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SOME CALCULUS WITH EXTENSIVE QUANTITIES: WAVE EQUATION

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Abstract. We take some first steps in providing a synthetic theory of distributions. In particular, we are interested in the use of distribution theory as foundation, not just as tool, in the study of the wave equation.

1. Introduction

The aim of this paper is to contribute to a synthetic theory of distributions. The sense in which we understand "synthetic" in this context is that we place ourselves in a setting (category) where everything is smooth (differentiable). Now distributions are sometimes thought of as very non-smooth functions, like the Heaviside function, or the Dirac delta. We take the viewpoint, stressed by Lawvere, that distributions are *extensive* quantities, where functions are intensive ones. It is only by a spurious comparison with functions that distributions seem non-smooth.

A main assumption about the category in which we work is that it is cartesian closed, meaning that function "spaces", and hence some of the methods of functional analysis, are available.

This viewpoint also makes it quite natural to formulate the wave equation as an evolution equation, i.e. an *ordinary* differential equation describing the evolution over time of any initial distribution, so it is an ordinary differential equation with values in the vector space of distributions.

The main construction in the elementary theory of the wave equation is the construction of the fundamental solution: the description of the evolution of a point (Dirac) distribution over time. (Other solutions with other initial states may then by obtained by *convolution* of the given initial state with the fundamental solution; we shall not go here into this classical technique.)

To say that *distributions* are extensive quantities implies that they transform covariantly. To say that *functions* are intensive quantities implies that they transform contravariantly. Distributions are here construed, following Schwartz, as linear functionals on the space of (smooth) functions. But since all functions in the synthetic context are smooth, as well as continuous, there is no distinction between distributions and Radon

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measures.

So we consider a cartesian closed category $\mathbf{E}$ with finite limits, in which there is given a commutative ring object R, to be thought of as the real number line.

Already on this basis, one can define the vector space $\mathcal{D}'_c(M)$ of distributions of compact support on M, for each object $M \in \mathbf{E}$, namely the object of R-linear maps $R^M \to R$ ("vector space" in this context means R-module).

We shall assume that elementary differential calculus for functions $R \to R$ is available, as in all models of SDG, cf. [4], [11], [8], etc. We shall also assume some integral calculus, but only in the weakest possible sense, namely we assume

Integration Axiom: For every $\psi : R \to R$, there is a unique $\Psi : R \to R$ with $\Psi' = \psi$ and with $\Psi(0) = 0$.

Note that we do not assume any order $\leq$ on R, so that "intervals" $[a, b] \subseteq R$ do not make sense as subsets. "Intervals", on the contrary, will be construed as *distributions*: for $a, b \in R$, $[a, b]$ denotes the distribution

$$\psi \mapsto \int_a^b \psi(x)\ dx$$

The right hand side here of course means $\Psi(b) - \Psi(a)$, where Ψ is the primitive of ψ given by the integration axiom. (This weak form of integration axiom holds in some of the very simple models of SDG, like in the topos classifying the theory of commutative rings.)

Finally, for the specific treatment of the wave equation, we need that the trigonometric functions cos and sin should be present. We assume that they are given as part of the data, and that they satisfy $\cos^2 + \sin^2 = 1$, and $\cos' = -\sin$, $\sin' = \cos$. Also as part of the data, we need specified an element $\pi \in R$ so that $\cos \pi = -1, \cos 0 = 1$.

Except for the Taylor Series/ Formal Solutions considerations in the end of the paper, the setting does not depend on the "nilpotent infinitesimals" of SDG, but could also be, say, that of Froelicher-Kriegl [2], or Grothendieck's "Smooth Topos".

We would also like to remark that one can probably construct such smooth toposes in which *no* non-trivial distribution of compact support has a density function, or equivalently, *no* function (other than 0) gives rise to a distribution of compact support; our description of fundamental solutions to the wave equation would not be affected. An example seems to be the topos classifying C^ω-algebras, where C^ω is the algebraic theory of entire real or complex analytic functions[1].

2. Generalities on distributions

We want to apply parts of the general theory of ordinary differential equations to some of the basic equations of mathematical physics, the wave and heat equations[2].

[1] Added in proof: This has now been verified by the second author, see a forthcoming note.

[2] We do not discuss the heat equation in the present paper; we hope to return to it and improve the version of [6].

This takes us by necessity to the realm of distributions. Not primarily as a technique, but because of the nature of these equations: they model evolution through time of (say) a *heat distribution*. A distribution is an *extensive* quantity, and does not necessarily have a density *function*, which is an *intensive* quantity; the most important of all distributions, the point distributions (or Dirac distributions), for instance, do not.

