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NOTIONS OF FLATNESS RELATIVE TO A GROTHENDIECK TOPOLOGY

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ABSTRACT. Completions of (small) categories under certain kinds of colimits and exactness conditions have been studied extensively in the literature. When the category that we complete is not left exact but has some weaker kind of limit for finite diagrams, the universal property of the completion is usually stated with respect to functors that enjoy a property reminiscent of flatness. In this fashion notions like that of a left covering or a multilimit merging functor have appeared in the literature. We show here that such notions coincide with flatness when the latter is interpreted relative to (the internal logic of) a site structure associated to the target category. We exploit this in order to show that the left Kan extensions of such functors, along the inclusion of their domain into its completion, are left exact. This gives in a very economical and uniform manner the universal property of such completions. Our result relies heavily on some unpublished work of A. Kock from 1989. We further apply this to give a pretopos completion process for small categories having a weak finite limit property.

1. Introduction—Basic concepts

Flatness of a functor from a small category into the category of sets is taken in the literature to mean either the left exactness of its left Kan extension along the Yoneda embedding or, equivalently, to mean that (the dual of) the category of elements of the functor is filtered. This second way of looking at flatness can be stated as a property which is expressible in geometric logic in the language of functors on the given category. In particular flatness amounts to the conjunction of the following three sentences:

(i) $\bigvee_{C\in\mathcal{C}} \exists x\colon C \;\; (x = x)$

(ii) $\forall x'\colon C' \;\; \forall x''\colon C'' \; \bigvee_{C} \; \bigvee_{\substack{u'\colon C\to C' \\ u''\colon C\to C''}} \exists x\colon C \;\; (u'(x) = x' \wedge u''(x) = x'')$

(iii) $\forall x\colon C' \;\; (u(x) = v(x) \to \; \bigvee_{C} \; \bigvee_{\substack{w\colon C\to C' \\ u\circ w=v\circ w}} \exists y\colon C \;\; (w(y) = x))$

This, in turn, manifests that flatness is a notion that is relative to the internal logic of a category where geometric logic can be interpreted. In that sense flatness is a property

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that can be attributed to functors with values in possibly small categories when they come equipped with a Grothendieck topology.

On the other hand a certain pattern usually repeats itself when one is studying completions of categories under certain kinds of colimits or exactness properties. According to this pattern one starts with a (usually small) category that may fail to have finite limits but still has some weak kind of limit (e.g multilimits or weak limits) and completes it under some class of colimits (e.g coproducts) or exactness conditions (e.g forming the exact completion of the category). Then the universal property of the completion holds with respect to functors that relate the prescribed (though not universal) cones of the domain category to the limiting cone in the target category. Thus notions such as that of a multilimit merging functor [HT1] or of a left covering functor [CV] (also called "flat" in [HT2]) have appeared in the literature. There the authors convey the feeling that these notions are notions of flatness, mostly based on the fact that when the domain category has finite limits then the functors in question are left exact. We try here to make that feeling more precise. Following the remark of the previous paragraph we explain how the various kinds of properties of functors appearing in the literature correspond to the standard notion of flatness when interpreted in the internal logic of a site structure that is naturally carried by the target category. So, for example, when studying the exact completion, the target category, being an exact category, is equipped with a Grothendieck topology generated by the coverings consisting of single regular epimorphisms. The notion of left covering functor from a category with weak limits to an exact category turns out to be that of a flat functor with respect to the Grothendieck topology just described.

Furthermore, when we start with a suitable kind of functor from a category with some sort of weak finite limits, the universally induced functor from the completion to the target category is obtained as a left Kan extension. The target category possesses some kind of colimits that are sufficient for computing the universally induced functor, at each object of the completion, via the usual colimit formula for Kan extensions. When is that extension a left exact functor? In some unpublished work of 1989 [AK] A. Kock tackles the following question: Consider a functor from a small category into a cocomplete category that is equipped with a Grothendieck topology. Assume that the functor is flat with respect to the internal logic of the target category. Is its left Kan extension, along its Yoneda embedding into the (full) presheaf category, left exact? The answer is yes provided that the colimits used in the calculation of the Kan extension are what A. Kock calls *postulated*. This means that, in the internal logic of the target category, the colimits in question are computed as in sets: Every (generalized) element of the colimit comes from an element of the components of the colimit and two such elements of the components are identified in the colimit if they are connected by a compatibility zig-zag (this description is expressible in geometric logic). More precisely, given a diagram $D\colon \mathcal{C} \to \mathcal{D}$ in a site $(\mathcal{D}, j)$ with a cocone $(L, l\colon D \Rightarrow L)$ we say that the cocone is postulated if the following two sentences hold in the internal logic of the site

