Theory and Applications of Categories, Vol. 11, No. 15, 2003, pp. 337–352.

EXPONENTIABILITY IN CATEGORIES OF LAX ALGEBRAS

Dedicated to Nico Pumplün on the occasion of his seventieth birthday

MARIA MANUEL CLEMENTINO, DIRK HOFMANN AND WALTER THOLEN

ABSTRACT. For a complete cartesian-closed category $\mathbf{V}$ with coproducts, and for any pointed endofunctor T of the category of sets satisfying a suitable Beck-Chevalley-type condition, it is shown that the category of lax reflexive $(T, \mathbf{V})$-algebras is a quasitopos. This result encompasses many known and new examples of quasitopoi.

1. Introduction

Failure to be cartesian closed is one of the main defects of the category of topological spaces. But often this defect can be side-stepped by moving temporarily into the quasitopos hull of **Top**, the category of pseudotopological (or Choquet) spaces, see for example [11, 14, 7]. A pseudotopology on a set X is most easily described by a relation $\mathfrak{x} \to x$ between ultrafilters $\mathfrak{x}$ on X and points x in X, the only requirement for which is the *reflexivity* condition $\dot{x} \to x$ for all $x \in X$, with $\dot{x}$ denoting the principal ultrafilter on x. In this setting, a topology on X is a pseudotopology which satisfies the *transitivity* condition

$$\mathfrak{X} \to \mathfrak{y} \ \& \ \mathfrak{y} \to z \ \Rightarrow \ m(\mathfrak{X}) \to z$$

for all $z \in X$, $\mathfrak{y} \in UX$ (the set of ultrafilters on X) and $\mathfrak{X} \in UUX$; here the relation $\to$ between UX and X has been naturally extended to a relation between UUX and UX, and $m = m_X : UUX \to UX$ is the unique map that gives U together with $e_X(x) = \dot{x}$ the structure of a monad $\mathsf{U} = (U, e, m)$. Barr [2] observed that the two conditions, reflexivity and transitivity, are precisely the two basic laws of a lax Eilenberg-Moore algebra when one extends the **Set**-monad U to a lax monad of Rel(**Set**), the category of sets with relations as morphisms. In [9] Barr's presentation of topological spaces was extended to include Lawvere's presentation of metric spaces as $\mathbf{V}$-categories with $\mathbf{V} = \overline{\mathbb{R}}_+$, the extended real half-line. Thus, for any symmetric monoidal category $\mathbf{V}$ with coproducts preserved by the tensor product, and for any **Set**-monad T that suitably extends from **Set**-maps to

The authors acknowledge partial financial assistance by Centro de Matemática da Universidade de Coimbra/FCT, Unidade de Investigação e Desenvolvimento Matemática e Aplicações da Universidade de Aveiro/FCT and NSERC.

Received by the editors 2003-01-23 and, in revised form, 2003-09-04.

Transmitted by Richard Wood. Published on 2003-09-18.

2000 Mathematics Subject Classification: 18C20, 18D15, 18A05, 18B30, 18B35.

Key words and phrases: lax algebra, partial product, locally cartesian-closed category, quasitopos.

all $\mathbf{V}$-matrices (or "$\mathbf{V}$-relations", with ordinary relations appearing for $\mathbf{V} = \mathbf{2}$, the two-element chain), the paper [9] develops the notion of reflexive and transitive $(\mathsf{T}, \mathbf{V})$-algebra, investigates the resulting category $\mathrm{Alg}(\mathsf{T}, \mathbf{V})$, and presents many examples, in particular $\mathbf{Top} = \mathrm{Alg}(\mathsf{U}, \mathbf{2})$.

The purpose of this paper is to show that dropping the transitivity condition leads us to a quasitopos not only in the case of **Top**, but rather generally. In order to define just reflexive $(\mathsf{T}, \mathbf{V})$-algebras, one indeed needs neither the tensor product of $\mathbf{V}$ (just the "unit" object) nor the "multiplication" of the monad T. Positively speaking then, we start off with a category $\mathbf{V}$ with coproducts and a distinguished object I in $\mathbf{V}$ and any pointed endofunctor T of **Set** and define the category $\mathrm{Alg}(T, \mathbf{V})$. Our main result says that when $\mathbf{V}$ is complete and locally cartesian closed and a certain Beck-Chevalley condition is satisfied, also $\mathrm{Alg}(T, \mathbf{V})$ is locally cartesian closed (Theorem 3.7).

Defining reflexive $(T, \mathbf{V})$-algebras for the "truncated" data T, $\mathbf{V}$ entails a considerable departure from [9], as it is no longer possible to talk about the bicategory $\mathrm{Mat}(\mathbf{V})$ of $\mathbf{V}$-matrices. The missing tensor product prevents us from being able to introduce the (horizontal) matrix composition; however, "whiskering" by **Set**-maps (considered as 1-cells in $\mathrm{Mat}(\mathbf{V})$) is still well-defined and well-behaved, and this is all that is needed in this paper.

We explain the relevant properties of $\mathrm{Mat}(\mathbf{V})$ in Section 2 and define the needed Beck-Chevalley condition. Briefly, this condition says that the comparison map that "measures" the extent to which the T-image of a pullback diagram in **Set** still is a pullback diagram must be cofully faithful when considered a 1-cell in $\mathrm{Mat}(\mathbf{V})$. Having presented our main result, at the end of Section 3 we show that this condition is equivalent to asking T to preserve pullbacks *or*, if $\mathbf{V}$ is thin (i.e., a preordered class), to transform pullbacks into weak pullback diagrams (barring trivial choices for I and $\mathbf{V}$). In certain cases, (BC) turns out to be even a necessary condition for local cartesian closedness of $\mathrm{Alg}(T, \mathbf{V})$, see 3.10. In Section 4 we show how to construct limits and colimits in $\mathrm{Alg}(T, \mathbf{V})$ in general, and Section 5 presents the construction of partial map classifiers, leading us to the theorem stated in the Abstract. A list of examples follows in Section 6.

Acknowledgements. We dedicate this paper in great admiration to Nico Pumplün with whom the third-named author started off studying Lawvere-Linton theories and monads in the late sixties. The authors are also grateful to Ross Street and Richard Wood for pointing them to their respective articles [15] and [3].

