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THE STRONG AMALGAMATION PROPERTY AND (EFFECTIVE) CODESCENT MORPHISMS

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ABSTRACT. Codescent morphisms are described in regular categories which satisfy the so-called strong amalgamation property. Among varieties of universal algebras possessing this property are, as is known, categories of groups, not necessarily associative rings, M-sets (for a monoid M), Lie algebras (over a field), quasi-groups, commutative quasi-groups, Steiner quasi-groups, medial quasi-groups, semilattices, lattices, weakly associative lattices, Boolean algebras, Heyting algebras. It is shown that every codescent morphism of groups is effective.

1. Introduction

Throughout the paper, we use "(effective) codescent morphism" to mean "(effective) codescent morphism with respect to the *basic fibration*". Namely, let $\mathcal{C}$ be a category with pushouts and let $p : B \longrightarrow E$ be its morphism. There is an adjunction

$$B/\mathcal{C} \underset{p_*}{\overset{p^!}{\rightleftarrows}} E/\mathcal{C}$$

between coslice categories with the left adjoint p_* pushing out along p and the right adjoint $p^!$ composing with p from right. The induced comonad on $E/\mathcal{C}$ gives rise to the Eilenberg-Moore category of coalgebras (we denote it by Codes(p)) equipped with the comparision functor

$$\Phi_p : B/\mathcal{C} \longrightarrow \text{Codes }(p).$$

Recall that objects of Codes(p) (called codescent data with respect to p) are triples (C, γ, ξ), with $C \in \text{Ob } \mathcal{C}$ and γ, ξ being morphisms $E \longrightarrow C$ and $C \longrightarrow C \sqcup_B E$, respectively, such that the following equalities hold (see Figs.1 and 2):

$$\xi\,\gamma = \pi_2, \tag{1.1}$$

$$(1_C, \gamma)\xi = 1_C, \tag{1.2}$$

$$(\pi_1 \sqcup_B 1_E)\xi = (\xi \sqcup_B 1_E)\xi, \tag{1.3}$$

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while the functor Φ_p sends each $f : B \longrightarrow D$ to

$$\big(D \sqcup_B E, \pi_2', \pi_1' \sqcup_B 1_E\big),$$

where π_1' and π_2' are pushouts of p and f, respectively, along each other.

$$\begin{tikzcd}
B \arrow[r, "p"] \arrow[d, "p"'] & E \arrow[dd, "\pi_2"] \arrow[dddr, bend left, "\gamma"] & \\
E \arrow[d, "\gamma"'] \arrow[r, phantom, "ph"] & & \\
C \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\xi"'] \arrow[ddrr, bend right, "1_C"'] & C \sqcup_B E \arrow[dr, "(1_C,\gamma)"'] & \\
& & C
\end{tikzcd}$$

Fig. 1

$$\begin{tikzcd}
B \arrow[r, "p"] \arrow[d, "p"'] & E \arrow[dd, "\pi_2"] \arrow[dddr] & \\
E \arrow[d, "\gamma"'] & & \\
C \arrow[r, shift left, "\pi_1"] \arrow[r, shift right, "\xi"'] \arrow[d, shift right, "\pi_1"'] \arrow[d, shift left, "\xi"] & C \sqcup_B E \arrow[dr, dashed, shift left, "\xi \sqcup_B 1_E"] \arrow[dr, dashed, shift right, "\pi_1 \sqcup_B 1_E"'] & \\
C \sqcup_B E \arrow[rr] & & (C \sqcup_B E) \sqcup_B E
\end{tikzcd}$$

Fig. 2

p is called a codescent (effective codescent) morphism if Φ_p is full and faithful (an equivalence of categories), i.e. if p_* is precomonadic (comonadic).

It is well-known that if $\mathcal{C}$ is finitely cocomplete, then codescent morphisms are precisely couniversal regular monomorphisms, i.e. morphisms whose pushouts along any morphisms are regular monomorphisms [JT].