(1) $\forall x : L \bigvee_{C\in\mathcal{C}} \exists y : D(C) \;\; (l_C(y) = x)$ and

(2)

$$\forall x : D(C)\ \forall y : D(C') \quad \Bigg(l_C(x) = l_{C'}(y) \longrightarrow \bigvee_{\mathcal{Z}(D(C),D(C'))} \exists z_1 : D(C_2)\ \dots\ \exists z_n : D(C_{2n})$$

$$(D(i_{1,2})(z_1) = x\ \wedge D(i_{2,3})(z_1) = D(i_{4,3})(z_2)\ \wedge \dots \wedge\ D(i_{2n+1,2n})(z_n) = y) \Bigg),$$

where the (infinite) disjunction runs over all the zig-zags

$$\begin{tikzcd} & D(C_2) \arrow[dl, "D(i_{1,2})"'] \arrow[d, "D(i_{2,3})"] & & D_{2n} \arrow[d, "D(i_{2n-1,2n})"'] \arrow[drr, "D(i_{2n,2n+1})"] & & \\ D(C) = D(C_1) & D(C_3) & \dots & D(C_{2n-1}) & & D_{2n+1} = D(C') \end{tikzcd}$$

from $D(C)$ to $D(C')$. If the topology j is subcanonical then every postulated cocone is actually a colimit cocone ([AK], Prop. 1.1), but the cocones that we will deal with here will be known to be colimiting beforehand. The main result from [AK] that we will use in the sequel is

1.1. Theorem. [AK, Corollary 3.2] *Let $(\mathcal{D}, j)$ be a cocomplete, finitely complete, subcanonical site and let $F : \mathcal{C} \to \mathcal{D}$ be a flat functor. Let $D : \mathbb{I} \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ be a finite diagram. If the colimits used for constructing $(\mathrm{Lan}_y F)(D_i)$ are postulated, for all $i \in \mathbb{I}$, then $\mathrm{Lan}_y F$ preserves the limit of the diagram D.*

In the setup of the various completions that we discussed above the colimits used for calculating the left Kan extension are (in each case) postulated (with respect to the topology associated with each completion setup). Thus we have a uniform and quite economical argument for the left exactness of the universally induced functor from the completion. More than that we show that the topology arising in each completion setup is the coarsest, among the subcanonical ones, for which the respective colimits are postulated. Finally, applying the mechanism of relative flatness we discuss a pretopos completion process for small categories with the property that finite diagrams in them have a finite family of weakly final cones.

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2. Instances of flatness

We investigate here the kind of functors that arise in connection with the universal properties of the regular and the exact completion of a weakly lex category, on the one hand, and with the sum completion of a familially lex category, on the other hand. We exhibit both as cases of flatness with respect to a suitably chosen topology.

Recall that a *weak limit* for a diagram D in a category is a cone W such that every other cone for D factors (in a not necessarily unique way) through W. A category is *weakly lex* if it has weak limits for all finite diagrams into it. A functor $F\colon \mathcal{C} \to \mathcal{D}$ from a weakly lex category to an exact one is *left covering* [CV] (flat, in the terminology of [HT2]) if, whenever $D\colon \mathbb{I} \to \mathcal{C}$ is a finite diagram in $\mathcal{C}$ with weak limit W, the canonically induced arrow $F(W) \to \lim(F \circ D)$ is a regular epimorphism.

Recall, further, that in a regular (and in particular in an exact) category regular epimorphisms are stable under pullback, thus there is a Grothendieck topology on such a category which is generated by single regular epimorphisms (such morphisms form a basis for a topology). Let us denote by j_{reg} that topology. It is well known that j_{reg} is a subcanonical topology.

