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SEVERAL CONSTRUCTIONS FOR FACTORIZATION SYSTEMS

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Abstract. The paper develops the previously proposed approach to constructing factorization systems in general categories. This approach is applied to the problem of finding conditions under which a functor (not necessarily admitting a right adjoint) "reflects" factorization systems. In particular, a generalization of the well-known Cassidy-Hébert-Kelly factorization theorem is given. The problem of relating a factorization system to a pointed endofunctor is considered. Some relevant examples in concrete categories are given.

1. Introduction

The problem of relating a factorization system on a category $\mathcal{C}$ to an adjunction

$$\mathcal{C} \underset{I}{\overset{H}{\rightleftarrows}} \mathcal{X} \ , \tag{1.1}$$

was thoroughly considered by C. Cassidy, M. Hébert and G. M. Kelly in [CHK]. The well-known theorem of these authors states that in the case of a finitely well-complete category $\mathcal{C}$ the pair of morphism classes

$$\big(I^{-1}(\text{Iso } \mathcal{X}), \big(H(\text{Mor } \mathcal{X})\big)^{\uparrow\downarrow}\big) \tag{1.2}$$

is a factorization system on $\mathcal{C}$ (recall that for any morphism class $\mathbb{N}$ the symbol $\mathbb{N}^{\uparrow}$ (resp. $\mathbb{N}^{\downarrow}$) denotes the class of all morphisms f with $f \downarrow n$ (resp. $n \downarrow f$) for all $n \in \mathbb{N}$). Subsequently, G. Janelidze posed the problem whether this is still the case if Iso $\mathcal{X}$ and Mor $\mathcal{X}$ in (1.2) are replaced by any morphism classes $\mathbb{E}$ and $\mathbb{M}$, respectively, constituting a factorization system on $\mathcal{X}$. This problem is dealt with in the author's recent works [Z1], [Z2]. For instance, in [Z1] it is shown[1] that the pair

$$\big(I^{-1}(\mathbb{E}), (H(\mathbb{M}))^{\uparrow\downarrow}\big) \tag{1.3}$$

is a factorization system on $\mathcal{C}$ for any factorization system $(\mathbb{E}, \mathbb{M})$ on $\mathcal{X}$ if (1.1) is a semi-left-exact reflection in the sense of [CHK], or, equivalently, an admissible reflection in the sense of [J1].

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[1]This result was further improved in [Z2] by omitting the requirement that H^C be faithful.

In [Z2] we proposed an approach of transporting factorization systems from $\mathcal{X}$ to $\mathcal{C}$, which sheds new light on the above-mentioned problem. In this paper we develop our approach and apply it to the same problem, but in a more general situation, namely, to the problem whether the pair of morphism classes

$$\big(I^{-1}(\mathbb{E}), (I^{-1}(\mathbb{E}))^{\downarrow}\big) \tag{1.4}$$

is a factorization system, where

$$I : \mathcal{C} \longrightarrow \mathcal{X}$$

is any functor, not necessarily admitting a right adjoint. We prove that the *pair* (1.4) *is a factorization system in any of the following cases:*

– when the induced functor

$$I^C : \mathcal{C}/C \longrightarrow \mathcal{X}/I(C) \tag{1.5}$$

between slice categories admits a right adjoint H^C for each $C \in \mathrm{Ob}\,\mathcal{C}$ and all H^C are full (this, in particular, includes the case of a fibration I; we point out that here $\mathcal{C}$ is not required to have pullbacks or any other (co)limits);

– when $\mathcal{C}$ admits pullbacks, $\mathcal{X} = \mathcal{C}$, I is equipped with the structure of a pointed endofunctor (with certain restrictions) and $\mathbb{M} \subset \mathrm{Mon}\,\mathcal{X}$. In this connection, we should make a special mention of the work [JT2] by G. Janelidze and W. Tholen, where the problem of relating a factorization system to a pointed endofunctor is investigated. Note that our treatment of this problem differs from that of [JT2];

– when $\mathcal{C}$ is complete and well-powered, each I^C in (1.5) has a right adjoint and $\mathbb{M} \subset \mathrm{Mon}\,\mathcal{X}$;

and, finally,

– when I has a right adjoint with various versions of the requirements given in [Z1], [Z2] and in this paper.

In particular, we give the following generalization of the Cassidy-Hébert-Kelly theorem:

Let $\mathcal{C}$ be a complete and well-powered category and let $(\mathbb{E}, \mathbb{M})$ *be any factorization system on $\mathcal{X}$ with $\mathbb{M} \subset \mathrm{Mon}\,\mathcal{X}$; then the pair* (1.3) *is a factorization system on $\mathcal{C}$ for any adjunction* (1.1)*.*

The results obtained in the paper are illustrated by examples in concrete categories. Namely, new factorization systems are constructed on the categories of Abelian groups (more generally, on any Abelian category), groups with unary operators, locally compact Hausdorff Abelian groups with compact sets of compact elements, Heyting algebras, topological spaces over a given space and so on.

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2. Preliminaries

We use the terms "factorization system", "prefactorization system" and "object orthogonal to a morphism" in the sense of P. J. Freyd and G. M. Kelly [FK]. Namely, a *factorization system* on a category $\mathcal{C}$ is a pair of morphism classes $(\mathbb{E}, \mathbb{M})$ such that

(i) $\mathbb{E}$, $(\mathbb{M})$ is right-closed (left-closed) under composition with isomorphisms, i.e., $ie \in \mathbb{E}$ $(mi \in \mathbb{M})$ whenever $e \in \mathbb{E}$ $(m \in \mathbb{M})$ and i is an isomorphism;

(ii) for each $e \in \mathbb{E}$ and $m \in \mathbb{M}$ we have $e \downarrow m$, which means that for each commutative square

$$\begin{tikzcd} \cdot \arrow[r, "e"] \arrow[d, "\alpha"'] & \cdot \arrow[d, "\beta"] \\ \cdot \arrow[r, "m"] & \cdot \end{tikzcd}$$

there exists a unique δ, for which $\alpha = \delta e$ and $\beta = m\delta$;

(iii) every morphism α admits an $(\mathbb{E}, \mathbb{M})$-factorization, i.e., there are morphisms $e \in \mathbb{E}$ and $m \in \mathbb{M}$ with $\alpha = me$.

Every factorization system $(\mathbb{E}, \mathbb{M})$ satisfies $\mathbb{E}^{\downarrow} = \mathbb{M}$ and $\mathbb{M}^{\uparrow} = \mathbb{E}$. A pair $(\mathbb{E}, \mathbb{M})$ satisfying these equalities is called a *prefactorization system* on $\mathcal{C}$.

Let us recall some needed definitions and results from [CHK]. Let $\mathcal{C}$ have a terminal object $\mathbf{1}$. Consider the partially ordered collection of prefactorization systems $(\mathbb{E}, \mathbb{M})$ on $\mathcal{C}$ satisfying the condition:

($*$) every morphism with codomain $\mathbf{1}$ admits an $(\mathbb{E}, \mathbb{M})$-factorization,

with usual order, i.e., $(\mathbb{E}, \mathbb{M}) \leq (\mathbb{E}', \mathbb{M}')$ iff $\mathbb{M} \subset \mathbb{M}'$. This collection is regarded as a category and denoted by $\mathrm{PFS}_*(\mathcal{C})$. Let $\mathrm{FRS}(\mathcal{C})$ be the collection of full replete reflective subcategories of $\mathcal{C}$, ordered by the inclusion and also regarded as a category. There is an adjunction

$$\mathrm{FRS}\,(\mathcal{C}) \underset{\Phi_{\mathcal{C}}}{\overset{\Psi_{\mathcal{C}}}{\rightleftarrows}} \mathrm{PFS}_*(\mathcal{C}) \ ,$$

where the right adjoint Ψ_c maps each $(\mathbb{E}, \mathbb{M})$ to the full subcategory of objects C for which the unique morphism $C \to \mathbf{1}$ lies in $\mathbb{M}$, while

$$\Phi_{\mathcal{C}}(\mathcal{X}) = \big(I^{-1}(\mathrm{Iso}\ \mathcal{X}), (I^{-1}(\mathrm{Iso}\ \mathcal{X}))^{\downarrow}\big)$$

for a full replete reflective subcategory $\mathcal{X}$ with the reflector $I : \mathcal{C} \longrightarrow \mathcal{X}$. The well-known result from [CHK] asserts that if $\mathcal{C}$ is finitely well-complete, then $\Phi_{\mathcal{C}}(\mathcal{X})$ is not only a prefactorization system, but, what is more, a factorization system as well.

One has $\Psi_{\mathcal{C}}\,\Phi_{\mathcal{C}} = 1_{\mathrm{FRS}\,(\mathcal{C})}$. The pair $\Phi_{\mathcal{C}}\,\Psi_{\mathcal{C}}(\mathbb{E}, \mathbb{M})$ is called the *reflective interior* of a prefactorization system $(\mathbb{E}, \mathbb{M})$ from $\mathrm{PFS}_*(\mathcal{C})$. Like [CHK] we denote it by $(\overset{\circ}{\mathbb{E}}, \overset{\circ}{\mathbb{M}})$. The morphisms of $\overset{\circ}{\mathbb{E}}$ are characterized as follows:

$$\alpha \in \overset{\circ}{\mathbb{E}} \quad \text{if and only if} \quad e\alpha \in \mathbb{E} \quad \text{for some} \quad e \in \mathbb{E}. \tag{2.1}$$

$(\mathbb{E}, \mathbb{M})$ is called *reflective* if $\overset{\circ}{\mathbb{E}} = \mathbb{E}$ (or, equivalently, $\overset{\circ}{\mathbb{M}} = \mathbb{M}$).

Let $\mathcal{C}$ be finitely complete, and let $\mathcal{X}$ be a full reflective subcategory with reflector I and unit η. The reflection of $\mathcal{C}$ onto $\mathcal{X}$ is said to be *semi-left-exact* if for any pullback diagram

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d] & X \arrow[d] \\ C \arrow[r, "\eta_C"] & IC \end{tikzcd}$$

with $X \in \mathrm{Ob}\, \mathcal{X}$, $I(\alpha)$ is an isomorphism. The equivalent notion is that of an *admissible* reflection [J1]. It is defined as a reflection such that all right adjoints H^C in the induced adjunctions between slice categories

$$\mathcal{C}/C \underset{I^C}{\overset{H^C}{\rightleftarrows}} \mathcal{X}/I(C)$$

are full and faithful. The reflection of $\mathcal{C}$ onto $\mathcal{X}$ is called *simple* [CHK], or *direct* [BG], [H] if, for any morphism $f : B \longrightarrow C$, the image under I of the canonical morphism in the diagram

$$\begin{tikzcd} B \arrow[dr, dashed] \arrow[drr, "\eta_B"] \arrow[ddr, "f"'] & & \\ & P \arrow[r] \arrow[d] \arrow[dr, phantom, "pb"] & IB \arrow[d, "If"] \\ & C \arrow[r, "\eta_C"'] & IC \end{tikzcd}$$

is an isomorphism. Every semi-left-exact reflection is simple. The converse fails to be valid.