2. $\mathbf{V}$-matrices

2.1. Let $\mathbf{V}$ be a category with coproducts and a distinguished object I. A $\mathbf{V}$*-matrix* (or $\mathbf{V}$*-relation*) r from a set X to a set Y, denoted by $r : X \nrightarrow Y$, is a functor $r : X \times Y \to \mathbf{V}$, i.e. an $X \times Y$-indexed family $(r(x, y))_{x,y}$ of objects in $\mathbf{V}$. With X, Y fixed, such $\mathbf{V}$-matrices form the objects of a category $\mathrm{Mat}(\mathbf{V})(X, Y)$, the morphisms $\varphi : r \to s$ of which are natural transformations, i.e. families $(\varphi_{x,y} : r(x, y) \to s(x, y))_{x,y}$ of morphisms in $\mathbf{V}$;

briefly,

$$\mathrm{Mat}(\mathbf{V})(X,Y) = \mathbf{V}^{X\times Y}.$$

2.2. Every **Set**-map $f : X \to Y$ may be considered as a $\mathbf{V}$-matrix $f : X \nrightarrow Y$ when one puts

$$f(x,y) = \begin{cases} I & \text{if } f(x) = y, \\ 0 & \text{else,} \end{cases}$$

with 0 denoting a fixed initial object in $\mathbf{V}$. This defines a functor

$$\mathbf{Set}(X,Y) \longrightarrow \mathrm{Mat}(\mathbf{V})(X,Y),$$

of the discrete category $\mathbf{Set}(X,Y)$, and the question is: when do we obtain a full embedding, for all X and Y? Precisely when

$$\mathbf{V}(I,0) = \emptyset \text{ and } |\mathbf{V}(I,I)| = 1 \tag{$*$}$$

as one may easily check. In the context of a cartesian-closed category $\mathbf{V}$, we usually pick for I a terminal object 1 in $\mathbf{V}$, and then condition $(*)$ is equivalently expressed as

$$0 \not\cong 1 \tag{$**$}$$

preventing $\mathbf{V}$ from being equivalent to the terminal category.

2.3. While in this paper we do not need the horizontal composition of $\mathbf{V}$-matrices in general, we do need the composites sf and gr for maps $f : X \to Y$, $g : Y \to Z$ and $\mathbf{V}$-relations $r : X \nrightarrow Y$, $s : Y \nrightarrow Z$, defined by

$$\begin{aligned} (sf)(x,z) &= s(f(x),z), \\ (gr)(x,z) &= \sum_{y\,:\,g(y)=z} r(x,y), \end{aligned}$$

for $x \in X$, $z \in Z$; likewise for morphisms $\varphi : r \to r'$ and $\psi : s \to s'$. Hence, we have the "whiskering" functors

$$-f : \mathrm{Mat}(\mathbf{V})(Y,Z) \to \mathrm{Mat}(\mathbf{V})(X,Z),$$

$$g- : \mathrm{Mat}(\mathbf{V})(X,Y) \to \mathrm{Mat}(\mathbf{V})(X,Z),$$

giving $\mathrm{Mat}(\mathbf{V})$ the structure of an equipment in the sense of [3]. Accordingly, the horizontal composition with **Set**-maps from either side is associative up to coherent isomorphisms whenever defined; hence, if $h : U \to X$ and $k : Z \to V$, then

$$(sf)h = s(fh) \quad \text{and} \quad k(gr) \cong (kg)r.$$

Although $\mathrm{Mat}(\mathbf{V})$ falls short of being a bicategory, even a sesquicategory [15], we refer to sets as 0-cells of $\mathrm{Mat}(\mathbf{V})$, $\mathbf{V}$-matrices as its 1-cells, and natural transformations between them as its 2-cells.

2.4. The transpose $r^\circ : Y \nrightarrow X$ of a $\mathbf{V}$-matrix $r : X \nrightarrow Y$ is defined by $r^\circ(y,x) = r(x,y)$ for all $x \in X$, $y \in Y$. Obviously $r^{\circ\circ} = r$, and with

$$(sf)^\circ = f^\circ s^\circ,\ (gr)^\circ = r^\circ g^\circ$$

we can also introduce whiskering by transposes of **Set**-maps from either side, also for 2-cells.

A **Set**-map $f : X \to Y$ gives rise to 2-cells

$$\eta : 1_X \to f^\circ f,\ \ \varepsilon : ff^\circ \to 1_Y$$

satisfying the triangular identities $(\varepsilon f)(f\eta) = 1_f$, $(f^\circ\varepsilon)(\eta f^\circ) = 1_f$.

As a consequence one obtains adjoints to the whiskering functors of 2.3:

$$\frac{t \to sf}{tf^\circ \to s} \qquad \frac{gr \to t}{r \to g^\circ t}$$

for all r, s, f, g as in 2.3 and $t : X \nrightarrow Z$; briefly,

$$(-f^\circ) \dashv (-f), \qquad (g-) \dashv (g^\circ -).$$

2.5. For a functor $T : \mathbf{Set} \to \mathbf{Set}$, we denote by $\kappa : TW \to U$ the comparison map from the T-image of the pullback $W := Z \times_Y X$ of (g,f) to the pullback $U := TZ \times_{TZ} TX$ of (Tg, Tf)

$$\begin{tikzcd}
TW \arrow[dr, dashed, "\kappa"] \arrow[drr, bend left, "Tk"] \arrow[ddr, bend right, "Th"'] & & \\
& U \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & TX \arrow[d, "Tf"] \\
& TZ \arrow[r, "Tg"] & TY.
\end{tikzcd} \tag{1}$$

We say that the **Set**-functor T satisfies the *Beck-Chevalley Condition (BC)* if the 1-cell κ is cofully faithful; that is, if the "whiskering" functor $-\kappa : \mathrm{Mat}(\mathbf{V})(U,S) \to \mathrm{Mat}(\mathbf{V})(TW,S)$ is full and faithful, for every set S.

In the next section we will relate this condition with other known formulations of the Beck-Chevalley condition.

3. Local cartesian closedness of $\mathrm{Alg}(T,\mathbf{V})$

3.1. Let (T,e) be a pointed endofunctor of **Set** and $\mathbf{V}$ category with coproducts and a distinguished object I. A *lax (reflexive)* $(T,\mathbf{V})$*-algebra* (X,a,η) is given by a set X, a 1-cell $a : TX \nrightarrow X$ and a 2-cell $\eta : 1_X \to ae_X$ in $\mathrm{Mat}(\mathbf{V})$. The 2-cell η is completely determined by the $\mathbf{V}$-morphisms

$$\eta_x := \eta_{x,x} : I \longrightarrow a(e_X(x), x),$$

$x \in X$. As we shall not change the notation for this 2-cell, *we write* (X, a) *instead of* (X, a, η). A *(lax) homomorphism* $(f, \varphi) : (X, a) \to (Y, b)$ of $(T, \mathbf{V})$-algebras is given by a map $f : X \to Y$ in **Set** and a 2-cell $\varphi : fa \to b(Tf)$ which must preserve the units: $(\varphi e_X)(f\eta) = \eta f$. The 2-cell φ is completely determined by a family of $\mathbf{V}$-morphisms

$$f_{\mathfrak{x},x} : a(\mathfrak{x}, x) \longrightarrow b(Tf(\mathfrak{x}), f(x)),$$

$x \in X$, $\mathfrak{x} \in TX$, and preservation of units now reads as $f_{e_X(x),x}\eta_x = \eta_{f(x)}$ for all $x \in X$. For simplicity, *we write* f *instead of* (f, φ), and when we write

$$f_{\mathfrak{x},x} : a(\mathfrak{x}, x) \longrightarrow b(\mathfrak{y}, y)$$

this automatically entails $\mathfrak{y} = Tf(\mathfrak{x})$ *and* $y = f(x)$; these are the $\mathbf{V}$-*components* of the homomorphism f. Composition of (f, φ) with $(g, \psi) : (Y, b) \to (Z, c)$ is defined by

$$(g, \psi)(f, \varphi) = (gf, (\psi(Tf))(g\varphi))$$

which, in the notation used more frequently, means

$$(gf)_{\mathfrak{x},x} = (a(\mathfrak{x}, x) \xrightarrow{f_{\mathfrak{x},x}} b(\mathfrak{y}, y) \xrightarrow{g_{\mathfrak{y},y}} c(\mathfrak{z}, z)).$$

We obtain the category $\mathrm{Alg}(T, \mathbf{V})$ (denoted by $\mathrm{Alg}(T, e; \mathbf{V})$ in [9]).