2.1. Proposition. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor from a category with weak finite limits to an exact category. Then F is flat with respect to the Grothendieck topology j_{reg} if and only if it is left covering.*

Proof. Assume F is left covering. Then we show the internal validity in the logic of $(\mathcal{D}, j_{\mathrm{reg}})$ of the sentences (i), (ii) and (iii) in the Introduction that describe flatness:

(i) For all $D \in \mathcal{D}$,

$$D \Vdash \bigvee_{C \in \mathcal{C}} \exists x\colon C \;\; (x = x)$$

Let C be a weak terminal object in $\mathcal{C}$. Then $F(C) \to 1_{\mathcal{D}}$ is a regular epi. Pulling it back along the unique $D \to 1_{\mathcal{D}}$ we get a j_{reg}-cover of D, $\{D' \to D\}$ and a D'-element of $F(C)$ as required.

(ii) For all $D \in \mathcal{D}$,

$$D \Vdash \forall x'\colon C' \;\; \forall x''\colon C'' \bigvee_{C} \bigvee_{\substack{u'\colon C \to C' \\ u''\colon C \to C''}} \exists x\colon C \;\; (u'(x) = x' \wedge u''(x) = x'')$$

Given a pair of D-elements, $x'\colon D \to F(C')$, $x''\colon D \to F(C'')$, let $(C, < u', u'' >)$ be a weak product of C' and C''. The canonically induced $F(C) \to F(C') \times F(C'')$ is a regular epi. Pulling it back along $< x', x'' >\colon D \to F(C') \times F(C'')$ we get a cover $\{\epsilon\colon D' \to D\}$ of D and a D'-element $x\colon D' \to F(C)$ such that $F(u') \circ x = x' \circ \epsilon$ and $F(u'') \circ x = x'' \circ \epsilon$, as required.

(iii) For all $D \in \mathcal{D}$, all $u,\ v\colon C' \rightrightarrows C''$

$$D \Vdash \forall x\colon C' \;\; (u(x) = v(x) \to \bigvee_{C} \bigvee_{\substack{w\colon C \to C' \\ u \circ w = v \circ w}} \exists y\colon C \;\; (w(y) = x))$$

Given a D-element $x\colon D \to F(C')$ such that $F(u) \circ x = F(v) \circ v$ take a weak equalizer (C, w) of $u,\ v$ in $\mathcal{C}$. The canonical $F(C) \to Eq(Fu, Fv)$ to the equalizer of $Fu,\ Fv$ is a regular epi. Pull back the factorization $\bar{x}$ of x through the equalizer to obtain a cover

$\{\epsilon: D' \to D\}$ and a D'-element $y: D' \to F(C)$.

$$\begin{tikzcd}
& F(C) \arrow[dr] \arrow[drr, bend left, "F(w)"] & & & \\
D' \arrow[ur, bend left, "y"] \arrow[dr, bend right, "\epsilon"'] & & Eq(Fu,Fv) \arrow[r] & F(C') \arrow[r, shift left, "F(u)"] \arrow[r, shift right, "F(v)"'] & F(C'') \\
& D \arrow[ur] \arrow[urr, bend right, "x"'] & & &
\end{tikzcd}$$

We have that $F(w) \circ y = x \circ \epsilon$, as required.

Conversely assume that F is flat with respect to the topology j_{reg}. Consider a diagram $D: \mathbb{I} \to \mathcal{C}$, a weak limit W for it and the canonical comparison morphism $\delta: FW \to \lim(F \circ D)$ in $\mathcal{D}$. The sentence

$$\forall x_i : D_i(\text{comp}(x_i) \to \bigvee_{W \in \text{Cone}(D)} \exists t : W \bigwedge_i (w_i(t) = x_i))$$

holds in the internal logic of $(\mathcal{D}, j_{\text{reg}})$. Here $\text{comp}(x_i)$ is an abbreviation for the conjunction of equations $D(\alpha)(x_i) = x_j$, where $\alpha: i \to j$ is an arrow in $\mathbb{I}$ and $< W, w_i >$ runs over the cones for D. This sentence is, in particular, forced by the object $\lim(F \circ D)$. Since the hypothesis of the implication is satisfied by the limiting cone $< \lim(F \circ D), x_i >$, there exists a covering $\epsilon: V \to \lim(F \circ D)$ and a cone $< W, w_i >$ for D, together with an element $t: V \to FW$ such that, for all $i \in \mathbb{I}$, $Fw_i \circ t = x_i \circ \epsilon$. The cone may be taken to be a weakly limiting one. We have now, for all i, $x_i \circ \delta \circ t = x_i \circ \epsilon$ thus, we get a factorization of the regular epi ϵ as $\epsilon = \delta \circ t$. The factorization of the regular epi ϵ takes place in an exact category so we may conclude that δ is a regular epi. (Consider the factorization of δ as a regular epi followed by a mono and use the fact that ϵ, being a regular epi in an exact category, is extremal. We conclude that the mono part of the factorization of δ is an iso.) ∎