As mentioned in the Introduction, in this paper we develop the procedure proposed in [Z2] for transporting factorization systems from a category $\mathcal{X}$ to a category C. Let us recall its essence. We deal not with adjunctions (1.1), but, more generally, with families of adjunctions between slice categories

$$\mathcal{A} : \left(\mathcal{C}/C \underset{I^C}{\overset{H^C}{\rightleftarrows}} \mathcal{X}/T(C) \right)_{C \in \mathrm{Ob}\, \mathcal{C}}, \tag{2.2}$$

(here T is any mapping $\mathrm{Ob}\, \mathcal{C} \longrightarrow \mathrm{Ob}\, \mathcal{X}$ and H^C are right adjoints). The arguments justifying why we do so are as follows. A factorization system $(\mathbb{E}, \mathbb{M})$ on $\mathcal{X}$ gives the full reflective subcategory $\mathbb{M}/T(C)$ of $\mathcal{X}/T(C)$ for each C; recall that the objects of $\mathbb{M}/T(C)$ are the morphisms $X \to T(C)$ which lie in $\mathbb{M}$. If all H^C are full, then $H^C(\mathbb{M}/T(C))$ are full reflective subcategories of $\mathcal{C}/C$, and the point is that sometimes they induce a factorization system on $\mathcal{C}$. More generally, if H^C is not necessarily full, then $H^C(\mathbb{M}/T(C))$ can be replaced by the full subcategory $\mathcal{C}_C$ of $\mathcal{C}/C$ generated by $H^C(\mathrm{Ob}\, \mathbb{M}/T(C))$. If $\mathcal{C}_C$

turns out to be reflective for any C, then we may again obtain a factorization system on $\mathcal{C}$, which we denote by $(\mathbb{E}_{\mathcal{A}}, \mathbb{M}_{\mathcal{A}})$. Let us clarify that

$$\mathbb{M}_{\mathcal{A}} = \bigcup_{C \in \mathrm{Ob}\, \mathcal{C}} \mathbb{M}_C,$$

where $\mathbb{M}_C$ consists of all morphisms isomorphic to the images of the objects of $\mathbb{M}/T(C)$ under H^C, while

$$\mathbb{E}_A = \mathbb{M}_A^{\uparrow}\ .$$

Let $f : B \longrightarrow C$ be a morphism in $\mathcal{C}$, and let

$$I^C(f) = me$$

be the $(\mathbb{E}, \mathbb{M})$-factorization of $I^C(f)$. We can consider e as a morphism $I^C(f) \longrightarrow m$ in $\mathcal{X}/T(C)$; therefore it has the adjunct e_f under the corresponding adjunction from (2.2). Clearly,

$$f = m_f\, e_f, \tag{2.3}$$

where $m_f = H^C(m)$. In [Z2] various sufficient conditions are given for $(\mathbb{E}_A, \mathbb{M}_A)$ to be a factorization system (then $(\mathbb{E}, \mathbb{M})$ is called *locally transferable along* (2.2)). Note that it is (2.3) that frequently gives the $(\mathbb{E}_A, \mathbb{M}_A)$-factorization of morphisms.

The considered procedure naturally led to the notion of a local factorization system introduced in [Z2]. By definition, a *local factorization system* with respect to some class of objects $\mathcal{C}'$ (or, shortly, a $\mathcal{C}'$-factorization system) is a pair of morphism classes $(\mathbb{E}, \mathbb{M})$, satisfying the axioms (i), (ii) above and

(iii′) every morphism α with codomain in $\mathcal{C}'$ admits an $(\mathbb{E}, \mathbb{M})$-factorization;

(iv) for each $m \in \mathbb{M}$, codom $m \in \mathcal{C}'$;

(v) for each $e \in \mathbb{E}$ there exists $m \in \mathbb{M}$ with codom $e =$ dom m.

Clearly, Ob $\mathcal{C}$-factorization systems are just usual factorization systems on $\mathcal{C}$. Local factorization systems possess most of the properties of factorization systems. In particular, $\mathbb{M}/C$ is a reflective subcategory of $\mathcal{C}/C$ for each $C \in \mathcal{C}'$. Hence we can carry out the procedure not only in the case where $(\mathbb{E}, \mathbb{M})$ is a (usual) factorization system on $\mathcal{X}$, but also in a more general case where it is a local factorization system with respect to a class $\mathcal{X}'$ of objects such that $T(\mathrm{Ob}\, \mathcal{C}) \subset \mathcal{X}'$. Our previous notations related to the procedure remain in force for this general case as well.

Finally, we wish to recall two results that are repeatedly used in this paper. The first of them is well-known and proved, for instance, in [T], [CHK].

Let $\mathcal{C}$ be an arbitrary category.

2.1. THEOREM. *Let $\mathbb{N}$ be a class of morphisms in $\mathcal{C}$, closed under composition; let $\mathcal{C}$ admit all pullbacks (along any morphism) of morphisms in $\mathbb{N}$, and all intersections of morphisms in $\mathbb{N}$; and let these pullbacks and intersections again belong to $\mathbb{N}$. Then $(\mathbb{N}^{\uparrow}, \mathbb{N})$ is a factorization system and $\mathbb{N} \subset \mathrm{Mon}\, \mathcal{C}$.* [2]

2.2. PROPOSITION. [Z2] *Let $(\mathcal{C}_C)_{C \in \mathrm{Ob}\, \mathcal{C}}$ be a family of full replete reflective subcategories of slice categories $\mathcal{C}/C$ with reflectors*

$$r^C : \mathcal{C}/C \longrightarrow \mathcal{C}_C$$

and units ζ^C, and let

$$\mathcal{M} = \bigcup_{C \in \mathrm{Ob}\, \mathcal{C}} \mathrm{Ob}\, \mathcal{C}_C.$$

The following conditions are equivalent:

(i) *the pair $(\mathcal{M}^{\uparrow}, \mathcal{M})$ is a factorization system on $\mathcal{C}$;*

(ii) *for each pair of objects C, C' of $\mathcal{C}$ and each $f \in \mathrm{Ob}\, \mathcal{C}/C$, if $\zeta^C_f : \alpha \longrightarrow \beta$ is a morphism in $\mathcal{C}/C'$, then $r^{C'}(\zeta^C_f)$ is an isomorphism.*

When these conditions are satisfied, we have

$$\mathcal{M}^{\uparrow} = \bigcup_{C \in \mathrm{Ob}\, \mathcal{C}} \{ i\, \zeta^C_f | f \in \mathrm{Ob}\, \mathcal{C}/C, \ \ i \ \text{ is isomorphism} \}. \tag{2.4}$$

If $\mathcal{C}$ admits pullbacks, then each of the above conditions is equivalent to

(iii) *$\mathcal{M}$ is stable under pullbacks, and for each object C, each $f \in \mathrm{Ob}\, \mathcal{C}/C$ and $C' =$ codom e_f, the morphism $r^{C'}(\zeta^C_f)$ is an isomorphism.*

If, in addition, $\mathcal{C}$ has all intersections of monomorphisms and $\mathrm{Ob}\, \mathcal{C}_C \subset \mathrm{Mon}\, \mathcal{C}$ for any C, then we have one more equivalent condition:

(iv) *$\mathcal{M}$ is stable under pullbacks and is closed under composition and intersection.*

Throughout the paper, when there is no risk that confusion might arise, we use the same notation for different morphisms in a slice category provided that they have the same underlying morphism from the entire category (as is, for instance, the case with Proposition 2.2 (ii)).

[2] In fact, as is shown in [T], for any morphism class $\mathbb{N}$, the existence of all intersections of morphisms from $\mathbb{N}$ is sufficient for the fulfillment of the condition $\mathbb{N} \subset \mathrm{Mon}\, \mathcal{C}$.

3. The Initial Case

In this section, unless it is specified otherwise, $\mathcal{C}$ and $\mathcal{X}$ always denote arbitrary categories without any requirement related to the existence of (co)limits.

Let $T : \mathrm{Ob}\,\mathcal{C} \longrightarrow \mathrm{Ob}\,\mathcal{X}$ be any mapping, and let $\mathcal{A}$ be any family of adjunctions

$$\left(\mathcal{C}/C \underset{I^C}{\overset{H^C}{\rightleftarrows}} \mathcal{X}/T(C) \right)_{C \in \mathrm{Ob}\,\mathcal{C}} \tag{3.1}$$

with right adjoints H^C and counits ε^C. Suppose $(\mathbb{E}, \mathbb{M})$ is any factorization system on $\mathcal{X}$. If H^C is full, $H^C(\mathbb{M}/T(C))$ is a reflective subcategory of $\mathcal{C}/C$ and its units are $e_f : f \longrightarrow m_f$. Applying Proposition 2.2, we obtain

3.1. Proposition. *Let all H^C be full. Then the following conditions are equivalent:*

(i) $(\mathbb{E}_{\mathcal{A}}, \mathbb{M}_{\mathcal{A}})$ *is a factorization system;*

(ii) *for all objects C, C' of $\mathcal{C}$ and each $f \in \mathrm{Ob}\,\mathcal{C}/C$, if $e_f : \alpha \longrightarrow \beta$ is a morphism in $\mathcal{C}/C'$, then there exists an $\mathcal{C}/C'$-isomorphism $e_\alpha \approx e_\beta e_f$ for $e_\alpha : \alpha \longrightarrow m_\alpha$ and $e_\beta : \beta \longrightarrow m_\beta$;*

(iii) *for all objects C, C' of $\mathcal{C}$ and each $f \in \mathrm{Ob}\,\mathcal{C}/C$, if $e_f : \alpha \longrightarrow \beta$ is a morphism in $\mathcal{C}/C'$, then there exists a morphism $e : I^{C'}(\beta) \longrightarrow x$ in $\mathcal{X}/T(C')$ such that both e and $eI^{C'}(e_f)$ lie in $\mathbb{E}$;*

(iv) *for all objects C, C' of $\mathcal{C}$ and each $f \in \mathrm{Ob}\,\mathcal{C}/C$, if $e_f : \alpha \longrightarrow \beta$ is a morphism in $\mathcal{C}/C'$, then we have $I^{C'}(e_f) \in \mathring{\mathbb{E}}$, where $(\mathring{\mathbb{E}}, \mathring{\mathbb{M}})$ is the reflective interior of $(\mathbb{E}, \mathbb{M})$ considered as a factorization system on $\mathcal{X}/T(C')$;*

(v) *for each morphism f, we have $e_f \in \mathbb{E}_{\mathcal{A}}$.*

If, in addition, $\mathcal{C}$ admits pullbacks and each I^C preserves the terminal object, then (i)–(v) *are also equivalent to each of the following conditions:*

(vi) $\mathbb{M}_{\mathcal{A}}$ *is stable under pullbacks and for each $f \in \mathrm{Ob}\,\mathcal{C}/C$, the object $C' = \mathrm{codom}\, e_f$ and the corresponding morphism $e_f : e_f \longrightarrow 1_{C'}$, we have $I^{C'}(e_f) \in \mathbb{E}$;*

(vii) $\mathbb{M}_{\mathcal{A}}$ *is stable under pullbacks and for each $f \in \mathrm{Ob}\,\mathcal{C}/C$ and the object $C' = \mathrm{codom}\, e_f$, we have $I^{C'}(e_f) \in \mathbb{E}$;*

(viii) $\mathbb{M}_{\mathcal{A}}$ *is stable under pullbacks, and for each $C \in \mathrm{Ob}\,\mathcal{C}$ and each $f \in \mathrm{Ob}\,\mathcal{C}/C$, the morphism m_{e_f} is an isomorphism.*

Proof. Observe in the first place that one always has

$$\text{(iii)} \Leftrightarrow \text{(iv)} \Rightarrow \text{(v)} \Rightarrow \text{(i)}.$$

Indeed, the equivalence (iii)$\Leftrightarrow$(iv) follows from (2.1). For (iii)$\Rightarrow$(v), note that $e_f \downarrow H^{C'}(m')$ means precisely the orthogonality of an object $H^{C'}(m')$ of $\mathcal{C}/C'$ to all $\mathcal{C}/C'$-morphisms, the underlying morphisms of which are e_f. This in turn is equivalent to the condition that m' is orthogonal to the images of the above-mentioned $\mathcal{C}/C'$-morphisms under $I^{C'}$. The latter condition clearly holds. (v)$\Rightarrow$(i) is obvious, since $f = m_f\, e_f$ for any $f \in \operatorname{Mor} \mathcal{C}$.