As stressed by Lawvere in [9], functions are *intensive* quantities, and transform *contravariantly*; distributions are *extensive* quantities and transform *covariantly*. For functions, this is the fact that the "space" of functions on M, R^M is contravariant in M, by elementary cartesian-closed category theory. Similarly, the "space" of distributions of compact support on M is a subspace of R^{R^M} (carved out by the R linearity condition), and so for similar elementary reasons is covariant in M.

Let us make the formula for covariant functorality $\mathcal{D}'_c$ explicit. Let $f : M \to N$ be a map. The map $\mathcal{D}'_c(f) : \mathcal{D}'_c(M) \to \mathcal{D}'_c(N)$—which may also be denoted f_*—is described by declaring

$$\langle f_*(T), \phi \rangle = \langle T, \phi \circ f \rangle, \tag{1}$$

where T is a distribution on M, and ϕ is a function on N. The brackets denote evaluation of distributions on functions. If we similarly denote the value of the contravariant functor $M \mapsto R^M$ on a map f by f^*, the defining equation for f_* goes $\langle f_*(T), \phi \rangle = \langle T, f^*(\phi) \rangle$.

We note that $\mathcal{D}'_c(M)$ is an R-linear space, and all maps $f_* : \mathcal{D}'_c(M) \to \mathcal{D}'_c(N)$ are R-linear. Also $\mathcal{D}'_c(M)$ is a *Euclidean* vector space V, meaning that the basic differential calculus in available, for instance that the basic axiom of SDG holds; we return to this in Section 2.

For any distribution T of compact support on M, one has its *Total*, which is just the number $\langle T, 1 \rangle \in R$, where 1 denotes the function on M with constant value 1. Since $f^*(1) = 1$ for any map f, it follows that f_* preserves Totals. (Alternatively, let $\mathbf{1}$ denotes the terminal object (= one-point set). Since $\mathcal{D}'_c(\mathbf{1}) \cong R$ canonically, the Total of T may also be described as $!_*(T)$, where $! : M \to \mathbf{1}$ is the unique such map. Then preservation of Totals follows from functorality and from uniqueness of maps into $\mathbf{1}$.)

Recall that a distribution T on M may be multiplied by any function $g : M \to R$, by the recipe

$$\langle g \cdot T, \phi \rangle = \langle T, g \cdot \phi \rangle. \tag{2}$$

A basic result in one-variable calculus is "integration by substitution". We present it here in pure "distribution" form; note that no assumption on monotonicity or even bijectivity of the "substitution" g is made.

2.1. Proposition. *Given any function $g : R \to R$, and given $a, b \in R$. Then, as distributions on R,*

$$g_*(g' \cdot [a, b]) = [g(a), g(b)].$$

PROOF. Let ψ be a test function, and let Ψ be a primitive of it, $\Psi' = \psi$. So $\langle [g(a), g(b)], \psi\rangle = \Psi(g(b)) - \Psi(g(a))$. On the other hand, by the chain rule, $\Psi \circ g$ is a primitive of $g' \cdot (\psi \circ g)$, and so

$$\Psi(g(a)) - \Psi(g(b)) = \langle [a, b], g' \cdot (\psi \circ g)\rangle = \langle g' \cdot [a, b], \psi \circ g\rangle = \langle g_*(g' \cdot [a, b]), \psi\rangle.$$

The external product of distributions of compact support is defined as follows. If P is a distribution on M, and Q a distribution on N, we get a distribution $P \times Q$ on $M \times N$, by

$$\langle P \times Q, \psi\rangle = \langle P, [m \mapsto \langle Q, \psi(m, -)\rangle]\rangle.$$

In general, the external product construction $\times$ will not be the same as the external product construction $\overline{\times}$ given by[3]

$$\langle P \mathbin{\overline{\times}} Q, \psi\rangle = \langle Q, [n \mapsto \langle P, \psi(-, n)\rangle]\rangle.$$

However, if $[a, b]$ and $[c, d]$ are intervals (viewed as distributions on R, as described above), $[a, b] \times [c, d] = [a, b] \mathbin{\overline{\times}} [c, d]$, as distributions on R^2, by an application of Fubini's Theorem, (which holds in the context here—it is a consequence of equality of mixed partial dervatives). Distributions arising in this way on R^2, we call *rectangles*. The evident generalization to higher dimensions, we call *boxes*. We have

$$\langle [a, b] \times [c, d], \psi\rangle = \int_a^b \int_c^d \psi(x, y)\; dy\; dx,$$

in traditional notation. Notice that we can define *the boundary* of the box $[a, b] \times [c, d]$ as the obvious distribution on R^2,

$$(p^2_c)_*[a, b] + (p^1_b)_*[c, d] - (p^2_d)_*[a, b] - (p^1_a)_*[c, d]$$

where $p^2_c(x) = (x, c)$, $p^1_b(y) = (b, y)$, etc. ∎

By a *singular* box in an object M, we understand the data of a map $\gamma : R^2 \to M$ and a box $[a, b] \times [c, d]$ in R^2, and similarly for singular intervals and singular rectangles. Such a singular box gives rise to a distribution on M, namely $g_*([a, b] \times [c, d])$.

