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MOORE CATEGORIES

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Abstract. In 1970, M. Gerstenhaber introduced a list of axioms defining Moore categories in order to develop the Baer Extension Theory. In this paper, we study some implications between the axioms and compare them with more recent notions, showing that, apart from size restrictions, a Moore category is a pointed, strongly protomodular and Barr-exact category with cokernels.

1. Introduction

For several years many category theorists were focused on defining an axiomatic context that would reflect the properties of groups and rings as the abelian categories do for abelian groups and modules. The difficulty found in weakening the axioms of abelian categories contributed for the arising of many different approaches, from the 1950's through the 1970's. Some were designed to represent a good context for non-abelian homology, such as Moore categories, while others were developed to capture more or less algebraic properties, such as Barr-exact Maltsev categories. So, unlike the abelian case, there was no outstanding theory that could be considered as a "good" generalization of groups.

In [Bourn, 1991], the author introduced the notion of a protomodular category, whose outstanding example is the category of groups. Later in [Bourn, 2000], he defined the notion of strong protomodular categories which capture some more group-like properties. Also based on protomodular categories, the new concept of semi-abelian categories appeared in [Janelidze, Márki, Tholen, 2002]. At this time the "old" complicated axioms from the earlier years were compared with more recent notions, establishing the existence of many disguised similarities.

Since recent notions had already appeared much earlier, although in disguised forms, we are interested in analyzing Moore categories. This notion was introduced as a category suitable for developing the Baer Extension Theory in [Gerstenhaber, 1970]. Having in mind the next higher cohomology group, containing the obstructions to extensions problems, he was concerned in giving a good context for the cohomology of groups. For these reasons, the categorical setting given is based on the categories of groups and rings. Like most of the theories of this time, the definition of a Moore category is also given by long list of entangled axioms. But, after the successful comparison of the "old" and "new" theories done for the semi-abelian categories, we expect to achieve some simplifications by

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translating the old axioms into more recent notions and discarding the needless axioms by studying their dependencies. As shown in the sequel, a Moore category turns out to be a pointed, strongly protomodular and Barr-exact category with cokernels. We conclude by presenting other examples of Moore categories besides the categories of groups and rings.

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2. Moore Categories

We denote kernels by $\rhd\!\!\longrightarrow$ and cokernels by $\longrightarrow\!\!\rhd$. The notion of a *normal monomorphism* will be used in the sense of protomodularity, i.e. with respect to an equivalence relation (Section 4). We write $k \dashv R$ when a monomorphism k is normal to an equivalence relation R. The kernel equivalence relation of a morphism f is represented by $R[f]$. A short exact sequence $A \rhd\!\!\xrightarrow{k} B \xrightarrow{p}\!\!\rhd Q$, where $k = \ker(p)$ and $p = \operatorname{coker}(k)$, will also be called an *extension* (of Q by A). Moreover, if $p \cdot s = 1_Q$, then we call the diagram $A \rhd\!\!\xrightarrow{k} B \underset{s}{\overset{p}{\rightleftarrows}} Q$ a *split extension*.

The notion of a Moore category introduced in [Gerstenhaber, 1970] is the following:

2.1. Definition. A category $\mathcal{C}$ with zero object, kernels and cokernels is called a *Moore category* if the following axioms hold:

M1.1 (3×3 lemma) Consider a commutative diagram

$$\begin{array}{ccccc}
A & \xrightarrow{k} & B & \xrightarrow{p} & Q \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
A' & \xrightarrow{k'} & B' & \xrightarrow{p'} & Q' \\
{\scriptstyle\alpha'}\downarrow & & \downarrow{\scriptstyle\beta'} & & \downarrow{\scriptstyle\gamma'} \\
A'' & \xrightarrow[k'']{} & B'' & \xrightarrow[p'']{} & Q''
\end{array}$$

where the three rows and two columns are extensions. If the third column is a zero morphism, then it is also an extension;

