Theory and Applications of Categories, Vol. 11, No. 9, 2003, pp. 212–214.

CHARACTERIZATION OF POINTED VARIETIES OF UNIVERSAL ALGEBRAS WITH NORMAL PROJECTIONS

ZURAB JANELIDZE

ABSTRACT. We characterize pointed varieties of universal algebras in which $(A \times B)/A \approx B$, i.e. all product projections are normal epimorphisms.

1. DEFINITION. *We will say that a pointed category* $\mathbf{C}$ *has normal projections if every product projection* $A \times B \to B$ *in* $\mathbf{C}$ *is a normal epimorphism.*

Equivalently, for any two objects A and B in such a category $\mathbf{C}$, forming the product $A \times B$ and then factoring it by $A \approx A \times 0$ results in B. In particular, every Jónsson-Tarski variety of universal algebras [3] (considered as a category) has this property; the same is true for the pointed subtractive varieties in the sense of Ursini [4].

The purpose of this paper is to characterize pointed varieties with normal projections (Theorem 3 below).

Before stating the theorem, we make a simple reformulation of Definition 1.

2. PROPOSITION. *Let* $\mathbf{C}$ *be a pointed variety. The following conditions are equivalent:*

(a) $\mathbf{C}$ *has normal projections;*

(b) there exists a natural number n*, such that for all* A *and* B *in* $\mathbf{C}$*, and for all* $a \in A$ *and* $b \in B$*,* $((a, b), (0, b)) \in R^n$*, where* R *is the reflexive homomorphic relation on* $A \times B$ *generated by the set* $\{((a', 0), (a'', 0)) |\ a' = 0 \text{ or } a'' = 0\}$*;*

(c) let $F[x]$ *be the free algebra in* $\mathbf{C}$ *generated by* x*; there exists a natural number* n *such that* $((x, x), (0, x)) \in Q^n$*, where* Q *is the reflexive homomorphic relation on* $F[x] \times F[x]$ *generated by the set* $\{((x, 0), (0, 0)), ((0, 0), (x, 0))\}$*.*

Moreover, the number n *in (b) and in (c) can be supposed to be the same.*

3. THEOREM. *A pointed variety* $\mathbf{C}$ *has normal projections if and only if the corresponding theory contains*

- *unary terms* $t_1, ..., t_m$ *and* $u_1, ..., u_m$*;*
- $(m + 2)$*-ary terms* $v_1, ..., v_n$*;*

Received by the editors 2003-01-08 and, in revised form, 2003-05-14.
Transmitted by Walter Tholen. Published on 2003-05-20.
2000 Mathematics Subject Classification: 18A20, 18A30, 08B05, 08B25.

and the following identities hold in $\mathbf{C}$*:*

- $x = v_1(t_1(x), ..., t_m(x), x, 0)$;
- $v_{i+1}(t_1(x), ..., t_m(x), x, 0) = v_i(t_1(x), ..., t_m(x), 0, x)$ *for each* $i \in \{1, ..., n-1\}$;
- $0 = v_n(t_1(x), ..., t_m(x), 0, x)$;
- $x = v_i(u_1(x), ..., u_m(x), 0, 0)$ *for each* $i \in \{1, ..., n\}$.

Moreover, for each n *this characterizes the pointed varieties of universal algebras which satisfy 2(b) (for the same* n*).*

