Theory and Applications of Categories, Vol. 13, No. 10, 2004, pp. 164–168.

FUNCTORIAL CONCEPTS OF COMPLEXITY FOR FINITE AUTOMATA

For Aurelio, an exacting colleague and a treasured friend since 1972, when he was one of the unfailingly enthusiastic four who traveled weekly from Milan to Perugia for mathematical discussions "fuori programma".

F. WILLIAM LAWVERE

ABSTRACT. Some unsolved problems about the classifying topos for Boolean algebras, as well as about the axiomatic arithmetic of finite combinatorial toposes, are closely connected with some simple distinctions between finite automata.

1. Adequacy of Measurement and Boolean Algebra

An automaton Q determines the following notion of perceived element y for any set X: For any partitioning f of X into Q parts, $y(f)$ specifies the state q into which the "element" is placed by f, but in a way that is natural with respect to the state transition rule, i.e. $y(wf) = wy(f)$ for all of the w acting on Q. Say that Q measures X if every perceived element of X comes from a (unique) actual element x in X via evaluation: $y(f) = f(x)$ for all f. (This is opposite to the set theorists' use of "X has measurable cardinality" which on the contrary is equivalent to the condition that X has some non-actual Q-perceived point, where Q is a given infinite set with all actors.) There is a three-element automaton that measures all finite sets X.

Denoting by F the category of all finite non-empty sets and arbitrary maps between them, the product-preserving functors $F \to$ Sets correspond exactly to Boolean algebras, making explicit that all jobs done by such algebras can be described as natural shufflings of such partitions. Because any non-empty finite set is a retract of a power of a fixed Q, a Boolean algebra is determined by the functor's restriction to the part of F consisting of powers of Q, and traditionally one took $Q = 2$; however, as pointed out in the first paragraph, in some contexts it is more effective to take $Q = 3$, because then only unary "propositional" operations w (not binary or higher ones) need be considered. Explicitly, if Q is considered as the integers modulo 3, the three polynomial state transitions $0, t \pm (1 - t^2)$ suffice to insure the naturality, and hence by the above assertion the actuality, of perceptions.

Received by the editors 2003-08-18.
Published on 2004-12-05.
2000 Mathematics Subject Classification: 05C35, 11B39, 11U10, 18B20, 18B25.
Key words and phrases: Automata, measurable cardinals, Aufhebung, finite topos, axiomatic arithmetic.

2. A Classifying Topos and the Aufhebung of Complexity

By contrast, the topos of all functors $F^{op} \to$ Sets constitutes the classifier for Boolean algebras in the sense that its points correspond uniquely to such algebras. The objects of this topos can be presented by infinitary positive formulas in the theory of Boolean algebras (the topos as a whole is not itself Boolean). This classifying topos contains as a cartesian closed reflective subcategory the category of all groupoids (= small categories in which every morphism is invertible) just as the category of all small categories is similarly contained in the simplicial topos of all functors from Delta^{op} to sets where Delta is the category of non-empty finite totally ordered sets and all order-preserving maps between them. A striking reciprocity is that for a small category C, the topos of C-automata is Boolean iff C is a groupoid; elementary algebra shows that in the group case every C-automaton Q is the disjoint sum of connected ones and that the connected ones are small in number ("small" can also be interpreted to mean "finite"), and in fact the converse holds: the only small categories C for which the number of connected C-automata is small are the groupoids. For example, the only Boolean toposes of finite automata are those for which all state transitions are invertible.

In case $C \subset D \subset F$ are finite, retract closed, full subcategories of the category F of nonempty finite sets, a right D-automaton Q might be considered to have mere C-complexity if it were adjointly induced up from its restriction Q_0 to C. But there are two senses of adjoint induction (unlike the case where C and D are themselves groups and linear automata are being considered), namely the left and right Kan quantifiers relative to the restriction process. Any Q which restricts to the same Q_0 lies between (in the sense of canonical maps of D-automata) these two extensions as follows:

$$Q_0 \otimes_C D \to Q \to \mathrm{Hom}_C(D, Q_0)$$

Note that up to splitting idempotents, we can assume that D is the monoid of all endomaps of a given finite set and that C is the monoid of all endomaps of a certain retract of that set, determined by an idempotent in D.