M1.2 In the commutative diagram

$$\begin{array}{ccccc}
A & \rhd\!\!\xrightarrow{k} & B & \xrightarrow{p}\!\!\rhd & Q \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} & & \downarrow\cong{\scriptstyle\gamma} \\
A' & \rhd\!\!\xrightarrow[k']{} & B' & \xrightarrow[p']{}\!\!\rhd & Q'
\end{array}$$

where both rows are extensions, the composition $k' \cdot \alpha$ is a kernel;

M2.1 $\mathcal{C}$ has pullbacks;

M2.2 The pullback of a cokernel is a cokernel;

M2.3 The intersection of two kernels is a kernel;

M3 Given a split extension $A \overset{k}{\rightarrowtriangle} B \underset{s}{\overset{p}{\rightleftarrows}} Q$, the pair of morphisms (k, s) is jointly epimorphic;

M4.1 There is a representative set under the equivalence relation for the extensions of Q by A;

M4.2 The equivalence classes of subobjects of an object form a set.

We first note that a Moore category has finite limits, since it is pointed and has pullbacks. So it is unnecessary to refer the kernels in the beginning of the definition.

In the next sections we are focused on proving that a Moore category is a pointed, strongly protomodular and Barr-exact category with cokernels. During this process we shall see that M2.3 and M3 are redundant.

3. Protomodularity

We will start by analyzing the properties obtained from M1.1.

3.1. Definition. A category $\mathcal{C}$ with pullbacks is called *protomodular* if the change of base functors, with respect to the fibration

$$\begin{array}{rccl} \pi : & \mathrm{Pt}\mathcal{C} & \longrightarrow & \mathcal{C}, \\ & B \underset{s}{\overset{p}{\rightleftarrows}} Q & \longmapsto & Q \quad \text{where } p \cdot s = 1_Q \end{array}$$

are conservative.

There are several alternative definitions for protomodularity. We will use the following three:

3.2. Proposition. [Proposition 7 of [Bourn, 1991]] *A category with pullbacks is protomodular if and only if the* **pullback cancellation property** *holds: for every diagram where $p \cdot s = 1_Q$ and the downward squares are commutative*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & {\scriptstyle 1}\ {\scriptstyle p}\downarrow\uparrow{\scriptstyle s}\ {\scriptstyle 2} & & \downarrow \\ P & \longrightarrow & Q & \longrightarrow & R, \end{array}$$

if $\boxed{1}$ *and* $\boxed{1}\boxed{2}$ *are pullbacks, then* $\boxed{2}$ *is a pullback.*

When $\mathcal{C}$ is pointed and has pullbacks, protomodularity can also be characterized by the **split short five lemma:** given a commutative diagram where the rows are split extensions,

$$\begin{array}{ccccc} A & \overset{k}{\rightarrowtail} & B & \underset{s}{\overset{p}{\rightleftarrows}} & Q \\ {\scriptstyle\alpha}\downarrow\cong & & \downarrow{\scriptstyle\beta} & & \cong\downarrow{\scriptstyle\gamma} \\ A' & \underset{k'}{\rightarrowtail} & B' & \underset{s'}{\overset{p'}{\rightleftarrows}} & Q', \end{array} \tag{1}$$

if α and γ are isomorphisms, then so is β (Theorem 2.3 (h) of [Bourn, Janelidze, 1998].).

A pair of morphisms $(x : X \longrightarrow A, y : Y \longrightarrow A)$ is *jointly strongly epimorphic* whenever a monomorphism $j : J \rightarrowtail A$ is an isomorphism provided that its pullbacks along x and y are isomorphisms. In the presence of equalizers, this notion implies that (x, y) is jointly epimorphic.

3.3. Lemma. [see [Bourn, 2001], p. 781] *A category with pullbacks is protomodular if and only if for every pullback diagram with $p \cdot s = 1_Q$*

$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ \downarrow & \lrcorner & {\scriptstyle p}\downarrow\uparrow{\scriptstyle s} \\ P & \longrightarrow & Q, \end{array}$$

the pair (a, s) is jointly strongly epimorphic.