3.2. Let $\mathbf{V}$ be finitely complete. The pullback (W, d) of $f : (X, a) \to (Z, c)$ and $g : (Y, b) \to (Z, c)$ in $\mathrm{Alg}(T, \mathbf{V})$ is constructed by the pullback $W = X \times_Z Y$ in **Set** and a family of pullback diagrams in $\mathbf{V}$, as follows:

$$\begin{tikzcd} d(\mathfrak{w}, w) \arrow[r, "f'_{\mathfrak{w},w}"] \arrow[d, "g'_{\mathfrak{w},w}"'] & b(\mathfrak{y}, y) \arrow[d, "g_{\mathfrak{y},y}"] \\ a(\mathfrak{x}, x) \arrow[r, "f_{\mathfrak{x},x}"] & c(\mathfrak{z}, z) \end{tikzcd}$$

for all $w \in W$; hence,

$$d(\mathfrak{w}, w) = a(Tg'(\mathfrak{w}), g'(w)) \times_c b(Tf'(\mathfrak{w}), f'(w))$$

in $\mathbf{V}$, where $g' : W \to X$ and $f' : W \to Y$ are the pullback projections in **Set**. For each $w = (x, y)$ in W, we define $\eta_w := < \eta_x, \eta_y >$.

3.3. Every set X carries the *discrete* $(T, \mathbf{V})$-*structure* e_X°. In fact, the 2-cell $\eta : 1_X \to e_X^\circ e_X$ making (X, e_X°) a $(T, \mathbf{V})$-algebra is just the unit of the adjunction $e_X \dashv e_X^\circ$ in $\mathrm{Mat}(\mathbf{V})$. Now $X \mapsto (X, e_X^\circ)$ defines the left adjoint of the forgetful functor

$$\mathrm{Alg}(T, \mathbf{V}) \longrightarrow \mathbf{Set}$$

since every map $f : X \to Y$ into a $(T, \mathbf{V})$-algebra (Y, b) becomes a homomorphism $f : (X, e_X^\circ) \to (Y, b)$; indeed the needed 2-cell $fe_X^\circ \to b(Tf)$ is obtained from the unit 2-cell $\eta : 1 \to be_Y$ with the adjunction $e_X \dashv e_X^\circ$: it is the mate of $f\eta : f \to be_Y f = b(Tf)e_X$. In pointwise notation, for

$$f_{\mathfrak{x},x} : e_X^\circ(\mathfrak{x}, x) \longrightarrow b(\mathfrak{y}, y)$$

one has $f_{\mathfrak{x},x} = 1_I$ if $e_X(x) = \mathfrak{x}$; otherwise its domain is the initial object 0 of $\mathbf{V}$, i.e. it is *trivial*.

3.4. We consider the discrete structure in particular on a one-element set 1. Then, for every $(T, \mathbf{V})$-algebra (X, a), an element $x \in X$ can be equivalently considered as a homomorphism $x : (1, e_1^\circ) \to (X, a)$ whose only non-trivial component is the unit $\eta_x : I \to a(e_X(x), x)$.

3.5. Assume $\mathbf{V}$ to be complete and locally cartesian closed. For a homomorphism $f : (X, a) \to (Y, b)$ and an additional $(T, \mathbf{V})$-algebra (Z, c) we form a substructure of the partial product of the underlying **Set**-data (see [10]), namely

$$\begin{tikzcd} Z & Q \arrow[l, "\mathrm{ev}"'] \arrow[r, "q"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ & P \arrow[r, "p"] & Y, \end{tikzcd} \tag{2}$$

with

$$P = Z^f = \{(s, y) \,|\, y \in Y,\ s : (X_y, a_y) \to (Z, c)\},$$

$$Q = Z^f \times_Y X = \{(s, x) \,|\, x \in X,\ s : (X_{f(x)}, a_{f(x)}) \to (Z, c)\},$$

where $(X_y = f^{-1}y, a_y)$ is the domain of the pullback

$$i_y : (X_y, a_y) \longrightarrow (X, a)$$

of $y : (1, e_1^\circ) \to (Y, b)$ along f. Of course, p and q are projections, and ev is the evaluation map. We must find a structure $d : TP \nrightarrow P$ which, together with a 2-cell η, will make these maps morphisms in $\mathrm{Alg}(T, \mathbf{V})$.

For $(s, y) \in P$ and $\mathfrak{p} \in TP$, in order to define $d(\mathfrak{p}, (s, y))$, consider each pair $x \in X$ and $\mathfrak{q} \in TQ$ with $f(x) = y$ and $Tf'(\mathfrak{q}) = \mathfrak{p}$ and form the partial product

$$\begin{tikzcd} c(\mathfrak{z}, s(x)) & c(\mathfrak{z}, s(x))^{f_{\mathfrak{x},x}} \times_b a(\mathfrak{x}, x) \arrow[l, "\tilde{\mathrm{ev}}_{\mathfrak{q},x}"'] \arrow[r] \arrow[d] & a(\mathfrak{x}, x) \arrow[d, "f_{\mathfrak{x},x}"] \\ & c(\mathfrak{z}, s(x))^{f_{\mathfrak{x},x}} \arrow[r, "\tilde{p}_{\mathfrak{q},x}"] & b(\mathfrak{y}, y) \end{tikzcd} \tag{3}$$

in $\mathbf{V}$, where $\mathfrak{z} = T\mathrm{ev}(\mathfrak{q})$, and then the multiple pullback $d(\mathfrak{p}, (s, y))$ of the morphisms $\tilde{p}_{\mathfrak{q},x}$ in $\mathbf{V}$, as in:

$$\begin{tikzcd} & c(\mathfrak{z}, s(x))^{f_{\mathfrak{x},x}} \arrow[dr, "\tilde{p}_{\mathfrak{q},x}"] & \\ d(\mathfrak{p}, (s, y)) \arrow[ur, "\pi_{\mathfrak{q},x}"] \arrow[rr, "p_{\mathfrak{p},(s,y)}"'] & & b(\mathfrak{y}, y). \end{tikzcd}$$