By "differential operator" on an object M, we here understand just an R-linear map $D : R^M \to R^M$. If D is such an operator, and T is a distribution on M, we define $D(T)$ by

$$\langle D(T), \psi\rangle := \langle T, D(\psi)\rangle,$$

and in this way, D becomes a linear operator $\mathcal{D}'_c(M) \to \mathcal{D}'_c(M)$.

In particular, if X is a vector field on M, one defines the directional derivative $D_X(T)$ of a distribution T on M by the formula

$$\langle D_X(T), \psi\rangle = \langle T, D_X(\psi)\rangle. \tag{3}$$

[3]In fact, the two external product formations described here provide the covariant functor $\mathcal{D}'_c(-)$ with two structures of monoidal functor $\mathbf{E} \to \mathbf{E}$, in fact, they are the monoidal structures that arise because $\mathcal{D}'_c(-)$ is a strong functor on $\mathbf{E}$ with a monad structure, [3], [5].

This in particular applies to the vector field $\partial/\partial x$ on R, and reads here $\langle T', \psi\rangle = \langle T, \psi'\rangle$ (ψ' denoting the ordinary derivative of the function ψ). (This is at odds with the minus sign which is usually put in into the definition of T', but it will cause no confusion—we are anyway considering second order operators, where there is no discrepancy.)

The following proposition is an application of the covariant functorality of the functor $\mathcal{D}'_c$, which will be used in connection with the wave equation in dimension 2. We consider the (orthogonal) projection $p : R^3 \to R^2$ onto the xy-plane; Δ denotes the Laplace operator in the relevant R^n, so for R^3, Δ is $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$.

2.2. Proposition. *For any distribution S (of compact support) on R^3,*

$$p_*(\Delta(S)) = \Delta(p_*(S)).$$

(The same result holds for any orthogonal projection p of R^n onto any linear subspace; the proof is virtually the same, if one uses invariance of Δ under orthogonal transformations.)

Proof. For any $\psi : R^2 \to R$,

$$\Delta(p^*\psi)) = p^*(\Delta(\psi)),$$

namely $\partial^2\psi/\partial x^2 + \partial^2\psi/\partial y^2$. From this, the Proposition follows purely formally. ∎

3. Calculus in Euclidean vector spaces

Recall that a *vector space* in the present context just means an R-module. A vector space E is called *Euclidean* if differential and integral calculus for functions $R \to E$ is available. An axiomatic account is given in [4], [11], [8] and other places. The coordinate vector spaces are Euclidean, but so are also the vector spaces R^M, and $\mathcal{D}'_c(M)$ for any M. To describe for instance the ("time") derivative $\dot{f}$ of a function $f : R \to \mathcal{D}'_c(M)$, we put

$$\langle \dot{f}(t), \psi\rangle = \frac{d}{dt}\langle f(t), \psi\rangle.$$

Similarly, from the integration axiom for R, one immediately proves that $\mathcal{D}'_c(M)$ satisfies the integration axiom, in the sense that for any $h : R \to \mathcal{D}'_c(M)$, there exists a unique $H : R \to \mathcal{D}'_c(M)$ satisfying $H(0) = 0$ and $H'(t) = h(t)$ for all t. In particular, if $h : R \to \mathcal{D}'_c(M)$, the "integral" $\int_a^b h(u)\ du$ makes sense (as $H(b) - H(a)$), and the Fundamental Theorem of Calculus holds, almost by definition.

As a particular case of special importance, we consider a *linear* vector field on a Euclidean R-module V. To say that the vector field is linear is to say that its principal-part formation $V \to V$ is a linear map, Γ, say. We have then the following version of a classical result. By a *formal* solution for an ordinary differential equation, we mean a solution defined on the set D_∞ of nilpotent elements in R (these form a subgroup of $(R, +)$).

3.1. PROPOSITION. *Let a linear vector field on a Euclidean vector space V be given by the linear map $\Gamma : V \to V$. Then the unique formal solution of the corresponding differential equation, i.e., the equation $\dot{F}(t) = \Gamma(F(t))$ with initial position v, is the map $D_\infty \times V \to V$ given by*

$$(t, v) \mapsto e^{t \cdot \Gamma}(v), \tag{4}$$

where the right hand side here means the sum of the following "series" (which has only finitely many non-vanishing terms, since t is assumed nilpotent):

$$v + t\Gamma(v) + \frac{t^2}{2!}\Gamma^2(v) + \frac{t^3}{3!}\Gamma^3(v) + \ldots$$

(Here of course $\Gamma^2(v)$ means $\Gamma(\Gamma(v))$, etc.)