One of the goals of descent theory is to characterize (effective) codescent morphisms in particular varieties of universal algebras. There are several results obtained along this line. For instance, this problem has been completely solved for commutative rings with units (Joyal-Tierney (unpublished), [M]).

In this paper we describe codescent morphisms in those varieties of algebras (more generally, regular categories) which satisfy the strong amalgamation property and investigate the problem of the effectiveness of codescent morphisms in the case of groups.

Recall that the strong amalgamation property means that for any pushout square

$$\begin{tikzcd}
{} \arrow[r, "m"] \arrow[d, "\alpha"'] & {} \arrow[d, "\alpha'"] \\
{} \arrow[r, "m'"] & {}
\end{tikzcd}$$

with monomorphisms m, α, the morphisms m', α' are also monomorphisms and, moreover, we have

$$\mathrm{Im}\ m' \cap \mathrm{Im}\ \alpha' = \mathrm{Im}\ m'\alpha.$$

Among the varieties of universal algebras possessing this property are, as is known, categories of groups, not necessarily associative rings, M-sets (for a monoid M), Lie algebras (over a field), quasi-groups, commutative quasi-groups, Steiner quasi-groups, medial quasi-groups, semilattices, lattices, weakly associative lattices, Boolean algebras, Heyting algebras. We show that a monomorphism $p : B \rightarrowtail E$ of a variety of this kind is codescent if and only if

For any congruence R on B and its closure R' in E one has

$$R' \cap (B \times B) = R.$$

This in particular implies that every (regular) monomorphism of M-sets, semilattices, Boolean algebras and Heyting algebras is codescent. Note that the latter result related to the case of Boolean algebras was obtained earlier by Makkai (an unpublished work).

The main result of this paper asserts that

Every codescent morphism of groups is effective.

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2. The Strong Amalgamation Property and Codescent Morphisms

Before we begin our discussion, let us recall some definitions [KMPT].

Let $\mathcal{C}$ be a category equipped with some proper factorization system $(\mathbb{E}, \mathbb{M})$, i.e., a factorization system which satisfies the conditions $\mathbb{E} \subset \mathrm{Epi}\ \mathcal{C}$ and $\mathbb{M} \subset \mathrm{Mon}\ \mathcal{C}$.

$\mathcal{C}$ is said to satisfy the amalgamation (transferability; congruence extension) property if for each span

$$\begin{tikzcd} {} \arrow[r, "m"] \arrow[d, "\alpha"'] & {} \\ {} & \end{tikzcd} \tag{2.1}$$

with $m, \alpha \in \mathbb{M}$ ($m \in \mathbb{M}$; $m \in \mathbb{M}$ and $\alpha \in \mathbb{E}$) there exists a commutative square

$$\begin{tikzcd} {} \arrow[r, "m"] \arrow[d, "\alpha"'] & {} \arrow[d, "\alpha'"] \\ {} \arrow[r, "m'"] & {} \end{tikzcd} \tag{2.2}$$

with $m', \alpha' \in \mathbb{M}$ ($m' \in \mathbb{M}$; $m' \in \mathbb{M}$). $\mathcal{C}$ is said to satisfy the strong amalgamation property (the intersection property of amalgamation) if any span (2.1) with $m, \alpha \in \mathbb{M}$ (any commutative diagram (2.2) with $m, \alpha, m', \alpha' \in \mathbb{M}$) admits a pullback

$$\begin{tikzcd} {} \arrow[r, "m"] \arrow[d, "\alpha"'] & {} \arrow[d, "\alpha''"] \\ {} \arrow[r, "m''"'] & {} \end{tikzcd} \tag{2.3}$$

with $m'', \alpha'' \in \mathbb{M}$. In the case of a variety $\mathcal{C}$ of universal algebras and $\mathbb{M} = \mathrm{Mon}\ \mathcal{C}$, this definition is equivalent to that of the strong amalgamation property given in the Introduction, as follows from the arguments given below.