3.4. Proposition. *A pointed category satisfying* M2.1 *and* M1.1 *is protomodular.*

Proof. Consider the diagram (1) and apply M1.1 to the commutative diagram

$$\begin{array}{ccccc} 0 & = & 0 & = & 0 \\ {\scriptstyle 0_A}\downarrow & & \downarrow{\scriptstyle 0_B} & & \downarrow{\scriptstyle 0_Q} \\ A & \overset{k}{\rightarrowtail} & B & \overset{p}{\longrightarrow} & Q \\ {\scriptstyle\alpha}\downarrow\cong & & \downarrow{\scriptstyle\beta} & & \cong\downarrow{\scriptstyle\gamma} \\ A' & \underset{k'}{\rightarrowtail} & B' & \underset{p'}{\longrightarrow} & Q' \end{array}$$

to conclude that the second column is an extension, thus β is an isomorphism. ∎

At this point, in a category satisfying M2.1 and M1.1, we may identify regular epimorphisms with cokernels, one of the well known properties of pointed protomodular categories (Corollary 14 of [Bourn, 1991].).

3.5. Remark. A pointed category satisfying M2.1 and M1.1 also satisfies M3, by Lemma 3.3 for $P = 0$.

4. Barr-exactness

We will use some of the properties of normal monomorphisms in pointed protomodular categories to prove the Barr-exactness of a Moore category.

We say that a morphism $f : X \longrightarrow A$ is *normal* to an equivalence relation $(r_1 \quad r_2) : R \rightarrowtail A \times A$ if X is an equivalence class of R, i.e. if there exists a morphism $\tilde{f}$ such that the first diagram commutes and the second is a pullback:

$$\begin{array}{ccc} X \times X & \xrightarrow{\tilde{f}} & R \\ \| & & \downarrow \\ X \times X & \xrightarrow[f \times f]{} & A \times A \end{array} \qquad \begin{array}{ccc} X \times X & \xrightarrow{\tilde{f}} & R \\ {\scriptstyle \pi_1}\downarrow & \lrcorner & \downarrow{\scriptstyle r_1} \\ X & \xrightarrow[f]{} & A. \end{array}$$

A normal morphism is always a monomorphism (Lemma 1 of [Bourn, 2001].) and it is a kernel if and only if $R = R[g]$, for some morphism g (Proposition 4 of [Bourn, 2001].).

Given two equivalence relations $(r_1 \quad r_2) : R \rightarrowtail A \times A$ and $(s_1 \quad s_2) : S \rightarrowtail A \times A$, the double relation $R\square S$ is given by the pullback

$$\begin{array}{ccc} R\square S & \longrightarrow & S \times S \\ \downarrow & \lrcorner & \downarrow{\scriptstyle (s_1 \quad s_2)\times(s_1 \quad s_2)} \\ R \times R & \xrightarrow[(r_1\times r_1 \quad r_2\times r_2)]{} & A \times A \times A \times A. \end{array}$$

Using elements, $(x, y, z, w) \in R\square S$ if and only if $(x, y), (z, w) \in R$ and $(x, z), (y, w) \in S$, which is denoted by

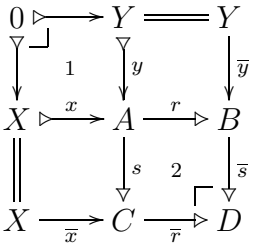
$$\begin{array}{ccc} x & R & y \\ S & & S \\ z & R & w. \end{array}$$

4.1. Theorem. [Theorem 11 of [Bourn, 2001]] *In a pointed protomodular category, consider* $(X \rightarrowtail A) \dashv R$ *and* $(Y \rightarrowtail A) \dashv S$. *When* $X \wedge Y = 0$, *there is a unique normal monomorphism* $\gamma : X \times Y \longrightarrow A$ *such that* $\gamma \cdot (1 \quad 0) = x$ *and* $\gamma \cdot (0 \quad 1) = y$. *Moreover, for the double relation* $R\square S$ *the morphism*

$$\begin{array}{ccc} R\square S & \longrightarrow & A \times A \\ x\, R\, y & & \\ S \quad S & \longmapsto & (x, w) \\ z\, R\, w & & \end{array}$$

