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ON EXTENSIONS OF LAX MONADS

Dedicated to Aurelio Carboni on the occasion of his sixtieth birthday

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Abstract. In this paper we construct extensions of **Set**-monads – and, more generally, of lax **Rel**-monads – into lax monads of the bicategory Mat($\mathbf{V}$) of generalized $\mathbf{V}$-matrices, whenever $\mathbf{V}$ is a well-behaved lattice equipped with a tensor product. We add some guiding examples.

Introduction

Extensions of **Set**-monads into lax monads in bicategories of generalized matrices have been used recently to study categories of lax algebras [5, 10, 9], generalizing Barr's [1] description of topological spaces as lax algebras for the ultrafilter monad and Lawvere's [16] description of metric spaces as $\mathbf{V}$-categories for $\mathbf{V}$ the extended real half-line. The recent interest in this area had its origin in the use of the description of topological spaces via ultrafilter convergence to characterize special classes of continuous maps, such as effective descent morphisms, triquotient maps, exponentiable maps, quotient maps and local homeomorphisms [18, 15, 4, 7, 14, 6].

In this area, one of the difficulties one has to deal with is the construction of lax extensions of **Set**-monads into a larger bicategory. Contrarily to the extensions studied so far, with *ad-hoc* constructions, here we present a uniform construction of an extension of a **Set**-monad, satisfying (BC), into a lax monad of the bicategory Mat($\mathbf{V}$) of generalized $\mathbf{V}$-matrices. This construction consists of three steps: first we apply Barr's extension of the monad into the category **Rel** of relations (in Section 1) and then we extend this into Mat($\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$) and finally into Mat($\mathbf{V}$) (as described in Section 3). This construction includes, for instance, Clementino-Tholen construction of an extension of the ultrafilter monad in case $\mathbf{V}$ is a lattice (Example 6.4). The techniques used here can be used also to extend lax monads from **Rel** into Mat($\mathbf{V}$). We find particularly interesting the presentation of the Hausdorff metric on subsets of a metric space as an extension of the lax powerset monad (Example 6.3).

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1. From **Set** to **Rel**

1.1. The 2-category **Rel**. We recall that **Rel** has as objects *sets* and as morphisms $r : X \nrightarrow Y$ *relations* $r \subseteq X \times Y$ (or equivalently $r : X \times Y \to \mathbf{2}$). With the hom-sets $\mathbf{Rel}(X, Y)$ partially ordered by inclusion, **Rel** is a 2-category.

Using its natural involution $(\ )^\circ$, that assigns to each relation $r : X \nrightarrow Y$ its inverse $r^\circ : Y \nrightarrow X$, and the embedding $\mathbf{Set} \hookrightarrow \mathbf{Rel}$, it is easily seen that every relation r can be written as $g \cdot f^\circ$:

$$\begin{tikzcd} X \arrow[rr, "r", "\shortmid"{marking}] & & Y \\ & r \arrow[ul, "f"] \arrow[ur, "g"'] & \end{tikzcd} \tag{1}$$

where f and g are the projections.

1.2. Barr's extension. In order to extend a **Set**-monad (T, η, μ) into **Rel**, Barr [1] defined first $\overline{T}(f^\circ) := (Tf)^\circ$ for any map f, and then made use of the factorization (1) of the relation $r : X \nrightarrow Y$ to define

$$\overline{T}r := Tg \cdot \overline{T}f^\circ,$$

that does not depend on the chosen factorization and extends naturally to 2-cells. Hence the following diagram

$$\begin{tikzcd} \mathbf{Set} \arrow[r, "T"] \arrow[d, hook] & \mathbf{Set} \arrow[d, hook] \\ \mathbf{Rel} \arrow[r, "\overline{T}"] & \mathbf{Rel} \end{tikzcd}$$

is commutative.

Barr proved that $\overline{T} : \mathbf{Rel} \to \mathbf{Rel}$ is an *op-lax* functor and that the natural transformations η and μ become *op-lax* in **Rel**; that is:

- $\overline{T}(r \cdot s) \leq \overline{T}r \cdot \overline{T}s$ for any pair of composable relations r, s;
- for every $r : X \nrightarrow Y$, one has

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[d, "r"', "\shortmid"{marking}] \arrow[dr, phantom, "\leq"] & TX \arrow[d, "\overline{T}r", "\shortmid"{marking}] \\ Y \arrow[r, "\eta_Y"] & TY \end{tikzcd} \qquad \begin{tikzcd} T^2X \arrow[r, "\mu_X"] \arrow[d, "\overline{T}^2r"', "\shortmid"{marking}] \arrow[dr, phantom, "\leq"] & TX \arrow[d, "\overline{T}r", "\shortmid"{marking}] \\ T^2Y \arrow[r, "\mu_Y"] & TY. \end{tikzcd}$$

1.3. The role of the Beck-Chevalley Condition. As Barr pointed out, this extension may fail to be a functor. The missing inequality depends on the behaviour of the functor $T : \mathbf{Set} \to \mathbf{Set}$: it holds if and only if T satisfies the *Beck-Chevalley Condition (BC)*, that is, if $(Tf)^\circ \cdot Tg = Tk \cdot (Th)^\circ$ for every pullback diagram

$$\begin{tikzcd} W \arrow[r, "k"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ Z \arrow[r, "g"] & Y \end{tikzcd}$$

in **Set**. (Under the Axiom of Choice, (BC) is equivalent to the preservation of weak pullbacks.)

Theorem. *For a functor $T : \mathbf{Set} \to \mathbf{Set}$, the following assertions are equivalent:*

i. There is a (unique) 2-functor $\overline{T} : \mathbf{Rel} \to \mathbf{Rel}$, preserving the involution, that extends T;

ii. T satisfies the Beck-Chevalley Condition. ■

We also have:

Proposition. *Given functors $S, T : \mathbf{Set} \to \mathbf{Set}$ satisfying (BC) and a natural transformation $\varphi : S \to T$, the following assertions are equivalent:*

i. $\overline{\varphi} : \overline{S} \to \overline{T}$ is a natural transformation;

*ii. for every map $f : X \to Y$, the **Set**-diagram*

$$\begin{tikzcd} SX \arrow[r, "\varphi_X"] \arrow[d, "Sf"'] & TX \arrow[d, "Tf"] \\ SY \arrow[r, "\varphi_Y"] & TY \end{tikzcd}$$

satisfies (BC), i.e. $(Tf)^\circ \cdot \varphi_Y = \varphi_X \cdot (Sf)^\circ$. ■

2. The extended setting: Mat($\mathbf{V}$) and lax monads

Throughout we will be concerned with the construction of *lax* extensions of a **Set**-monad to more general 2-categories. In this section we describe the 2-categories as well as the lax axioms for a monad we will deal with.

2.1. The category of $\mathbf{V}$-matrices. We consider a *complete and cocomplete lattice* $\mathbf{V}$ as a category, and assume that it is *symmetric monoidal-closed*, with tensor product $\otimes$ and unit $k_{\mathbf{V}}$. We denote its initial and terminal objects by $\bot_{\mathbf{V}}$ and $\top_{\mathbf{V}}$, respectively. The 2-category Mat($\mathbf{V}$) has as objects *sets* and as 1-cells $r : X \nrightarrow Y$ $\mathbf{V}$-*matrices*, that is, maps $r : X \times Y \to \mathbf{V}$; given $r, r' : X \nrightarrow Y$, there is a (unique) 2-cell $r \to r'$ if, for every $(x, y) \in X \times Y$, $r(x, y) \leq r'(x, y)$ in $\mathbf{V}$. Composition of 1-cells $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$ is given by *matrix multiplication*, i.e.

$$s \cdot r(x, z) = \bigvee_{y \in Y} r(x, y) \otimes s(y, z),$$

for every $x \in X$ and $z \in Z$. Further information about this category can be found in [2] and [10].