3.6. We define the 2-cell $\eta : 1_P \to de_P$ componentwise. Let $(s, y) \in P$ and consider each $x \in X$ and $\mathfrak{q} \in TQ$ with $f(x) = y$ and $Tf'(\mathfrak{q}) = e_P(s, y) = T(s, y)e_1$ (where $(s, y) : 1 \to P$). Consider the pullback $j_y : X_y \to Q$ of $(s, y) : 1 \to P$ along f' in $\mathbf{Set}$; whence, $j_y(x) = s(x)$. By (BC) there is $\mathfrak{x} \in TX_y$ such that $Tj_y(\mathfrak{x}) = \mathfrak{q}$ and $T!(\mathfrak{x}) = e_1(*)$ (where $! : X_y \to 1$ and $*$ is the only point of 1). Since $\mathrm{ev}j_y = s$, we may form the diagram

$$\begin{tikzcd} c(\mathfrak{z}, s(x)) & a_y(\mathfrak{x}, x) \arrow[l, "s_{\mathfrak{x},x}"'] \arrow[r, "(i_y)_{\mathfrak{x},x}"] \arrow[d] & a(\mathfrak{x}, x) \arrow[d, "f_{\mathfrak{x},x}"] \\ & I \arrow[r, "\eta_y"] & b(e_Y(y), y) \end{tikzcd}$$

in $\mathbf{V}$, where $\mathfrak{z} = T\mathrm{ev}(\mathfrak{q}) = Ts(\mathfrak{x})$, and the square is a pullback. The universal property of (3) guarantees the existence of $\tilde{\eta}_{\mathfrak{q},x} : I \to c(\mathfrak{z}, s(x))^{f_{\mathfrak{x},x}}$ such that $\tilde{p}_{\mathfrak{q},x}\tilde{\eta}_{\mathfrak{q},x} = \eta_y$ and $\tilde{\mathrm{ev}}_{\mathfrak{q},x}(\tilde{\eta}_{\mathfrak{q},x} \times_b 1) = s_{\mathfrak{x},x}$. Then, with the multiple pullback property, the morphisms $\tilde{\eta}_{\mathfrak{q},x}$ define jointly $\eta_{(s,y)} : I \to d(e_P(s, y), (s, y))$.

3.7. Theorem. *If the pointed* $\mathbf{Set}$*-functor T satisfies (BC) and* $\mathbf{V}$ *is complete and locally cartesian closed, then also* $\mathrm{Alg}(T, \mathbf{V})$ *is locally cartesian closed.*

Proof. Continuing in the notation of 3.5 and 3.6, we equip Q with the lax algebra structure $r : TQ \nrightarrow Q$ that makes the square of diagram (2) a pullback diagram in $\mathrm{Alg}(T, \mathbf{V})$. Then the 2-cell defined by

$$r(\mathfrak{q}, (s, x)) \xrightarrow{\pi_{\mathfrak{q},x} \times_b 1} c(\mathfrak{z}, s(x))^{f_{\mathfrak{x},x}} \times_b a(\mathfrak{x}, x) \xrightarrow{\tilde{\mathrm{ev}}_{\mathfrak{q},x}} c(\mathfrak{z}, s(x))$$

makes $\mathrm{ev} : (Q, r) \to (Z, c)$ a homomorphism.

In order to prove the universal property of the partial product, given any other pair $(h : (L, u) \to (Y, b), k : (M, v) \to (Z, c))$, where $M := L \times_Y X$, we consider the map $t : L \to P$, defined by $t(l) := (s_l, h(l))$, with

$$((X_{h(l)}, a_{h(l))}) \xrightarrow{s_l} (Z, c)) = ((X_{h(l)}, a_{h(l)}) \xrightarrow{j_l} (M, v) \xrightarrow{k} (Z, c)),$$

where j_l is the pullback of $l : (1, e_1^\circ) \to (L, u)$ along $f'' : (M, v) \to (L, u)$. We remark that in the commutative diagram

$$\begin{tikzcd}
Z & & Q \arrow[ll, "\mathrm{ev}"'] \arrow[rr, "q"] \arrow[dd, "f'"] & & X \arrow[dd, "f"] \\
& M \arrow[ul, "k"] \arrow[ur, "t'"] \arrow[dd, "f''"'] & & X_{h(l)} \arrow[ll, "j_l"'] \arrow[ur, "i_{h(l)}"] \arrow[dd] & \\
& & P \arrow[rr, "p"] & & Y \\
& L \arrow[ur, "t"] & & 1 \arrow[ll, "l"] \arrow[ur, "h(l)"'] &
\end{tikzcd}$$

every vertical face of the cube is a pullback in **Set**.

Now, for each $l \in L$ and $\mathfrak{l} \in L$ we define $t_{\mathfrak{l},l} : u(\mathfrak{l}, l) \to d(Tt(\mathfrak{l}), t(l))$ componentwise. Since $\mathrm{ev}t' = k$ we observe that Tk factors through the comparison map $\kappa : TM \to TL \times_{TP} TQ$, defined by the diagram

$$\begin{tikzcd}
TM \arrow[dr, dashed, "\kappa"] \arrow[drr, bend left, "Tt'"] \arrow[ddr, bend right, "Tf''"'] & & \\
& TL \times_{TP} TQ \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & TQ \arrow[d, "Tf'"] \\
& TL \arrow[r, "Tt"] & TP;
\end{tikzcd}$$

that is $Tk = (T\mathrm{ev})(Tt') = (T\mathrm{ev})\pi_2\kappa$. Since also kv factors through κ, i.e., $kv = k\tilde{v}\kappa$, with (BC) we conclude that the 2-cell $kv \to c(Tk)$ is of the form

$$\begin{tikzcd}
M \arrow[r, "\kappa"] & TL \times_{TP} TQ \arrow[r, bend left=60, "k\tilde{v}"{name=U}] \arrow[r, bend right=60, "(T\mathrm{ev})\pi_2"'{name=D}] & Z. \arrow[from=U, to=D, Rightarrow, "\varphi"]
\end{tikzcd}$$