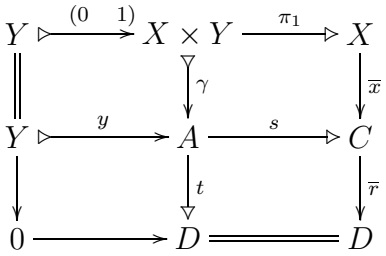
is a monomorphism and presents $R\square S$ *as an equivalence relation on* A *such that* $RS = SR = R \vee S = R\square S$ *and* $\gamma \dashv (R\square S)$.

4.2. Lemma. *In a pointed category satisfying* M2.1 *and* M1.1*, given a commutative diagram*



where the first and second lines (rows and columns) are extensions, $\boxed{1}$ *is a pullback and* $\boxed{2}$ *is a pushout, then all rows and columns are extensions. Moreover, there exists a unique morphism* $\gamma : X \times Y \longrightarrow A$ *such that* $\gamma \cdot (1 \quad 0) = x$, $\gamma \cdot (0 \quad 1) = y$ *and* $X \times Y \rightarrowtriangle^{\gamma} A \xrightarrow{\overline{r} \cdot s} D$ *is an extensions.*

Proof. We have $x \dashv R[r]$, $y \dashv R[s]$ such that $X \wedge Y = 0$. Applying Theorem 4.1, there exists $\gamma : X \times Y \rightarrowtail A$ such that $\gamma \cdot (1 \quad 0) = x$, $\gamma \cdot (0 \quad 1) = y$ and $\gamma \dashv (R[r] \vee R[s])$. Since $\boxed{2}$ is a pushout, we have $R[r] \vee R[s] = R[t]$, for the cokernel $t = \overline{r} \cdot s$, thus $\gamma = \ker(t)$. Applying M1.1 to the commutative diagram



where all rows, the first and second columns are extensions and the third column is a zero sequence, we prove that $X \rightarrowtriangle^{\overline{x}} C \xrightarrow{\overline{r}} D$ is an extension. Finally, we conclude that $Y \rightarrowtriangle^{\overline{y}} B \xrightarrow{\overline{s}} D$ is an extension by applying M1.1 to the original diagram. ∎

4.3. Lemma. *In a pointed category with cokernels satisfying* M2.1 *and* M1.1*, any monomorphism* $\overline{y}$ *that factorizes as* $\overline{y} = r \cdot y$*, with* r *a cokernel and* y *a kernel, is also a kernel.*

Proof. Consider $x = \ker(r)$, $s = \operatorname{coker}(y)$ and $\overline{s} = \operatorname{coker}(\overline{y})$. From $\overline{s} \cdot r \cdot y = 0$, we get a unique morphism $\overline{r}$ such that $\overline{r} \cdot s = \overline{s} \cdot r$. We have the diagram of Lemma 4.2 with $\boxed{1}$ a pullback, since $\overline{y}$ is a monomorphism, and $\boxed{2}$ a pushout, since $\overline{s} = \operatorname{coker}(\overline{y})$. ∎

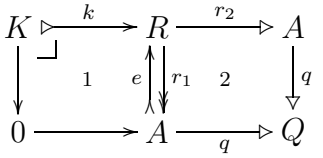
4.4. Proposition. *In a pointed category with cokernels satisfying* M2.1 *and* M1.1*, reflexive relations are effective equivalence relations.*

PROOF. Suppose $(r_1 \quad r_2) : R \rightarrowtail A \times A$ is a reflexive relation, i.e. there exists a monomorphism $e : A \rightarrowtail R$ such that $r_1 \cdot e = r_2 \cdot e = 1_A$. For $k = \ker(r_1)$, the pair (k, e) is jointly epimorphic.