Rel is a crucial example of a 2-category of this sort, obtained when $\mathbf{V} = \mathbf{2} = \{\bot, \top\}$, with $\otimes = \wedge$. The monoidal map $\mathbf{2} \hookrightarrow \mathbf{V}$, with $\bot \mapsto \bot_{\mathbf{V}}$ and $\top \mapsto k_{\mathbf{V}}$ naturally gives rise to an embedding $\mathbf{Rel} \hookrightarrow$ Mat($\mathbf{V}$) whenever $\bot_{\mathbf{V}} \neq k_{\mathbf{V}}$, condition assumed from now on. By *relation* in Mat($\mathbf{V}$) we mean any $\mathbf{V}$-matrix with entries $\bot_{\mathbf{V}}$ and $k_{\mathbf{V}}$; that is, any image of a relation by this embedding.

2.2. Lax monads. Here we propose a definition of lax monad different from Barr's [1]; namely, we assume that the functor is *lax* (and not necessarily op-lax).

By a *lax monad* (T,η,μ) in $\mathrm{Mat}(\mathbf{V})$ we mean:

– a *lax functor* $T : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$ (so that $1_{TX} \le T1_X$ and $Ts \cdot Tr \le T(s\cdot r)$ for composable $\mathbf{V}$-matrices r, s), and

– *op-lax natural transformations* $\eta : 1_{\mathrm{Mat}(\mathbf{V})} \to T$ and $\mu : T^2 \to T$,

such that $\mu \cdot T\mu \le \mu \cdot \mu T$, $\mathrm{Id}\,T \le \mu \cdot T\eta \le T\,\mathrm{Id}$ and $\mathrm{Id}\,T \le \mu \cdot \eta T \le T\,\mathrm{Id}$; that is, for every set X,

$$
\begin{tikzcd}
T^3X \arrow[r, "\mu_{TX}"] \arrow[d, "T\mu_X"'] \arrow[dr, phantom, "\le"] & T^2X \arrow[d, "\mu_X"] \\
T^2X \arrow[r, "\mu_X"] & TX
\end{tikzcd}
\qquad
\begin{tikzcd}
 & TX \arrow[d, shift right, "T\eta_X"'] \arrow[d, shift left, "\eta_{TX}"] \arrow[ddl, phantom] & \\
 \arrow[r, phantom, "\le"] & T^2X \arrow[d, "\mu_X"] \arrow[r, phantom, "\le"] & {} \\
 & TX &
\end{tikzcd}
\tag{2}
$$

with the curved arrows $TX \xrightarrow{1_{TX}} TX$ (left) and $TX \xrightarrow{T1_X} TX$ (right).

We point out that this definition does not coincide with Bunge's [3], although the only differences occur in the right-hand diagram. In the final section we present an example of a lax monad (in our sense) which is not a lax monad à la Bunge (see Example 6.3).

3. The strategy

3.1. The monoidal closed category $\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$. It is straightforward to check that the formula:

$$
f \otimes g(v) = \bigvee_{v',v''\,:\,v'\otimes v'' \ge v} f(v') \wedge g(v''), \tag{3}
$$

for any $f, g \in \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$ and $v \in \mathbf{V}$, defines a tensor product in $\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$ that preserves joins, with unit element

$$
\begin{array}{rcl}
k : \mathbf{V}^{\mathrm{op}} & \to & \mathbf{2} \\
v & \mapsto & \begin{cases} \top & \text{if } v \le k_{\mathbf{V}} \\ \bot & \text{elsewhere.} \end{cases}
\end{array}
$$

(We point out that this tensor product is a particular case of Day's *convolution* [11].) Symmetry of this tensor is also inherited from symmetry of the tensor product of $\mathbf{2}$ and $\mathbf{V}$, so that we have:

Proposition. *Given a symmetric monoidal closed lattice* $\mathbf{V}$*, formula* (3) *gives a symmetric monoidal closed structure on* $\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$.

We remark that, in case the tensor product in $\mathbf{V}$ is its categorical product, then $f \otimes g = f \wedge g$ as well.

3.2. $\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$-MATRICES VERSUS $\mathbf{V}^{\mathrm{op}}$-INDEXED RELATIONS. The embedding

$$\begin{array}{rcl} E:\mathbf{2} & \to & \mathbf{2}^{\mathbf{V}^{\mathrm{op}}} \\ w & \mapsto & \begin{array}{rcl} E(w):\mathbf{V}^{\mathrm{op}} & \to & \mathbf{2} \\ v & \mapsto & E(w)(v)=\left\{\begin{array}{ll} w & \text{if } v\le k_{\mathbf{V}} \\ \perp & \text{elsewhere}\end{array}\right.\end{array}\end{array}$$

preserves the tensor product, the unit element and infima. It preserves suprema if and only if $k_{\mathbf{V}}=\top_{\mathbf{V}}$. Therefore, as detailed in [10], it induces a 2-functor

$$E:\mathrm{Mat}(\mathbf{2})\to\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}).$$

Denoting the set of functors from $\mathbf{A}$ to $\mathbf{B}$ by $[\mathbf{A},\mathbf{B}]$, the composition of the natural bijections

$$[X\times Y,[\mathbf{V}^{\mathrm{op}},\mathbf{2}]]\cong[X\times Y\times\mathbf{V}^{\mathrm{op}},\mathbf{2}]\cong[\mathbf{V}^{\mathrm{op}}\times X\times Y,\mathbf{2}]\cong[\mathbf{V}^{\mathrm{op}},[X\times Y,\mathbf{2}]]$$

assigns to any $\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$-matrix $a:X\times Y\to\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$ a $\mathbf{V}^{\mathrm{op}}$-indexed family of relations $(a_v:X\times Y\to\mathbf{2})_{v\in\mathbf{V}}$, defined by

$$a_v(x,y)=a(x,y)(v).$$

LEMMA. *For $a,a':X\nrightarrow Y$, $b:Y\nrightarrow Z$ in $\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$ and $v,v'\in\mathbf{V}$, one has:*

a. $v\le v'\ \Rightarrow\ a_v\ge a_{v'}$;

b. $a\le a'\ \Rightarrow\ a_v\le a'_v$;

c. $b_v\cdot a_{v'}\le(b\cdot a)_{v\otimes v'}$;

d. if $\otimes=\wedge$ *in* $\mathbf{V}$, *then* $b_v\cdot a_v=(b\cdot a)_v$.