Now let us assume that each H^C is full.

(i)$\Leftrightarrow$(ii). By Proposition 2.2, $(\mathbb{E}_{\mathcal{A}}, \mathbb{M}_{\mathcal{A}})$ is a factorization system if and only if $r^{C'}(e_f) : m_\alpha \to m_\beta$ is an isomorphism; here $r^{C'}$ is the reflector $\mathcal{C}/C' \to H^{C'}(\mathbb{M}/T(C'))$. It remains to observe that $e_\beta e_f = r^{C'}(e_f) e_\alpha$.

(ii)$\Rightarrow$(iii) follows from the fact that, as is shown in [Z2], for every $f : B \longrightarrow C$ and the corresponding morphism $e_f : f \longrightarrow m_f$ in $\mathcal{C}/C$, we have

$$I^C(e_f) \in \mathbb{E}. \tag{3.2}$$

(3.2) is implied by the equality $e = \varepsilon^C_m I^C(e_f)$ and the fact that ε^C_m is a split monomorphism.

Let $\mathcal{C}$ admit pullbacks and each I^C preserve the terminal object.

(iii)$\Rightarrow$(vi): If $\beta = 1_{C'}$, then the morphism e in (iii) is a split monomorphism. Therefore $I^{C'}(e_f) \in \mathbb{E}$.

(vi)$\Rightarrow$(vii) and (vii)$\Rightarrow$(viii) are obvious. (viii)$\Leftrightarrow$(i) follows from Proposition 2.2. ∎

3.2. Remark. If the conditions of Proposition 3.1 are fulfilled, then $\mathbb{E}_{\mathcal{A}}$ consists precisely of morphisms isomorphic to e_f for some $f \in \operatorname{Mor} \mathcal{C}$. Moreover, the $(\mathbb{E}_{\mathcal{A}}, \mathbb{M}_{\mathcal{A}})$ - factorization of f is $f = m_f\, e_f$.

3.3. Remark. The equivalence of (i), (ii), (v) and (viii) holds if (3.1) satisfies a milder condition than the fullness of all H^C:

The mapping

$$H^C : \operatorname{Mor}(m, m) \longrightarrow \operatorname{Mor}(H^C(m), H^C(m)) \tag{3.3}$$

is surjective for any $m \in \mathbb{M}/T(C)$ such that there exists $e \in \mathbb{E}$ with $me \in I^C(\mathcal{C}/C)$.

The proof employs Lemma 4.1 of Section 4.

Let all H^C be full. Then (3.2) implies that the statement (iii) of Proposition 3.1 holds if the following condition is satisfied (we call it the *condition of compatibility with respect to* $\mathbb{E}$):

For all $C, C' \in \operatorname{Ob} \mathcal{C}$ and $h \in \operatorname{Mor} \mathcal{C}$, if $h : \alpha_1 \longrightarrow \beta_1$ and $h : \alpha_2 \longrightarrow \beta_2$ are morphisms in $\mathcal{C}/C$ and $\mathcal{C}/C'$, respectively, then

$$I^C(h) \in \mathbb{E} \text{ if and only if } I^{C'}(h) \in \mathbb{E}.$$

This observation immediately gives rise to

3.4. Theorem. *Let*

$$I : \mathcal{C} \longrightarrow \mathcal{X} \tag{3.4}$$

be any functor, and let $(\mathbb{E}, \mathbb{M})$ *be a factorization system on* $\mathcal{X}$. *Suppose, for each* $C \in \mathrm{Ob}\,\mathcal{C}$, *the induced functor*

$$I^C : \mathcal{C}/C \longrightarrow \mathcal{X}/I(C) \tag{3.5}$$

has a right adjoint H^C, *and let all* H^C *be full. Then the pair of morphism classes*

$$\big(I^{-1}(\mathbb{E}), (I^{-1}(\mathbb{E}))^{\downarrow}\big) \tag{3.6}$$

is a factorization system on $\mathcal{C}$.

Proof. We only have to show that $\mathbb{E}_{\mathcal{A}} = I^{-1}(\mathbb{E})$. The inclusion $\mathbb{E}_{\mathcal{A}} \subset I^{-1}(\mathbb{E})$ follows from (3.2), while the converse inclusion is obtained from the following assertion which is easy to prove. ∎

3.5. Lemma. *Let* $\mathcal{A}$ *be a family of adjunctions* (3.1), *and let* $h \in \mathrm{Mor}\,\mathcal{C}$. *If for any* $C \in \mathrm{Ob}\,\mathcal{C}$ *we have* $I^C(h) \in \mathbb{E}$ *whenever* $h : \alpha \longrightarrow \beta$ *is a morphism in* $\mathcal{C}/C$, *then* $h \in \mathbb{E}_{\mathcal{A}}$.

We wish to emphasize the fact that in Theorem 3.4 there is no requirement for the existence of any (co)limits in $\mathcal{C}$. If $\mathcal{C}$ admits pullbacks and I has a right adjoint H, then, as known, each I^C has a right adjoint H^C (H^C pulls back the H-image of morphisms along the corresponding unit). Therefore Theorem 3.4 generalizes

3.6. Theorem. [Z1,Z2] *Let* $\mathcal{C}$ *have pullbacks, and let*

$$\mathcal{C} \underset{I}{\overset{H}{\rightleftarrows}} \mathcal{X} \tag{3.7}$$

be an adjunction. If each H^C *is full, then the pair of morphism classes*

$$\big(I^{-1}(\mathbb{E}), (H(\mathbb{M}))^{\uparrow\downarrow}\big) \tag{3.8}$$

is a factorization system on $\mathcal{C}$.

3.7. Remark. The class $(H(\mathrm{Mor}\,\mathcal{X}))^{\uparrow\downarrow}$ is described in [CHK] for the case of semi-left-exact reflection (3.7). It is shown that it consists precisely of those morphisms $f : B \longrightarrow C$ for which the square

$$\begin{array}{ccc} B & \xrightarrow{\eta_B} & IB \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle If} \\ C & \xrightarrow{\eta_C} & IC \end{array}$$

is a pullback (here η is the unit of (3.7)). Recall that such morphisms are called trivial coverings in G. Janelidze's Galois theory [J1], [J2]. In [Z1] we give an analogous characterization of morphisms from $(H(\mathbb{M}))^{\uparrow\downarrow}$ if, again, (3.7) is a semi-left-exact reflection: we have to restrict ourselves only to the trivial coverings which are mapped into $\mathbb{M}$.

3.8. Example. Take the variety **Heyting** of Heyting algebras and its reflective subcategory **Boole** of Boolean algebras. The reflector

$$\mathbf{Heyting} \longrightarrow \mathbf{Boole} \tag{3.9}$$

maps every H to the Boolean algebra $I(H)$ of all regular elements of H (i.e., elements $a \in H$ for which the equality $\neg\neg a = a$ holds), while the unit η_H maps a to $\neg\neg a$. Note that the homomorphicity of η_H follows from the fact that $I(H)$ is a reflective subcategory of H (regarded as a category) with reflector η_H [B]. It can be verified that the considered reflection is semi-left-exact. "Reflecting" ($\mathbb{E}$=*surj.hom.*, $\mathbb{M}$ =*monos* (*inj.hom.*)) from **Boole**, we obtain the factorization system (3.8) on **Heyting**:

$f : H_1 \longrightarrow H_2$ lies in $I^{-1}(\mathbb{E})$ precisely when for each regular $b \in H_2$ there exists a regular $a \in H_1$ such that $f(a) = b$;
$f : H_1 \longrightarrow H_2$ lies in $(H(\mathbb{M}))^{\uparrow\downarrow}$ precisely when it satisfies the following two properties:

(i) *the restriction of f on regular elements of H_1 is injective;*

(ii) *for each regular element a of H_1 and each $b \in H_2$ such that $\neg\neg b = f(a)$ there exists a unique $a' \in H_1$ with $\neg\neg a' = a$ and $f(a') = b$.*

3.9. Example. Let $\mathcal{C}$ be an Abelian category, and let (V, ε) (or, shortly, V) be a copointed endofunctor with ε_C being a monomorphism and $V(C/V(C)) = 0$ for each $C \in \mathrm{Ob}\,\mathcal{C}$. Consider the reflective subcategory $\mathcal{X}$ of those C for which $V(C) = 0$. Recall that the reflector

$$I : \mathcal{C} \longrightarrow \mathcal{X} \tag{3.10}$$

maps C to $C/V(C)$, while $\eta_C = \mathrm{coker}\ \varepsilon_C$.

Clearly, the pair $(\mathbb{E}, \mathbb{M})$, where $\mathbb{E}$ consists of all $\mathcal{C}$-epimorphisms in $\mathcal{X}$ and $\mathbb{M}$ consists of all monomorphisms, is a factorization system on $\mathcal{X}$. We have:

$f : C \to C'$ lies in $I^{-1}(\mathbb{E})$ precisely when $\mathrm{Im}\ f + V(C') = C'$.

Indeed, consider the commutative diagram

$$\begin{tikzcd}
\mathrm{Im}\ f \arrow[d, "i_1"'] \arrow[ddr, tail, "m"] & C \arrow[l, two heads, "e"'] \arrow[r, two heads] \arrow[dd, "f"] & C/V(C) \arrow[dd, "I(f)"] \\
\mathrm{Im}\ f \oplus V(C') \arrow[dr, dashed, "s"] & & \\
V(C') \arrow[u, "i_2"] \arrow[r, tail, "\varepsilon_{C'}"'] & C' \arrow[r, two heads] & C'/V(C)
\end{tikzcd}$$

with evident e, m and s. If $I(f)$ is an epimorphism, then for any $\alpha : A \to C'$ we have morphisms β, γ, δ such that $\eta_{C'}\alpha = I(f)\beta$, $\eta_C\gamma = \beta$ and $\alpha - si_1e\gamma = \varepsilon_{C'}\delta = si_2\delta$, whence $\alpha = s(i_1e\gamma + i_2\delta)$, and therefore s is an epimorphism. The converse is verified similarly.

It is proved in [JT2] that if $\mathcal{C}$ is the category of (left) R-modules (R is a commutative ring with unit), then the reflection of $\mathcal{C}$ onto $\mathcal{X}$ is semi-left-exact if and only if V is idempotent, i.e., εV is an isomorphism. It can be verified that the latter assertion remains true for *arbitrary Abelian* $\mathcal{C}$.

Let V be idempotent. In order to calculate the class $(H(\mathbb{M}))^{\uparrow\downarrow}$ we observe that for any trivial covering m there exists θ such that $m\theta = \varepsilon_C$. Consider an arbitrary monomorphism m satisfying the latter equality for some θ. As is easy to see, $V(m)$ is an isomorphism. Let us show that the right square in the commutative diagram

$$\begin{tikzcd} V(C) \arrow[r, tail, "\varepsilon_C"] \arrow[d, "V(m)", "\approx"'] & C \arrow[r, two heads, "\text{coker } \varepsilon_C"] \arrow[d, tail, "m"] & C/V(C) \arrow[d, "I(m)"] \\ V(C') \arrow[r, tail, "\varepsilon_{C'}"] & C' \arrow[r, two heads, "\text{coker } \varepsilon_{C'}"] & C'/V(C') \end{tikzcd}$$

is a pullback. To this end, consider arbitrary φ and ψ with $I(m)\varphi = \text{coker } \varepsilon_C \cdot \psi$. We have φ' and ψ' such that $\varphi = \text{coker } \varepsilon_C \cdot \varphi'$ and $\varepsilon_{C'}\psi' = m\varphi' - \psi$. Then $m(\varphi' - \varepsilon_C V(m)^{-1}\psi') = \psi$ and coker $\varepsilon_C(\varphi' - \varepsilon_C V(m)^{-1}\psi') = \varphi$. Next, we show that $I(m)$ is a monomorphism. Indeed, there exist morphisms ρ and τ such that $\ker I(m) = \text{coker } \varepsilon_C \cdot \rho$ and $m\rho = \varepsilon_{C'}\tau$, where $\ker m' = \text{coker } \varepsilon_C \cdot \varepsilon_C V(m)^{-1}\tau = 0$. We obtain:

$f : C \longrightarrow C'$ lies in $(H(\mathbb{M}))^{\uparrow\downarrow}$ precisely when f is a monomorphism and $V(C') \subset C$ (i.e., there exists θ with $f\theta = \varepsilon_{C'}$).