PROOF. We have to prove that $\dot{F}(t) = \Gamma(F(t))$. We calculate the left hand side by differentiating the series term by term (there are only finitely many non-zero terms):

$$\Gamma(v) + \frac{2t}{2!} \cdot \Gamma^2(v) + \frac{3t^2}{3!}\Gamma^3(v) + ... = \Gamma(v + t \cdot \Gamma(v) + \frac{t^2}{2!} \cdot \Gamma^2(v) + ...)$$

using linearity of Γ. But this is just Γ applied to $F(t)$. ∎

There is an analogous result for second order differential equations of the form $\ddot{F}(t) = \Gamma(F(t))$ (with Γ linear); the proof is similar and we omit it:

3.2. PROPOSITION. *The formal solution of this second order differential equation $\ddot{F} = \Gamma F$, with initial position v and initial velocity w, is given by*

$$F(t) = v + t \cdot w + \frac{t^2}{2!}\Gamma(v) + \frac{t^3}{3!}\Gamma(w) + \frac{t^4}{4!}\Gamma^2(v) + \frac{t^5}{5!}\Gamma^2(w) + ...$$

We shall need the following result ("change-of-variable Lemma"); for $V = R$, it is identical to Proposition 2.1, and the proof is in any case the same.

3.3. PROPOSITION. *Given $f : R \to V$, where V is a Euclidean vector space, and given $g : R \to R$. Then for any a, $b \in R$,*

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Linear maps between Euclidean vector spaces preserve differentiation and integration of functions $R \to V$; we shall explicitly need the following particular assertion

3.4. PROPOSITION. *Let $F : V \to W$ be a linear map between Euclidean vector spaces. Then for any $f : R \to V$,*

$$F(\int_a^b f(t)\,dt) = \int_a^b F(f(t))\,dt$$

.

4. Spheres and balls as distributions

Let S be a distribution in R^n; ultimately, it will be the unit sphere, see below. We describe some families of distributions derived from it. Let $t \in R$ (not necessarily $t > 0$—we haven't even assumed an order relation on R). We then have the homothety "multiplying by t from R^n to R^n", which we denote H^t, so

$$H^t(\underline{x}) = t \cdot \underline{x},$$

for any $\underline{x} \in R^n$.

We are going to use the covariant functorality of $\mathcal{D}'_c$ with respect to these maps H^t. Note that for any distribution T on R^n,

$$H^0_*(T) = Total(T) \cdot \delta(\underline{0}), \tag{5}$$

where $\delta(\underline{0})$ denotes the Dirac distribution at $\underline{0} \in R^n$, given by $\langle \delta(\underline{0}), \psi \rangle = \psi(\underline{0})$. We put

$$S^t := H^t_*(S).$$

It has the same Total as S, but its support[4] is larger (e.g. for $t = 2$, it is 2^{n-1} times as big as that of S). So if S is the unit sphere, S^t is "the *diluted sphere of radius* t". We also want an *undiluted* sphere of radius t; we put

$$S_t := t^{n-1} \cdot S^t.$$

Note that in dimension 1, $S_t = S^t$.

The *ball of radius 1* is made up from shells (undiluted spheres) "of radius u ($0 \leq u \leq 1$)" (heuristically!), motivating us to put

$$B := \int_0^1 S_u\,du,$$

using integration in $\mathcal{D}'_c(R^n)$. Let $t \in R$. We put

$$B^t := H^t_*(B).$$

It has the same Total as B, but its support is larger ("if $t > 1$"—heuristically), so if S is the unit sphere, B is "*the diluted ball of radius* t" (think of the expanding universe). We also want an *undiluted* ball of radius t; we put

$$B_t := t^n \cdot B^t.$$

We then have

[4] We haven't here introduced the notion of support of a distribution, and only use the word here for motivating the word "diluted".

4.1. Proposition. *For all $t \in R$,*

$$B_t = \int_0^t S_v \, dv.$$

Proof.

$$\begin{aligned} B_t &= t^n \cdot B^t = t^n \cdot H^t_*(B) \\ &= t^n \cdot H^t_*(\int_0^1 S_u \, du) \\ &= t^n \cdot \int_0^1 H^t_*(S_u) \, du \end{aligned}$$

(by Proposition 3.4)

$$\begin{aligned} &= t^n \int_0^1 H^t_*(u^{n-1} \cdot H^u_*(S)) \, du \\ &= t^n \cdot \int_0^1 u^{n-1} H^{t \cdot u}_*(S) \, du \\ &= \int_0^1 (t \cdot u)^{n-1} \cdot H^{t \cdot u}_*(S) \, t \, du \\ &= \int_0^t v^{n-1} H^v_*(S) \, dv \end{aligned}$$

(by change-of-variable Lemma (Proposition 3.3), with $v := t \cdot u$), which is $\int_0^t S_v \, dv$, as claimed. ∎

We now give explicit defining formulae for S in dimensions 1, 2 and 3. These are of course standard integral formulae in disguise—explicit integral formulae come by applying the definitions, and then integral formulae for S^t, S_t and B_t may be derived (using Proposition 4.1 and related arguments)—we give some of these formulae below.