PROOF. The only non-trivial assertion is (d). When $\otimes=\wedge$, for each $x\in X$ and $z\in Z$,

$$\begin{array}{rcl}(b\cdot a)_v(x,z) & = & (b\cdot a)(x,z)(v)\\ & = & \bigvee_{y\in Y}(a(x,y)\wedge b(y,z))(v)\\ & = & \bigvee_{y\in Y}a(x,y)(v)\wedge b(y,z)(v)\\ & = & b_v\cdot a_v(x,z).\end{array}$$

∎

3.3. THE YONEDA EMBEDDING. We consider now the *Yoneda embedding*

$$\begin{array}{rcl} \mathsf{Y}:\mathbf{V} & \to & \mathbf{2}^{\mathbf{V}^{\mathrm{op}}} \\ v & \mapsto & \begin{array}{rcl} \mathsf{Y}(v):\mathbf{V}^{\mathrm{op}} & \to & \mathbf{2} \\ u & \mapsto & \begin{cases} \top & \text{if } u\le v \\ \bot & \text{otherwise,} \end{cases} \end{array} \end{array}$$

and its left adjoint

$$\begin{array}{rcl} L:\mathbf{2}^{\mathbf{V}^{\mathrm{op}}} & \to & \mathbf{V} \\ f & \mapsto & \bigvee\{v\in\mathbf{V}\ ;\ f(v)=\top\}. \end{array}$$

PROPOSITION. *The functors* Y *and* L *are strict monoidal functors.*

PROOF. The functor Y is monoidal: from $\mathsf{Y}(k_{\mathbf{V}})(v)=\top$ if and only if $v\le k_{\mathbf{V}}$, it follows that $\mathsf{Y}(k_{\mathbf{V}})=k$; moreover,

$$\begin{array}{rcl} (\mathsf{Y}(v)\otimes\mathsf{Y}(v'))(u)=\top & \Leftrightarrow & \displaystyle\bigvee_{r\otimes s\ge u}\mathsf{Y}(v)(r)\wedge\mathsf{Y}(v')(s)=\top \\ & \Leftrightarrow & \exists r,s\in\mathbf{V}\ :\ r\otimes s\ge u,\ r\le v \text{ and } s\le v' \\ & \Leftrightarrow & u\le v\otimes v' \ \Leftrightarrow\ \mathsf{Y}(v\otimes v')(u)=\top. \end{array}$$

The functor L is monoidal, since:

$$L(k)=\bigvee\{v\in\mathbf{V}\ ;\ k(v)=\top\}=k_{\mathbf{V}},\ \text{and}$$

$$\begin{array}{rcl} L(f)\otimes L(g) & = & \bigvee\{r\in\mathbf{V}\ ;\ f(r)=\top\}\otimes\bigvee\{s\in\mathbf{V}\ ;\ g(s)=\top\} \\ & = & \bigvee\{r\otimes s\ ;\ r,s\in\mathbf{V},\ f(r)=\top=g(s)\} \\ & = & \bigvee\{v\in\mathbf{V}\ ;\ (f\otimes g)(v)=\top\}\ =\ L(f\otimes g). \end{array}$$

■

These two strict monoidal functors induce a pair of lax functors

$$\mathrm{Mat}(\mathbf{V}) \underset{L}{\overset{\mathsf{Y}}{\rightleftarrows}} \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}),$$

L being in fact a 2-functor.

3.4. THE USE OF THE EMBEDDINGS TO CONSTRUCT THE EXTENSION. The construction of the extension of a lax monad (T,η,μ) in $\mathbf{Rel}=\mathrm{Mat}(\mathbf{2})$ into $\mathrm{Mat}(\mathbf{V})$ we will describe in the next two sections consists of two steps.

First we use the interpretation of a $\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$-matrix as a $\mathbf{V}^{\mathrm{op}}$-indexed family of relations and the embedding described in 3.2, obtaining a commutative diagram

$$\begin{tikzcd} \mathrm{Mat}(\mathbf{2}) \arrow[r, "T"] \arrow[d, "E"'] & \mathrm{Mat}(\mathbf{2}) \arrow[d, "E"] \\ \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[r, "\widehat{T}"] & \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}). \end{tikzcd} \tag{4}$$

Secondly, we use the adjunction $L \dashv \mathsf{Y}$ of Section 3.3 to transfer a lax monad (S, δ, ν) in $\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$ into $\mathrm{Mat}(\mathbf{V})$, defining $\widetilde{S} := LS\mathsf{Y}$ and showing that, under some conditions on $\mathbf{V}$, the following diagram

$$\begin{tikzcd} \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[r, "S"] \arrow[d, "L"'] & \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[d, "L"] \\ \mathrm{Mat}(\mathbf{V}) \arrow[r, "\widetilde{S}"] & \mathrm{Mat}(\mathbf{V}) \end{tikzcd} \tag{5}$$

is commutative and $(\widetilde{S}, \widetilde{\delta}, \widetilde{\nu})$ is a lax monad in $\mathrm{Mat}(\mathbf{V})$.

Finally, gluing these constructions, since LE is the embedding $\mathbf{Rel} \hookrightarrow \mathrm{Mat}(\mathbf{V})$, we obtain a commutative diagram

$$\begin{tikzcd} \mathbf{Rel} \arrow[r, "T"] \arrow[d, hook, "E"'] \arrow[dd, bend right] & \mathbf{Rel} \arrow[d, hook, "E"] \arrow[dd, bend left] \\ \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[r, "\widehat{T}"] \arrow[d, "L"'] & \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[d, "L"] \\ \mathrm{Mat}(\mathbf{V}) \arrow[r, "\widetilde{\widehat{T}}"] & \mathrm{Mat}(\mathbf{V}), \end{tikzcd}$$

and corresponding op-lax natural transformations $\widetilde{\widehat{\eta}}$ and $\widetilde{\widehat{\mu}}$.

We point out that in the construction sketched in diagram (4), and described in the next section, one can replace, without much effort, the lattice $\mathbf{2}$ by a general symmetric monoidal closed category. Moreover, in the construction (5) carried out in Section 5 one can easily replace the monoidal adjunction $L \dashv \mathsf{Y}$ by any other such adjunction.

4. From $\mathrm{Mat}(\mathbf{2})$ to $\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$

4.1. EXTENSION OF A LAX ENDOFUNCTOR. Given a lax functor $T : \mathrm{Mat}(\mathbf{2}) \to \mathrm{Mat}(\mathbf{2})$, for each $a : X \nrightarrow Y$ and $(\mathfrak{x}, \mathfrak{y}) \in TX \times TY$, we define

$$\widehat{T}a(\mathfrak{x}, \mathfrak{y})(v) := T(a_v)(\mathfrak{x}, \mathfrak{y}).$$

THEOREM. *Let $T : \mathbf{Rel} \to \mathbf{Rel}$ be a lax functor.*

a. The assignments $X \mapsto \widehat{T}X := TX$ and $a \mapsto \widehat{T}a$ define a lax functor $\widehat{T} : \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \to \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$ such that

$$\begin{tikzcd} \mathbf{Rel} \arrow[r, "T"] \arrow[d, hook, "E"'] \arrow[dr, phantom, "\geq"] & \mathbf{Rel} \arrow[d, hook, "E"] \\ \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[r, "\widehat{T}"] & \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}); \end{tikzcd}$$

that is, the functors $\widehat{T}E$ and ET agree on objects and, for each relation $r : X \nrightarrow Y$, $ET(r) \leq \widehat{T}E(r)$.

b. *$\widehat{T}$ preserves the involution whenever T does.*

c. *If $k_{\mathbf{V}} = \top_{\mathbf{V}}$ or T preserves the $\perp$-relation, then $\widehat{T}$ is an extension of T, that is, the following diagram commutes*

$$\begin{tikzcd}
\mathbf{Rel} \arrow[r, "T"] \arrow[d, hook, "E"'] & \mathbf{Rel} \arrow[d, hook, "E"] \\
\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[r, "\widehat{T}"] & \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}).
\end{tikzcd}$$

d. *If $\otimes = \wedge$, then $\widehat{T}$ is a (strict) 2-functor if T is one.*