For each $x \in X$ and $\mathfrak{q} \in TQ$ such that $f(x) = h(l)$ and $Tf'(\mathfrak{q}) = Tt(\mathfrak{l})$, let $\mathfrak{m} \in TM$ be such that $(Tf'')(\mathfrak{m}) = \mathfrak{l}$ and $(Tt')(\mathfrak{m}) = \mathfrak{q}$. In the diagram

$$\begin{tikzcd}
c(\mathfrak{z}, s_l(x)) & v(\mathfrak{m}, (l, x)) \arrow[l, "k_{\mathfrak{m},(l,x)}"'] \arrow[r] \arrow[d] & a(\mathfrak{x}, x) \arrow[d, "f_{\mathfrak{x},x}"] \\
& u(\mathfrak{l}, l) \arrow[r, "h_{\mathfrak{l},l}"] & b(\mathfrak{y}, y)
\end{tikzcd}$$

in $\mathbf{V}$ one has $\mathfrak{z} = (T\mathrm{ev})(\mathfrak{q})$ and the morphism $k_{\mathfrak{m},(l,x)}$ depends only on $\mathfrak{q}$ and $\mathfrak{l}$. Moreover, the square is a pullback, hence there is a $\mathbf{V}$-morphism $\tilde{t}_{\mathfrak{l},l} : u(\mathfrak{l}, l) \to c(\mathfrak{z}, s_l(x))^{f_{\mathfrak{x},x}}$ such that $\tilde{p}_{\mathfrak{q},x}\tilde{t}_{\mathfrak{l},l} = h_{\mathfrak{l},l}$ and $k_{\mathfrak{m},(l,x)}(\tilde{t}_{\mathfrak{l},l} \times_b 1) = \tilde{\mathrm{ev}}_{\mathfrak{q},x}$. With the multiple pullback property, the morphisms $\tilde{t}_{\mathfrak{l},l}$ define the unique 2-cell that makes $t : (L, u) \to (P, d)$ a homomorphism. ∎

If in the proof we take for (Y, b) the terminal object of $\mathrm{Alg}(T, \mathbf{V})$, that is, the pair $(1, \top)$ where the lax structure $\top$ is constantly equal to the terminal object of $\mathbf{V}$, we conclude:

3.8. Corollary. *If the pointed* **Set***-functor* T *satisfies (BC) and* $\mathbf{V}$ *is complete and cartesian closed, then also* $\mathrm{Alg}(T, \mathbf{V})$ *is cartesian closed.*

We explain now the strength of our Beck-Chevalley condition.

3.9. Proposition. *For* T *and* $\mathbf{V}$ *as in 2.5, let* $\mathbf{V}(I, 0) = \emptyset$. *Then:*

(a) *If* T *satisfies (BC), then* T *transforms pullbacks into weak pullbacks. The two conditions are actually equivalent when* $\mathbf{V}$ *is thin (i.e. a preordered class).*

(b) *If* $\mathbf{V}$ *is not thin, satisfaction of (BC) by* T *is equivalent to preservation of pullbacks by* T.

(c) *If* $\mathbf{V}$ *is cartesian closed, with* $I = 1$ *the terminal object, then* T *satisfies (BC) if and only if* $(Tf)^\circ Tg = Tk(Th)^\circ$, *for every pullback diagram*

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd} \tag{4}$$

in **Set**.

Proof. (a) Let $\kappa : TW \to U$ be the comparison map of diagram (1). By (BC) the 2-cell $\kappa\eta : \kappa \to \kappa\kappa^\circ\kappa$ is the image by $-\kappa$ of a 2-cell $\sigma : 1_U \to \kappa\kappa^\circ$. Hence, for each $u \in U$ there is a $\mathbf{V}$-morphism

$$I \to \kappa\kappa^\circ(u, u) = \sum_{\mathfrak{w} \in TW \,:\, \kappa(\mathfrak{w}) = u} \kappa(\mathfrak{w}, u).$$

Therefore the set $\{\mathfrak{w} \in TW \mid \kappa(\mathfrak{w}) = u\}$ cannot be empty, that is, κ is surjective.

If $\mathbf{V}$ is thin and κ is surjective, there is a (necessarily unique) 2-cell $1_U \to \kappa\kappa^\circ$. Then each 2-cell $\psi : \kappa r \to \kappa s$ induces a 2-cell $\varphi : r \to s$ defined by

$$r \xrightarrow{r\sigma} r\kappa\kappa^\circ \xrightarrow{\psi\kappa^\circ} s\kappa\kappa^\circ \xrightarrow{s\varepsilon} s$$

whose image under $-\kappa$ is necessarily ψ.

(b) If T preserves pullbacks, then κ is an isomorphism and (BC) holds.

Conversely, let T satisfy (BC) and let $\kappa : TW \to U$ be a comparison map as in (1). We consider $\mathfrak{w}_0$, $\mathfrak{w}_1 \in TW$ with $\kappa(\mathfrak{w}_0) = \kappa(\mathfrak{w}_1)$ and $\mathbf{V}$-morphisms $\alpha, \beta : v \to v'$ with $\alpha \neq \beta$, and define $r : U \times U \to \mathbf{V}$ by $r(u, u') = v$ and $s : U \times U \to \mathbf{V}$ by $s(u, u') = v'$.

The 2-cell $\psi : r\kappa \to s\kappa$, with $\psi_{\mathfrak{w},u} = \alpha$ if $\mathfrak{w} = \mathfrak{w}_0$ and $\psi_{\mathfrak{w},u} = \beta$ elsewhere, factors through κ only if $\mathfrak{w}_0 = \mathfrak{w}_1$.

(c) For any commutative diagram (4) there is a 2-cell $kh^\circ \to f^\circ g$, defined by

$$kh^\circ \xrightarrow{\eta kh^\circ} f^\circ fkh^\circ = f^\circ ghh^\circ \xrightarrow{f^\circ g\varepsilon} f^\circ g,$$

which is an identity morphism in case the diagram is a pullback.

If T satisfies (BC) and $\mathbf{V}$ is not thin, the equality $Tk(Th)^\circ = (Tf)^\circ Tg$ follows from (b). If $\mathbf{V}$ is thin, then in the diagram (1) the 2-cell $\sigma : 1 \to \kappa\kappa^\circ$ considered in (a) gives rise to a 2-cell

$$(Tf)^\circ Tg = \pi_2\pi_1^\circ \xrightarrow{\pi_2\sigma\pi_1^\circ} \pi_2\kappa\kappa^\circ\pi_1^\circ = Tk(Th)^\circ,$$

and the equality follows.

Conversely, the equality $(Tf)^\circ Tg = Tk(Th)^\circ$ guarantees the surjectivity of κ, hence (BC) follows in case $\mathbf{V}$ is thin, by (a). If $\mathbf{V}$ is not thin, we first observe that a coproduct $\sum_X I$ is isomorphic to I only if X is a singleton, due to the cartesian closedness of $\mathbf{V}$. Now, $(Tf)^\circ Tg = Tk(Th)^\circ$ means that, for every $\mathfrak{z} \in TZ$ and $\mathfrak{x} \in TX$ with $Tg(\mathfrak{z}) = Tf(\mathfrak{x})$,

$$\begin{aligned} I = Tf(\mathfrak{x}, Tg(\mathfrak{z})) &= Tf^\circ Tg(\mathfrak{z},\mathfrak{x}) = TkTh^\circ(\mathfrak{z},\mathfrak{x}) \\ &= \sum\{I \mid \mathfrak{w} \in TW \,:\, Tk(\mathfrak{w}) = \mathfrak{x} \;\&\; Th(\mathfrak{w}) = \mathfrak{z}\}. \end{aligned}$$

From this equality we conclude that there exists exactly one such $\mathfrak{w}$, i.e. $TW = TZ \times_{TY} TX$. ∎