Clearly, $\mathcal{C}$ possesses the strong amalgamation property if and only if it satisfies both the amalgamation and the intersection property of amalgamation. If $\mathcal{C}$ admits finite products, then the transferability property is equivalent to the amalgamation and congruence extension properties [2]. The latter statement remains true if we replace "finite products" by "pushouts". In that case all the definitions presented above can be reformulated. Namely, the amalgamation (transferability; congruence extension) property is equivalent to the requirement that the class $\mathbb{M}$ be stable under pushouts along $\mathbb{M}$-morphisms (any morphisms; $\mathbb{E}$-morphisms), while the strong amalgamation property is satisfied if and only if the amalgamation property is fulfilled and pushout (2.2) is also a pullback square for any $m, \alpha \in \mathbb{M}$.

In what follows we make use of the following known result.

2.1. Theorem. *Let $\mathcal{C}$ admit pushouts. Then the following conditions are equivalent:*

(i) *$\mathcal{C}$ satisfies the intersection property of amalgamation;*

(ii) *$\mathbb{M}$ consists of all regular monomorphisms.*

If, in addition, $\mathcal{C}$ satisfies the amalgamation property, then each of (i), (ii) *is equivalent to*

(iii) *every epimorphism lies in $\mathbb{E}$, and hence $\mathbb{E} = \mathrm{Epi}\ \mathcal{C}$.*

The implication (i)⇒(ii) is obvious. (ii)⇒(i) is proved in [K], [Ri], [T], while the equivalence (i)⇔(iii) is proved in [Ri].

Most of investigations related to the question whether concrete categories satisfy the above-discussed properties deal with the case, where

$$\mathbb{M} = \text{all monomorphisms}. \tag{2.4}$$

From now on *we shall follow assumption* (2.4).

2.2. Proposition. *Let $\mathcal{C}$ be a regular category with pushouts and let $\mathcal{C}$ satisfy the strong amalgamation property. Then the following conditions are equivalent for a monomorphism $f : B \rightarrowtail E$:*

(i) *f is a codescent morphism;*

(ii) *f is a couniversal regular monomorphism;*

(iii) *f is a $\{$Regular epis$\}$-couniversal regular monomorphism*[3]*;*

(iv) *f is an $\{$Epis$\}$-couniversal monomorphism;*

[2]We have to apply the transferability property to both "sides" and after that to take the product.

[3]Similarly to the notion of a couniversal regular monomorphism, for arbitrary morphism classes $\mathcal{E}$ and $\mathcal{M}$ one can define a $\mathcal{E}$-couniversal $\mathcal{M}$-morphism as a morphism whose pushout along any $\mathcal{E}$-morphism lies in $\mathcal{M}$.

(v) *for any kernel pair* $R \overset{\alpha_1}{\underset{\alpha_2}{\rightrightarrows}} B$ *on* B *and the kernel pair* $R' \overset{\alpha'_1}{\underset{\alpha'_2}{\rightrightarrows}} E$ *of the coequalizer of* $(f\alpha_1, f\alpha_2)$, *the square*

$$\begin{tikzcd} R \arrow[r, dashed] \arrow[d, "(\alpha_1,\alpha_2)"'] & R' \arrow[d, "(\alpha'_1,\alpha'_2)"] \\ B \times B \arrow[r, tail, "f\times f"'] & E \times E \end{tikzcd} \tag{2.5}$$

is a pullback.

Each (*regular*) *monomorphism of* $\mathcal{C}$ *is a codescent morphism if and only if* $\mathcal{C}$ *satisfies additionally the congruence extension* (*or, equivalently, the transferability property*).