Dimension 1 $S := \delta(1) + \delta(-1)$

Dimension 2 $S := \mathrm{cis}_*([0, 2\pi])$, where $\mathrm{cis} : R \to R^2$ is the map $\theta \mapsto (\cos\theta, \sin\theta)$.

Dimension 3 $S := s \cdot \mathrm{sph}_*([0, 2\pi] \times [0, \pi])$, where $s : R^2 \to R$ is the function $(\theta, \phi) \mapsto \sin\phi$, and where sph is "the spherical coordinates map" $R^2 \to R^3$ given by

$$(\theta, \phi) \mapsto (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi). \tag{6}$$

In dimension 2, for instance, we have

$$\langle S^t, \psi \rangle = \int_0^{2\pi} \psi(t \cdot \cos\theta, t \cdot \sin\theta) \, d\theta,$$

$$\langle S_t, \psi \rangle = \int_0^{2\pi} t \cdot \psi(t \cdot \cos\theta, t \cdot \sin\theta)\, d\theta,$$

and so by Proposition 4.1,

$$\langle B_t, \psi \rangle = \int_0^t \left[\int_0^{2\pi} u \cdot \psi(u \cdot \cos\theta, u \cdot \sin\theta)\, d\theta \right] du,$$

which the reader may want to rearrange, using Fubini, into the standard formula for integration in polar coordinates over the disk of radius t; but note we have no assumptions like "$t > 0$".

Note also that $B_0 = 0$, whereas S^0 and B^0 are constants times the Dirac distribution at the origin $\underline{0}$ (use (5)). The constants are the "area" of the unit sphere, or the "volume" of the unit ball, in the appropriate dimension. Explicitly,

$$S^0 = 2 \cdot \delta(\underline{0}),\ 2\pi \cdot \delta(\underline{0}),\ 4\pi \cdot \delta(\underline{0}), \tag{7}$$

and

$$B^0 = 2 \cdot \delta(\underline{0}),\ \pi \cdot \delta(\underline{0}),\ \frac{4\pi}{3} \cdot \delta(\underline{0}) \tag{8}$$

in dimensions 1,2, and 3, respectively.

We shall also have occasion to consider the distribution (of compact support) $t \cdot S^t$ on R^3 as well as its projection $p_*(t \cdot S^t)$ on the xy-plane (using functorality of $\mathcal{D}'_c$ with respect to the projection map $p : R^3 \to R^2$).

We insert for reference two obvious "change of variables" equations. Recall that $H_t : R^n \to R^n$ is the homothetic transformation "multiplying by t". We have, for any vector field $\mathbf{F}$ on R^n (viewed, via principal part, as a map $R^n \to R^n$):

$$\operatorname{div}(\mathbf{F} \circ H_t) = t \cdot (\operatorname{div} \mathbf{F}) \circ H_t, \tag{9}$$

and

$$t^n \int_{B_1} \phi \circ H_t = \int_{B_t} \phi. \tag{10}$$

5. Divergence Theorem for Unit Sphere

The Main Theorem of vector calculus is Stokes' Theorem: $\int_{\partial\gamma} \omega = \int_\gamma d\omega$, for ω an $(n-1)$-form, γ a suitable n-dimensional figure (with appropriate measure on it) and $\partial\gamma$ its geometric boundary. In the synthetic context, the theorem holds at least for any singular cubical chain $\gamma : I^n \to M$ (I^n the n-dimensional coordinate cube), because the theorem may then be reduced to the fundamental theorem of calculus, which is the only way integration enters in the elementary synthetic context; measure theory not being available therein. For an account of Stokes' Theorem in this context, see [11] p.139. Below, we shall apply the result not only for singular *cubes* as in loc.cit., but also for singular *boxes*, like the usual $(\gamma : R^2 \to R^2, [0, 2\pi] \times [0, 1])$, "parametrizing the unit disk B by polar coordinates",

$$\gamma(\theta, r) = (r \cos\theta, r \sin\theta). \tag{11}$$

We shall need from vector calculus the Gauss-Ostrogradsky "Divergence Theorem"

$$\text{flux of } \mathbf{F} \text{ over } \partial\gamma = \int_\gamma (\text{divergence of } \mathbf{F}),$$

with $\mathbf{F}$ a vector field, for the geometric "figure" $\gamma =$ the unit ball in R^n. For the case of the unit ball in R^n, the reduction of the Divergence Theorem to Stokes' Theorem is a matter of the *differential* calculus of vector fields, differential forms, inner products etc. (See e.g. [7] p. 204). For the convenience of the reader, we recall the case $n = 2$.