Thus, when V is idempotent, the above-described morphism classes constitute a factorization system on $\mathcal{C}$.

3.10. Remark. We still can characterize morphisms of the class $(I^{-1}(\mathbb{E}))^{\downarrow}$ in Theorem 3.4 if I is not necessarily a left adjoint in an admissible (or, equivalently, semi-left-exact) reflection, but if, nevertheless, each I^C has a full and faithful right adjoint. As is known [JT1], the latter condition is equivalent to requiring that I be a fibration. In that case, the pair of morphism classes

$$\big(I^{-1}(\mathbb{E}), \Theta_{\mathbb{M}}\big), \tag{3.11}$$

where $\Theta_{\mathbb{M}}$ is the class of Cartesian morphisms over all $\mathbb{M}$-morphisms, is a factorization system on $\mathcal{C}$.

3.11. Example. Let us consider the category $\mathcal{C}$ of commutative rings R with units, which have precisely one prime ideal (equivalently, each element of which is either invertible or nilpotent; another equivalent condition is that the quotient of R by its nilradical is a field [AM]). We will first verify the existence of pullbacks along an injective homomorphism $g : R \rightarrowtail R'$ in $\mathcal{C}$. For this, consider any $h : R'' \longrightarrow R'$ from $\mathcal{C}$ and the pullback in the category of rings

$$\begin{tikzcd} P \arrow[r, "h'"] \arrow[d, tail, "g'"'] & R \arrow[d, tail, "g"] \\ R'' \arrow[r, "h"'] & R' \end{tikzcd}$$

Each element a of P is either nilpotent or invertible in R''. If b is its inverse, then $h(b)$ is the inverse of $h'(a)$, and hence $h'(a)$ is not nilpotent (without loss of generality it can be assumed that $0 \neq 1$ in R'). Therefore $h'(a)$ has the inverse in R, which clearly should coincide with $h(b)$. This implies that $b \in P$. Consequently, P is an object of $\mathcal{C}$.

It is obvious that the subcategory **Fld** of fields is reflective in $\mathcal{C}$. Therefore the functor $I^C : \mathcal{C}/C \longrightarrow \mathbf{Fld}/I(C)$ has a right adjoint for any C. One can easily observe that the corresponding counit ε^C is an isomorphism and therefore I is a fibration.

Take the following factorization system $(\mathbb{E}, \mathbb{M})$ on **Fld**: $\mathbb{E}$ consists of all algebraic extensions and $\mathbb{M}$ consists of extensions $F \rightarrowtail F'$ for which F is algebraically closed in F'. Then, as is easy to verify:

$f : R \longrightarrow R'$ lies in $I^{-1}(\mathbb{E})$ precisely when for each $a \in R'$ there exists a polynomial $\alpha \in f(R)[x]$, at least one coefficient of which does not belong to N' and is such that $\alpha(a) \in N'$;

$f : R \longrightarrow R'$ lies in $(H(\mathbb{M}))^{\uparrow\downarrow}$ precisely when f is injective and R includes the nilradical N' of R' as well as all elements which are the roots of some polynomial $\alpha \in R[x]$, at least one coefficient of which does not belong to N'.

Theorem 3.4 may turn out to be conveniently applicable if $\mathcal{C}$ is well-powered and all morphisms of $\mathcal{C}$ and $\mathcal{X}$ are monomorphisms because in that case both slice categories in (3.5) are posets and the adjoint functor theorem might be easily used.

3.12. Example. It is well known how to transport factorization systems from a category $\mathcal{C}$ to a slice category $\mathcal{C}/C$. (In passing, let us point out that this construction gives an example of factorization systems of form (3.11) for the relevant fibration.) Below we will generalize this construction in the particular case of the category **Top** of topological spaces and special factorization systems on it.

Let us first adopt the following convention: the subscript Emb in the notation of topological categories indicates the subcategory of all objects and (only) embeddings of an entire category.

Let Y be a topological space and let Y' be an arbitrary subspace. Consider the functor

$$I : \big(\mathbf{Top}/Y\big)_{\mathrm{Emb}} \longrightarrow \mathbf{Top}_{\mathrm{Emb}}$$

mapping each $\alpha : X \longrightarrow Y$ to X' in the pullback

$$\begin{tikzcd} X' \arrow[r, tail, "i_\alpha"] \arrow[d, "\alpha'"'] & X \arrow[d, "\alpha"] \\ Y' \arrow[r, tail] & Y \end{tikzcd}$$

Clearly, $(\mathbf{Top}/Y)_{\mathrm{Emb}}/\alpha$ is isomorphic to $\mathbf{Top}_{\mathrm{Emb}}/X$ and the induced functor

$$I^{\alpha} : \mathbf{Top}_{\mathrm{Emb}}/X \longrightarrow \mathbf{Top}_{\mathrm{Emb}}/X'$$

is merely a change-of-base functor i_{α}^{*} for subspaces. We can apply the adjoint functor theorem to it[3] since both $\mathbf{Top}_{\mathrm{Emb}}/X$ and $\mathbf{Top}_{\mathrm{Emb}}/X'$ are complete lattices and I^{α} preserves arbitrary joins. Moreover, I^{α} is surjective. This implies [B] that its right adjoint is full. Using Theorem 3.4, we obtain the factorization system $(I^{-1}(\mathbb{E}), (I^{-1}(\mathbb{E}))^{\downarrow})$ on $(\mathbf{Top}/Y)_{\mathrm{Emb}}$ for any factorization system $(\mathbb{E}, \mathbb{M})$ on $\mathbf{Top}_{\mathrm{Emb}}$ (for instance, for (*isomorphisms, embeddings*)). Combining $(I^{-1}(\mathbb{E}), (I^{-1}(\mathbb{E}))^{\downarrow})$ with the factorization system (*surj. cont. mappings, embeddings*) on $\mathbf{Top}/Y$, we get the factorization system $(\mathbb{E}_Y, \mathbb{M}_Y)$ on $\mathbf{Top}/Y$:

$f : \alpha \longrightarrow \beta$ lies in $\mathbb{E}_Y$ precisely when $f(X) \cap \beta^{-1}(Y') \rightarrowtail \beta^{-1}(Y')$ lies in $\mathbb{E}$;

$f : \alpha \longrightarrow \beta$ lies in $\mathbb{M}_Y$ precisely when $f : X \rightarrowtail Z$ is an embedding, $\alpha^{-1}(Y') \rightarrowtail \beta^{-1}(Y')$ lies in $\mathbb{M}$ and X contains the complement of $\beta^{-1}(Y')$ in Z; here $X = \operatorname{dom} \alpha$ and $Z = \operatorname{dom} \beta$.

Note that when $Y' = Y$, $(\mathbb{E}_Y, \mathbb{M}_Y)$ is obtained in the usual way from a certain factorization system on $\mathbf{Top}$, namely, from $(\widetilde{\mathbb{E}}, \mathbb{M})$, where $\widetilde{\mathbb{E}}$ consists precisely of continuous mappings which can be factorized as a composition of a surjective continuous mapping and some $\mathbb{E}$-morphism.

3.13. Example. To some factorization systems on the category **Haus** of Hausdorff topological spaces we will relate factorization systems on the category of the so-called V-Hausdorff spaces defined below. Here (V, ε) is a copointed endofunctor on $\mathbf{Top}_{\mathrm{Emb}}$ and we require of (V, ε) to satisfy the following property: for each embedding $f : X \rightarrowtail Y$ the square

$$\begin{tikzcd} V(X) \arrow[r, tail, "\varepsilon_X"] \arrow[d, tail, "V(f)"'] & X \arrow[d, tail, "f"] \\ V(Y) \arrow[r, tail, "\varepsilon_Y"'] & Y \end{tikzcd}$$

is a pullback. Let $\mathcal{C}$ be the full subcategory of $\mathbf{Top}_{\mathrm{Emb}}$ containing those spaces Y for which $V(Y)$ is Hausdorff. We call such spaces V-Hausdorff. Consider V as a functor

$$V : \mathcal{C} \longrightarrow \mathbf{Haus}_{\mathrm{Emb}} .$$

Again, both $\mathcal{C}/Y$ and $\mathbf{Haus}_{\mathrm{Emb}}/Y$ are complete lattices and

$$V^{Y} : \mathcal{C}/Y \longrightarrow \mathbf{Haus}_{\mathrm{Emb}}/V(Y) ,$$

[3] The adjoint functor theorem cannot be applied directly to I since $(\mathbf{Top}/Y)_{\mathrm{Emb}}$ does not admit coequalizers.

clearly preserves arbitrary joins. Moreover, V^Y is surjective. These arguments provide the factorization system $(V^{-1}(\mathbb{E}), (V^{-1}(\mathbb{E}))^{\downarrow})$ on $\mathcal{C}$ for any factorization system $(\mathbb{E}, \mathbb{M})$ on $\mathbf{Haus}_{\mathrm{Emb}}$. Keeping in mind the factorization system (*surj. cont. mappings, embeddings*) on the category of all V-Hausdorff spaces (with all continuous mappings), we get a new factorization system $(\mathbb{E}_V, \mathbb{M}_V)$ on it. Taking into account the structure of H^Y (namely, $H^Y(\,Z \rightarrowtail V(Y)) = (\,Z \cup (Y \backslash V(Y)) \rightarrowtail Y)$, we obtain:

$f : X \longrightarrow Y$ lies in $\mathbb{E}_V$ precisely when $f(X) \cap V(Y) \longrightarrow V(Y)$ lies in $\mathbb{E}$;
$f : X \longrightarrow Y$ lies in $\mathbb{M}_V$ precisely when f is an embedding and $X \cap V(Y) \rightarrowtail V(Y)$ lies in $\mathbb{M}$ and X contains the complement of $V(Y)$ in Y.

3.14. Example. $\mathbf{Haus}_{\mathrm{Emb}}$ in Example 3.13 can be replaced by $\mathbf{CompHaus}_{\mathrm{Emb}}$, while (*surj. cont. mappings, embeddings*) by (*dense mappings, closed embeddings*). But then for V^Y to preserve arbitrary joins $\bigvee$, we have to restrict ourselves only to the spaces Y satisfying the infinite distributive law

$$X \cap \Big(\bigvee_{i \in I} X_i\Big) = \bigvee_{i \in I} (X \cap X_i) \tag{3.12}$$

for the closed-set lattice of Y. In this way, for each factorization system $(\mathbb{E}, \mathbb{M})$ on $\mathbf{CompHaus}_{\mathrm{Emb}}$, we obtain the factorization system $(\mathbb{E}_V, \mathbb{M}_V)$ on the category of all topological spaces Y which satisfy (3.12) and are such that $V(Y)$ is compact and Hausdorff. We have:

$f : X \longrightarrow Y$ lies in $\mathbb{E}_V$ precisely when $\overline{f(X)} \cap V(Y) \rightarrowtail V(Y)$ lies in $\mathbb{E}$;

$f : X \longrightarrow Y$ lies in $\mathbb{M}_V$ precisely when f is a closed embedding, $\overline{X} \cap V(Y) \rightarrowtail V(Y)$ lies in $\mathbb{M}$ and $\overline{X}$ is the largest element in the closed-set lattice of Y among those Z for which $Z \cap V(Y) \subset \overline{X} \cap V(Y)$.