Proof. According to Proposition 2, we have to characterize pointed varieties for which $((x, x), (0, x)) \in Q^n$, i.e. for which there exist $s_1, ..., s_{n+1} \in F[x]$ such that $s_1 = x, s_{n+1} = 0$, and for each $i \in \{1, ..., n\}$ the pair $((s_i, x), (s_{i+1}, x))$ is in Q. On the other hand, $((s_i, x), (s_{i+1}, x)) \in Q$ if and only if for some $(m+2)$-ary term v_i and unary terms $t_1, ..., t_m$, $u_1, ..., u_m \in F[x]$, one has the equalities $(s_i, x) = v_i((t_1, u_1), ..., (t_m, u_m), (x, 0), (0, 0))$ and $(s_{i+1}, x) = v_i((t_1, u_1), ..., (t_m, u_m), (0, 0), (x, 0))$. Moreover, we can assume that the t's, u's and the number m are the same for each $i \in \{1, ..., n\}$. Writing the equalities above separately for the components of pairs, we obtain $s_i = v_i(t_1, ..., t_m, x, 0)$, $s_{i+1} = v_i(t_1, ..., t_m, 0, x)$, $x = v_i(u_1, ..., u_m, 0, 0)$. Since s's are expressed by v's, we may omit them; after this the identities become exactly as in the formulation of the theorem. ∎

4. Example. Let $\mathbf{C}$ be a pointed variety with normal projections, for which we could take $n = 1$ in Theorem 3. Then, the theory corresponding to $\mathbf{C}$ has unary terms $t_1, ..., t_m, u_1, ..., u_m$ and an $(m + 2)$-ary term v, which satisfy the identities

$$x = v(t_1(x), ..., t_m(x), x, 0), \;\; x = v(u_1(x), ..., u_m(x), 0, 0), \;\; 0 = v(t_1(x), ..., t_m(x), 0, x).$$

When the unary terms are either x or 0, an easy argument shows that we could rewrite these identities as

$$x = w(x, 0, x, x, 0), \;\; x = w(x, x, 0, 0, 0), \;\; 0 = w(x, 0, x, 0, x).$$

If the term $w = w(x_1, x_2, x_3, x_4, x_5)$ depends only on the first variable x_1 and the last variable x_5, then we can write $w(x_1, x_2, x_3, x_4, x_5) = x_1 + x_5$ and our identities become

$$x = x + 0, \;\; 0 = x + x;$$

in this case the variety $\mathbf{C}$ becomes nothing but a pointed subtractive variety in the sense of Ursini [4]. On the other hand, $w(x_1, x_2, x_3, x_4, x_5) = x_2 + x_4$ would give

$$x = 0 + x, \;\; x = x + 0$$

which defines a Jónsson-Tarski variety [3].

5. Remark. It is known that every semi-abelian category has normal projections (see Condition SA*3a in [2]). More generally, every pointed category in which every pair of canonical morphisms $(1_A, 0) : A \to A \times B$, $(0, 1_B) : B \to A \times B$ is jointly epimorphic has this property. In particular, this is the case for the unital categories in the sense of Bourn [1].

References

[1] D. Bourn, Mal'cev categories and fibration of pointed objects, Applied Categorical Structures 4, 1996, 302-327.

[2] G. Janelidze, L. Márki and W. Tholen, Semiabelian categories, Journal of Pure and Applied Algebra 168, 2002, 367-386.

[3] B. Jónsson and A. Tarski, Direct decompositions of finite algebraic systems, Notre Dame Mathematical Lectures, 1947, Notre Dame, Indiana.

[4] A. Ursini, On subtractive varieties, I, Algebra Universalis 31, 1994, 204-222.

The Faculty of Mechanics and Mathematics,
Ivane Javakhishvili State University of Tbilisi,
1 Chavchavadze Ave.,
380028 Tbilisi, Georgia
Email: `zurab@gol.ge`

This article may be accessed via WWW at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/11/9/11-09.{dvi,ps}`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

The method of distribution of the journal is via the Internet tools `WWW/ftp`. The journal is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION. Individual subscribers receive (by e-mail) abstracts of articles as they are published. Full text of published articles is available in .dvi, Postscript and PDF. Details will be e-mailed to new subscribers. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

INFORMATION FOR AUTHORS. The typesetting language of the journal is TeX, and LaTeX is the preferred flavour. TeX source of articles for publication should be submitted by e-mail directly to an appropriate Editor. They are listed below. Please obtain detailed information on submission format and style files from the journal's WWW server at `http://www.tac.mta.ca/tac/`. You may also write to `tac@mta.ca` to receive details by e-mail.