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CHARACTERIZATION OF PROTOMODULAR VARIETIES OF UNIVERSAL ALGEBRAS

DOMINIQUE BOURN AND GEORGE JANELIDZE

Abstract. Protomodular categories were introduced by the first author more than ten years ago. We show that a variety $\mathcal{V}$ of universal algebras is protomodular if and only if it has 0-ary terms $e_1, \ldots, e_n$, binary terms $t_1, \ldots, t_n$, and (n+1)-ary term t satisfying the identities $t(x, t_1(x, y), \ldots, t_n(x, y)) = y$ and $t_i(x, x) = e_i$ for each $i = 1, \ldots, n$.

1. Introduction

Protomodular categories were first introduced in [2]; their role in algebra, and various further developments are also described in [3]-[6]. Recall that if $\mathbb{C}$ is a category and B is any object in it, then $\mathrm{Pt}(B)$ denotes the *category of points* in the slice category $\mathbb{C}/B$, i.e. the category whose objects are the triples (A, α, β) in which $\alpha : A \to B$ and $\beta : B \to A$ are morphisms in $\mathbb{C}$ with $\alpha.\beta = 1_B$, and whose morphisms are the commutative triangles between such points over B. When $\mathbb{C}$ has finite limits, any morphism $p : E \to B$ in $\mathbb{C}$ determines a pullback functor p^*:

$$p^* : \mathrm{Pt}(B) \to Pt(E) \tag{1.1}$$

Then the category $\mathbb{C}$ is said protomodular when, for every morphism p, the functor p^* is conservative, i.e. reflects isomorphisms. Whenever $\mathbb{C}$ has an initial object 0, it obviously suffices to require the functor (1.1) to reflect isomorphisms just for the initial object $E = 0$. And then, if $\mathbb{C}$ is pointed (and so $0 = 1$ in $\mathbb{C}$), that requirement transforms into the so-called *Split Short Five Lemma*.

In particular, the category of groups is protomodular [2]. A simple means of producing new examples comes from the fact that every category that admits a pullback preserving conservative functor from it into a protomodular category, is protomodular itself. Therefore any variety of groups with additional algebraic structure (like rings and modules or algebras over rings, etc.) also is protomodular. Thanks to the Yoneda embedding, the same is true for the internal (such) structures in any category with finite limits (see [2]). Moreover any protomodular category being Maltsev [3], we have immediately the second part of the following string of inclusions, whose first part will be a consequence of our main theorem: $\mathcal{K}_1 \subset \mathcal{K}_2 \subset \mathcal{K}_3$, where:

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1. $\mathcal{K}_1$ is the class of Varieties of algebras having 0-ary 1 and binary $\circ$ and $\backslash$ with $x \circ (x \backslash y) = y$ and $x \backslash x = 1$, see [8] and also [1].
2. $\mathcal{K}_2$ is the class of Protomodular varieties
3. $\mathcal{K}_3$ is the class of Varieties of algebras having a ternary p with $p(x, y, y) = x = p(y, y, x)$ (Maltsev varieties)

The main purpose of this paper is to show that protomodular varieties have itself a syntactical characterization, namely the following:

1.1. Theorem. *A variety $\mathcal{V}$ of universal algebras is protomodular if and only if it has 0-ary terms $e_1, \ldots, e_n$, binary terms $t_1, \ldots, t_n$, and (n+1)-ary term t satisfying the identities $t(x, t_1(x, y), \ldots, t_n(x, y)) = y$ and $t_i(x, x) = e_i$ for each $i = 1, \ldots, n$.*

Intuitively one could think of t as a "generalized multiplication" having n "divisions" $t_1, \ldots, t_n$ and n corresponding "units" $e_1, \ldots, e_n$. We do not exclude the case $n = 0$, with the variety $\mathcal{V} = \{0, 1\}$. Indeed, in this case, the identities of the theorem reduce to $t(x) = y$, whose non empty models are singletons.

Note also that we have:

$$t(x, t_1(y, y), \ldots, t_n(y, y)) = t(x, e_1, \ldots, e_n) = t(x, t_1(x, x), \ldots, t_n(x, x)) = x \tag{1.2}$$

and so:

$$p(x, y, z) = t(x, t_1(y, z), \cdots, t_n(y, z)) \tag{1.3}$$

is a Maltsev term.

While this paper was in preparation, *semiabelian categories* were introduced in [6]; we repeat from [6] that a variety of universal algebras is semiabelian if and only if it is pointed and protomodular—and that the Theorem 1.1 therefore also characterizes the semiabelian varieties with moreover $e_1 = \cdots = e_n = 0$, since those are pointed.

After [6] has already appeared, the authors of [6] and of the present article have found several papers of A. Ursini and other universal algebraists from which we have learned the following:

- The terms and identities we are using are well known in universal algebra in the special case $e_1 = \cdots = e_n = 0$ which contains the semi abelian case but not the non pointed protomodular case. The varieties having such terms were studied by A. Ursini in [9] under the name *BIT speciale* and in [10] under the name *classically ideal determined varieties.*
- E. Beutler has shown that the BIT speciale varieties are the same as the so-called *C-coherent varieties* (see [1], Proposition 2.3 $(i) \Leftrightarrow (iii)$), from which (in the pointed case, and once the concept of protomodular category is introduced !) our main result easily follows.

On the other hand, the referee suggested to us to mention also the related work of K. Fichtner [7].