4. When $\mathbb{M} \subset \mathrm{Mon}\, \mathcal{X}$

Even if the condition of the fullness of all H^C fails to be fulfilled, to the given data (3.1) and a factorization system $(\mathbb{E}, \mathbb{M})$ on $\mathcal{X}$ we can relate the full replete subcategory $\mathcal{C}_C$ of $\mathcal{C}/C$ induced by the class $H^C(\mathbb{M}/C)$ of $\mathcal{C}/C$-objects and sometimes turning out to be reflective. To find the corresponding condition, we will first consider a general situation.

4.1. Lemma. *Let*

$$\mathcal{C} \underset{I}{\overset{H}{\rightleftarrows}} \mathcal{X} \tag{4.1}$$

be an adjunction with unit η and let $\mathcal{Y}$ be a reflective subcategory of $\mathcal{X}$ with reflector

$$r : \mathcal{X} \longrightarrow \mathcal{Y} \tag{4.2}$$

and unit ζ. Suppose $\mathcal{B}$ denotes the full replete subcategory of $\mathcal{C}$ given by the class $H(\mathrm{Ob}\ \mathcal{Y})$ of objects. The following conditions are equivalent:

(i) *$\mathcal{B}$ is reflective and the units of the reflection are the compositions*

$$C \xrightarrow{\eta_C} HIC \xrightarrow{H(\zeta_{IC})} HrIC\ ; \tag{4.3}$$

(ii) *for each $C \in \mathrm{Ob}\ \mathcal{C}$ and each $Y \in \mathrm{Ob}\ \mathcal{Y}$ the mapping*

$$H : \mathrm{Mor}\,(rIC, Y) \longrightarrow \mathrm{Mor}\,(H(rIC), H(Y))$$

is surjective;

(iii) *$\mathcal{B}$ is reflective and for each $C \in \mathrm{Ob}\ \mathcal{C}$ the mapping*

$$H : \mathrm{Mor}\,(rIC, rIC) \longrightarrow \mathrm{Mor}\,(H(rIC), H(rIC)) \tag{4.4}$$

is surjective.

If every split $\mathcal{C}$-epimorphism of the form $H(\alpha)$ with $\alpha \in \mathrm{Mor}\ \mathcal{Y}$ is an isomorphism, then each of (i)–(iii) *is also equivalent to*

(iv) *$\mathcal{B}$ is reflective.*

Proof. We first observe that, since (4.3) are precisely the units of the composition of adjunctions (4.1) and (4.2), they are always weakly universal with respect to $\mathcal{B}$.

(i)⇒(ii). For each $g : HrIC \longrightarrow HY$, we have a morphism $\alpha : rIC \longrightarrow Y$ such that $gH(\zeta_{IC})\eta_C = H(\alpha)H(\zeta_{IC})\eta_C$. If $\mathcal{B}$ is reflective, then $g = H(\alpha)$.

(ii)⇒(iii). We verify that composition (4.3) is universal. Indeed, if $g_1H(\zeta_{IC})\eta_C = g_2H(\zeta_{IC})\eta_C$, then we have $g_1 = g_2$ since both g_i have the form $H(\beta_i)$, .

(iii)⇒(i). If $\mathcal{B}$ is reflective and the units of this reflection are $\xi_C : C \longrightarrow H(Y_C)$, then there exist morphisms $\alpha : rIC \longrightarrow Y_C$ and $g : H(Y_C) \longrightarrow HrIC$ such that $g\xi_C = H(\zeta_{IC})\eta_C$ and $H(\alpha)H(\zeta_{IC})\eta_C = \xi_C$. Hence

$$H(\alpha)g = 1_{H(Y_C)} \tag{4.5}$$

and $(gH(\alpha))H(\zeta_{IC})\eta_C = H(\zeta_{IC})\eta_C$. Since $gH(\alpha) = H(\beta)$ for some β, we obtain $\beta = 1_{rIC}$. Hence $gH(\alpha) = 1_{HrIC}$.

(iv)⇒(i) trivially follows from equality (4.5) under the relevant condition on the H-image of $\mathcal{Y}$-morphisms. ∎

Now we can formulate our next statement.

4.2. Proposition. *Let $\mathbb{M} \subset \mathrm{Mon}\ \mathcal{X}$. The following conditions are equivalent:*

(i) *$(\mathbb{E}_\mathcal{A}, \mathbb{M}_\mathcal{A})$ is a factorization system;*

(ii) *for all objects C, C' of $\mathcal{C}$ and each $f \in \mathrm{Ob}\ \mathcal{C}/C$, if $e_f : \alpha \longrightarrow \beta$ is a morphism in $\mathcal{C}/C'$, then there exists a $\mathcal{C}/C'$-isomorphism $e_\alpha \approx e_\beta e_f$ for $e_\alpha : \alpha \longrightarrow m_\alpha$ and $e_\beta : \beta \longrightarrow m_\beta$, and the mapping*

$$H^C : \mathrm{Mor}\,(m, m') \longrightarrow \mathrm{Mor}\,(m_f, H^C(m'))\ ^4 \tag{4.6}$$

is surjective for any $m' \in \mathbb{M}/T(C)$;

[4] Recall that, as usual, m is an $\mathbb{M}$-morphism in the $(\mathbb{E}, \mathbb{M})$-factorization of f.

(iii) *for each morphism f, we have $e_f \in \mathbb{E}_{\mathcal{A}}$.*

If, moreover, $\mathcal{C}$ has pullbacks, then each of (i)–(iii) *is equivalent to the following condition:*

(iv) $\mathbb{M}_{\mathcal{A}}$ *is stable under pullbacks, and for each $C \in \operatorname{Ob}\mathcal{C}$, $f \in \operatorname{Ob}\mathcal{C}/C$ and $m' \in \mathbb{M}/T(C)$, the mapping* (4.6) *is surjective and m_{e_f} is an isomorphism.*

If, in addition, $\mathcal{C}$ has all intersections of monomorphisms, then we have one more equivalent condition:

(v) $\mathbb{M}_{\mathcal{A}}$ *is stable under pullbacks and is closed under composition and intersection.*

Proof. (i)⇒(ii). We first apply Lemma 4.1, taking into account the easily verified fact that H^C preserves monomorphisms considered as objects of the slice category $\mathcal{X}/T(C)$, and conclude that (4.6) is surjective and the unit of the obtained reflection $\mathcal{C}/C \longrightarrow \mathcal{C}_C$ is given by $e_f : f \longrightarrow m_f$. Then we use Proposition 2.2.

(ii)⇒(iii) and the equivalence (i)⇔(iv) follow again from Lemma 4.1 and Proposition 2.2, while (i)⇔(v) follows from Theorem 2.1. ∎

4.3. Remark. Under the conditions of Proposition 4.2 both the structure of $\mathbb{E}_{\mathcal{A}}$ and the form of the $(\mathbb{E}_{\mathcal{A}}, \mathbb{M}_{\mathcal{A}})$-factorization of morphisms coincide with those described in Section 3.

The results obtained above will be applied below to the problem of relating a factorization system to a pointed endofunctor.

From now on it is assumed that $\mathcal{C}$ *admits pullbacks*. Let (T, η) be a pointed endofunctor on $\mathcal{C}$. Consider the family of adjunctions

$$\mathcal{D} : \Big(\mathcal{C}/C \underset{\eta_C!}{\overset{\eta_C^*}{\rightleftarrows}} \mathcal{C}/T(C) \Big)_{C \in \operatorname{Ob}\mathcal{C}}, \tag{4.7}$$

where η_C^* is the usual change-of-base functor and $\eta_C!$ is the corresponding composition functor. Observe that in that case $\mathbb{M}_{\mathcal{D}}$ contains precisely the pullbacks of $\mathbb{M}$-morphisms along the corresponding η for any factorization system $(\mathbb{E}, \mathbb{M})$ on $\mathcal{C}$. Moreover, for each $f : B \longrightarrow C$ in $\mathcal{C}$, the morphism e_f is the canonical one in the commutative diagram

$$\begin{tikzcd} B \arrow[dr, dashed, "e_f"] \arrow[drr, "e"] \arrow[ddr, "f"'] & & \\ & P \arrow[r, "\sigma"'] \arrow[d, "m_f"] & X \arrow[d, "m"] \\ & C \arrow[r, "\eta_C"'] & TC \end{tikzcd} \tag{4.8}$$

where me is the $(\mathbb{E}, \mathbb{M})$-factorization of $\eta_C f$ and the internal square is a pullback.

4.4. Lemma. *The class $\mathbb{M}_{\mathcal{D}}$ is stable under pullbacks.*

Proof. For a given pullback

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "n"'] & B \arrow[d, "m'"] \\ C \arrow[r, "\alpha"'] & D \end{tikzcd}$$

with $m' \in \mathbb{M}_{\mathcal{D}}$, we have the commutative diagram

$$\begin{tikzcd} A \arrow[r, "\beta"] \arrow[d, "n"'] \arrow[drr, dashed, "\xi"'] & B \arrow[rr] \arrow[d, "m'"] & & K \arrow[d, "m"] \\ & & S \arrow[ur] \arrow[dd, "m''"] & \\ C \arrow[r, "\alpha"] \arrow[drr, "\eta_C"'] & D \arrow[rr, "\eta_D"] & & TD \\ & & TC \arrow[ur, "T\alpha"'] & \end{tikzcd} \tag{4.9}$$

for some $m \in \mathbb{M}$ and ξ, where the right-hand rear square and the right-hand lateral square are pullbacks, and therefore the left-hand lateral square is also a pullback; moreover, $m'' \in \mathbb{M}$. ∎

4.5. Theorem. *If $(\mathbb{E}, \mathbb{M})$ is a factorization system on $\mathcal{C}$ with $\mathbb{M} \subset \operatorname{Mon} \mathcal{C}$ and there holds the condition*

$$\textit{if} \quad m \in \mathbb{M}, \quad \textit{then} \quad T(m) \in \mathbb{M}, \tag{4.10}$$

then the pair $(\mathbb{E}_{\mathcal{D}}, \mathbb{M}_{\mathcal{D}})$ is also a factorization system on $\mathcal{C}$. If for each $C \in \operatorname{Ob} \mathcal{C}$, the pullback along η_C reflects isomorphisms for $\mathbb{M}$-morphisms, then $\mathbb{E}_{\mathcal{D}}$ consists precisely of those $f : A \longrightarrow B$ for which $\eta_B f \in \mathbb{E}$; if, moreover, $\eta_C \in \mathbb{E}$, then $(\mathbb{E}_{\mathcal{D}}, \mathbb{M}_{\mathcal{D}})$ coincides with the pair

$$\big(T^{-1}(\mathbb{E}), (T^{-1}(\mathbb{E}))^{\downarrow}\big). \tag{4.11}$$

Proof. Let us show that the condition (iii) of Proposition 4.2 is satisfied. By Proposition 2.6 of [CJKP] and Lemma 4.4, it suffices to verify the existence of a diagonal morphism in the commutative square

$$\begin{tikzcd} B \arrow[r, "e_f"] \arrow[d, "\alpha"'] & P \arrow[d, "1_P"] \\ A \arrow[r, "m'"'] & P \end{tikzcd}$$

for each $f : B \longrightarrow C$ and each $m' \in \mathbb{M}_D$. Consider the commutative diagram

$$\begin{tikzcd}
 & & B \arrow[dl, "\alpha"'] \arrow[dr, "e_f"] \arrow[drr, "e"] & & \\
 & A \arrow[rr, "m'"] \arrow[ddl] & & P \arrow[r, "\sigma"'] \arrow[d, "m_f"] \arrow[ddl, "\eta_P"'] & X \arrow[d, "m"] \arrow[dl, phantom, "\mathrm{(II)}"] \\
 & & \mathrm{(I)} & C \arrow[r, "\eta_C"'] & TC \\
S \arrow[rr, "m''"'] & & TP \arrow[urr, "Tm_f"'] & &
\end{tikzcd} \tag{4.12}$$

Here the squares (I), (II) are pullbacks and $m'' \in \mathbb{M}$. By condition (4.10) we have $T(m_f)m'' \in \mathbb{M}$. Hence $e \downarrow T(m_f)m''$, which implies the existence of a diagonal morphism $\delta : X \longrightarrow S$ for the bordering quadrangle in (4.12). We have $T(m_f)m''(\delta\sigma) = m\sigma = T(m_f)\eta_P$, whence $m''\delta\sigma = \eta_P$. The pullback (I) gives rise to the sought-for morphism $\delta' : P \longrightarrow A$.