Given a vector field $\mathbf{F}(x, y) = (F(x, y), G(x, y))$ in R^2, apply Stokes' Theorem to the differential form

$$\omega := -G(x, y)\, dx + F(x, y)\, dy$$

for the singular rectangle γ given by (11) above. Then, using the equational assumptions on $\cos, \sin$ and their derivatives, we have

$$\begin{cases} \gamma^*(dx) = \cos\theta\, dr - r \sin\theta\, d\theta \\ \gamma^*(dy) = \sin\theta\, dr + r \cos\theta\, d\theta \\ \gamma^*(dx \wedge dy) = r\,(dr \wedge d\theta) \end{cases}$$

Since $d\omega = (\partial G/\partial y + \partial F/\partial x)\, dx \wedge dy = \mathrm{div}(\mathbf{F})\, dx \wedge dy$, then

$$\gamma^*(d\omega) = \mathrm{div}(\mathbf{F}) r\,(dr \wedge d\theta)$$

On the other hand,

$$\gamma^*\omega = (F \sin\theta - G \cos\theta)\, dr + (Fr \cos\theta + Gr \sin\theta)\, d\theta, \tag{12}$$

(all F, G, and $\mathbf{F}$ to be evaluated at $(r\cos\theta, r\sin\theta)$). Therefore

$$\int_\gamma d\omega = \int_0^{2\pi} \int_0^1 \mathrm{div}(\mathbf{F}) r\, dr\, d\theta;$$

this is $\int_{B_1} \mathrm{div}(\mathbf{F})\ dA$. On the other hand by Stokes' Theorem $\int_\gamma d\omega = \int_{\partial\gamma} \omega$ which is a curve integral of the 1-form (12) around the boundary of the rectangle $[0, 2\pi] \times [0, 1]$. This curve integral is a sum of four terms corresponding to the four sides of the rectangle. Two of these (corresponding to the sides $\theta = 0$ and $\theta = 2\pi$) cancel, and the term corresponding to the side where $r = 0$ vanishes because of the r in $r\,(dr \wedge d\theta)$, so only the side with $r = 1$ remains, and its contribution is, with the correct orientation,

$$\int_0^{2\pi} (F(\cos\theta, \sin\theta) \cos\theta + G(\cos\theta, \sin\theta) \sin\theta)\, d\theta = \int_{S_1} \mathbf{F} \cdot \mathbf{n}\, ds$$

where $\mathbf{n}$ is the outward unit normal of the unit circle. This expression is the flux of $\mathbf{F}$ over the unit circle, which thus equals the divergence integral calculated above.

6. Time Derivatives of Expanding Spheres and Balls

We now combine vector calculus with the calculus of the basic ball and sphere distributions, as introduced in Section 3, to prove the following result:

6.1. THEOREM. *In R^n (for any n), we have, for any t,*

$$\frac{d}{dt}S^t = t \cdot \Delta(B^t),$$

(Δ = the Laplace operator).

PROOF. We consider the effect of the two expressions on an arbitrary function ψ. We have

$$\begin{aligned}
\langle \frac{d}{dt}S^t, \psi\rangle &= \langle S, \frac{d}{dt}(\psi \circ H^t)\rangle \text{ by various definitions} \\
&= \langle S, u \mapsto (\nabla\psi(H^t(u)) \cdot u\rangle \text{ by differential calculus} \\
&= \text{ flux over } S \text{ of } (\nabla\psi \circ H^t) \text{ by special property of } S \\
&= \langle B, \operatorname{div}(\nabla\psi \circ H^t)\rangle \text{ by divergence Theorem} \\
&= t\langle B, \operatorname{div}(\nabla\psi) \circ H^t\rangle \text{ by (9)} \\
&= t\langle B, (\Delta\psi) \circ H^t\rangle \text{ by definition of } \Delta \\
&= t\langle B^t, \Delta(\psi)\rangle = t\langle \Delta(B^t), \psi\rangle \text{ by various definitions}
\end{aligned}$$

from which the result follows. ■

We collect some further information about t-derivatives of some of the t-parametrized distributions considered. From Proposition 4.1 and the Fundamental Theorem of Calculus, we immediately derive

$$\frac{d}{dt}(B_t) = S_t. \tag{13}$$

In dimension 1, we have

$$\frac{d}{dt}(S_t) = \Delta(B_t); \tag{14}$$

for,

$$\frac{d}{dt}\langle S_t, \psi\rangle = \frac{d}{dt}\langle \psi(t) + \psi(-t)\rangle = \psi'(t) - \psi'(-t),$$

whereas

$$\langle \Delta B_t, \psi\rangle = \langle B_t, \psi''\rangle = \int_{-t}^{t} \psi''(t)\,dt,$$

and the result follows from the Fundamental Theorem of Calculus.