3.10. Finally we remark that, in some circumstances, the 2-categorical part of (BC) is essential for local cartesian-closedness of $\mathrm{Alg}(T, \mathbf{V})$. Indeed, if $\mathbf{V}$ is extensive [4], T transforms pullback diagrams into weak pullback diagrams and $\mathrm{Alg}(T, \mathbf{V})$ is locally cartesian closed, then T satisfies (BC), as we show next. To check (BC) we consider a 2-cell $\psi : r\kappa \to s\kappa$, with $\kappa : TW \to U$ the comparison map of diagram (1) and $r, s : U \to S$. We need to check that $\psi = \varphi\kappa$ for a unique 2-cell $\varphi : r \to s$. This 2-cell exists, and it is unique if and only if

$$\forall \mathfrak{w}_0, \mathfrak{w}_1 \in TW \;\; \forall s \in S \;\; \kappa(\mathfrak{w}_0) = \kappa(\mathfrak{w}_1) \;\;\Rightarrow\;\; \psi_{\mathfrak{w}_0,s} = \psi_{\mathfrak{w}_1,s}.$$

For $v := r(\kappa(\mathfrak{w}_0), s)$ and $v' := s(\kappa(\mathfrak{w}_0), s)$, and $\alpha := \psi_{\mathfrak{w}_0,s}$ and $\beta = \psi_{\mathfrak{w}_1,s}$, we want to show that $\alpha = \beta$.

For that, in the pullback diagram (4) we consider structures a, b, c, d, on X, Y, Z and W respectively, constantly equal to $I + v$, with $\eta : I \to I + v$ the coproduct injection. For d' constantly equal to $I + v'$, in the diagram

$$\begin{tikzcd} (W, d') & (W, d) \arrow[l, "(\mathrm{id},\varepsilon)"'] \arrow[r, "(k,1)"] \arrow[d, "(h,1)"'] & (X, a) \arrow[d, "(f,1)"] \\ & (Z, c) \arrow[r, "(g,1)"] & (Y, b) \end{tikzcd}$$

we define ε by:

$$\varepsilon_{\mathfrak{w},w} = \begin{cases} 1+\alpha & \text{if } \mathfrak{w} = \mathfrak{w}_0, \\ 1+\beta & \text{elsewhere.} \end{cases}$$

The square is a pullback. Hence the morphism $(\mathrm{id}, \varepsilon)$ factors through the partial product via $t \times_Y \mathrm{id}$, with $t : Z \to P$. Since the 2-cell of $t \times_Y \mathrm{id}$ is obtained by a pullback construction and $\kappa(\mathfrak{w}_0) = \kappa(\mathfrak{w}_1)$, its 2-cell "identifies" $\mathfrak{w}_0$ and $\mathfrak{w}_1$, hence $\varepsilon_{\mathfrak{w}_0,w} = \varepsilon_{\mathfrak{w}_1,w}$, that is, $1+\alpha = 1+\beta$. Therefore $\alpha = \beta$, by extensitivity of $\mathbf{V}$.

4. (Co)completeness of the category $\mathrm{Alg}(T, \mathbf{V})$

4.1. We assume $\mathbf{V}$ to be complete and cocomplete. The construction of limits in $\mathrm{Alg}(T, \mathbf{V})$ reduces to a combined construction of limits in $\mathbf{Set}$ and $\mathbf{V}$, as we show next.

The limit of a functor

$$\begin{array}{rcl} F : \mathbf{D} & \to & \mathrm{Alg}(T, \mathbf{V}) \\ D & \mapsto & (FD, a_D) \\ D \xrightarrow{f} E & \mapsto & (FD, a_D) \xrightarrow{Ff} (FE, a_E) \end{array}$$

is constructed in two steps.

First we consider the composition of F with the forgetful functor into $\mathbf{Set}$

$$\mathbf{D} \xrightarrow{\quad F \quad} \mathrm{Alg}(T, \mathbf{V}) \xrightarrow{\qquad} \mathbf{Set}, \tag{5}$$

and construct its limit in $\mathbf{Set}$

$$(L \xrightarrow{\quad p^D \quad} FD)_{D \in \mathbf{D}}.$$

Then, we define the $(T, \mathbf{V})$-algebra structure $a : TL \to\!\!\!\!\!\shortmid\;\, L$, that is the map $a : TX \times X \to \mathbf{V}$, pointwise. For every $\mathfrak{l} \in TL$ and $l \in L$, we consider now the functor

$$\begin{array}{rcl} F_{\mathfrak{l},l} : \mathbf{D} & \to & \mathbf{V} \\ D & \mapsto & a_D(Tp^D(\mathfrak{l}), p^D(l)) \\ D \xrightarrow{f} E & \mapsto & a_D(Tp^D(\mathfrak{l}), p^D(l)) \xrightarrow{Ff_{Tp^D(\mathfrak{l}),p^D(l)}} a_E(Tp^E(\mathfrak{l}), p^E(l)) \end{array}$$

and its limit in $\mathbf{V}$

$$(a(\mathfrak{l}, l) \xrightarrow{\quad p^D_{\mathfrak{l},l} \quad} a_D(Tp^D(\mathfrak{l}), p^D(l)))_{D \in \mathbf{D}}.$$

This equips $p^D : (L, a) \to (FD, a_D)$ with a 2-cell $p^D a \to a_D T p^D$.

By construction

$$(L, a) \xrightarrow{\quad p^D \quad} (FD, a_D) \tag{6}$$

is a cone for F. To check that it is a limit, let

$$(Y, b) \xrightarrow{\quad g^D \quad} (FD, a_D)$$

be a cone for F. By construction of (L, p^D), there exists a map $t : Y \to L$ such that $p^D t = g^D$ for each $D \in \mathbf{D}$. For each $\mathfrak{y} \in TY$ and $y \in Y$,

$$b(\mathfrak{y}, y) \xrightarrow{g^D_{\mathfrak{y},y}} a_D(Tp^D(Tt(\mathfrak{y})), p^D(t(y)))$$

is a cone for the functor $F_{Tt(\mathfrak{y}),t(y)}$. Hence, by construction of $a(Tt(\mathfrak{y}), t(y))$, there exists a unique $\mathbf{V}$-morphism $t_{\mathfrak{y},y}$ making the diagram

$$\begin{tikzcd}
a(Tt(\mathfrak{y}), t(y)) \arrow[r, "p^D_{\mathfrak{y},y}"] & a_D(Tp^D(Tt(\mathfrak{y})), p^D(t(y))) \\
b(\mathfrak{y}, y) \arrow[u, dashed, "t_{\mathfrak{y},y}"] \arrow[ur, "g^D_{\mathfrak{y},y}"'] &
\end{tikzcd}$$

commutative. These $\mathbf{V}$-morphisms define pointwise the unique 2-cell $gb \to p^D a$.