If $\mathcal{C}$ *is* (*Barr-*)*exact, then one can replace the first "kernel pair" in* (v) *by "equivalence relation".*

Proof. Let us first indicate some properties of the considered categories $\mathcal{C}$. The pair (*Regular epis, Monos*) is a factorization system on $\mathcal{C}$. Moreover, by Theorem 2.1 each monomorphism as well as each epimorphism is regular and for any diagram

$$\begin{tikzcd} & {} \arrow[d, "e"] \\ {} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & {} \end{tikzcd} \tag{2.6}$$

with e being an epimorphism there exists a diagram

$$\begin{tikzcd} {} \arrow[r, shift left, "\overline{\alpha}"] \arrow[r, shift right, "\overline{\beta}"'] \arrow[d, "\overline{e}"'] & {} \arrow[d, "e"] \\ {} \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & {} \end{tikzcd} \tag{2.7}$$

such that $\overline{e}$ is also an epimorphism and $e\overline{\alpha} = \alpha\overline{e}$, $e\overline{\beta} = \beta\overline{e}$ [4].

The equivalences (ii)⇔(iii)⇔(iv) obviously follow from the presented properties. (i)⇔(ii) is the well-known fact proved, for instance, in [JT].

(iv)⇒(v): Let e be a coequalizer of (α_1, α_2). Then e', the pushout of e along f, is a

[4]We first take a pullback e_1 of e along α and then the pullback e_2 of e along βe_1. e' is given by the composition $e_2 e_1$.

coequalizer of $(f\alpha_1, f\alpha_2)$. Consider the diagram

$$\begin{tikzcd}
A \arrow[ddr, shift right, "\beta_1"'] \arrow[ddr, shift left, "\beta_2"] \arrow[drr, "\gamma"] & & \\
 & R \arrow[r, dashed, "f'"] \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\alpha_2"] & R' \arrow[d, shift right, "\alpha'_1"'] \arrow[d, shift left, "\alpha'_2"] \\
 & B \arrow[r, tail, "f"] \arrow[d, "e"'] & E \arrow[d, "e'"] \\
 & C \arrow[r, dashed, "g"] & C'
\end{tikzcd} \tag{2.8}$$

with any β_1, β_2, γ such that $f\beta_1 = \alpha'_1\gamma$ and $f\beta_2 = \alpha'_2\gamma$. Then $ge\beta_1 = e'f\beta_1 = e'f\beta_2 = ge\beta_2$, whence $e\beta_1 = e\beta_2$, since g being the pushout of f along a (regular) epimorphism is a monomorphism. Since (α_1, α_2) is a kernel pair of e, we have a unique morphism $\delta : A \longrightarrow R$ such that $\beta_1 = \alpha_1\delta$ and $\beta_2 = \alpha_2\delta$. Moreover, $\alpha'_1 f'\delta = f\alpha_1\delta = f\beta_1 = \alpha'_1\gamma$ and, similarly, $\alpha'_2 f'\delta = \alpha'_2\gamma$, so that $f'\delta = \gamma$.

(v)⇒(iv): Let $g\alpha = g\beta$ and consider $\overline{e}, \overline{\alpha}, \overline{\beta}$ from diagram (2.7). Clearly, one has $e'f\overline{\alpha} = e'f\overline{\beta}$, which gives a morphism θ with $\overline{\alpha} = \alpha_1\theta$ and $\overline{\beta} = \alpha_2\theta$. Therefore $e\overline{\alpha} = e\overline{\beta}$, from which it follows that $\alpha = \beta$. ■

For varieties of universal algebras the condition (v) of Proposition 2.2 clearly takes the form:

For any congruence R on B and its closure R' in E one has

$$R' \cap (B \times B) = R.$$

2.3. Example. As is known, the varieties of M-sets (for a given monoid M) [KMPT], groups [S], not necessarily associative rings [Di], Lie algebras (over a given field) [R], quasi-groups [YK1], commutative quasi-groups [YK], Steiner quasi-groups [KMPT], medial quasi-groups [JK], semilattices [HK], lattices [Y], [G], weakly associative lattices [FG], Boolean algebras [DY] and Heyting algebras [D] satisfy the strong amalgamation property. Moreover, the categories of M-sets [KMPT], semillatices [KMPT], [BL], Boolean algebras [Si] and Heyting algebras [D] possess the congruence extension property as well. In each case we give the reference (according to [KMPT]) where the corresponding property is determined for the first time.