Let us recall the definition of a lextensive category. A category $\mathcal{C}$ is *extensive* if it has finite sums and for any two objects X, Y in $\mathcal{C}$, the functor

$$(\mathcal{C}/X) \times (\mathcal{C}/Y) \to \mathcal{C}/(X+Y),$$

sending an object $(f: C \to X, g: D \to Y)$ to $(f + g: C + D \to X + Y)$ (and defined on arrows in an obvious way), is an equivalence of categories. An extensive category with finite limits is called *lextensive.*

Let us further recall that a diagram $D: \mathbb{J} \to \mathcal{C}$ in a category has a *multilimit* $\{X_i | i \in I\}$ if the cone functor $\text{cone}(D): \mathcal{C}^{\text{op}} \to \text{Set}$ is isomorphic to the coproduct of representables $\coprod_i \hom(-, X_i)$. We say that a category is *familially lex* if it has multilimits for all finite diagrams into it. A functor $F: \mathcal{C} \to \mathcal{D}$ from a familially lex category into a left exact one *merges* ([HT1]) a multilimit $\{X_i | i \in I\}$ of a diagram D into $\mathcal{C}$ if the coproduct $\coprod_i F(X_i)$ exists in $\mathcal{D}$ and the canonically induced arrow $\coprod_i F(X_i) \to \lim(F \circ D)$ is an isomorphism.

In a lextensive category coproducts are stable under pullback thus coproduct injections generate a topology on the category. More precisely a family of arrows $\{X_i \to X \mid i \in I\}$ is a basic covering if $\coprod_i X_i \cong X$. Let us call the topology in question j_{sum}. It is easy to see that this topology is subcanonical.

2.2. PROPOSITION. *Let $F: \mathcal{C} \to \mathcal{D}$ be a functor from a familially lex category to a lextensive category. Then F is j_{sum} -* flat *if and only if it is merging multilimits of finite diagrams.*

PROOF. Similar to the above one. ■

Another case that will be of interest to us later on is that of flatness with respect to the precanonical topology j_{prec} on a pretopos, i.e the topology whose coverings are given by finite epimorphic families. In particular we want to identify the external version of it when the source category of the functor $F: \mathcal{C} \to \mathcal{E}$ into a pretopos has some very weak finite completeness property. We shall return to this in the final section, where we treat a general pretopos completion process.

3. Some left exact Kan extensions

3.1. PROPOSITION. *In an exact category $\mathcal{D}$ coequalizers of equivalence relations are j_{reg}- postulated.*

PROOF. Given a coequalizer diagram

$$R \underset{v}{\overset{u}{\rightrightarrows}} D \xrightarrow{q} Q$$

we want to show the validity in the internal logic of $(\mathcal{D}, j_{\mathrm{reg}})$ of the sentences

(i) $\forall x : Q\ \exists y : D\ (q(y) = x)$

(ii) $\forall x : D\ \forall y : D\ (q(x) = q(y) \to \bigvee_{\mathcal{Z}} \exists z_1 : R\ ...\ \exists z_n : R\ (x = u(z_1) \wedge v(z_1) = u(z_2) \wedge ... \wedge v(z_n) = y))$

Every $E \in \mathcal{D}$ forces (i) because given an E-element $x: E \to Q$ we pull q back along x and we find a cover $\{\epsilon: E' \to E\}$ and an E'- element $y: E' \to D$ such that $q \circ y = x \circ \epsilon$, as required.