For each $l \in L$, $\eta_l : I \to a(e_L(l), l)$ is the morphism induced by the cone

$$(\eta^D_{p^D(l),p^D(l)} : I \to a_D(e_{FD}(p^D(l)), p^D(l)))_{D \in \mathbf{D}}.$$

4.2. Cocompleteness. To construct the colimit of a functor $F : \mathbf{D} \to \mathrm{Alg}(T, \mathbf{V})$ we first proceed analogously to the limit construction. That is, we form the colimit in **Set**

$$(FD \xrightarrow{i^D} Q)_{D \in \mathbf{D}}$$

of the functor (5).

To construct the structure $c : TQ \nrightarrow Q$, for each $\mathfrak{q} \in TQ$ and $q \in Q$, we consider the functor $F^{\mathfrak{q},q} : \mathbf{D} \to \mathbf{V}$, with

$$F^{\mathfrak{q},q}(D) = \sum \{a_D(\mathfrak{x}, x) \,|\, Ti^D(\mathfrak{x}) = \mathfrak{q},\ i^D(x) = q\},$$

and, for $f : D \to E$, the morphism $F^{\mathfrak{q},q}(f) : F^{\mathfrak{q},q}(D) \to F^{\mathfrak{q},q}(E)$ is induced by

$$a_D(\mathfrak{x}, x) \xrightarrow{Ff_{\mathfrak{x},x}} a_E(Tf(\mathfrak{x}), f(x)) \longrightarrow \sum \{a_E(\mathfrak{y}, y) \,|\, Ti^E(\mathfrak{y}) = \mathfrak{q},\ i^E(y) = q\} = F^{\mathfrak{q},q}(E).$$

and denote by $\tilde{c}(\mathfrak{q}, q)$ the colimit of $F^{\mathfrak{q},q}$. If $\mathfrak{q} \neq e_Q(q)$ for $q \in Q$, then $\tilde{c}(\mathfrak{q}, q)$ is in fact the structure $c(\mathfrak{q}, q)$ on the colimit. For $\mathfrak{q} = e_Q(q)$, the multiple pushout

$$\begin{tikzcd}
 & & \tilde{c}(e_Q(q), q) \arrow[ddr] & \\
 & a_D(e_{FD}(x), x) \arrow[ur, "i^D_{e_{FD}(x),x}"] & & \\
I \arrow[ur, "\eta_x"] \arrow[rrr, "\eta_q"] & & & c(e_Q(q), q),
\end{tikzcd}$$

defines $c(e_Q(q), q)$, with $D \in \mathbf{D}$ and $x \in FD$ such that $i^D(x) = q$.

5. Representability of partial morphisms

5.1. Let $\mathcal{S}$ be a pullback-stable class of morphisms of a category $\mathbf{C}$. An $\mathcal{S}$-*partial map from X to Y* is a pair $(X \xleftarrow{s} U \longrightarrow Y)$ where $s \in \mathcal{S}$. We say that $\mathcal{S}$ *has a classifier* if there is a morphism true : $1 \to \tilde{1}$ in $\mathcal{S}$ such that every morphism in $\mathcal{S}$ is, in a unique way, a pullback of true; $\mathbf{C}$ *has $\mathcal{S}$-partial map classifiers* if, for every $Y \in \mathbf{C}$, there is a morphism $\mathrm{true}_Y : Y \to \tilde{Y}$ in $\mathcal{S}$ such that every $\mathcal{S}$-partial map $(X \xleftarrow{s} U \longrightarrow Y)$ from X to Y can be uniquely completed so that the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "s"'] & Y \arrow[d, "\mathrm{true}_Y"] \\ X \arrow[r, dashed] & \tilde{Y}. \end{tikzcd}$$

is a pullback.

From Corollary 4.6 of [10] it follows that:

5.2. Proposition. *If $\mathcal{S}$ is a pullback-stable class of morphisms in a finitely complete locally cartesian-closed category $\mathbf{C}$, then the following assertions are equivalent:*

(i) *$\mathcal{S}$ has a classifier;*

(ii) *$\mathbf{C}$ has $\mathcal{S}$-partial map classifiers.*

■

5.3. Our goal is to investigate whether the category $\mathrm{Alg}(T, \mathbf{V})$ has $\mathcal{S}$-partial map classifiers, for the class $\mathcal{S}$ of extremal monomorphisms. For that we first observe:

5.4. Lemma. *An $\mathrm{Alg}(T, \mathbf{V})$-morphism $s : (U, c) \to (X, a)$ is an extremal monomorphism if and only if the map $s : U \to X$ is injective and, for each $\mathfrak{u} \in TU$ and $u \in U$, $s_{\mathfrak{u},u} : c(\mathfrak{u}, u) \to a(\mathfrak{x}, x)$ is an isomorphism in $\mathbf{V}$.*

5.5. Proposition. *In $\mathrm{Alg}(T, \mathbf{V})$ the class of extremal monomorphisms has a classifier.*

Proof. For $\tilde{1} = (1 + 1, \tilde{\top})$, where $\tilde{\top}$ is pointwise terminal, we consider the inclusion true : $1 \to \tilde{1}$ onto the first summand. For every extremal monomorphism $s : (U, c) \to (X, a)$, we define $\chi_U : (X, a) \to \tilde{1}$ with $\chi_U : X \to 1 + 1$ the characteristic map of $s(U)$, and the 2-cell constantly $! : a(\mathfrak{x}, x) \to 1$. Then the diagram below

$$\begin{tikzcd} (U, s) \arrow[r, "!"] \arrow[d, "s"'] & 1 \arrow[d, "\mathrm{true}"] \\ (X, a) \arrow[r, "\chi_U"] & \tilde{1} \end{tikzcd}$$

is a pullback diagram; it is in fact the unique possible diagram that presents s as a pullback of true. ■

Using Theorem 3.7 and Proposition 5.5, we conclude that:

5.6. Theorem. *If the pointed* **Set***-functor* T *satisfies (BC) and* $\mathbf{V}$ *is a complete and cocomplete locally cartesian closed category, then* $\mathrm{Alg}(T,\mathbf{V})$ *is a quasitopos.* ∎

5.7. Remark. Representability of (extremal mono)-partial maps can also be proved directly, and in this way one obtains a slight improvement of Theorem 5.6: $\mathrm{Alg}(T,\mathbf{V})$ is a quasi-topos whenever T satisfies (BC) and $\mathbf{V}$ is a complete and cocomplete cartesian closed category, not necessarily locally so.

6. Examples.

6.1. We start off with the trivial functor T which maps every set to a terminal object 1 of **Set**. T preserves pullbacks. Choosing for I the top element of any (complete) lattice $\mathbf{V}$ we obtain with $\mathrm{Alg}(T,\mathbf{V})$ nothing but the topos **Set**. This shows that local cartesian closedness of $\mathbf{V}$ is not a necessary condition for local cartesian closedness of $\mathrm{Alg}(T,\mathbf{V})$. We also note that T does not carry the structure of a monad.