The composition $r_2 \cdot k$ is a monomorphism, since $r_2 \cdot k \cdot x = r_2 \cdot k \cdot y$ implies $(r_1 \quad r_2) \cdot k \cdot x = (r_1 \quad r_2) \cdot k \cdot y$, allowing us to conclude that $x = y$. This monomorphism $r_2 \cdot k$ factors by a cokernel and a kernel, thus by Lemma 4.3 is also a kernel. Consider $q = \operatorname{coker}(r_2 \cdot k)$. We have $q \cdot r_1 = q \cdot r_2$ by the fact that (k, e) is jointly epimorphic and

$$q \cdot r_1 \cdot k = 0 = q \cdot r_2 \cdot k$$
$$q \cdot r_1 \cdot e = q = q \cdot r_2 \cdot e.$$

Applying the pullback cancellation property of protomodular categories to the commutative diagram



we conclude that $\boxed{2}$ is a pullback. ∎

As a consequence, in a category satisfying M2.1 and M1.1, we may identify kernels with normal monomorphisms.

4.5. REMARK. A pointed category with cokernels satisfying M2.1 and M1.1 also satisfies M2.3, since the stability for finite intersections holds for normal monomorphisms ($f \dashv R$ and $g \dashv S$, imply that $(f \wedge g) \dashv (R \wedge S)$).

To prove that a Moore category is regular, we will use the pullback property of M2.2.

4.6. PROPOSITION. *A pointed category with cokernels satisfying* M2.1*,* M1.1 *and* M2.2 *is protomodular and regular.*

PROOF. Based on Proposition 3.2 of [Janelidze, Márki, Tholen, 2002], if $\mathcal{C}$ is a pointed category with kernels, cokernels of kernels, such that the pullback of a cokernel is a cokernel and $\ker(f) = 0$ if and only if f is a monomorphism, then $\mathcal{C}$ has a pullback-stable (cokernel, monomorphism)-factorization system (the pullback of a (cokernel, monomorphism)-factorization is also such a factorization). ∎

Conversely, a pointed protomodular and regular category satisfies M1.1 (Theorem 12 of [Bourn, 2001]) and the pullback of a cokernel is a cokernel, since they coincide with the regular epimorphisms.

4.7. PROPOSITION. *A pointed category with cokernels satisfying* M2.1*,* M1.1 *and* M2.2 *is protomodular and Barr-exact.*

PROOF. By Propositions 4.6 and 4.4. ∎

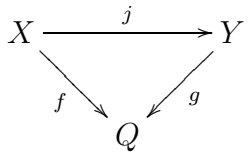
5. Strong Protomodularity

Condition M1.2 is finally used in order to establish strong protomodularity.

5.1. Definition. A category $\mathcal{C}$ with finite limits is called *strongly protomodular* if the change of base functors, with respect to the fibration $\pi : \mathrm{Pt}\mathcal{C} \longrightarrow \mathcal{C}$, are left exact, conservative and reflect normal monomorphisms.

We have seen that normal monomorphisms in $\mathcal{C}$ are kernels. Normal monomorphisms in the category of pointed objects have the following characterization:

5.2. Proposition. [Proposition 2.1 of [Bourn, 2000]] *When a category $\mathcal{C}$ is quasi-pointed and protomodular, a map*



is a normal monomorphisms in $\mathcal{C}/Q$ if and only if $j \cdot \ker(f)$ is a normal monomorphism in $\mathcal{C}$. The same result holds in $\mathrm{Pt}\mathcal{C}[Q]$.

So, j is a normal monomorphism in $\mathcal{C}/Q$ (or $\mathrm{Pt}\mathcal{C}[Q]$) if and only if $j \cdot \ker(f)$ is a kernel in $\mathcal{C}$.