2. Protomodularity in algebraic language

Let $\mathcal{V}$ be a variety of universal algebras. Consider a diagram in $\mathcal{V}$ of the form:

$$
\begin{tikzcd}
E \times_B A' \arrow[rr] \arrow[dr, "p^*(f)"] \arrow[dd, shift right] & & A' \arrow[dr, "f"] \arrow[dd, shift right, "\alpha'"'] & \\
& E \times_B A \arrow[rr] \arrow[dl, shift right] & & A \arrow[dl, shift right, "\alpha"'] \\
E \arrow[rr, "p"'] \arrow[uu, shift right] \arrow[ur, shift right] & & B \arrow[uu, shift right, "\beta'"'] \arrow[ur, shift right, "\beta"'] &
\end{tikzcd}
\tag{2.1}
$$

where:

- $f : (A', \alpha', \beta') \to (A, \alpha, \beta)$ is a morphism in $\mathrm{Pt}(B)$;
- $p : E \to B$ is any morphism in $\mathcal{V}$;
- other horizontal arrows are the appropriate pullback projections;
- the left hand triangle represents the image $p^*(f)$ of f under the pullback functor (1.1).

The definition of protomodularity says: $\mathcal{V}$ is protomodular if for each such diagram we have, for any map $f : A' \to A$:

$$p^*(f) \text{ is an isomorphism } \Rightarrow f \text{ is an isomorphism} \tag{2.2}$$

Let us begin by the following:
Observation: (a) Since the category $\mathcal{V}$ is exact, it is sufficient to require (2.2) only when f is a monomorphism. Indeed, applying this weaker requirement first to the diagonal $A' \rightarrowtail A' \times_A A'$ and then to f itself, we obtain:

$$p^*(f) \text{ is an iso } \Rightarrow p^*(A' \rightarrowtail A' \times_A A') \text{ is an iso } \Rightarrow (A' \rightarrowtail A' \times_A A') \text{ is an iso}$$
$$\Rightarrow f : A' \to A \text{ is a mono } \Rightarrow f \text{ is an iso}$$

where the first implication holds because p^* preserves pullbacks.

(b) Again, since $\mathcal{V}$ is exact, the implication (2.2) automatically holds when p is a regular epimorphism (= surjective map). Therefore requiring (2.2) we may also assume that p is a monomorphism.

(c) Since the free algebra $A[\emptyset]$ on the empty set is the initial object in $\mathcal{V}$, it is sufficient to require (2.2) for $E = A[\emptyset]$. Moreover, (b) tells us that we can replace $A[\emptyset]$ by its image C in B, which of course is the subalgebra in B generated by all constants.

Since monomorphisms in $\mathcal{V}$ are nothing but subalgebra injections (up to isomorphism), we obtain the following proposition, in which (1) $\Leftrightarrow$ (2) follows from our Observation while (2) $\Leftrightarrow$ (3) is obvious:

2.1. Proposition. *The following conditions on a variety $\mathcal{V}$ of universal algebras are equivalent:*

1. *$\mathcal{V}$ is protomodular.*

2. *Let $B \subset A' \subset A$ be in $\mathcal{V}$ (the inclusions are of course supposed to be homomorphisms), $\alpha : A \to B$ a homomorphism with $\alpha(b) = b$ for each $b \in B$, and K the inverse image under α of the subalgebra in B generated by all constants. If A' contains K, then $A' = A$.*

3. *Let A be in $\mathcal{V}$, B a subalgebra in A, $\alpha : A \to B$ a homomorphism with $\alpha(b) = b$ for each $b \in B$, and K the inverse image under α of the subalgebra in B generated by constants. Then A is generated by B and K.*

3. Proof of Theorem 1.1.

Suppose there are $e_1, \ldots, e_n, t_1, \ldots, t_n, t$ as in the formulation of Theorem 1.1. In order to prove that $\mathcal{V}$ is protomodular, we will prove 2.1(3)—essentially by repeating a simple argument, well known for groups. For an arbitrary element $a \in \mathcal{V}$, we have:

$$a = t(\alpha(a), t_1(\alpha(a), a), \ldots, t_n(\alpha(a), a))$$

and since:

$$\alpha(t_i(\alpha(a), a)) = t_i(\alpha(a), \alpha(a)) = e_i,$$

the element $t_i(\alpha(a), a)$ is in K, for each $(i = 1, \ldots, n)$. Therefore a belongs to the subalgebra generated by B and K, as desired.
Conversely, suppose $\mathcal{V}$ satisfies the condition 2.1(3). We take:

- $A = A[x, y]$, the free algebra in $\mathcal{V}$ on two generators x and y;
- $B = A[x] =$ the subalgebra of A generated by x;
- $\alpha : A \to B$ the homomorphism defined by $\alpha(x) = \alpha(y) = x$.

Then since the algebra A is generated by B and K, and B is generated by x, the element y can be presented in A as:

$$y = t(x, k_1, \ldots, k_n)$$

for some $k_1, \ldots, k_n$ in K and (n+1)-ary term t. Moreover, since K is a subalgebra in $A[x, y]$, there exist binary terms $t_1, \ldots, t_n$ with $k_i = t_i(x, y)$ for each $i = 1, \ldots, n$. And furthermore, since all $t_i(x, x) = \alpha(t_i(x, y)) = \alpha(k_i)$ belong to the subalgebra in A generated by constants, there exist 0-ary terms $e_1, \ldots, e_n$ with $t_i(x, x) = e_i$ for each $i = 1, \ldots, n$.

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Université du Littoral, 50 rue F. Buisson, BP699
62228 Calais, France

Math. Institute of the Georgian Academy of Sciences, 1 Alexidze str.,
380093 Tbilisi, Georgia
Email: bourn@lmpa.univ-littoral.fr, gjanel@math.acnet.ge

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