To prove the second part of the theorem, recall that $\mathbb{E}_{\mathcal{D}}$ consists precisely of morphisms of the form ie_f, where i is an isomorphism. Consider diagram (4.8), but this time for e_f instead of f. Since $e_f = m_{e_f} e_{e_f}$ is the $(\mathbb{E}_{\mathcal{D}}, \mathbb{M}_{\mathcal{D}})$-factorization of e_f, we have that m_{e_f} is an isomorphism. Therefore so is m, and $\eta_C e_f \in \mathbb{E}$. The converse is obvious. From the commutativity of the diagram

$$\begin{tikzcd}
C \arrow[r, "\eta_C"'] \arrow[d, "f"'] & TC \arrow[d, "T(f)"] \\
C' \arrow[r, "\eta_{C'}"'] & TC'
\end{tikzcd} \tag{4.13}$$

it follows that $(\mathbb{E}_{\mathcal{D}}, \mathbb{M}_{\mathcal{D}})$ coincides with (4.11). ∎

4.6. Example. As is known [M], the set S_G of all compact elements (i.e., of elements a of a topological group G, such that the smallest closed subgroup $\overline{gp(a)}$ of G containing a is compact) is a closed subgroup of G if G is locally compact Hausdorff and Abelian. If $f : G \longrightarrow G'$ is a continuous homomorphism of such groups and $a \in S_G$, then $f(\overline{gp(a)})$ is compact and hence closed. Therefore $\overline{gp(f(a))} \subset f(\overline{gp(a)})$, whence $f(a) \in S_{G'}$. These arguments provide the pointed endofuctor $T(G) = G/S_G$ (with η_G being the projection $G \longrightarrow G/S_G$) on the category $\mathcal{C}$ of locally compact Hausdorff Abelian groups with compact sets of compact elements (the latter requirement is needed for our further purposes).

To verify that $\mathcal{C}$ has finite products, we observe that $S_{G\times G'} = S_G \times S_{G'}$ for any Hausdorff topological groups G and G'. Indeed, if $(a, a') \in S_{G\times G'}$, then the projection $\pi : G \times G' \to G$ maps $\overline{gp(a, a')}$ into a closed subset of G. This readily implies that $\overline{gp(a)} = \pi\overline{gp(a, a')}$ and $a \in S_G$. The converse inclusion is obvious. Since the closed subsets of a locally compact space are also locally compact, $\mathcal{C}$ admits equalizers and thus $\mathcal{C}$ is finitely complete.

Obviously, ($\mathbb{E}$ =*dense homom.,* $\mathbb{M}$ = *closed embeddings*) is a factorization system on $\mathcal{C}$. We will show that T preserves closed embeddings. Let $f : G \rightarrowtail G'$ be a mapping of this kind, and let $a \in G$ be a compact element of G'. Then $\overline{gp'(a)} \subset G$, whence $\overline{gp(a)}$ (equal to $\overline{gp'(a)}$) is also compact. Therefore $T(f)$ is injective. Since S_G is compact, the mapping η_G is closed. From the commutativity of (4.13) we conclude that $T(f)$ is also closed.

Thus all the conditions of Theorem 4.5 are satisfied and we obtain the factorization system (4.11) on $\mathcal{C}$. We have:

$f : G \longrightarrow G'$ lies in $T^{-1}(\mathbb{E})$ precisely when $\overline{f(G) + S_{G'}}$ $(= \overline{f(G)} + S_{G'})$ coincides with G';
$f : G \longrightarrow G'$ lies in $(T^{-1}(\mathbb{E}))^{\downarrow}$ precisely when f is a closed embedding and $S_{G'} \subset G$.

4.7. Remark. One can easily verify that Theorem 4.5 can be generalized to the case where $(\mathbb{E}, \mathbb{M})$ is not necessarily a (usual) factorization system, but, more generally, an arbitrary local factorization system with respect to a class $\mathcal{C}'$ of objects such that $T(\mathrm{Ob}\,\mathcal{C}) \subset \mathcal{C}'$ – we only have to replace " $m \in \mathbb{M}$ " in (4.10) by " $m \in \mathbb{M}_{\mathcal{D}}$ ".

4.8. Remark. In [JT2], among many other things, the pair $(\mathcal{E}_T, \mathcal{M}_T)$ of morphism classes is related to a pointed endofunctor (T, η). Recall that $\mathcal{E}_T$ consists precisely of *T-vertical morphisms,* i.e., of morphisms $f : B \longrightarrow C$ for which there exists a morphism d such that the diagram

$$\begin{tikzcd} C \arrow[r, "d"] \arrow[d, "1_C"'] & TB \arrow[d, "Tf"] \\ C \arrow[r, "\eta_C"] & TC \end{tikzcd}$$

is a pullback, while $\mathcal{M}_T$ consists precisely of *trivial T-coverings,* i.e., of morphisms $f : B \longrightarrow C$ for which the diagram

$$\begin{tikzcd} B \arrow[r, "\eta_B"'] \arrow[d, "f"'] & TB \arrow[d, "Tf"] \\ C \arrow[r, "\eta_C"'] & TC \end{tikzcd}$$

is a pullback. It is proved in [JT2] that $(\mathcal{E}_T, \mathcal{M}_T)$ is a factorization system and coincides with

$$\big(T^{-1}(\mathrm{Iso}\,\mathcal{C}), \big(T^{-1}(\mathrm{Iso}\,\mathcal{C})\big)^{\downarrow}\big)$$

if T is induced by a simple reflection.

In view of Theorem 4.5 there arises a question whether the pair $(\mathcal{E}_T, \mathcal{M}_T)$ is equal to $(\mathrm{Iso}\,\mathcal{C}_{\mathcal{D}}, \mathrm{Mor}\,\mathcal{C}_{\mathcal{D}})$ and if not, whether it is equal to $(\mathbb{E}_{\mathcal{D}}, \mathbb{M}_{\mathcal{D}})$ for some $(\mathbb{E}, \mathbb{M})$. Note that if $T\eta$ is an isomorphism and ηT is a monomorphism, then $\mathcal{M}_T = \mathrm{Mor}\,\mathcal{C}_{\mathcal{D}}$ if and only if η is an isomorphism. Indeed, η_C obviously lies in $\mathrm{Mor}\,\mathcal{C}_{\mathcal{D}}$ and if it lies also in $\mathcal{M}_T$, then it is the pullback of $T(\eta_C)$ along η_{TC} and is therefore an isomorphism. This in particular implies that $\mathcal{E}_T \neq \mathrm{Iso}\,\mathcal{C}_D$ in general.

However, sometimes $(\mathcal{E}_T, \mathcal{M}_T)$ may coincide with $(\mathbb{E}_{\mathcal{D}}, \mathbb{M}_{\mathcal{D}})$ for $\mathbb{E} = \{i\eta_C | C \in \mathrm{Ob}\ \mathcal{C},\ i$ is isomorphism$\}$ and $\mathbb{M} = \{T(f)i | f \in \mathrm{Mor}\ \mathcal{C},\ i$ is isomorphism$\}$ if $(\mathbb{E}, \mathbb{M})$ is a $T(\mathrm{Ob}\ \mathcal{C})$-factorization system (this happens if, for instance, either T is induced by a reflection or each η_C is an epimorphism and T is idempotent, i.e., ηT is an isomorphism). More precisely,

$$\mathcal{M}_T = \mathbb{M}_{\mathcal{D}}$$

if and only if for each $f : B \longrightarrow C$ the pullback of $T(f)$ along η_C lies in $\mathcal{M}_T$. This condition is studied in detail in [JT2]. We observe that if, in addition, the canonical morphism in the diagram

$$\begin{tikzcd} B \arrow[dr, dashed] \arrow[drr, "\eta_B"] \arrow[ddr, "f"'] & & \\ & P \arrow[r] \arrow[d] \arrow[dr, phantom, "pb"] & TB \arrow[d, "Tf"] \\ & C \arrow[r, "\eta_C"'] & TC \end{tikzcd}$$

lies in $\mathcal{E}_T$ for any f and $\mathcal{C}$ has a terminal object preserved by T, then, according to [JT2], $(\mathcal{E}_T, \mathcal{M}_T)$ is a factorization system and consequently

$$\mathcal{E}_T = \mathbb{E}_{\mathcal{D}}.$$

4.9. Remark. The requirement $\mathbb{M} \subset \mathrm{Mon}\ \mathcal{C}$ is essential for the validity of the first part of Theorem 4.5. Indeed, on the category of Abelian groups let us consider the pointed endofunctor defined by $T(G) = G/2G$. It is obvious that a monomorphism $f : G \rightarrowtail G'$ lies in $\mathrm{Mor}\, \mathcal{C}_{\mathcal{D}}$ if and only if G contains $2G'$. This implies that $\mathrm{Mor}\, \mathcal{C}_D$ is not closed under composition since it contains, for instance, the inclusions $4\mathbb{Z} \rightarrowtail 2\mathbb{Z}$ and $2\mathbb{Z} \rightarrowtail \mathbb{Z}$, but does not contain $4\mathbb{Z} \rightarrowtail \mathbb{Z}$.

Furthermore, requirement (4.10) is also essential for the validity of Theorem 4.5 since, as is easy to verify, $\mathrm{Mon}\, \mathcal{C}_{\mathcal{D}} = \mathrm{Mon}\, \mathcal{C} \cap \mathrm{Mor}\, \mathcal{C}_{\mathcal{D}}$.

From Remark 4.8 we conclude that the equality of $(\mathbb{E}_{\mathcal{D}}, \mathbb{M}_{\mathcal{D}})$ with (4.11) shown in the second part of Theorem 4.5 does not hold in general even if $\mathbb{M} \subset \mathrm{Mon}\ \mathcal{C}$.

5. The Case of Complete and Well-Powered Categories

According to Remark 4.9, it is by no means true that the pair $(\mathbb{E}_{\mathcal{A}}, \mathbb{M}_{\mathcal{A}})$ is always a factorization system even if a given category $\mathcal{C}$ satisfies a number of natural conditions and $\mathbb{M} \subset \mathrm{Mon}\ \mathcal{C}$. From this remark it follows in particular that $(\mathbb{E}_{\mathcal{A}}, \mathbb{M}_{\mathcal{A}})$ is, in general, not even a prefactorization system. This leads us to considering a new pair of morphism classes

$$\big(\mathbb{E}_{\mathcal{A}}, \mathbb{M}_{\mathcal{A}}^{\uparrow\downarrow}\big), \tag{5.1}$$

which sometimes turns out to be a factorization system, while $(\mathbb{E}_{\mathcal{A}}, \mathbb{M}_{\mathcal{A}})$ does not (see Examples 5.7 and 5.8).