5.3. Proposition. *A pointed category with cokernels satisfying* M2.1, M1.1 *and* M1.2 *is strongly protomodular.*

Proof. Since the category is pointed and protomodular, it suffices to prove that the change of base functors

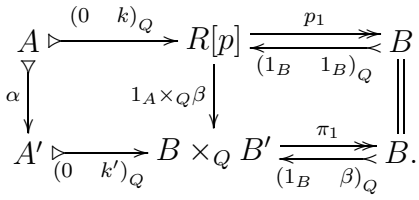
$$
\begin{array}{rccc}
(0_Q)^* : & \mathrm{Pt}\mathcal{C}[Q] & \longrightarrow & \mathcal{C} \\
& B \underset{s}{\overset{p}{\rightleftarrows}} Q & \longmapsto & A = \ker(p) \\
& {\scriptstyle\beta}\downarrow \qquad \| & & \downarrow{\scriptstyle\alpha} \\
& B' \underset{s'}{\overset{p'}{\rightleftarrows}} Q & & A' = \ker(p')
\end{array}
$$

reflect normal monomorphisms. This is given by M1.2. ∎

Conversely, we have:

5.4. Proposition. *A pointed, strongly protomodular and Barr-exact category with cokernels satisfies* M1.2.

PROOF. The diagram of M1.2 induces the commutative diagram



Since the category is strongly protomodular and α is a kernel, the composite $(0 \quad k')_Q \cdot \alpha = 1_A \times_Q \beta \cdot (0 \quad k)_Q$ is a kernel. By Lemma 4.3 applied to the monomorphism $k' \cdot \alpha = \pi_2 \cdot ((0 \quad k')_Q \cdot \alpha)$ is a kernel. ∎

Our results can be gathered into the following characterization:

5.5. THEOREM. *Consider the following conditions for a category $\mathcal{C}$:*

(i) *$\mathcal{C}$ is a Moore category;*

(ii) *$\mathcal{C}$ is pointed, strongly protomodular and Barr-exact with cokernels.*

Then (i) *implies* (ii).

5.6. REMARK. The reverse implication of Theorem 5.5 holds provided that $\mathcal{C}$ satisfies M4.1 and M4.2, which is in particular the case when $\mathcal{C}$ is a variety. Condition M4.2 holds because varieties are well-powered. To see that condition M4.1 is satisfied, we use a consequence of Theorem 1.1 in [Bourn, Janelidze, 2002] that characterizes these varieties by the existence of binary terms $t_1, \cdots, t_n$ and an $(n+1)$-ary term t such that $t(x, t_1(x,y), \cdots, t_n(x,y)) = y$ and $t_i(x,x) = 0$, for all $i = 1, \cdots, n$. Note that $x = y$ if and only if $t_i(x,y) = 0$, for all $i = 1, \cdots, n$, since

$$y = t(x, t_1(x,y), \cdots, t_n(x,y)) = t(x, 0, \cdots, 0) = t(x, t_1(x,x), \cdots, t_n(x,x)) = x.$$

Given an extension $A \overset{k}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} Q$ of Q by A, we have an equivalence relation

$$x \sim y \Leftrightarrow p(x) = p(y) \Leftrightarrow t_i(x,y) \in A, \text{for all } i = 1, \cdots, n.$$

So y is in the equivalence class of x implies that $y = t(x, t_1(x,y), \cdots, t_n(x,y))$ is in $t(\{x\} \times A \times \cdots \times A)$, thus the equivalence class of x has at most the cardinality of A^n. Finally, the cardinality of B is less or equal than the cardinality of $Q \times A^n$, proving that the equivalence classes of extensions of Q by A form a set.

6. Mo(o)re Examples

1. Grp, Rng.

2. Abelian categories. They are pointed and have cokernels by definition. They are also essentially affine, since they are additive and have finite limits (Corollary 5 of [Bourn, 1991]), thus strongly protomodular ([Bourn, 2000]). Finally, they are Barr-exact by the Tierney equation.