3. Effective Codescent Morphisms of Groups

Proposition 2.2 gives rise to

3.1. Theorem. *Let* $p : B \rightarrowtail E$ *be a monomorphism of groups.* p *is a codescent morphism if and only if for any normal subgroup* N *of* B *and its normal closure* N' *in* E *one has*

$$N' \cap B = N.$$

When B *is a normal subgroup of* E*, this is equivalent to the requirement that any normal subgroup of* B *be normal in* E *as well.*

Before giving our main result, let us recall some well-known facts related to free products of groups with an amalgamated subgroup [Ku]. Let H be a group, and $(H \overset{\varphi_i}{\rightarrowtail} G_i)_{i\in I}$ be a family of groups equipped with monomorphisms φ_i. Further, let G be a group containing (isomorphic copies of) all G_i, and let

$$G_i \cap G_j = H \tag{3.1}$$

for $i \neq j$ (here we identify every $h \in H$ with all $\varphi_i(h)$). For any $i \in I$ and any right coset of G_i by H, except for H itself, we choose a representative. We denote the set of all chosen representatives by A. G is a free product of $(G_i)_{i\in I}$ with the amalgamated subgroup H if and only if every element of G can be uniquely written as a product

$$h\, a_1\, a_2 \cdots a_n, \tag{3.2}$$

where $n \geq 0$, $h \in H$, all a_j lie in A and no two a_j, a_{j+1} belong to the same G_i [5]. Form (3.2) is called canonical. It is easy to see how an element

$$a'_1\, a'_2 \cdots a'_m \tag{3.3}$$

of G (taken in the uncancellable form) can be reduced to the canonical form. Indeed, if $m = 1$ and $a'_m \in H$, then (3.3) is already the desired one. If again $m = 1$, but $a'_m \in G_{i_m} \backslash H$, then we consider the representation

$$a'_m = h\, \overline{a'_m} \tag{3.4}$$

with $h \in H$ and $\overline{a'_m} \in G_{i_m} \cap A$; (3.4) is the canonical form for (3.3). Let $m > 1$ and $a'_2\, a'_3 \cdots a'_m$ be already reduced to the required form

$$a'_2\, a'_3 \cdots a'_m = h'\, a_1\, a_2 \cdots a_n$$

with $a_j \in G_{i'_j} \cap A$. If $a'_1 \in H$, then

$$(a'_1\, h')a_1\, a_2 \cdots a_n$$

[5] In fact, this criterion remains true even without requirement (3.1)—we only have to replace "set of representatives" by "system of representatives" and then to specify what kind of products (3.2) and the uniqueness is meant here.

is clearly the canonical form for (3.3). If $a'_1 \in G_{i_1} \backslash H$, then we consider the representation

$$a'_1\, h' = h'' \overline{a'_1},$$

where $h'' \in H$ and $\overline{a'_1} \in G_{i_1} \cap A$. If $i_1 \neq i'_1$, then

$$h''\, \overline{a'_1} a_1\, a_2 \cdots a_n$$

is the canonical form for (3.3). If $i_1 = i'_1$ and $\overline{a'_1}\, a_1 \in H$, then

$$(h''\, \overline{a'_1}\, a_1)\, a_2 \cdots a_n$$

is the required representation for (3.3). If $i_1 = i'_1$ and $\overline{a'_1}\, a_1 \notin H$, then we have

$$\overline{a'_1}\, a_1 = h'''\, \overline{a'_1 a_1}$$

for some $h''' \in H$ and $\overline{a'_1 a_1} \in G_{i_1} \cap A$; the representation

$$h'''\, \overline{a'_1 a_1}\, a_2 \cdots a_n$$

is the desired form for (3.3).