5.1. Proposition. *Let $\mathcal{C}$ be a complete and well-powered category and let* $(\mathbb{E}, \mathbb{M})$ *be a factorization system on* $\mathcal{X}$ *with* $\mathbb{M} \subset \mathrm{Mon}\ \mathcal{X}$*. Suppose* $\mathcal{A}$ *satisfies the condition of compatibility with respect to* $\mathbb{E}$ *(see Section* 3*) and* $I^C(\alpha)$ *is an isomorphism for each one in* $\mathcal{C}$*. Then the pair* $(\mathbb{E}_{\mathcal{A}}, \mathbb{M}_{\mathcal{A}}^{\uparrow\downarrow})$ *is a factorization system on* $\mathcal{C}$*,* $\mathbb{M}_{\mathcal{A}}^{\uparrow\downarrow} \subset \mathrm{Mon}\ \mathcal{C}$ *and*

$$\mathbb{E}_{\mathcal{A}} = \bigcup_{C \in \mathrm{Ob}\ \mathcal{C}} I^{C^{-1}}(\mathbb{E}). \tag{5.2}$$

5.2. Remark. We do not specify whether $\mathbb{E}$ in (5.2) is taken as a class of objects or a class of morphisms in $\mathcal{X}/T(C)$ since, by virtue of Lemma 3.5 and the compatibility condition, in both cases $\mathbb{E}_{\mathcal{A}}$ in the left-hand part of (5.2) is the same.

Proof of Proposition 5.1. Let $f : B \longrightarrow C$ be any morphism of $\mathcal{C}$. We already have the factorization

$$f = m_f\, e_f. \tag{5.3}$$

Recall that $m_f = H^C(m)$ and m is an $\mathbb{M}$-morphism in the $(\mathbb{E}, \mathbb{M})$-factorization $I^C(f) = me$. Let $\mathbb{N}$ be the morphism class $\mathbb{M}_{\mathcal{A}}^{\uparrow\downarrow} \cap \mathrm{Mon}\ \mathcal{C}$. By Theorem (2.1), $(\mathbb{N}^{\uparrow}, \mathbb{N})$ is a factorization system and therefore $e_f = nf'$ with $n \in \mathbb{N}$ and $f' \in \mathbb{N}^{\uparrow}$. It suffices to show that $f' \in \mathbb{E}_{\mathcal{A}}$. Again consider factorization (5.3), this time assuming that $f \in \mathbb{N}^{\uparrow}$. Then $m_f \in \mathbb{N}$ and thus $f \downarrow m_f$. This implies that m_f is a split epimorphism and thus m_f is an isomorphism. Since $I^C(m_f) = m\varepsilon^C_m$ and $m \in \mathrm{Mon}\ \mathcal{C}$, we conclude that m is also an isomorphism . Hence $I^C(f) \in \mathbb{E}$ and, consequently, the image of $f : f \to 1_C$ under I^C lies in $\mathbb{E}$. By the condition of compatibility and Lemma 3.5, we obtain that $f \in \mathbb{E}_{\mathcal{A}}$. Actually, we have proved that $\mathbb{N}^{\uparrow} = \mathbb{E}_{\mathcal{A}}$ and therefore $\mathbb{M}_{\mathcal{A}}^{\uparrow\downarrow} \subset \mathrm{Mon}\ \mathcal{C}$. The above arguments also prove (5.2). ■

Proposition 5.1 immediately gives rise to

5.3. Theorem. *Let* $\mathcal{C}$ *be a complete and well-powered category and let*

$$I : \mathcal{C} \longrightarrow \mathcal{X}$$

be any functor such that for each $C \in \mathrm{Ob}\ \mathcal{C}$ *the induced functor*

$$I^C : \mathcal{C}/C \longrightarrow \mathcal{X}/I(C)$$

admits a right adjoint. Then, for each factorization system $(\mathbb{E}, \mathbb{M})$ *on* $\mathcal{X}$ *with* $\mathbb{M} \subset \mathrm{Mon}\ \mathcal{X}$*, the pair of morphism classes*

$$\big(I^{-1}(\mathbb{E}), (I^{-1}(\mathbb{E}))^{\downarrow}\big)$$

is a factorization system on $\mathcal{C}$*.*

In particular, we have

5.4. THEOREM. *Let $\mathcal{C}$ be both complete and well-powered. For each adjunction*

$$\mathcal{C} \underset{I}{\overset{H}{\leftrightarrows}} \mathcal{X}$$

and each factorization system $(\mathbb{E}, \mathbb{M})$ *on* $\mathcal{X}$ *with* $\mathbb{M} \subset \mathrm{Mon}\ \mathcal{X}$ *the pair of morphism classes*

$$\big(I^{-1}(\mathbb{E}), (H(\mathbb{M}))^{\uparrow\downarrow}\big) \tag{5.4}$$

is a factorization system on $\mathcal{C}$*.*

PROOF. Just recall that, according to [Z1], (5.4) is a prefactorization system. ∎

To describe morphisms from $(H(\mathbb{M}))^{\uparrow\downarrow}$, we give

5.5. LEMMA. *Let $\mathcal{C}$ be a category with pullbacks, and let $\mathbb{E}$ be any class of morphisms closed under composition and satisfying the following cancellation property:*

$$\textit{if} \quad e = \beta\alpha \quad \textit{and} \quad e \in \mathbb{E}, \quad \textit{then} \quad \beta \in \mathbb{E}.$$

Suppose $(\mathbb{E}', \mathbb{M}')$ *is a factorization system on* $\mathcal{C}$ *such that* $\mathbb{E}' \subset \mathbb{E}$*. Then the following conditions for a* $\mathcal{C}$*-morphism* m *are equivalent:*

(i) $m \in \mathbb{E}^{\downarrow}$*;*

(ii) *if the pullback* α *of* m *along some morphism lies in* $\mathbb{E}$*, then* α *is an isomorphism;*

(iii) *if the pullback* α *of* m *along some* $\mathbb{M}'$*-morphism lies in* $\mathbb{E}$*, then* α *is an isomorphism;*

(iv) $m \in \mathbb{M}'$ *and* (iii) *holds.*

PROOF. (i)⇒(ii), (ii)⇒(iii), (i)⇒(iv) and (iv)⇒(iii) are obvious. We verify (iii)⇒(i). Consider the commutative diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & B \arrow[d, "g"] \\ C \arrow[r, "m"'] & D \end{tikzcd} \tag{5.5}$$

with $e \in \mathbb{E}$. Factorizing g by $(\mathbb{E}', \mathbb{M}')$ and taking the corresponding pullback, we obtain the diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, dashed, "h"'] & B \arrow[d, "e'"] \\ P \arrow[r, "q"'] \arrow[d, "m''"'] & K \arrow[d, "m'"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

From the upper square one has $q \in \mathbb{E}$, whence q is an isomorphism. The rest of the proof is obvious. ∎

Note that the class $I^{-1}(\mathbb{E})$ from Theorems 5.3 and 5.4 satisfies the cancellation property of Lemma 5.5 since $(I^{-1}(\mathbb{E}))^{\downarrow} \subset \mathrm{Mon}\,\mathcal{C}$.

5.6. Example. It is well-known that the pair of morphism classes $(\mathbb{E}, \mathbb{M})$, where $\mathbb{E}$ consists of all integral homomorphisms, while $\mathbb{M}$ consists of monomorphisms $f : G \rightarrowtail G'$ for which G is integrally closed in G', is a factorization system on the category **Rng** of commutative rings with units. Theorem 5.4 allows us to transport $(\mathbb{E}, \mathbb{M})$ along the adjunction

$$\mathbf{Ab} \underset{I}{\overset{H}{\leftrightarrows}} \mathbf{Rng}\ , \tag{5.6}$$

where **Ab** is the category of Abelian groups and $I(G)$ is the group ring over an Abelian group G.

We have:

$f : G \longrightarrow G'$ lies in $I^{-1}(\mathbb{E})$ precisely when for each $x \in G'$ there exists $n \geq 1$ such that $x^n \in \mathrm{Im}\, f$[5].

Indeed, if $x^n = f(a)$, then x clearly is integral over $I(f)(I(G))$ and now it suffices to observe that elements of G' are generators of $I(G')$. For the converse, suppose $I(G')$ is integral over $I(f)(I(G))$. Since $\mathrm{Im}\, f$ generates $I(f)(I(G))$ as an Abelian group, each $x \in G'$ satisfies an equality of the form

$$x^n + b_1 x^{n-1} + \cdots + b_{n-1} x + b_n = 0, \tag{5.7}$$

where every b_i is written as

$$b_i = k_{i1} f(a_{i1}) + k_{i2} f(a_{i2}) + \cdots + k_{im_i} f(a_{im_i}). \tag{5.8}$$

Here $k_{ij} \in \mathbb{Z}$ and $a_{ij} \in G$. But $I(G')$ is free over G' (as an Abelian group), hence, from (5.7) and (5.8) we obtain $x^n = f(a_{ij})x^k$ for some i, j and $0 \leq k < n$. Therefore $x^{n-k} \in \mathrm{Im}\, f$.

Applying Lemma 5.5, the class of morphisms diagonalized with $I^{-1}(\mathbb{E})$ can be easily calculated. Namely, we have:

$(H(\mathbb{M}))^{\uparrow\downarrow}$ consists of those monomorphisms $f : G \rightarrowtail G'$ for which G contains all $x \in G'$ with the property $x^n \in G$ for some $n \geq 1$.

Note that the latter class coincides with $\mathbb{M}_{\mathcal{A}}$ for the family $\mathcal{A}$ induced by (5.6). Indeed, any f from $(H(\mathbb{M}))^{\uparrow\downarrow}$ is a pullback of the inclusion $R \rightarrowtail I(G')$ along the corresponding unit, where R is the integral closure of $I(G)$ in $I(G')$.

[5] Here we use the multiplicative abbreviation for the operation of an Abelian group G so as not to confuse it with the operation of addition in $I(G)$.

5.7. Example. We return to Example 3.9 and follow its notation. In particular, $\mathcal{C}$ denotes an Abelian category with a copointed endofunctor (V, ε) such that ε_C is a monomorphism and $V(C/V(C)) = 0$ for any $C \in \mathrm{Ob}\,\mathcal{C}$. We have seen that if, in addition, (V, ε) is idempotent, then $(I^{-1}(\mathbb{E}), (H(\mathbb{M}))^{\uparrow\downarrow})$ is a factorization system on $\mathcal{C}$. Theorem 5.4 implies that this is the case even if (V, ε) is not idempotent, but $\mathcal{C}$ is complete and well-powered. The class $I^{-1}(\mathbb{E})$ has already been calculated in Example 3.9. The class $(H(\mathbb{M}))^{\uparrow\downarrow}$ is not, in general, described as in that example (take the copointed endofunctor $V(G) = 2G$ on the category of Abelian groups and then apply arguments similar to those of Remark 4.9). To characterize its morphisms, observe in the first place that all of them are monomorphisms. Applying Lemma 5.5, we obtain:

For a monomorphism $f : C \rightarrowtail C'$ *the following conditions are equivalent:*

(i) $f \in (H(\mathbb{M}))^{\uparrow\downarrow}$;

(ii) *if* $g : S \rightarrowtail C'$ *is a subobject of* C' *such that*

$$S = (C \cap S) + V(S), \tag{5.9}$$

then $S \subset C$ (*equivalently* $V(S) \subset C$).

It is obvious that V preserves subobjects. From now on it is assumed that V *also preserves finite joins in the lattices of subobjects* and hence is additive.

Each of (i) *and* (ii) *is implied by the following condition:*
(iii) $V^l(C') \subset C$ *for some* $l \geq 1$.
If the chain of subobjects

$$C' \supset V(C') \supset V^2(C') \supset \dots \tag{5.10}$$

is stabilized at the k*-th step, then* (i)–(iii) *are equivalent and, moreover, we have one more equivalent condition:*
(iv) $V^k(C') \subset C$.

(iii)⇒(ii): From (5.9) we obtain

$$S = (C \cap S) + V(C \cap S) + V^2(C \cap S) + \cdots + V^{l-1}(C \cap S) + V^l(S),$$

whence $S \subset C$.