The equation (14) implies the following equation if $n = 1$; we shall prove that it also holds if $n \geq 2$:

$$t \cdot \frac{d}{dt}(S_t) = (n-1)S_t + t \cdot \Delta(B_t). \tag{15}$$

For, differentiate $S_t = t^{n-1} \cdot S^t$ to get

$$\frac{d}{dt}(S_t) = (n-1)t^{n-2} \cdot S^t + t^{n-1}\frac{d}{dt}(S^t),$$

which by Theorem 6.1 and the definition of B_t in terms of B^t is $= (n-1)t^{n-2} \cdot S^t + \Delta(B_t)$. Multiplying this equation by t and using the defining equation $S_t = t^{n-1}S^t$ gives the result.

We we shall finally argue that

$$t \cdot \frac{d}{dt}(B^t) = S^t - nB^t. \tag{16}$$

For, differentiating the defining equation $B_t = t^n \cdot B^t$ gives $d/dtB_t = nt^{n-1} \cdot B^t + t^n \cdot d/dtB^t$. Now the left hand side here is S_t, by (13), so we conclude that $t^{n-1} \cdot S^t = nt^{n-1} \cdot B^t + t^n \cdot d/dtB^t$. If t were invertible, we would conclude by cancelling t^{n-1} in this equation. But since the equation holds for *all* t, we may cancel it in any case: a consequence of the integration axiom is the Lavendhomme Cancellation Principle, which says that if $t \cdot g(t) = 0$ for *all* t, then $g(t) = 0$ for all t, see [8] Ch.1 Prop. 15. Applying this principle $n-1$ times then yields (16).

7. Wave equation

Let Δ denote the Laplace operator $\sum \partial^2/\partial x_i^2$ on R^n. We shall consider the wave equation (WE) in R^n, (for $n = 1, 2, 3$),

$$\frac{d^2}{dt^2}Q(t) = \Delta Q(t) \tag{17}$$

as a second order ordinary differential equation on the Euclidean vector space $\mathcal{D}'_c(R^n)$ of distributions of compact support; in other words, we are looking for functions

$$Q : R \to \mathcal{D}'_c(R^n)$$

so that for all $t \in R$, $\ddot{Q}(t) = \Delta(Q(t))$ (viewing Δ as a map $\mathcal{D}'_c(R^n) \to \mathcal{D}'_c(R^n)$.)

Consider a function $f : R \to V$, where V is a Euclidean vector space (we are interested in $V = \mathcal{D}'_c(R^n)$) . Then we call the pair of vectors in V consisting of $f(0)$ and $\dot{f}\ (0)$ the *initial state* of f. We can now, for each of the cases $n = 1$, $n = 3$, and $n = 2$ describe fundamental solutions to the wave equations. (The case $n = 2$ is less explicit, and is derived "by projection" from the one in dimension 3.) By *fundamental solutions*, we mean solutions whose initial state is either a constant times $(\delta(\underline{0}, 0))$, or a constant times $(0, \delta(\underline{0})$.

7.1. THEOREM. *In dimension 1: The function* $R \to \mathcal{D}'_c(R)$ *given by*

$$t \mapsto S^t (= S_t)$$

is a solution of the WE; its initial state is $2(\delta(\underline{0}), 0)$.

The function $R \to \mathcal{D}'_c(R)$ *given by*

$$t \mapsto B_t$$

is a solution of the WE with initial state $2(0, \delta(\underline{0}))$.

PROOF. We have $d/dt(B_t) = S_t$ by (13), and $d/dt(S_t) = \Delta(B_t)$, by (14). This establishes the WE for B_t. Since Δ and d/dt commute, it therefore follows that WE also holds for $d/dt(B_t) = S_t$. The initial position of the solution S_t is $S_0 = S^0 = 2\delta(\underline{0})$, by (7), and the initial velocity $\Delta(B_0)$ by (14), which is 0 since $B_0 = 0$. The initial state of the solution B_t is $B_0 = 0$, and the initial velocity is S_0, as we already calculated, so is $= 2\delta(\underline{0})$. ∎

7.2. THEOREM. *In dimension 3: The function* $R \to \mathcal{D}'_c(R^3)$ *given by*

$$t \mapsto S^t + t^2\Delta(B^t)$$

is a solution of the WE with initial state $4\pi\delta(\underline{0}), 0)$. *The function* $R \to \mathcal{D}'_c(R^3)$ *given by*

$$t \mapsto t \cdot S^t$$

is a solution of the WE with initial state $4\pi(0, \delta(\underline{0}))$.