Now it is clear that if $a'_{j_1}, a'_{j_2}, \ldots, a'_{j_k}$ are precisely those elements in (3.3) which do not lie in H and if $i_{j_r} \neq i_{j_{r+1}}$ (!) for any $1 \leq r \leq k-1$, then in the canonical form of (3.3) we have exactly k factors (except, perhaps, for a coefficient from H) and these factors belong to $G_{i_{j_1}}, G_{i_{j_2}}, \ldots, G_{i_{j_k}}$, respectively. Indeed, in each step of the reduction we either pick out an H-coefficient from some a'_j or perform the relevant multiplication. Clearly, each a'_{j_r} is changed again by an element from $G_{i_{j_r}} \backslash H$, while no two $a_{j_r}, a_{j_{r+1}}$ can "cancel" each other.

3.2. PROPOSITION. *For any monomorphism $p : B \rightarrowtail E$ the action of Φ_p on objects is surjective up to an isomorphism.*

PROOF. Let (C, γ, ξ) be codescent data with respect to p. The pushout in Fig. 1 is the concatenation of the following two pushouts:

$$\begin{tikzcd}
B \arrow[r, tail, "p"] \arrow[d] & E \arrow[d] \\
\gamma(B) \arrow[r, tail, "p'"] \arrow[d, tail, "i"'] & E/(\mathrm{Ker}\ \gamma \cap B)' \arrow[d, tail, "i'"] \\
C \arrow[r, tail, "\pi_1"] & C \sqcup_{\gamma(B)} E/(\mathrm{Ker}\ \gamma \cap B)'
\end{tikzcd} \tag{3.5}$$

Here $(\mathrm{Ker}\ \gamma \cap B)'$ is the normal closure of $(\mathrm{Ker}\ \gamma \cap B)$ in E. From (1.2) we conclude that π_1 and thus both p' and i' are monomorphisms.

From (1.1) we have

$$\xi(\gamma(E)) = \pi_2(E). \tag{3.6}$$

On the other hand, by (1.2) ξ is a split monomorphism. Therefore the homomorphism $\gamma(E) \longrightarrow \pi_2(E) \approx E/(\mathrm{Ker}\ \gamma \cap B)'$ is an isomorphism and its inverse is the morphism induced by γ. This in particular implies that $(\mathrm{Ker}\ \gamma \cap B)' = \mathrm{Ker}\ \gamma$ and thus diagram (3.5) takes the form

$$\begin{tikzcd} B \arrow[r, tail, "p"'] \arrow[d] & E \arrow[d] \\ \gamma(B) \arrow[r, tail, "p'"] \arrow[d, tail, "i"'] & \gamma(E) \arrow[d, tail, "i'"] \\ C \arrow[r, tail, "\pi_1"] & C \sqcup_{\gamma(B)} \gamma(E) \end{tikzcd}$$

If $C = \gamma(E)$, then $(C, \gamma, \xi) \approx \Phi^p(B \longrightarrow \gamma(B))$.

Suppose now that $C \neq \gamma(E)$. Without loss of generality it can be assumed that p is not an isomorphism. Since we intend to work with the free product of the groups $G_1 = C$ and $G_2 = \gamma(E)$ (as well as with the free product of the groups $G_1 = C$, $G_2 = \gamma(E)$ and G_3 being another copy of $\gamma(E)$) with the amalgamated subgroup $\gamma(B)$, we choose a representative from each right coset (different from $\gamma(B)$) of G_1 and G_2 (in G_3 we choose the same representatives as in G_2). Let $c \in C \backslash \gamma(E)$ and let the canonical form of $\xi(c)$ be written as

$$\xi(c) = \gamma(b)\, a_1\, a_2 \cdots a_n. \tag{3.7}$$

We show that if $a_i \in G_1$ for some i, then

$$\xi(a_i) \in G_1,$$

whence, by (1.2)