If, for the same T, we choose $\mathbf{V} = \mathbf{Set}$, then $\mathrm{Alg}(T,\mathbf{Set})$ is the formal coproduct completion of the category $\mathbf{Set}_*$ of pointed sets, i.e. $\mathrm{Alg}(T,\mathbf{Set}) \cong \mathrm{Fam}(\mathbf{Set}_*)$.

6.2. Let $T = \mathrm{Id}$, $e = \mathrm{id}$. Considering for $\mathbf{V}$ as in [9] the two-element chain $\mathbf{2}$, the extended half-line $\overline{\mathbb{R}}_+ = [0,\infty]$ (with the natural order reversed), and the category **Set**, one obtains with $\mathrm{Alg}(T,\mathbf{V})$ the category of

- sets with a reflexive relation
- sets with a fuzzy reflexive relation
- reflexive directed graphs,

respectively.

More generally, if we let $TX = X^n$ for a non-negative integer n, with the same choices for $\mathbf{V}$ one obtains

- sets with a reflexive $(n+1)$-ary relation
- sets with a fuzzy reflexive $(n+1)$-ary relation
- reflexive directed "multigraphs" given by sets of vertices and of edges, with an edge having an ordered n-tuple of vertices as its source and a single edge as its target; reflexivity means that there is a distinguished edge $(x,\cdots,x) \to x$ for each vertex x.

Note that the case $n = 0$ encompasses Example 6.1.

6.3. For a fixed monoid M, let T belong to the monad T arising from the adjunction

$$\mathbf{Set}^M \underset{\longrightarrow}{\overset{\longleftarrow}{\perp}} \mathbf{Set},$$

i.e. $TX = M \times X$ with $e_X(x) = (0, x)$, with 0 neutral in M (writing the composition in M additively). T preserves pullbacks. The quasitopos $\mathrm{Alg}(T, \mathbf{Set})$ may be described as follows. Its objects are "M-normed reflexive graphs", given by a set X of vertices and sets $a(x, y)$ of edges from x to y which come with a "norm" $v_{x,y} : a(x, y) \to M$ for all $x, y \in X$; there is a distinguished edge $1_x : x \to x$ with $v_{x,x}(1_x) = 0$. Morphisms must preserve the norm. Of course, for trivial M we are back to directed graphs as in 6.2.

It is interesting to note that if one forms $\mathrm{Alg}(\mathsf{T}, \mathbf{Set})$ for the (untruncated) monad T (see [9]), then $\mathrm{Alg}(\mathsf{T}, \mathbf{Set})$ is precisely the comma category $\mathbf{Cat}/M$, where M is considered a one-object category; its objects are categories which come with a norm function v for morphisms satisfying $v(gf) = v(g) + v(f)$ for composable morphisms f, g.

6.4. Let $T = U$ be the ultrafilter functor, as mentioned in the Introduction. U transforms pullbacks into weak pullback diagrams. Hence, for $\mathbf{V} = \mathbf{2}$ we obtain with $\mathrm{Alg}(T, \mathbf{2})$ the quasitopos of pseudotopological spaces, and for $\mathbf{V} = \overline{\mathbb{R}}_+$ the quasitopos of (what should be called) quasiapproach spaces (see [9, 8]). If we choose for $\mathbf{V}$ the extensive category $\mathbf{Set}$, then the resulting category $\mathrm{Alg}(U, \mathbf{Set})$ is a rather naturally defined supercategory of the category of ultracategories (as defined in [9]) but fails to be locally cartesian closed, according to 3.9(b) and 3.10.

References

[1] J. Adámek, H. Herrlich and G.E. Strecker, *Abstract and concrete categories* (Wiley Interscience, New York 1990).

[2] M. Barr, Relational algebras, in: *Springer Lecture Notes in Math.* **137** (1970), pp. 39-55.

[3] A. Carboni, G.M. Kelly, D. Verity and R. Wood, A 2-categorical approach to change of base and geometric morphisms II, *Theory Appl. Categories* **4** (1998) 82-136.

[4] A. Carboni, S. Lack and R.F.C. Walters, Introduction to extensive and distributive categories, *J. Pure Appl. Algebra* **84** (1993) 145-158.

[5] M. M. Clementino and D. Hofmann, Triquotient maps via ultrafilter convergence, *Proc. Amer. Math. Soc.* **130** (2002) 3423-3431.

[6] M. M. Clementino and D. Hofmann, Topological features of lax algebras, *Applied Cat. Struct.* **11** (2003) 267-286.

[7] M. M. Clementino, D. Hofmann and W. Tholen, The convergence approach to exponentiable maps, *Port. Math.* **60** (2003) 139-160.

[8] M. M. Clementino, D. Hofmann and W. Tholen, One setting for all: metric, topology, uniformity, approach structure, *Appl. Categ. Structures* (to appear).

[9] M. M. Clementino and W. Tholen, Metric, Topology and Multicategory – A common approach, *J. Pure Appl. Algebra* **179** (2003) 13-47.

[10] R. Dyckhoff and W. Tholen, Exponentiable morphisms, partial products and pullback complements, *J. Pure Appl. Algebra* **49** (1987) 103-116.

[11] H. Herrlich, E. Lowen-Colebunders and F. Schwarz, Improving **Top**: **PrTop** and **PsTop**, in: *Category Theory at Work* (Heldermann Verlag, Berlin 1991), pp 21-34.

[12] F.W. Lawvere, Metric spaces, generalized logic, and closed categories, *Rend. Sem. Mat. Fis. Milano* **43** (1973) 135-166.

[13] S. Mac Lane, *Categories for the Working Mathematician*, 2nd. ed. (Springer, New York 1998).

[14] J. Reiterman and W. Tholen, Effective descent maps of topological spaces, *Top. Appl.* **57** (1994) 53-69.

[15] R. Street, Categorical structures, in: *Handbook of Algebra*, vol.1 (editor M. Hazewinkel) (Elsevier, Amsterdam 1996), pp. 529-577.

Dep. de Matemática, Univ. de Coimbra, 3001-454 Coimbra, PORTUGAL

Dep. de Matemática. Univ. de Aveiro, 3810-193 Aveiro, PORTUGAL

Dep. of Math. and Stat, York University, Toronto, CANADA M3J 1P3

Email: `mmc@mat.uc.pt`
`dirk@mat.ua.pt`
`tholen@mathstat.yorku.ca`

This article may be accessed via WWW at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/11/15/11-15.{dvi,ps}`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

The method of distribution of the journal is via the Internet tools `WWW/ftp`. The journal is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION. Individual subscribers receive (by e-mail) abstracts of articles as they are published. Full text of published articles is available in .dvi, Postscript and PDF. Details will be e-mailed to new subscribers. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

INFORMATION FOR AUTHORS. The typesetting language of the journal is TEX, and LATEX is the preferred flavour. TEX source of articles for publication should be submitted by e-mail directly to an appropriate Editor. They are listed below. Please obtain detailed information on submission format and style files from the journal's WWW server at `http://www.tac.mta.ca/tac/`. You may also write to `tac@mta.ca` to receive details by e-mail.