PROOF. We calculate first d/dt of $t \cdot S^t$, using Theorem 6.1:

$$\frac{d}{dt}(t \cdot S^t) = S^t + t^2 \cdot \Delta(B^t), \tag{18}$$

and so by Theorem 6.1 again,

$$\begin{aligned} \frac{d^2}{dt^2}(t \cdot S^t) &= t \cdot \Delta(B^t) + 2 \cdot t \cdot \Delta(B^t) + t^2 \cdot \Delta(\frac{d}{dt}B^t) \\ &= 3 \cdot t \cdot \Delta(B^t) + t \cdot \Delta(t \cdot \frac{d}{dt}B^t) \\ &= 3 \cdot t \cdot \Delta(B^t) + t \cdot \Delta(S^t - 3B^t), \end{aligned}$$

using (16), and now by linearity of Δ, the terms involving $\Delta(B^t)$ cancel, so we are left with the equation

$$\frac{d^2}{dt^2}(t \cdot S^t) = \Delta(t \cdot S^t), \tag{19}$$

which establishes WE for $t \cdot S^t$.

Since d/dt and Δ commute, and since $t \cdot S^t$ is a solution, then so is its t-derivative (calculated in (18) above), i.e. $S^t + t^2 \cdot \Delta(B^t)$ is a solution. The assertions about initial position and velocity follow from (7), (using Theorem 6.1 to calculate the initial velocity of the solution $S^t + t^2\Delta(B^t)$). ∎

Recall that we considered the orthogonal projection $p : R^3 \to R^2$. Applying covariant functorality, we get for any distribution Q on R^3 of compact support a distribution $p_*(Q)$ on R^2, also of compact support.

7.3. THEOREM. *In dimension 2: The function $R \to \mathcal{D}'_c(R^2)$ given by*

$$t \mapsto p_*(S^t + t^2\Delta(B^t))$$

is a fundamental solution of the WE in dimension 2; its initial state is $4\pi(\delta(\underline{0}), 0)$. The function $R \to \mathcal{D}'_c(R^2)$ given by

$$t \mapsto p_*(t \cdot S^t)$$

is a fundamental solution of the WE in dimension 2; its initial state is $4\pi(0, \delta(\underline{0}))$.

(Note: The S^t and B^t in the statement of the theorem are those of R^3.)

PROOF. The fact that the distributions in question are solutions of the WE is immediate from the Proposition 2.2 ("p_* commutes with Δ") and from the fact that $p_* : \mathcal{D}'_c(R^3) \to \mathcal{D}'_c(R^2)$ is linear, and hence commutes with formation of d/dt; also, $\mathcal{D}'_c(p)$ sends Dirac distribution at $\underline{0} \in R^3$ to Dirac distribution at $\underline{0} \in R^2$, so the initial values and velocities are as claimed. ∎

An explicit integral expression for the two fundamental solutions here, obtained by projection, requires more assumptions, in particular, a square root formation, as is known from classical descriptions of the solutions in terms of "Poisson's kernel". We may express this by saying that the distributional solutions presented exist under our weak assumptions, but that they are not presented by functions (densities).

We haven't touched the notion of support, but when defined (in a context where it makes sense), the two fundamental solutions , S^t and $t \cdot S^t$ in dimension 1 and 3 will have support only on the geometric sphere of radius t (which is of "codimension" 1), whereas the solution $p_*(t \cdot S^t)$ will have support in the direct image in R^2 of $S^t \subseteq R^3$, and be of codimension 0. This accounts for the Huygens Principle that in a 2-dimensional world, sounds cannot be sharp signals, cf. e.g. [15] p. 227.

One might of course also derive one-dimensional fundamental solutions by orthogonal projection along $q : R^3 \to R$. Since fundamental solutions are unique modulo constants, we conclude that $q_*(S^t + t^2\Delta(B^t))$ is proportional to the 1-dimensional S^t (whose support is a 2-point set).

Combining Theorem 7.2 with Proposition 3.2, we can obtain information about S^t, and other spheres and balls, for nilpotent t. As examples, we shall prove

7.4. PROPOSITION. *If $t^5 = 0$, then in dimension 1,*

$$B_t = 2[t \cdot \delta(\underline{0}) + \frac{t^3}{3!} \cdot \delta(\underline{0})''],$$

and in dimension 3,

$$S^t = 4\pi[t \cdot \delta(\underline{0}) + \frac{t^3}{3!} \cdot \Delta(\delta(\underline{0}))].$$

Proof. We prove the second assertion only. (The proof of the first one is similar, using Theorem 7.1.) We already observed in (7) that, in dimension 3,

$$S^0 = 4\pi \cdot \delta(\underline{0}).$$

Now the two expressions above are both solutions to WE with initial state $(0, 4\pi\delta(\underline{0}))$—the left hand side by Theorem 7.2, and the right hand side by Proposition 3.2, with $\Gamma = \Delta$, $v = 0$, $w = 4\pi\delta(\underline{0})$. ∎

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