Concerning (ii), given any pair of E- elements x, y of D such that $q \circ x = q \circ y$, since in an exact category (u, v) is the kernel pair of q, we get a factorization of (x, y) through (u, v), in other words an E-element $z: E \to R$ such that $u \circ z = x$ and $v \circ z = y$, as required. ■

3.2. PROPOSITION. *In a lextensive category $\mathcal{D}$ sums are j_{sum}- postulated.*

PROOF. Similar. ■

3.3. Lemma. *Let $k: \mathcal{C} \to \tilde{\mathcal{C}}$ be the inclusion of the small category $\mathcal{C}$ into a full subcategory of $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ (i.e the Yoneda embedding factors as $y = i \circ k$), let $\mathcal{D}$ be a category equipped with a subcanonical Grothendieck topology j and $F: \mathcal{C} \to \mathcal{D}$ be a j-flat functor. Assume also that $\tilde{\mathcal{C}}$ is closed into $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ under finite limits and that, for all $X \in \tilde{\mathcal{C}}$, the category $k \downarrow X$ has a final subcategory $\mathcal{K}_X$ with the property that colimits of shape $\mathcal{K}_X$ exist in $\mathcal{D}$ and are j-postulated. Then the left Kan extension of F along k, $\mathrm{Lan}_k F$, exists and is left exact.*

Proof. The left Kan extension exists and can be computed as

$$\mathrm{Lan}_k F(X) = \mathrm{colim}(k \downarrow X \to \mathcal{C} \to \mathcal{D}) = \mathrm{colim}(\mathcal{K}_X \to \mathcal{C} \to \mathcal{D})$$

For the left exactness suppose that we are given a finite diagram $D: \mathcal{J} \to \tilde{\mathcal{C}}$. Then Kock's theorem stated in the introduction, along with the observation that for all $X \in \tilde{\mathcal{C}}$, $k \downarrow X \cong y \downarrow iX$, gives the following:

$$\begin{aligned}(\mathrm{Lan}_k F)(\mathrm{lim} D) &\cong \mathrm{colim}(k \downarrow \mathrm{lim} D \to \mathcal{C} \to \mathcal{D}) \\ &\cong \mathrm{colim}(y \downarrow i(\mathrm{lim} D) \to \mathcal{C} \to \mathcal{D}) \\ &\cong \mathrm{colim}(y \downarrow \mathrm{lim}(iD) \to \mathcal{C} \to \mathcal{D}) \\ &\cong \mathrm{lim}_{\mathcal{J}} \mathrm{colim}(y \downarrow iD_j) \to \mathcal{C} \to \mathcal{D}) \\ &\cong \mathrm{lim}_{\mathcal{J}} \mathrm{colim}(k \downarrow D_j) \to \mathcal{C} \to \mathcal{D}) \\ &\cong \mathrm{lim}_{\mathcal{J}} (\mathrm{Lan}_k F)(D_j)\end{aligned}$$

■

By an *exact completion* of a weakly lex category we mean a category $\mathcal{C}_{\mathrm{ex}}$ along with a fully faithful functor $k: \mathcal{C} \to \mathcal{C}_{\mathrm{ex}}$ such that every left covering functor $F: \mathcal{C} \to \mathcal{D}$ into an exact category factors up to natural isomorphism, essentially uniquely, through an exact functor $\bar{F}: \mathcal{C}_{\mathrm{ex}} \to \mathcal{D}$, as $F \cong \bar{F} \circ k$. Such a category is unique up to equivalence. Recall from [HT2] that for a small $\mathcal{C}$ the exact completion can be constructed as a full subcategory of the functor category $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ that fulfills the conditions of the above Lemma. The functor $\bar{F}$ in the definition of the exact completion is defined as a left Kan extension of F along the inclusion k. The same holds for the sum completion of a familially lex category [HT1]. In particular we get the following easy consequences of the above Lemma:

3.4. Corollary. *Let $F: \mathcal{C} \to \mathcal{D}$ be a left covering functor from a category with weak finite limits to an exact category. Then the left Kan extension of F along the inclusion $\mathcal{C} \to \mathcal{C}_{\mathrm{ex}}$ is an exact functor.*

Proof. The left Kan extension trivially preserves regular epis and is left exact by the two propositions above and the Corollary to Kock's theorem presented in the Introduction. ■

3.5. Corollary. *Let $F: \mathcal{C} \to \mathcal{D}$ be a functor that merges multilimits of finite diagrams from a familially lex category to a lextensive category with sums. Then the left Kan extension of F along the inclusion $\mathcal{C} \to \mathcal{C}_{\text{sum}}$ into its sum completion preserves finite limits (along with coproducts).*

Proof. Similar to the above one. ■

4. When are the colimits postulated?

Notice that since flatness with respect to a site structure amounts to the verification of geometric sentences in the internal logic of the site, in other words to the existence of coverings so that locally certain conditions hold, it follows that, whenever a topology j is coarser than k then j-flatness implies k-flatness. The same remark applies to the postulatedness of colimits.