$$\xi(a_i) = a_i.$$

Let the canonical form of $\xi(a_i)$ have the form

$$\xi(a_i) = \gamma(b')a'_1\, a'_2 \cdots a'_m.$$

Clearly, by (1.2) $m > 0$. Suppose that at least one a'_j lies in G_2. From (1.3) we obtain

$$\begin{aligned} &\gamma(b) \cdots a_{i-1}\, a_i\, a_{i+1} \cdots \\ &\qquad = \gamma(b) \cdots a_{i-1}\, \gamma(b')\, a'_1\, a'_2 \cdots a'_m\, a_{i+1} \cdots, \end{aligned} \tag{3.8}$$

where the elements in circles are considered as representatives of G_3. The left and right neighbors of any $\gamma(b'')$ (if they exist) in the right part of (3.8) lie in different G_i $(1 \leq i \leq 3)$. Therefore, according to the observation preceding this proposition, the canonical form of the right part of (3.8) necessarily contains a representative from G_2. On the other hand, the left part of (3.8) is already canonical and does not contain any representative from G_2, which is a contradiction.

Let C' be the smallest subgroup of C containing $\gamma(B)$ and all elements of G_1 which take part in representation (3.7) for some $c \in C\backslash\gamma(E)$. From (1.2) we obtain

$$c = \gamma(b)\, a_1' a_2' \cdots a_n',$$

where each a_i' coincides either with a_i or with $\gamma(a_i)$. Thus C is generated by C' and $\gamma(E)$. Moreover, as we have just proved

$$\xi\big|_{C'} = \pi_1\big|_{C'}.$$

Therefore, after choosing a set of representatives from the right cosets of C' and $\gamma(E)$ by $\gamma(B)$, we conclude that each element of C has precisely one canonical form. Thus $C = C' \sqcup_{\gamma(B)} \gamma(E)$ and $(C, \gamma, \xi) \approx \Phi^p(\gamma')$, where $\gamma' : B \longrightarrow C'$ is the homomorphism induced by γ. ∎

Proposition 3.2 immediately gives rise to

3.3. Theorem. *In the category of groups every codescent morphism is effective.*

3.4. Remark. It seems natural to ask what classes of effective codescent morphisms various comonadicity criteria determine in this case. In this connection, we only observe that for a monomorphism $p : B \rightarrowtail E$, the change-of-cobase functor p_* preserves (all) equalizers if and only if the intersection of any right and left cosets of E by B which are different from B, contains at most one element. The "if" part is easy to verify. For the "only if" part we consider the equalities $tb = b's$ and $tb_1 = b_1's$ with $t, s \in E\backslash B$, $b, b', b_1, b_1' \in B$ and choose a diagram

$$\begin{tikzcd} & B \arrow[dl, tail, "i_1"'] \arrow[d, tail, "i_2"] \arrow[dr, tail, "i_3"] & \\ C \arrow[r, tail, "\gamma"] & D \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & K \end{tikzcd}$$

with monomorphisms i_1, i_2, i_3 such that the bottom part is an equalizer diagram and, moreover, there exist $d, d' \in D\backslash B$ with $b^{-1}\alpha(d) = b_1^{-1}\beta(d)$ and $\alpha(d')b' = \beta(d')b_1'$. It is easy to see that for the element $x = d'td$ of $D \sqcup_B E$ we have $(\alpha \sqcup_B 1_E)(x) = (\beta \sqcup_B 1_E)(x)$. Hence $x = (\gamma \sqcup_B 1_E)(y)$ for some $y \in C \sqcup_B E$. If we choose representatives of the right cosets of C by B, then their images under γ can be taken as representatives of (some) right cosets of D. The canonical form of y necessarily ends in an element c from C. We have $\gamma(c) = b'''d$ for some $b''' \in B$ and therefore $\alpha(d) = \beta(d)$ and $b = b_1$, $b' = b_1'$.

In particular, when B is normal in E, p_* preserves equalizers if and only if B is trivial.

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