An open problem which seems important for the understanding of these combinatorics is the following: Given C, how much bigger must we take D in order that for all further enlargements to $D' \supseteq D$ the two extreme inductions give the same result:

$$Q_0 \otimes_C D' = \mathrm{Hom}_D(D', Q_0 \otimes_C D)$$

The latter common object would then be an equivariant retract of any D'-automaton Q whose C-restriction is Q_0.

3. Second order automata

Any topos can replace sets as an arena in which monoids, automata, measuring, etc., can be considered, for example the Boolean algebra classifier could so serve. However, in some toposes (such as this one) there is a generalized notion of monoid, motivated by the theory of higher order differential equations, which sometimes shares with monoids the topos-character of the actions. In any cartesian closed category a first-order action of an alphabet A on Q, i.e. a map $A \times Q \to Q$, can equivalently be viewed as a map $Q \to Q^A$ assigning to each state its destinies under all actions; in case there is a preferred point in A which we want to act as the identity, that can be expressed by requiring that the composite $Q \to Q^A \to Q$ is 1_Q (where the second map is evaluation at the point); invoking subjective infinity, we can even consider this as an action of a quotient monoid. The generalization starts not with just an alphabet, but with a given sub-alphabet $A_0 \hookrightarrow A$; an "action" on a configuration space Q is defined to mean a prolongation operator $Q^{A_0} \to Q^A$, i.e. an operation assigning to each $A_0 \to Q$ an extension to all of A. By analogy with physics and with higher-order ODEs, the term "state space" is best applied to Q^{A_0} (not to the configuration space Q) in this context; in order that an actor in A can determine a new configuration, there must be given not only the current configuration, but also a specification of what initial transition is already under way. (Actually, the notion of "current configuration" itself is only meaningful if we are given the additional structure of a specified point in A_0.) There can be derived, from such an action, an ordinary unary automaton with states Q^{A_0} and alphabet A^{A_0} as follows: For any "letter" σ and any state $x, (x \cdot \sigma)(t) = \underline{x}(\sigma(t))$ for t in A_0, where the underbar denotes the given prolongation operator.

For example, Fibonacci constructed a model, a key feature of which has become a metaphor for such "states of becoming" in other contexts: "the old is pregnant with the new". Indeed his dynamics is second order relative to the inclusion

$$A_0 = \{-1, 0\} \to \{\text{previous, present, next}\} = A$$

which interprets -1 as previous and 0 as present; a prolongation operator is determined by any binary operation on Q, which could be, for example, (truncated) addition. Among the non-identity elements of the alphabet A^{A_0} is the special σ defined by

$$\sigma(-1) = \text{ present}$$

$$\sigma(0) = \text{ next}$$

whose action brings the states in Q^{A_0} forward in time, generating by iteration the associated Fibonacci sequences. There is further structure here beyond first-order time: the other seven letters of this alphabet are also natural state actors on every Fibonaccian second-order automaton.

However, not every automaton with alphabet A^{A_0} (even with the inclusion acting as the identity) comes from a prolongation; indeed the prolongations do not usually form a topos, unless the sub-alphabet A_0 is a very special sort of object, one for which the exponentiation functor $(\ )^{A_0}$ has a further right adjoint. In the case of the Boolean algebra classifier, there is such an object for every finite set, but considered as a contractible object rather than as a discrete one! That is, there are two adjointly opposite embeddings of the category F of finite sets into the topos of all functors $F^{op} \to$ Sets: the discrete embedding assigns the associated constant functors, and the other interprets $S_0 \in F$ instead as the functor $A_0(n) = S_0^n$ (this topos has the remarkable property that codiscrete = representable). Then the right adjoint to $(\ )^{A_0}$ turns out to be the functor $(\ )^{1/A_0}$ assigning to each Q the new functor

$$(Q^{1/A_o})(n) = Q(n^{S_0})$$

For example, $S_0 = 2$ gives the contractible object which is the generic Boolean algebra $2^{(\ )}$ (in the usual 2-valued notation), and an inclusion of A_0 in some object A is equivalent to the choice of a special element of $A(2)$.