4.1. Proposition. *Let $\mathcal{E}$ be a pretopos and j a subcanonical Grothendieck topology on it such that coproducts and coequalizers of equivalence relations in $\mathcal{E}$ are j-postulated. Then j is finer than j_{prec}.*

Proof. Let $\{e_i: E_i \to E\}$ be a finite epimorphic family in $\mathcal{E}$ and R_e be the sieve on E generated by it. We want to show that sheafification with respect to j, $l_j: [\mathcal{E}^{\text{op}}, \text{Set}] \to \text{sh}(\mathcal{E}, j)$, sends the j_{prec}-dense arrow $R_e \to yE$ to an iso. In fact we will show that applying the $(-)^+$ construction once, with respect to j, will yield for all $X \in \mathcal{E}$ an isomorphism

$$R_e^+(X) = \operatorname*{colim}_{R \in \text{Cov}_j(X)} \text{hom}(R, R_e) \to \text{hom}_{\mathcal{E}}(X, E)$$

Notice here that, since j was assumed to be subcanonical, $yE^+(X) \cong \text{hom}_{\mathcal{E}}(X, E)$. In turn it suffices to show that the above canonical map is surjective. Recall how the above map works: It sends the element of the colimit represented by a natural transformation $\vartheta: R \to R_e$ to the unique extension $x: X \to E$ making the square

$$\begin{tikzcd} R \arrow[r, "k"] \arrow[d, "\vartheta"'] & X \arrow[d, "x"] \\ R_e \arrow[r, "i"] & E \end{tikzcd}$$

commutative. So given $x: X \to E$, we look for $\vartheta: R \to R_e$ making the above square commutative. But since $e: \coprod_{i=1}^n E_i \to E$ coequalizes its kernel pair $u, v: K \rightrightarrows \coprod_{i=1}^n E_i$ and coequalizers of equivalence relations are j- postulated we have that, for all $X \in \mathcal{E}$

$$X \Vdash \forall x: E \; \exists z: \coprod_{i=1}^{n} E_i \;\; (e(z) = x)$$

Thus we have that there exists a j-cover $T = \{\xi_t: X_t \to X\}$ of X and, for all t, elements $z_t: X_t \to \coprod_{i=1}^n E_i$ such that $e \circ z_t = x \circ \xi_t$. Then, since coproducts are j-postulated, we have for the X_t-elements z_t of $\coprod_{i=1}^n E_i$ that there exist, for all t, coverings $S_r = \{\xi_{t,r}: X_{t,r} \to X_t\}$ and, for all r elements $z_{t,r}: X_{t,r} \to E_i$ to some E_i such that: For all t and all r we have $e_i \circ z_{t,r} = e \circ \xi_t \circ \xi_{t,r} = x \circ \xi_t \circ \xi_{t,r}$. Let R now denote the composite cover $\{X_{t,r} \to X_t \to X\}$.

Having found the cover R we define a natural transformation $\vartheta: R \to R_e$: For any $\alpha: Z \to X$, which is necessarily of the form $\alpha = \xi_t \circ \xi_{t,r} \circ \alpha_{t,r}$, we define

$$\vartheta_Z(\alpha) = e_i \circ z_{t,r} \circ \alpha_{t,r} = x \circ \xi_t \circ \xi_{t,r} \circ \alpha_{t,r}$$

It is obvious that ϑ is well defined (the definition does not depend on the factorization of α through the $X_{t,r}$). The naturality of ϑ as well as the fact that $i \circ \vartheta = x \circ k$ follow immediately. ∎

In a similar manner, we have

4.2. Proposition. *Let $\mathcal{D}$ be an exact category and j a subcanonical Grothendieck topology on it such that coequalizers of equivalence relations in $\mathcal{D}$ are j-postulated. Then j is finer than j_{reg}.*

4.3. Proposition. *Let $\mathcal{D}$ be a lextensive category and j a subcanonical Grothendieck topology on it such that sums in $\mathcal{D}$ are j-postulated. Then j is finer than j_{sum}.*

5. A more general pretopos completion

We study here a free pretopos completion of a small category that may lack even weak finite limits but still satisfies the following very weak finite completeness property:

5.1. Definition. *A diagram $D: \mathbb{I} \to \mathcal{C}$ has an fm-limit (fm for finitely many) if there exists a finite family of cones for D which is weakly final, in the sense that every other cone for D factors through one in that family. This is equivalent to saying that the (contravariant) cone functor* cone(D) *is finitely generated in* $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$. *We say that $\mathcal{C}$ is fm-lex if it has fm-limits for* ***finite*** *diagrams.*

This notion can be traced in [SGA4], Exposé VI, Exercise 2.17 (c), in connection with the characterization of coherent presheaf toposes. It has been studied recently in [B], [BKR], [KRV]. More precisely, a presheaf topos $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ is coherent if and only if $\mathcal{C}$ has an fm-terminal object, fm-pullbacks and fm-equalizers in the obvious sense ([SGA4]). The fact that existence of fm-limits for these particular three kinds of diagrams is equivalent to the existence of fm-limits for all finite diagrams is shown in [B].

5.2. Proposition. *If $\mathcal{C}$ is a small fm-lex category then a functor into a pretopos, $F: \mathcal{C} \to \mathcal{E}$, is j_{prec}-flat if and only if it* merges fm-limits, *in the sense that if $\{P_k \mid k = 1, \dots, n\}$ is an fm-limit for the finite diagram $D: \mathbb{I} \to \mathcal{C}$, then the canonically induced $\coprod_{k=1}^n FP_k \to \lim F \circ D$ is epi.*

PROOF. It is shown in [BKR] that a functor $F: \mathcal{C} \to \mathcal{E}$, where now $\mathcal{E}$ is a topos, is flat (in the standard sense, i.e with respect to the canonical topology of the topos) if and only if it merges fm-limits. We may conclude considering the topos of sheaves on $\mathcal{E}$ for the precanonical topology and using the fact that the embedding $y: \mathcal{E} \hookrightarrow \mathrm{shv}(\mathcal{E}, j_{\mathrm{prec}})$ preserves finite limits, finite coproducts and coequalizers of equivalence relations ([El], D.3.3.9). Alternatively we may proceed to a straightforward verification of our claim in the spirit of our Proposition 2.1. ∎

5.3. THEOREM. *If $\mathcal{C}$ is a small fm-lex category then*

(i) the finite sum completion of $\mathcal{C}$, $\mathrm{fam}\,\mathcal{C}$, has weak finite limits;

(ii) the exact completion of its finite sum completion, $(\mathrm{fam}\,\mathcal{C})_{\mathrm{ex}}$, is a pretopos;

(iii) $(\mathrm{fam}\,\mathcal{C})_{\mathrm{ex}}$ enjoys the following universal property: For any pretopos $\mathcal{E}$ and any pretopos morphism from $(\mathrm{fam}\,\mathcal{C})_{\mathrm{ex}}$ to $\mathcal{E}$, composition with the two inclusions $k \circ \eta: \mathcal{C} \hookrightarrow (\mathrm{fam}\,\mathcal{C})_{\mathrm{ex}}$ induces an equivalence

$$j_{\mathrm{prec}}\text{-flat}(\mathcal{C}, \mathcal{E}) \;\cong\; \mathrm{Pretop}((\mathrm{fam}\,\mathcal{C})_{\mathrm{ex}}, \mathcal{E})$$

between j_{prec}-flat functors from $\mathcal{C}$ to $\mathcal{E}$ and pretopos morphisms from $(\mathrm{fam}\,\mathcal{C})_{\mathrm{ex}}$ to $\mathcal{E}$.

PROOF. (i) Obviously the object of $\mathrm{fam}\,\mathcal{C}$ given by an fm-terminal family of objects in $\mathcal{C}$ is a weak terminal object in $\mathrm{fam}\,\mathcal{C}$. Given a diagram $D: \mathbb{I} \to \mathcal{C}$ in $\mathcal{C}$ with fm-limit $(C_i)_{i=1}^n$ then the induced diagram $\eta \circ D: \mathbb{I} \to \mathrm{fam}\,\mathcal{C}$ has obviously $(C_i)_{i=1}^n$ as a weak limit. Coproducts commute with binary products in $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$, thus if $(C_k)_{k=1}^l$ is an fm-product for X_i and Y_j in $\mathcal{C}$ then $(C_{(i,j,k(i,j))})_{(i,j,k)}$ is a weak product for $(X_i)_{i=1}^n$ and $(Y_j)_{j=1}^m$ in $\mathrm{fam}\,\mathcal{C}$.