Proof. To prove (a), we have to show that $\widehat{T}(b) \cdot \widehat{T}(a) \leq \widehat{T}(b \cdot a)$, $1_{\widehat{T}X} \leq \widehat{T}1_X$ and that $ET \leq \widehat{T}E$. To show the first inequality, consider $a : X \nrightarrow Y$ and $b : Y \nrightarrow Z$ in $\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$, and $\mathfrak{x} \in TX$, $\mathfrak{z} \in TZ$ and $v \in \mathbf{V}$. Then:

$$\begin{aligned}
\widehat{T}b \cdot \widehat{T}a(\mathfrak{x},\mathfrak{z})(v) &= \left(\bigvee_{\mathfrak{y}\in TY} \widehat{T}a(\mathfrak{x},\mathfrak{y}) \otimes \widehat{T}b(\mathfrak{y},\mathfrak{z})\right)(v) \\
&= \bigvee_{\mathfrak{y}\in TY} \bigvee_{v'\otimes v''\geq v} \widehat{T}a(\mathfrak{x},\mathfrak{y})(v') \otimes \widehat{T}b(\mathfrak{y},\mathfrak{z})(v'') \\
&= \bigvee_{v'\otimes v''\geq v} (Tb_{v''} \cdot Ta_{v'})(\mathfrak{x},\mathfrak{z}) \\
&\leq \bigvee_{v'\otimes v''\geq v} T(b\cdot a)_{v'\otimes v''}(\mathfrak{x},\mathfrak{z}) \ \leq \ T(b\cdot a)_v(\mathfrak{x},\mathfrak{z}).
\end{aligned}$$

Now, for a relation $r : X \nrightarrow Y$,

$$\widehat{T}E(r) = \begin{cases} Tr & \text{if } v \leq k_{\mathbf{V}} \\ T\perp & \text{otherwise} \end{cases} \quad \text{while} \quad ET(r) = \begin{cases} Tr & \text{if } v \leq k_{\mathbf{V}} \\ \perp & \text{otherwise,} \end{cases}$$

hence $\widehat{T}E \geq ET$ follows. This inequality implies that $1_{\widehat{T}X} \leq \widehat{T}1_X$, since

$$\widehat{T}1_X = \widehat{T}E1_X \geq ET1_X \geq E1_{TX} = 1_{TX}.$$

The proofs of (b) and (c) are now straightforward. One concludes (d) using Lemma 3.2(d) in the calculation of $\widehat{T}b \cdot \widehat{T}a$ presented in the proof of (a). ∎

Now we prove some auxiliary results.

Lemma. *For $a : X \nrightarrow Y$ in $\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$, $r : Y \nrightarrow Z$, $s : W \nrightarrow X$ in $\mathbf{Rel}$, and $v \in \mathbf{V}$:*

a. *$(\widehat{T}a)_v = Ta_v$;*

b. *$(Er \cdot a)_v = r \cdot a_v$ and $(a \cdot Es)_v = a_v \cdot s$.*

PROOF. (a) is straightforward.

(b): For $x \in X$ and $z \in Z$,

$$\begin{aligned}(Er \cdot a)_v(x,z) &= (Er \cdot a)(x,z)(v) = \bigvee_{y \in Y} (a(x,y) \otimes Er(y,z))(v) \\ &= \bigvee_{v' \otimes v'' \geq v} \bigvee_{y \in Y} (a(x,y)(v') \otimes Er(y,z)(v'')).\end{aligned}$$

In this join it is enough to consider:

- $v'' \leq k_{\mathbf{V}}$, since elsewhere $Er(y,z)(v'') = \perp$ and the tensor product is $\perp$ as well, and
- $v' = v$, due to monotonicity of $a(x,y)$; hence,

$$(Er \cdot a)_v(x,z) = \bigvee_{y \in Y} a(x,y)(v) \otimes r(y,z) = (r \cdot a_v)(x,z).$$

The other equality is proved analogously. ■

PROPOSITION. *If $T : \mathbf{Rel} \to \mathbf{Rel}$ preserves composition on the left (right) with $r : Y \times Z \to \mathbf{2}$, then $\widehat{T}$ preserves composition with Er.*

PROOF. For any $a : X \nrightarrow Y$ in $\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$ and $v \in \mathbf{V}$,

$$\widehat{T}(Er \cdot a)_v = T((Er \cdot a)_v) = T(r \cdot a_v) = Tr \cdot Ta_v = Tr \cdot (\widehat{T}a)_v.$$

The stability under composition on the right has an analogous proof. ■

4.2. EXTENSION OF A LAX MONAD. Given a natural transformation $\alpha : S \to T$ between lax functors $S, T : \mathbf{Rel} \to \mathbf{Rel}$, we define $\widehat{\alpha} : \widehat{S} \to \widehat{T}$ by $\widehat{\alpha}_X := E\alpha_X$ for every set X; that is

$$\begin{aligned}\widehat{\alpha}_X : SX \times TX &\to \mathbf{2}^{\mathbf{V}^{\mathrm{op}}} \\ (\mathfrak{x}, \mathfrak{x}') &\mapsto \begin{array}{rcl} \widehat{\alpha}_X(\mathfrak{x}, \mathfrak{x}') : \mathbf{V}^{\mathrm{op}} & \to & \mathbf{2} \\ v & \mapsto & \begin{cases} \alpha_X(\mathfrak{x}, \mathfrak{x}') & \text{if } v \leq k_{\mathbf{V}} \\ \perp & \text{elsewhere.} \end{cases} \end{array}\end{aligned}$$

PROPOSITION. *Let $S, T : \mathbf{Rel} \to \mathbf{Rel}$ be lax functors. Then:*

a. $\widehat{\mathrm{Id}} = \mathrm{Id}$.

b. $\widehat{S} \cdot \widehat{T} = \widehat{S \cdot T}$.

c. If $\alpha : S \to T$ is a (lax, op-lax) natural transformation, so is $\widehat{\alpha} : \widehat{S} \to \widehat{T}$.

PROOF. Straightforward. ■

Theorem. *Let (T,η,μ) be a lax monad in* $\mathbf{Rel}$. *If $k_{\mathbf{V}} = \top_{\mathbf{V}}$ or T preserves the $\perp$-matrix, then $(\widehat{T},\widehat{\eta},\widehat{\mu})$ is a lax monad that extends the former one.*

Proof. We only have to check diagrams (2) for $(\widehat{T},\widehat{\eta},\widehat{\mu})$, which follow directly from the corresponding diagrams for (T,η,μ) once one observes that the former ones are obtained from the latter applying the 2-functor E. ∎

5. From $\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$ into $\mathrm{Mat}(\mathbf{V})$

5.1. Transfer of a lax endofunctor. Using the monoidal adjunction of 3.3, for a lax endofunctor S in $\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$, we define

$$\mathrm{Mat}(\mathbf{V}) \xrightarrow{\widetilde{S}} \mathrm{Mat}(\mathbf{V}) := (\mathrm{Mat}(\mathbf{V}) \xrightarrow{\mathsf{Y}} \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \xrightarrow{S} \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \xrightarrow{L} \mathrm{Mat}(\mathbf{V})).$$

Proposition. *Let $S : \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \to \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$ be a lax functor. Then $\widetilde{S} = LS\mathsf{Y}$ is such that*

$$\begin{tikzcd} \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[r, "S"] \arrow[d, "L"'] \arrow[dr, phantom, "\geq"] & \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \arrow[d, "L"] \\ \mathrm{Mat}(\mathbf{V}) \arrow[r, "\widetilde{S}"] & \mathrm{Mat}(\mathbf{V}). \end{tikzcd}$$

The inequality in the diagram becomes an equality whenever $S\mathsf{Y}L \leq \mathsf{Y}LS$.

