}_{\infty/\mathsf{st}(\mathbf{E})}$ *where* $\mathsf{st}(\mathbf{E})$ *is considered as included into* $\mathsf{st}(\mathbf{E}_\infty)$ *via* ι.

PROOF. We first give a more detailed description of the inverse limit $\mathbf{E}_\infty = \mathsf{Lim}_{U \in \mathsf{st}(\mathbf{E})} \mathbf{E}/U$. Its objects are cartesian sections of the fibration corresponding to the pseudo-functor $\mathbf{E}/(-) : \mathsf{st}(\mathbf{E})^{\mathsf{op}} \to \mathbf{Log}$. More explicitly, an object of $\mathbf{E}_\infty$ is a function A picking an object $A_U \in \mathbf{E}/U$ for every $U \in \mathsf{st}(\mathbf{E})$ and a (mono)morphism $A_{V,U} : A_V \to A_U$ with

$$\begin{tikzcd} A_V \arrow[r, tail, "A_{V,U}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A_U \arrow[d] \\ V \arrow[r, tail] & U \end{tikzcd}$$

for every pair $V \leq U$ in $\mathsf{st}(\mathbf{E})$ in a functorial way, i.e. satisfying $A_{U,U} = \mathrm{id}_{A_U}$ and $A_{V,U} \circ A_{W,V} = A_{W,U}$ whenever $W \leq V \leq U$. For $A, B \in \mathbf{E}_\infty$ a morphism from A to B is given by a family $(f_U : A_U \to B_U)_{U \in \mathsf{st}(\mathbf{E})}$ such that

$$\begin{tikzcd} A_V \arrow[r, tail, "A_{V,U}"] \arrow[d, "f_V"'] \arrow[dr, phantom, "\lrcorner"] & A_U \arrow[d, "f_U"] \\ B_V \arrow[r, tail, "B_{V,U}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & B_U \arrow[d] \\ V \arrow[r, tail] & U \end{tikzcd}$$

whenever $V \leq U$.

It is evident that $\mathbf{E}_\infty$ has finite limits which are constructed componentwise. A (canonical) terminal object 1 in $\mathbf{E}_\infty$ for example is given by $1_U = U$ and $1_{V,U}$ the unique morphism from V to U in $\mathbf{E}$.

For $A, B \in \mathbf{E}_\infty$ their exponential B^A is constructed by putting $(B^A)_U = {B_U}^{A_U}$ (exponential in $\mathbf{E}/U$) and for $V \leq U$ the map $(B^A)_{V,U} : {B_V}^{A_V} \to {B_U}^{A_U}$ is given as the transpose of $B_{V,U} \circ \widetilde{\mathsf{ev}}$ where $\widetilde{\mathsf{ev}} : {B_V}^{A_V} \times A_U \to B_V$ is the unique map with $\widetilde{\mathsf{ev}} \circ ({B_V}^{A_V} \times A_{V,U}) = \mathsf{ev} : {B_V}^{A_V} \times A_V \to B_V$ as in the diagram

$$\begin{tikzcd} {B_V}^{A_V} \times A_V \arrow[r, "\mathsf{ev}"] \arrow[d, "\cong", "{B_V}^{A_V} \times A_{V,U}"'] & B_V \arrow[r, tail, "B_{V,U}"] & B_U \\ {B_V}^{A_V} \times A_U \arrow[ur, "\widetilde{\mathsf{ev}}"'] & & \end{tikzcd}$$

Thus $\mathbf{E}_\infty$ is cartesian closed.

A subobject classifier Ω in $\mathbf{E}_\infty$ can be constructed by choosing for every $U \in \mathsf{st}(\mathbf{E})$ a generic mono $\mathsf{t}_U : 1_U \to \Omega_U$ in $\mathbf{E}/U$ and then defining for $V \leq U$ in $\mathsf{st}(\mathbf{E})$ the map $\Omega_{V,U} : \Omega_V \to \Omega_U$ as the unique morphism in $\mathbf{E}/U$ satisfying

$$\begin{tikzcd}
1_V \arrow[r, tail] \arrow[d, "\mathsf{t}_V"'] \arrow[dr, phantom, "\lrcorner" very near start] & 1_U \arrow[d, "\mathsf{t}_U"] \\
\Omega_V \arrow[r, tail, "\Omega_{V,U}"'] & \Omega_U
\end{tikzcd}$$

from which definition it is clear that $\Omega_{U,U} = \mathrm{id}_{\Omega_U}$ and $\Omega_{W,U} = \Omega_{V,U} \circ \Omega_{V,U}$ as

$$\begin{tikzcd}
1_W \arrow[r, tail] \arrow[d, "\mathsf{t}_W"'] \arrow[dr, phantom, "\lrcorner" very near start] & 1_V \arrow[r, tail] \arrow[d, "\mathsf{t}_V"'] \arrow[dr, phantom, "\lrcorner" very near start] & 1_U \arrow[d, "\mathsf{t}_U"] \\
\Omega_W \arrow[r, tail, "\Omega_{W,V}"'] & \Omega_V \arrow[r, tail, "\Omega_{V,U}"'] & \Omega_U
\end{tikzcd}$$

whenever $W \leq V \leq U$ in $\mathsf{st}(\mathbf{E})$. That $\Omega_{V,U}$ fits into a pullback diagram