Similarly, given a pair $f,\ g: X \rightrightarrows (Y_j)_{j=1}^n$ it is either equalized by the empty family or f and g are represented by arrows $f_j,\ g_j: X \rightrightarrows Y_j$ into the same component of the family, thus a weak equalizer for f and g in $\mathrm{fam}\,\mathcal{C}$ is the fm-equalizer of the latter pair in $\mathcal{C}$. A weak equalizer of general pair

$$f,\ g:\ (X_i)_{i=1}^m \rightrightarrows (Y_j)_{j=1}^n$$

is given by the object $(Z_{(i,k(i))})_{(i,k)}$, where $(Z_{k(i)})_k$ is a weak equalizer for

$$f_i,\ g_i:\ X_i \rightrightarrows (Y_j)_{j=1}^n.$$

This is due to the fact that in an extensive category, such as $\mathrm{fam}\,\mathcal{C}$, a weak equalizer of a pair $f,\ g: \coprod_i X_i \rightrightarrows Y$ is the coproduct of weak equalizers of

$$f \circ in_i,\ g \circ in_i:\ X_i \to X \rightrightarrows Y.$$

Weak terminals, weak binary products and weak equalizers suffice for the construction of weak finite limits (cf. [CV], Proposition 1).

(ii) We know that $\mathrm{fam}\,\mathcal{C}$ has weak finite limits, finite sums and is extensive, thus ([LV], Corollary 1.10) that $(\mathrm{fam}\,\mathcal{C})_{\mathrm{ex}}$ is extensive and exact, i.e a pretopos.

(iii) The equivalence in the statement of the theorem sends a j_{prec}-flat functor F to its left Kan extension along the embedding $k \circ \eta$. We need to know that the extension exists and is left exact. For that suffices to know that $\mathrm{Lan}_k(\mathrm{Lan}_\eta F)$ exists and for the latter that $\mathrm{Lan}_\eta F$ is j_{reg}-flat (equivalently, left covering, after Proposition 2.1).

We want to show that a certain functor from a weakly lex category to an exact one is left covering. For that it suffices to be left covering with respect to the terminal object, binary products and equalizers ([CV], Proposition 27). We show that $\mathrm{Lan}_\eta F$ is left covering with respect to binary products:

Let $(C_{(i,j,k(i,j))})_{(i,j,k)}$ be a weak product for $(X_i)_{i=1}^n$ and $(Y_j)_{j=1}^m$ in $\mathrm{fam}\,\mathcal{C}$. Then, as explained in the previous proof, for each pair of indices (i,j), $(C_k)_{k=1}^l$ is an fm-product for X_i and Y_j in $\mathcal{C}$. By Proposition 5.2, j_{prec}-flatness of F means that there is a regular epi

$$\coprod_{k=1}^{l} FC_k \to FX_i \times FY_j$$

thus there is a regular epi

$$\coprod_{i=1}^{n}\coprod_{j=1}^{m}\coprod_{k=1}^{l} FC_{k(i,j)} \to \coprod_{i=1}^{n}\coprod_{j=1}^{m}(FX_i \times FY_j)$$

i.e a regular epi

$$(\mathrm{Lan}_\eta F)(C_{(i,j,k(i,j))})_{(i,j,k)} \to (\mathrm{Lan}_\eta F)(X_i)_{i=1}^n \times (\mathrm{Lan}_\eta F)(Y_j)_{j=1}^m,$$

as required.

The case of equalizers and of the terminal object is similar, using the description of weak terminals and weak equalizers in $\mathrm{fam}\,\mathcal{C}$ given in (i). ∎

Remarks: 1. The first clause of the above result appears in the analysis of the existence of limits in cocompletions given in [KRV]. We include it here as we need the explicit description of the weak limits in $\mathrm{fam}\,\mathcal{C}$ for the proof of (iii).

2. As explained in the beginning of this section, for an fm-lex category $\mathcal{C}$, the topos $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ is coherent. Thus the full subcategory of coherent objects of the topos $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ is a pretopos. Using the description of coherent objects in a topos (see [El], D.3.3) we readily conclude that the pretopos in question is $(\mathrm{fam}\,\mathcal{C})_{\mathrm{ex}}$.

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