Proof. By the adjointness property, $LS \leq LS\mathsf{Y}L$, the required inequality. In addition, if $S\mathsf{Y}L \leq \mathsf{Y}LS$, then $LS\mathsf{Y}L \leq L\mathsf{Y}LS \leq LS$, and the equality follows. ∎

Analogously to the previous construction, we can easily check that:

Lemma. *For $S : \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \to \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$, if S preserves composition on the left (right) with a matrix $a : X \to\!\!\!\!\!\!\;\shortmid\; Y$, then so does $\widetilde{S}$, with a replaced by La.*

Proof. Indeed,

$$\begin{array}{rclcl} \widetilde{S}(b\cdot La) & = & LS\mathsf{Y}(b\cdot La) & \leq & LS\mathsf{Y}(L\mathsf{Y}b\cdot La) \\ & = & LS\mathsf{Y}L(\mathsf{Y}b\cdot a) & \leq & LS(\mathsf{Y}b\cdot a) \\ & = & LS\mathsf{Y}b\cdot LSa & \leq & LS\mathsf{Y}b\cdot LS\mathsf{Y}La \\ & = & \widetilde{S}b\cdot\widetilde{S}(La). & & \end{array}$$

∎

5.2. TRANSFER OF A LAX MONAD. First we analyse the behaviour of the construction with respect to the composition of functors and natural transformations. For any (lax, op-lax) natural transformation $\alpha : R \to S$ between lax functors $R, S : \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \to \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$, we define $\widetilde{\alpha} : \widetilde{R} \to \widetilde{S}$ by $\widetilde{\alpha}_X := L\alpha_{\mathsf{Y}X}$, for every set X; that is,

$$\widetilde{\alpha}_X : \widetilde{R}X \nrightarrow \widetilde{S}X, \quad (\mathfrak{x}, \mathfrak{x}') \mapsto \bigvee\{v \in \mathbf{V}\,;\, \alpha_X(\mathfrak{x}, \mathfrak{x}')(v) = \top\}.$$

LEMMA. *Let* $R, S : \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}) \to \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$ *be lax functors. Then:*

a. $\widetilde{\mathrm{Id}} = \mathrm{Id}$.

b. $\widetilde{RS} \le \widetilde{R}\widetilde{S}$, *with equality in case* $R\mathsf{Y}L \le \mathsf{Y}LR$.

c. If $\alpha : R \to S$ *is a (lax, op-lax) natural transformations, so is* $\widetilde{\alpha} : \widetilde{R} \to \widetilde{S}$.

PROOF. It is straightforward. ∎

THEOREM. *For each lax monad* (S, δ, ν) *in* $\mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$, $(\widetilde{S}, \widetilde{\delta}, \widetilde{\nu})$ *is a lax monad in* $\mathrm{Mat}(\mathbf{V})$ *provided that* $S\mathsf{Y}L \le \mathsf{Y}LS$.

PROOF. We already know that $\widetilde{S} = LS\mathsf{Y}$ is a lax functor, $\widetilde{\delta} = L\delta : L\mathsf{Y} = \mathrm{Id} \to \widetilde{S}$ and $\widetilde{\nu} = L\nu : \widetilde{SS} = \widetilde{S}\widetilde{S} \to \widetilde{S}$ are op-lax natural transformations. It remains to be shown that they fulfil the conditions of diagram (2): for each set X,

$$\begin{array}{rcl}
\widetilde{\nu}_X \cdot \widetilde{S}\widetilde{\nu}_X & = & L\nu_X \cdot LS\mathsf{Y}L\nu_X \;\le\; L\nu_X \cdot L\mathsf{Y}LS\nu_X \;=\; L(\nu_X \cdot S\nu_X)\\
& \le & L(\nu_X \cdot \nu_{SX}) \;=\; L\nu_X \cdot L\nu_{SX} \;=\; \widetilde{\nu}_X \cdot \widetilde{\nu}_{\widetilde{S}X};\\
\widetilde{\nu}_X \cdot \widetilde{S}\widetilde{\delta}_X & = & L\nu_X \cdot LS\mathsf{Y}L\delta_X \;\ge\; L\nu_X \cdot LS\delta_X \;=\; L(\nu_X \cdot S\delta_X)\\
& \ge & L1_{SX} \;\ge\; 1_{\widetilde{S}X};\\
\widetilde{\nu}_X \cdot \widetilde{S}\widetilde{\delta}_X & = & L\nu_X \cdot LS\mathsf{Y}L\delta_X \;\le\; L\nu_X \cdot L\mathsf{Y}LS\delta_X \;=\; L(\nu_X \cdot S\delta_X)\\
& \le & LS1_X \;\le\; \widetilde{S}L1_X \;=\; \widetilde{S}1_X;\\
\widetilde{\nu}_X \cdot \widetilde{\delta}_{\widetilde{S}X} & = & L(\nu_X \cdot \delta_{SX}) \;\le\; L1_{SX} \;=\; 1_{\widetilde{S}X}.
\end{array}$$

∎

5.3. AN EXTRA CONDITION ON $\mathbf{V}$. In order to guarantee that $S\mathsf{Y}L \le \mathsf{Y}LS$ we will impose an extra condition on $\mathbf{V}$ which we analyse in the sequel. It was used in [10] under the designation $\mathbf{V}$ *is* $\sqsubset$*-atomic*, and it was formulated there as: for all $u, v, w \in \mathbf{V}$,

a. $u \sqsubset v \le w \;\Rightarrow\; u \sqsubset w$,

b. $v = \bigvee At(v)$,

where $At(v) = \{u \in \mathbf{V}\,;\, u \sqsubset v\ \&\ \forall S \subseteq \mathbf{V}\ u \sqsubset \bigvee S\ \Rightarrow \exists s \in S\,:\, u \leq s\}$ is the set of *$\sqsubset$-atoms of v*.

It was pointed out to us by Dexue Zhang that this condition is equivalent to $\mathbf{V}$ being a *completely distributive lattice*, i.e. the functor $\bigvee : \mathbb{D}\mathbf{V} \to \mathbf{V}$, where $\mathbb{D}\mathbf{V}$ is the lattice of downsets of $\mathbf{V}$, has a left adjoint. (For the connection between this constructive formulation of complete distributivity and the classical one see [19].) In order to show this, we first observe that the lattice $\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$ is isomorphic to the lattice $\mathbb{D}\mathbf{V}$ and that the following diagram commutes:

$$\begin{tikzcd}
\mathbb{D}\mathbf{V} \arrow[rr, "\cong"] \arrow[dr, shift left, "\bigvee"] & & \mathbf{2}^{\mathbf{V}^{\mathrm{op}}} \arrow[dl, shift left, "L"] \\
& \mathbf{V} \arrow[ul, shift left, "\downarrow -"] \arrow[ul, phantom, "\top"] \arrow[ur, shift left, "\mathsf{Y}"] \arrow[ur, phantom, "\top"] . &
\end{tikzcd}$$

PROPOSITION. *The following conditions are equivalent:*

i. $\mathbf{V}$ *is $\sqsubset$-atomic for a transitive relation $\sqsubset$ in* $\mathbf{V}$*;*

ii. $\mathbf{V}$ *is $\sqsubset$-atomic for a relation $\sqsubset$ in* $\mathbf{V}$*;*

iii. $\mathbf{V}$ *is completely distributive;*

iv. there exists a family $(A(v))_{v\in\mathbf{V}}$ of subsets of $\mathbf{V}$ such that, for each $f \in \mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$ and $v \in \mathbf{V}$,

$$\mathsf{Y}L(f)(v) = \bigwedge_{u\in A(v)} f(u).$$

PROOF. (i) $\Rightarrow$ (ii) is obvious.