$$\begin{tikzcd}
\Omega_V \arrow[r, tail, "\Omega_{V,U}"] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \Omega_U \arrow[d] \\
V \arrow[r, tail, "i = 1_{V,U}"'] & U
\end{tikzcd}$$

can be seen as follows. As $i^* : \mathbf{E}/U \to \mathbf{E}/V$ is a logical functor it in particular preserves subobject classifiers. Thus, we have

$$\begin{tikzcd}
1_V \arrow[r, "\cong"] \arrow[d, "\mathsf{t}_V"'] \arrow[dr, phantom, "\lrcorner" very near start] & i^*1_U \arrow[r, tail] \arrow[d, "i^*\mathsf{t}_U"'] \arrow[dr, phantom, "\lrcorner" very near start] & 1_U \arrow[d, "\mathsf{t}_U"] \\
\Omega_V \arrow[r, "\chi", "\cong"'] \arrow[dr] & i^*\Omega_U \arrow[r, tail, "m"] \arrow[d] \arrow[dr, phantom, "\lrcorner" very near start] & \Omega_U \arrow[d] \\
& V \arrow[r, tail, "i"'] & U
\end{tikzcd}$$

The map χ is an isomorphism because $i^*\mathsf{t}_U$ is a subobject classifier in $\mathbf{E}/V$ as i^* is logical. Thus, we have $\Omega_{V,U} = m \circ \chi$ form which the claim follows.

This finishes the proof that $\mathbf{E}_\infty$ is a topos.

One easily sees that the projection $\pi_U : \mathbf{E}_\infty \to \mathbf{E}/U$ is equivalent to the logical functor $U^* : \mathbf{E}_\infty \to \mathbf{E}_\infty/U \simeq \mathbf{E}/U$. One can embed $\mathbf{E}$ into $\mathbf{E}_\infty$ via a functor $\mathcal{I}$ sending $A \in \mathbf{E}$ to

$\mathcal{I}(A) \in \mathbf{E}_\infty$ where $\mathcal{I}(A)_U = A \times U$ and $\mathcal{I}(A)_{V,U} = A \times i_{V,U}$ as illustrated in the diagram

$$\begin{tikzcd} A \times V \arrow[r, tail, "A \times i_{V,U}"] \arrow[d, "\pi_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \times U \arrow[d, "\pi_2"] \\ V \arrow[r, tail, "i_{V,U}"'] & U \end{tikzcd}$$

and $f : A \to B$ in $\mathbf{E}$ to $\mathcal{I}(f) : \mathcal{I}(A) \to \mathcal{I}(B)$ with $\mathcal{I}(f)_U = f \times U$. The functor U^* has a full and faithful left adjoint Σ_U equivalent to the restriction of $\mathcal{I}$ to $\mathbf{E}/U$. As U^* is logical and has a full and faithful left adjoint it follows that U^* has also a right adjoint Π_U which is also full and faithful.

Obviously, the the restriction of $\mathcal{I}$ to $\mathsf{st}(\mathbf{E})$ factors through $\mathsf{st}(\mathbf{E}_\infty)$ and is equivalent to $\iota : \mathbf{E} \to \mathbf{E}_\infty : U \mapsto (U \cap V)_{V \in \mathsf{st}(\mathbf{E})}$ as $U \times V \cong U \cap V$ for subterminals U, V. Thus, we have $\mathbf{E}/U \simeq \mathbf{E}_\infty/\iota(U)$ for every $U \in \mathsf{st}(\mathbf{E})$ from which it follows that $\mathbf{E} \simeq \mathbf{E}_{\infty/\mathsf{st}(\mathbf{E})}$ as desired. ∎

Notice that for defining the embedding $\mathcal{I} : \mathbf{E} \to \mathbf{E}_\infty$ we have used essentially that $\mathbf{E}$ has supports and binary products.

We observe that applying the topos completion procedure of Theorem 3.1 to the partial toposes of the Examples 1.7–1.9 gives rise to the toposes from which they are were constructed, i.e. $\mathbf{T}_\infty \simeq \mathsf{FinSet}^{\omega^{\mathrm{op}}}$, $\mathsf{Wrk}_\infty \simeq \mathbf{Set}^{\mathsf{Ord}^{\mathrm{op}}}$ and $\mathsf{Sh}_c(X)_\infty \simeq \mathsf{Sh}(X)$ for locally compact Hausdorff spaces X.

Finally we want to remark that the topos completion procedure of Theorem 3.1 extends to NNOs (Natural Numbers Objects) in the following way. If $\mathbf{E}$ is a partial topos such that for every $U \in \mathsf{st}(\mathbf{E})$ there is a NNO

$$U \xrightarrow{z_U} N_U \xrightarrow{s_U} N_U$$

in $\mathbf{E}/U$ then these can be organised into a single NNO in $\mathbf{E}_\infty$ choosing for $V \leq U$ the map $N_{V,U} : N_V \to N_U$ as the unique arrow making the diagram

$$\begin{tikzcd} V \arrow[r, tail] \arrow[d, "z_V"'] & U \arrow[d, "z_U"] \\ N_V \arrow[r, tail, "N_{V,U}"'] \arrow[d, "s_V"'] & N_U \arrow[d, "s_U"] \\ N_V \arrow[r, tail, "N_{V,U}"'] & N_U \end{tikzcd}$$

commute. Existence and uniqueness of $N_{V,U}$ follows from the fact that the pullback functor $V^* : \mathbf{E}/U \to \mathbf{E}/V$ preserves NNOs.

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