(ii)⇒(iv): We only observe that C contains any subobject S with $V(S) = S$.

We have already encountered the condition (iv) for $k = 1$ in Example 3.9, namely:

(v) $V(C') \subset C$.

Observe that monomorphisms satisfying (v) are precisely morphisms from $\mathbb{M}_{\mathcal{A}}$ for the evident family $\mathcal{A}$. Obviously, we have (v)⇒(i). As has been noted, the converse is true if V is idempotent, though it fails in general. Nevertheless, even for V described in the corresponding counter-example, under some additional conditions on f the equivalence (i)⇔(v) holds. More precisely, *Let $\mathcal{C}$ be the category of Abelian groups, and let $V(C) = nC$ $(n \geq 2)$. If one of the following conditions holds:*

(a) *C' is isomorphic to the product of cyclic groups of prime orders;*

(b) *nC' is a serving subgroup of C';*

(c) *$C + nC'$ is a serving subgroup of C',*

then each of the conditions (i) *and* (ii) *is equivalent to* (v)*.*

Indeed, if (b) holds, then $n^2C' = nC$ and we apply the preceding statement. Suppose $C + nC'$ is serving so that $n(C + nC') = (C + nC') \cap nC' = nC'$. Then the extension $C \rightarrowtail C + nC'$ clearly lies in $I^{-1}(\mathbb{E})$ and therefore $nC' \subset C$. It remains to observe that each C' from (a) satisfies (b) [FKS].

Finally, let us consider the case where $\mathcal{C}$ *is the category of* (*left*) *R-modules for any commutative ring R with unit, and V is any pointed endofunctor* (with the above-mentioned properties). If R is a field and C' is finitely generated, then chain (5.10) is obviously stabilized. For arbitrary R, we observe that if $x_1, x_2, \dots, x_m \in C'$, then the inclusion

$$C \cap \operatorname{mod}(x_1, x_2, \dots, x_m) \rightarrowtail \operatorname{mod}(x_1, x_2, \dots, x_m)$$

lies in $I^{-1}(\mathbb{E})$ if and only if there exist elements $y_{ij} \in V(Rx_j)$ $(1 \leq i,\ j \leq m)$ such that for any i we have

$$x_i + y_{i1} + y_{i2} + \cdots + y_{im} \in C; \tag{5.11}$$

here $\operatorname{mod}(x_1, x_2, \dots, x_m)$ denotes the submodule generated by $x_1, x_2, \dots, x_m$.

From this observation we immediately get:

Each of (i) *and* (ii) *implies the following conditions:*

(vi) *C contains all $x_1, x_2, \dots, x_m$ from C' satisfying* (5.11) *for some $y_{ij} \in V(Rx_j)$;*

(vii) *C contains all x from G' such that there exists an element $y \in V(Rx)$ with $x+y \in C$.*

If R is a principal ideal ring and C' is finitely generated, then (i), (ii) *and* (vi) *are equivalent. If, moreover, C' is a cyclic module, then these conditions are also equivalent to* (vii)*.*

5.8. Example. Let Σ be a nonempty set, and let $\mathcal{C}$ be the category of groups equipped with unary operators w_i for each $i \in \Sigma$. Applying the above results, we will relate a certain factorization system on $\mathcal{C}$ to any structure that makes Σ a commutative semigroup. In what follows it is assumed that Σ is equipped with this structure. First we observe that for any $(G, (w_i)_{i\in\Sigma})$ from $\mathcal{C}$ every w_i $(i \in \Sigma)$ induces the operator on $G/N(G)$, where $N(G)$ is the normal subgroup generated by all $n_{jk}(x) \equiv w_j w_k(x) w_{jk}(x^{-1})$, where $x \in G$ and $j, k \in \Sigma$. Indeed, we have

$$\begin{aligned} w_i\big(w_j w_k(x)\, w_{jk}(x^{-1})\big) &= \big[w_i w_j w_k(x)\, [w_i w_{jk}(x^{-1})\, w_{ijk}(x)]\, w_i w_j w_k(x^{-1})\big] \\ &\quad \cdot \big[w_i w_j w_k(x)\, w_{ij} w_k(x^{-1})\big] \cdot \big[w_{ij} w_k(x)\, w_{ijk}(x^{-1})\big] \end{aligned}$$

and all of the three factors lie in $N(G)$. This implies that the full subcategory $\mathcal{X}$ of all $(G,(w_i)_{i\in\Sigma})$ such that

$$w_j\, w_i = w_{ji} \tag{5.12}$$

for every $i, j \in \Sigma$, is reflective in $\mathcal{C}$.

The structure of a factorization system naturally arises on $\mathcal{X}$. The class $\mathbb{E}$ in it consists of morphisms $e : G \longrightarrow G'$ such that for each $x \in G'$ there exists $i \in \Sigma$ with $w_i(x) \in \mathrm{Im}\ e$, while $\mathbb{M}$ consists of monomorphisms $m : G \rightarrowtail G'$ which satisfy the following condition: if $w_i(x) \in G$ for some i, then $x \in G$. The $(\mathbb{E}, \mathbb{M})$-factorization of $f : G \longrightarrow G'$ is given as $G \longrightarrow H \rightarrowtail G'$, where $H = \{x \in G' | \exists i, w_i(x) \in \mathrm{Im}\ f\}$. (5.12) implies that H is a subgroup of G', is closed under the action of all operators and the inclusion $H \rightarrowtail G'$ lies in $\mathbb{M}$.

We apply Theorem 5.4 and extend $(\mathbb{E}, \mathbb{M})$ to the entire $\mathcal{C}$. We have:

$f : G \longrightarrow G'$ lies in $I^{-1}(\mathbb{E})$ precisely when for each $x \in G'$ there exists $i \in \Sigma$ such that $w_i(x) \in \mathrm{Im}\ f \cdot N(G')$.

The class $(H(\mathbb{M}))^{\uparrow\downarrow}$ contains only monomorphisms and from Lemma 5.5 we obtain:

The following conditions for a monomorphism $f : G \rightarrowtail G'$ are equivalent:

(i) *f lies in $(H(\mathbb{M}))^{\uparrow\downarrow}$;*

(ii) *for every subgroup S of G' closed under each w_i, if the inclusion $G \cap S \rightarrowtail S$ lies in $I^{-1}(\mathbb{E})$, then $S \subset G$.*

Similarly to Example 5.7, the class $(H(\mathbb{M}))^{\uparrow\downarrow}$ differs from the class $\mathbb{M}_{\mathcal{A}}$ which is determined by the following condition:

(iii) *$N(G') \subset G$ and for each $x \in G'$, if $w_i(x) \in G$ for some $i \in \Sigma$, then $x \in G$,*

and is not closed under composition (for a counter-example, we consider the case where Σ is trivial, G' is the cyclic group of order 4 and the unique operator w on G' maps *1* to *3*. The monomorphisms $\{0\} \rightarrowtail N(G')$ and $N(G') \rightarrowtail G'$ lie in $\mathbb{M}_{\mathcal{A}}$, while their composition does not).

If the chain of subgroups

$$G' \supset N(G') \supset N^2(G') \supset \dots$$

is stabilized at the k-th step and for any $i, j \in \Sigma$ we have

$$w_i w_j = w_j w_i \tag{5.13}$$

in G', then each of (i) *and* (ii) *implies the condition:*

(iv) *$N^k(G') \subset G$ and if $w_i(x) \in G$ for some $i \in \Sigma$, then $x \in G$.*

If, in addition, G is normal in G', then (i), (ii), (iv) *are equivalent and we have one more equivalent condition:*

(v) $N^l(G') \subset G$ *for some* $l \geq 1$ *and if* $w_i(x) \in G$ *for some* $i \in \Sigma$, *then* $x \in G$.

(ii)$\Rightarrow$(iv)*:* Let S be the smallest Σ-subgroup containing x. Clearly, it is generated (as a group) by x and elements of the form $w_{j_1} w_{j_2} \cdots w_{j_n}(x)$ for all $j_1, j_2, \ldots, j_n \in \Sigma$. (5.13) implies that $w_i(S) \subset G$, whence $S \subset G$ and $x \in G$.

(v)$\Rightarrow$(ii): Let S be a subgroup of G' closed under each w_i and such that the inclusion $G \cap S \rightarrowtail S$ lies in $I^{-1}(\mathbb{E})$. Let $x \in S$. Then

$$w_i(x) = a\, z_1\, z_2 \cdots z_m \tag{5.14}$$

for some $i \in \Sigma$, $a \in G$ and elements z_k of the form

$$s_k\, n_{i_k j_k}(x_k)\, s_k^{-1}$$

with $s_k, x_k \in S$. There exist $p_k, q_k \in \Sigma$ such that $w_{p_k}(s_k) = a_k\, n_k$ and $w_{q_k}(x_k) = b_k\, m_k$ for some $a_k, b_k \in G$ and $n_k, m_k \in N(S)$. Then, according to (5.13) we have

$$w_{p_k} w_{q_k}(z_k) = w_{q_k}(a_k\, n_k)\, w_{p_k}(n_{i_k j_k}(b_k\, m_k))\, w_{q_k}(n_k^{-1}\, a_k^{-1}).$$

But

$$n_{i_k j_k}(b_k\, m_k) = w_{i_k} w_{j_k}(b_k)\, n_{i_k j_k}(m_k)\, w_{i_k j_k}(b_k^{-1}).$$

Hence

$$\begin{aligned} w_{p_k} w_{q_k}(z_k) = {} & w_{q_k}(a_k) w_{q_k}(n_k) w_{p_k} w_{i_k} w_{j_k}(b_k)\, w_{p_k}(n_{i_k j_k}(m_k)) \\ & \cdot w_{p_k} w_{i_k j_k}(b_k^{-1})\, w_{q_k}(n_k^{-1})\, w_{q_k}(a_k^{-1}). \end{aligned}$$

Since G, $N(S)$ and $N^2(S)$ are closed under the action of all operators and G is normal in G', we have

$$w_{p_k} w_{q_k}(z_k) = a'_k\, n'_k\, b'_k\, \overline{n}_k\, {b'_k}^{-1}\, {n'_k}^{-1}\, {a'_k}^{-1} = a'_k\, c_k (n'_k\, \overline{n}_k\, {n'_k}^{-1}) d_k\, {a'_k}^{-1},$$

where $a'_k, b'_k, c_k, d_k \in G$, $n'_k \in N(S)$ and $\overline{n}_k \in N^2(S)$. Applying the normality of G several times, we obtain from (5.13) that

$$w_{p_1} w_{q_1} w_{p_2} w_{q_2} \cdots w_{p_m} w_{q_m} w_i(x) = bn \tag{5.15}$$

for some $b \in G$ and $n \in N^2(S)$.

If $x \in N(S)$, then (5.15) is equal to $w_{p_1 q_1 p_2 q_2 \cdots p_m q_m i}(x) n'$ for some $n' \in N^2(S)$, and hence

$$w_{p_1 q_1 p_2 q_2 \cdots p_k q_k i}(x) \in G \cdot N^2(S).$$

We have shown that if the inclusion $G \cap S \rightarrowtail S$ lies in $I^{-1}(\mathbb{E})$, then $G \cap N(S) \rightarrowtail N(S)$ and therefore $G \cap N^{l-1}(S) \rightarrowtail N^{l-1}(S)$ also lie in $I^{-1}(\mathbb{E})$. Hence, for any $x \in N^{l-1}(S)$, there exists $i \in \Sigma$ such that $w_i(x) \in G \cdot N^l(S) \subset G$, whence $x \in G$ and $N^{l-1}(S) \subset G$. Similar arguments show that $N^t(S) \subset G$ for any $t \geq 1$. This implies that $S \subset G$.

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