(ii) $\Rightarrow$ (iii): Given a relation $\sqsubset$ such that $\mathbf{V}$ is $\sqsubset$-atomic, we define the left adjoint $A : \mathbf{V} \to \mathbb{D}\mathbf{V}$ of the functor $\bigvee : \mathbb{D}\mathbf{V} \to \mathbf{V}$ by

$$A(v) := At(v) = \{u \in \mathbf{V}\,;\, u \sqsubset v\ \&\ \forall S \subseteq \mathbf{V}\ u \sqsubset \bigvee S\ \Rightarrow \exists s \in S\,:\, u \leq s\}.$$

By definition of $\sqsubset$-atomic, this is a monotone map such that $\bigvee \cdot A = \mathrm{Id}_{\mathbf{V}}$. Moreover, it is obvious that, for any $S \in \mathbb{D}\mathbf{V}$, $A \cdot \bigvee(S) \subseteq S$.

(iii) $\Rightarrow$ (iv): Given $A \dashv \bigvee$, the family of images $(A(v))_{v\in\mathbf{V}}$ satisfies the condition stated in (iv). Indeed, the adjunction gives

$$\forall v \in \mathbf{V}\ \ \forall S \in \mathbb{D}\mathbf{V}\ \ v \leq \bigvee S\ \Leftrightarrow\ A(v) \subseteq S.$$

Hence, one has

$$\begin{aligned}
\mathsf{Y}L(f)(v) = \top \quad &\Leftrightarrow \quad v \leq \bigvee\{w \in \mathbf{V}\,;\ f(w) = \top\} \\
&\Leftrightarrow \quad A(v) \subseteq \{w \in \mathbf{V}\,;\, f(w) = \top\} \\
&\Leftrightarrow \quad \forall u \in A(v)\ f(u) = \top.
\end{aligned}$$

(iv) $\Rightarrow$ (i): Let $\sqsubset$ be defined by

$$u \sqsubset v \;:\Leftrightarrow\; \exists w \in \mathbf{V} \;:\; u \in A(w) \text{ and } w \leq v.$$

Then it clearly satisfies condition (a) of the definition of $\sqsubset$-atomic. Moreover, for any $u \in A(v)$, $u \sqsubset v$.

To show that $\sqsubset$ is transitive, it is enough to notice that, since $\mathsf{Y}L\mathsf{Y} = \mathsf{Y}$, we have

$$\top = \mathsf{Y}(v)(v) = \mathsf{Y}L\mathsf{Y}(v)(v) \;\Rightarrow\; \bigwedge_{u \in A(v)} \mathsf{Y}(v)(u) = \top \;\Rightarrow\; \forall u \in A(v) \;:\; u \leq v.$$

To show equality (b) we first observe that $v = \bigvee A(v)$, as stated above. We then show that $A(v) \subseteq At(v)$. Let $u \in A(v)$ and $u \sqsubset \bigvee S$ for some subset S of $\mathbf{V}$. By definition of $\sqsubset$, there exists $v' \in \mathbf{V}$ such that $u \in A(v')$ and $v' \leq \bigvee S$. Let

$$\begin{array}{rcl} f : \mathbf{V}^{\mathrm{op}} & \to & \mathbf{2} \\ w & \mapsto & \begin{cases} \top & \text{if } \exists s \in S \;:\; w \leq s \\ \bot & \text{elsewhere.} \end{cases} \end{array}$$

Then $\mathsf{Y}L(f)(v') = \top$, since $v' \leq L(f) = \bigvee S$, and therefore $f(u) = \top$, that is, there exists $s \in S$ such that $u \leq s$ as claimed. ∎

Lemma. *If $\mathbf{V}$ is completely distributive, one has, for each $\mathbf{2}^{\mathbf{V}^{\mathrm{op}}}$-matrix $a : X \nrightarrow Y$ and each element v of $\mathbf{V}$,*

$$(\mathsf{Y}La)_v = \bigwedge_{u \in A(v)} a_u.$$

Proof. For each $x \in X$ and $y \in Y$, using the proposition above, we have

$$(\mathsf{Y}La)_v(x,y) = \mathsf{Y}L(a(x,y))(v) = \bigwedge_{u \in A(v)} a(x,y)(u) = \bigwedge_{u \in A(v)} a_u(x,y).$$

∎

5.4. The extension. Next we will show that $\widetilde{\widetilde{(\ )}}$ extends each lax monad in **Rel** into $\mathrm{Mat}(\mathbf{V})$. We start outlining this construction.

Let (T, η, μ) be a lax monad in **Rel**. Then $\widetilde{\widetilde{T}} : \mathrm{Mat}(\mathbf{V}) \to \mathrm{Mat}(\mathbf{V})$ agrees with T on objects. To define $\widetilde{\widetilde{T}}(a)$ for a morphism $a : X \nrightarrow Y$ in $\mathrm{Mat}(\mathbf{V})$, we consider the relation

$$\begin{array}{rcl} \mathsf{Y}a_v : X \times Y & \to & \mathbf{2} \\ (x,y) & \mapsto & \begin{cases} \top & \text{if } v \leq a(x,y) \\ \bot & \text{elsewhere;} \end{cases} \end{array}$$

it is straightforward that

$$\begin{array}{rcl} \widetilde{\widetilde{T}}(a) : TX \times TY & \to & \mathbf{V} \\ (\mathfrak{x}, \mathfrak{y}) & \mapsto & \bigvee\{v \in \mathbf{V} \,;\, T(\mathsf{Y}a_v)(\mathfrak{x}, \mathfrak{y}) = \top\}. \end{array}$$

In the op-lax natural transformations $\eta : \mathrm{Id} \to T$ and $\mu : T^2 \to T$, $\widetilde{\widehat{(\ )}}$ acts as

$$\begin{array}{rcl} X \times TX \overset{\eta_X}{\to} \mathbf{2} & \mapsto & X \times TX \overset{\widetilde{\widehat{\eta}}_X}{\to} \mathbf{V} \\ & & (x, \mathfrak{x}) \ \mapsto \ \begin{cases} k_{\mathbf{V}} & \text{if } \eta_X(x, \mathfrak{x}) = \top \\ \bot & \text{elsewhere,} \end{cases} \end{array}$$

$$\begin{array}{rcl} T^2X \times TX \overset{\mu_X}{\to} \mathbf{2} & \mapsto & T^2X \times TX \overset{\widetilde{\widehat{\mu}}_X}{\to} \mathbf{V} \\ & & (\mathfrak{X}, \mathfrak{x}) \ \mapsto \ \begin{cases} k_{\mathbf{V}} & \text{if } \mu_X(\mathfrak{X}, \mathfrak{x}) = \top \\ \bot & \text{elsewhere.} \end{cases} \end{array}$$