3. Heyting semilattices (definitions and omitted proofs in this example can be found in [Johnstone, 1982]). It is proven in [Johnstone] that the category HSLat of Heyting semilattices (meet-semilattices with implication) is semi-abelian.

   Before proving the strong protomodularity of HSLat, we first note that a morphism $\alpha : A \longrightarrow A'$ is a normal monomorphism in HSLat if and only if $\alpha(A)$ is a filter in A'. In fact, if $\alpha \dashv R$, then $(\alpha(a_1), \alpha(a_2)) \in R$, for every pair of elements (a_1, a_2) of A, and $(\alpha(a), a') \in R$ implies $a' \in \alpha(A)$. Since $1_{A'} = \alpha(1_A)$, we get $(1_{A'}, a') \in R$ if and only if $a' \in \alpha(A)$. Finally, $\alpha(A)$ is a filter in A' because $1_{A'} \in \alpha(A)$, α is closed for meets and given $a \in A, a' \in A'$ such that $\alpha(a) \leq a'$, we have

   $$\begin{aligned}(\alpha(a), 1_{A'}), (a', a') \in R &\Rightarrow (\alpha(a) \to a', 1_{A'} \to a') \in R \\ &\Rightarrow (1_{A'}, a') \in R \\ &\Rightarrow a' \in \alpha(A).\end{aligned}$$

   Conversely, if $\alpha(A)$ is a filter in A', then $S = \{(a'_1, a'_2) \in A' \times A' : a'_1 \to a'_2, a'_2 \to a'_1 \in \alpha(A)\}$ is an equivalence relation on A' (see [Johnstone, 1982] VI-2.10 for a proof which applies as such to Heyting semilattices). Moreover, $(1_{A'}, a') \in S$ if and only if $a' \in \alpha(A)$ and $(\alpha(a_1), \alpha(a_2)) \in S$, for every pair (a_1, a_2) of elements of A, since α preserves the implication. Hence, there exists a morphism $\tilde{\alpha} : A \times A \longrightarrow S$, with $\tilde{\alpha}(a_1, a_2) = (\alpha(a_1), \alpha(a_2))$. Furthermore, given $a \in A$ and $a' \in A'$ such that $(\alpha(a), a') \in S$, by transitivity we have $(1_{A'}, \alpha(a)), (\alpha(a), a') \in S$ implies $(1_{A'}, a') \in S$, so that $a' \in \alpha(A)$. Hence, $\alpha \dashv S$.

   For any Heyting semilattice Q, we must prove that the change of base functor of Proposition 5.3 reflects normal monomorphisms. Note that $A = \{b \in B : p(b) = 1_Q\}$, $k : A \hookrightarrow B$ is an inclusion and $\alpha(a) = \beta(a)$, for every element a of A. Suppose $\alpha(A) = \beta(A)$ is a filter in A'. Since A' is a filter in B', we conclude that $\beta \cdot k(A) = \beta(A)$ is a filter in B'.

4. $\mathrm{Grp}(\mathcal{C})$, $\mathrm{Rng}(\mathcal{C})$, for an elementary topos $\mathcal{C}$ with a natural number object. They are obviously pointed and the existence of cokernels is guaranteed by the natural number object (see [Johnstone, 2002], for instance). When $\mathcal{C}$ is finitely complete, the categories of internal groups and rings are strongly protomodular (Corollary 4.2 of [Bourn, 2000]). They are Barr-exact because $\mathcal{C}$ is also Barr-exact.

5. $\mathrm{Pt}\mathcal{C}[Q]$, for $\mathcal{C}$ strongly protomodular, Barr-exact with coequalizers. They are obviously pointed and coequalizers in $\mathcal{C}$ give cokernels in the category of pointed objects. Finally, since $\mathcal{C}$ is strongly protomodular and Barr-exact, the same holds for $\mathrm{Pt}\mathcal{C}[Q]$ (see Corollary 5.6 of [Bourn, 2000] for the strong protomodularity).

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