4. Finite Toposes and their Internal Theory

For purposes of this discussion, we have been considering that finite automata are objects in toposes of finite presheaves on finite categories. (Thus with this broad definition, graph theory and combinatorial topology are part of automata theory.) These toposes are simpler than Grothendieck toposes in several respects: (a) Any topos morphism between two such is actually essential (i.e. has an extra left adjoint), as is (b) any subtopos, i.e. given any Lawvere-Tierney modal operator, the associated sheafification functor has a left adjoint, giving rise to a "skeleton" comonad; in fact, (c) any subtopos of such is again such, i.e. although the existential and disjunctive conditions on an object which can be imposed with the help of a Grothendieck topology or a Lawvere-Tierney modal operator may be a useful way to express certain examples, the end result will nonetheless be another full presheaf topos, on a smaller site [1].

Finiteness implies that much more powerful principles are available than merely the higher-order Heyting logic of an arbitrary topos. The number theory of cohesive and variable "numbers" can be generated by making such principles explicit.

Peano's axioms as such would hold only in a discrete case, but consider Dedekind's: The condition

$$X \xrightarrow{\alpha} X \xrightarrow{\beta} X \quad \& \quad \beta\alpha = 1 \Rightarrow \alpha\beta = 1$$

(internalized by stating that a certain subobject (equalizer) of $X^X \times X^X$ is invariant under switching) can be imposed on all objects of a topos. A consequence is then the much stronger form of Dedekind finiteness which involves general subobjects, not just retracts: If a monomorphic endomap of Y is given, then on $X = P(Y)$, the induced map splits (by quantification), hence is an isomorphism by the imposition, but the inverse descends to yield that the given mono is an isomorphism. (Similarly, any given epimorphic endomap of any object is invertible.)

An open problem is whether the "Dedekind finite" toposes in this sense enjoy the properties a,b,c stated above for the smaller class of "actually finite" toposes. Note that although there is a huge number of incomparable nonstandard examples, the assumption of a given geometric morphism involves functors (adjoint) in both directions and hence a certain commensurability of domain and codomain. Does the finiteness principle of Dedekind or the essentiality of all modal operators imply that there exists an automaton

Q which can measure all objects (as in our opening discussion with $Q = 3$ for the case of abstract finite sets)? Of course, the converse is not true because toposes of infinite sets measured by a single infinite automaton are common in set theory (it suffices to exclude so-called "measurable" cardinals).

More formally, the measuring concept poses the question whether a certain monad T, constructed by double dualizing into Q, is actually the identity monad. One always has $Q = T(Q)$. A less internal version is the following: Let M be a transition monoid on $Q = 3$ which is adequate for finite abstract sets. Then for any finite monoid C, the topos of right C-automata is fixed by a monad obtained by a dual measuring whose processing takes place in the category of left $M \times C$ automata [2]. This is a combinatorial reflection of the common phenomenon that when the base category of less-structured spaces has a bounded duality, then so do many categories of spaces with additional geometrical structure.

References

[1] Artin, M., Faisceaux constructibles et supports propres, in SGA4, Tome 3, Springer Lecture Notes in Mathematics **305**, 1973, 1-461.

[2] Isbell, J., Small Adequate Subcategories, Journal of the London Mathematical Society **43**, 1968, 242-246.

University at Buffalo
Department of Mathematics
244 Mathematics Building
Buffalo, N.Y. 14260-2900
USA
Email: wlawvere@buffalo.edu

This article may be accessed via WWW at http://www.tac.mta.ca/tac/ or by anonymous ftp at ftp://ftp.tac.mta.ca/pub/tac/html/volumes/13/10/13-10.{dvi,ps}

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

The method of distribution of the journal is via the Internet tools `WWW/ftp`. The journal is archived electronically and in printed paper format.

Subscription information. Individual subscribers receive (by e-mail) abstracts of articles as they are published. Full text of published articles is available in .dvi, Postscript and PDF. Details will be e-mailed to new subscribers. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, `rrosebrugh@mta.ca`.

Information for authors. The typesetting language of the journal is TeX, and LaTeX2e is the preferred flavour. TeX source of articles for publication should be submitted by e-mail directly to an appropriate Editor. They are listed below. Please obtain detailed information on submission format and style files from the journal's WWW server at `http://www.tac.mta.ca/tac/`. You may also write to `tac@mta.ca` to receive details by e-mail.