THEOREM. *Let (T, η, μ) be a lax monad in* $\mathbf{Rel}$. *If* $\mathbf{V}$ *is completely distributive,* $k_{\mathbf{V}} = \top_{\mathbf{V}}$ *or T preserves the $\bot$-matrix, then $(\widetilde{\widehat{T}}, \widetilde{\widehat{\eta}}, \widetilde{\widehat{\mu}})$ is a lax monad in* $\mathrm{Mat}(\mathbf{V})$, *that extends the former one.*

PROOF. Using Theorems 4.1, 4.2, and Proposition 5.1 and Theorem 5.2, it is enough to show that $\widehat{T}\mathsf{Y}L \leq \mathsf{Y}L\widehat{T}$, whenever $\mathbf{V}$ is completely distributive. For $a : X \nrightarrow Y \in \mathrm{Mat}(\mathbf{2}^{\mathbf{V}^{\mathrm{op}}})$, $v \in \mathbf{V}$, $\mathfrak{x} \in TX$, $\mathfrak{y} \in TY$:

$$\begin{array}{rcl} (\widehat{T}\mathsf{Y}L(a))(\mathfrak{x}, \mathfrak{y})(v) & = & T(\mathsf{Y}L(a))_v(\mathfrak{x}, \mathfrak{y}) \ = \ T(\displaystyle\bigwedge_{u \in A(v)} a_u)(\mathfrak{x}, \mathfrak{y}) \\ & \leq & \displaystyle\bigwedge_{u \in A(v)} Ta_u(\mathfrak{x}, \mathfrak{y}) \ = \ \mathsf{Y}LTa(\mathfrak{x}, \mathfrak{y})(v). \end{array}$$

$\blacksquare$

COROLLARY. *Let (T, η, μ) be a monad in* $\mathbf{Set}$. *If T satisfies (BC),* $\mathbf{V}$ *is completely distributive, and $k_{\mathbf{V}} = \top_{\mathbf{V}}$ or $\top$ preserves the $\bot$-matrix, then $(\widetilde{\widehat{\overline{T}}}, \widetilde{\widehat{\overline{\eta}}}, \widetilde{\widehat{\overline{\mu}}})$ is a lax monad in* $\mathrm{Mat}(\mathbf{V})$, *that extends the given one. Moreover, if the tensor product $\otimes$ in* $\mathbf{V}$ *is its categorical product, then $\widetilde{\widehat{\overline{T}}}$ is in fact a functor.*

PROOF. The first assertion follows from Theorem 1.3 together with the theorem above. To show that $\widetilde{\widehat{\overline{T}}}$ is a functor in case $\otimes = \wedge$, we first show that, for each $a : X \nrightarrow Y$ and $b : Y \nrightarrow Z$ in $\mathrm{Mat}(\mathbf{V})$ and each $u, v \in \mathbf{V}$ with $u \in A(v)$, $\mathsf{Y}(b \cdot a)_v \leq \mathsf{Y}b_u \cdot \mathsf{Y}a_u$. Indeed, for every $x \in X$ and $z \in Z$,

$$\begin{array}{rcl} \mathsf{Y}(b \cdot a)(x, z)(v) = \top & \Leftrightarrow & v \leq (b \cdot a)(x, z) \\ & \Rightarrow & u \sqsubset (b \cdot a)(x, z) \ = \ \displaystyle\bigvee_{y \in Y} a(x, y) \wedge b(y, z) \\ & \Rightarrow & \exists y \in Y \ : \ u \leq a(x, y) \wedge b(y, z) \\ & \Leftrightarrow & \exists y \in Y \ : \ \mathsf{Y}a(x, y)(u) = \top = \mathsf{Y}b(y, z)(u) \\ & \Leftrightarrow & (\mathsf{Y}b \cdot \mathsf{Y}a)(x, z)(u) = \top. \end{array}$$

Finally, to check that $\widetilde{\widehat{\overline{T}}}(b \cdot a) \leq \widetilde{\widehat{\overline{T}}}b \cdot \widetilde{\widehat{\overline{T}}}a$, we consider $\mathfrak{x} \in TX$, $\mathfrak{z} \in TZ$ and $v \in A(\widetilde{\widehat{\overline{T}}}(b \cdot a)(\mathfrak{x}, \mathfrak{z}))$. Hence $\top = \overline{T}(\mathsf{Y}(b \cdot a)_v)(\mathfrak{x}, \mathfrak{z})$. Then, for any $u \in A(v)$, $\top = (\overline{T}\mathsf{Y}b_u \cdot \overline{T}\mathsf{Y}a_u)(\mathfrak{x}, \mathfrak{z})$. But then there exists $\mathfrak{y} \in TY$ such that $\overline{T}\mathsf{Y}b_u(\mathfrak{x}, \mathfrak{y}) = \top = \overline{T}\mathsf{Y}a_u(\mathfrak{y}, \mathfrak{z})$. Therefore $u \leq \widetilde{\widehat{\overline{T}}}(b) \cdot \widetilde{\widehat{\overline{T}}}(a)(\mathfrak{x}, \mathfrak{z})$ and then $\widetilde{\widehat{\overline{T}}}(b \cdot a) \leq \widetilde{\widehat{\overline{T}}}b \cdot \widetilde{\widehat{\overline{T}}}a$ as claimed. ∎

6. Examples

In this section we present examples of extensions. Our main examples are based on the category $\mathrm{Mat}(\overline{\mathbb{R}}_+)$, where $([0, \infty], \geq)$ is endowed with the tensor product $+$. We remark that in this situation the terminal object is also the unit element 0 and that $\overline{\mathbb{R}}_+$ is $>$-atomic (see [12] for further information about this sort of lattices), hence we may apply our results. For simplicity, we use the same notation for the given (lax) monad and its extension.

6.1. The identity monad. Barr's extension of the identity monad $(\mathrm{Id}, 1, 1)$ in **Set** into **Rel** gives the identity monad. The same occurs in the next step: its extension into $\mathrm{Mat}(\mathbf{V})$ as defined here is the identity monad. (We remark that this monad may have other lax extensions, as it is shown in [8].)

6.2. The powerset monad. The powerset monad (P, η, μ) in **Set** is defined by:

- P is the *powerset functor*, assigning to each set X its powerset PX and to each map its direct image,
- $\eta_X(x) = \{x\}$ for every $x \in X \in \mathbf{Set}$, and
- $\mu_X(\mathcal{A}) = \bigcup \mathcal{A}$ for every set $\mathcal{A}$ of subsets of X.

It is easy to check that the functor P satisfies (BC), hence this monad can be extended to **Rel**, with

$$A(Pr)B \ \Leftrightarrow\ \forall x \in A\ \exists y \in B\ :\ xry \text{ and } \forall y \in B\ \exists x \in A\ :\ xry.$$

For $\mathbf{V} = \overline{\mathbb{R}}_+$, $d : X \times Y \to \overline{\mathbb{R}}_+$, $A \subseteq X$ and $B \subseteq Y$, the extension $Pd(A, B)$ is defined by

$$\inf\{v \in \overline{\mathbb{R}}_+ \,|\, \forall x \in A\ \exists y \in B\ :\ d(x, y) \leq v \text{ and } \forall y \in B\ \exists x \in A\ :\ d(x, y) \leq v\}.$$

In case d is a premetric in X, $\widetilde{P}d$ is the *usual premetric* in PX.

6.3. The lax powerset monad. If we consider now $H : \mathbf{Rel} \to \mathbf{Rel}$ with $HX := PX$ the powerset of X and

$$A(Hr)B \text{ if for each } b \in B \text{ there exists } a \in A \text{ such that } a\,r\,b,$$

it is easy to check that $1_{HX} \leq H1_X$ and $Hr \cdot Hs \leq H(r \cdot s)$, hence H is a lax functor. We may equip H with the structure of a lax monad, considering the (strict) natural transformations $\eta : \mathrm{Id}_{\mathbf{Rel}} \to H$ and $\mu : H^2 \to H$, defined by

$$x(\eta_X)A \text{ if } x \in A \quad \text{and} \quad \mathcal{A}(\mu_X)A \text{ if } \bigcup \mathcal{A} \subseteq A,$$

for $x \in X$, $A \subseteq X$ and $\mathcal{A} \subseteq HX$. It is easy to check that, for every set X, $A, A' \subseteq X$ and $\mathfrak{A} \subseteq HHX$,

$$A(\mu_X \cdot \eta_{HX})A' \;\Leftrightarrow\; A(\mu_X \cdot H\eta_X)A' \;\Leftrightarrow\; A(H1_X)A' \;\Leftrightarrow\; A \subseteq A', \text{ and}$$

$$\mathfrak{A}(\mu_X \cdot H\mu_X)A \;\Leftrightarrow\; \mathfrak{A}(\mu_X \cdot \mu_{HX})A \;\Leftrightarrow\; \bigcup\bigcup \mathfrak{A} \subseteq A.$$

Hence, $1_{HX} \leq \mu_X \cdot \eta_{HX} = \mu_X \cdot H\eta_X = H1_X$ and $\mu_X \cdot H\mu_X = \mu_X \cdot \mu_{HX}$, and then (H, η, μ) is a lax monad in **Rel**. (We remark that it is not a lax monad in the sense of Bunge [3], since $\mu_X \cdot H\eta_X \not\leq 1_{HX}$.)

It has an interesting lax extension to $\mathrm{Mat}(\overline{\mathbb{R}}_+)$: given $d : X \nrightarrow Y$ in $\mathrm{Mat}(\overline{\mathbb{R}}_+)$, for each $A \subseteq X$ and $B \subseteq Y$,

$$Hd(A, B) = \inf\{v \geq 0 \,|\, A(Hd_v)B\} = \inf\{v \geq 0 \,|\, \forall x \in A\ \exists y \in B \,:\, d(x, y) \leq v\}.$$

For a premetric $d : X \nrightarrow X$, Hd assigns to each pair of subsets A, B of X, its *Hausdorff (non-symmetric) premetric*

$$d_H(A, B) = \sup_{x \in A} \inf_{y \in B} d(x, y).$$

This identification holds in the general case of a $\mathbf{V}$-matrix $d : X \nrightarrow Y$, considering d_H defined as above. Indeed, for $v \in \overline{\mathbb{R}}_+$,

$$\begin{aligned} \forall x \in A\ \exists y \in B \,:\, d(x, y) \leq v \;&\Rightarrow\; \forall x \in A\ \inf_{y \in B} d(x, y) \leq v \;\Rightarrow\; d_H(A, B) \leq v \\ &\Rightarrow\; d_H(A, B) \leq Hd(A, B). \end{aligned}$$

On the other hand,

$$\begin{aligned} d_H(A, B) < v \;&\Rightarrow\; \forall x \in A\ \inf_{y \in B} d(x, y) < v \;\Rightarrow\; \forall x \in A\ \exists y \in B \,:\, d(x, y) \leq v \\ &\Rightarrow\; Hd(A, B) \leq v. \end{aligned}$$

Hence, $d_H = Hd$ as claimed.

6.4. THE ULTRAFILTER MONAD. We consider now the ultrafilter monad (U, η, μ) in **Set**, with:

- the functor $U : \mathbf{Set} \to \mathbf{Set}$ such that UX is the set of ultrafilters of X for every set X, and $Uf(\mathfrak{x})$ the ultrafilter generated by $f(\mathfrak{x})$, for every map $f : X \to Y$ and every ultrafilter $\mathfrak{x}$ in X.

- $\eta_X : X \to UX$ assigns to each point x the principal ultrafilter

$$\dot{x} = \{A \subseteq X \,|\, x \in A\};$$

- $\mu_X : U^2X \to UX$ is the *Kowalsky multiplication*, i.e.

$$\mu_X(\mathfrak{X}) = \bigcup_{\mathcal{X}\in\mathfrak{X}} \bigcap_{\mathfrak{x}\in\mathcal{X}} \mathfrak{x}.$$

The functor U satisfies (BC), hence it has an extension in **Rel**, given by

$$\mathfrak{x}(Ur)\mathfrak{y} \;\Leftrightarrow\; r[\mathfrak{x}] \subseteq \mathfrak{y} \;\Leftrightarrow\; r^\circ[\mathfrak{y}] \subseteq \mathfrak{x},$$

for every relation $r : X \nrightarrow Y$, $\mathfrak{x} \in UX$ and $\mathfrak{y} \in UY$. This can be equivalently described by

$$\mathfrak{x}(Ur)\mathfrak{y} \;\Leftrightarrow\; \forall A \in \mathfrak{x} \;\; \forall B \in \mathfrak{y} \;\; \exists x \in A \;\; \exists y \in B \;:\; xry.$$

Its lax extension U to Mat($\mathbf{V}$) coincides with Clementino-Tholen lax extension [10] (which we will denote below by U'), as we show next.

For each $d : X \nrightarrow Y$ in Mat($\mathbf{V}$),

$$\begin{aligned} Ud(\mathfrak{x},\mathfrak{y}) &= \bigvee\{v \in \mathbf{V} \,|\, \mathfrak{x}(Ud_v)\mathfrak{y}\} \\ &= \bigvee\{v \in \mathbf{V} \,|\, \forall A \in \mathfrak{x} \;\; \forall B \in \mathfrak{y} \;\; \exists x \in A \;\; \exists y \in B \;:\; d(x,y) \geq v\}, \end{aligned}$$

while

$$U'd(\mathfrak{x},\mathfrak{y}) = \bigwedge_{A\in\mathfrak{x},B\in\mathfrak{y}} \; \bigvee_{x\in A,y\in B} d(x,y).$$

For each $v \in \mathbf{V}$ such that $\mathfrak{x}(Ud_v)\mathfrak{y}$, $v \leq \bigvee_{x\in A,y\in B} d(x,y)$, hence $v \leq U'd(\mathfrak{x},\mathfrak{y})$, and therefore $Ud(\mathfrak{x},\mathfrak{y}) \leq U'd(\mathfrak{x},\mathfrak{y})$.

If w is a $\sqsubset$-atom and $w \sqsubset U'd(\mathfrak{x},\mathfrak{y})$, then $w \sqsubset \bigvee_{x\in A,y\in B} d(x,y)$ for each $A \in \mathfrak{x}$ and $B \in \mathfrak{y}$. Hence there exists $x \in A$ and $y \in B$ such that $w \leq d(x,y)$, and therefore $w \leq Ud(\mathfrak{x},\mathfrak{y})$. Hence, $U'd(\mathfrak{x},\mathfrak{y}) \leq Ud(\mathfrak{x},\mathfrak{y})$ and the equality follows.

We point out that, although $U : \mathbf{Rel} \to \mathbf{Rel}$ is a (strict) functor, its extension U : Mat($\mathbf{V}$) $\to$ Mat($\mathbf{V}$) is not always op-lax. It is the case when $\mathbf{V} = ([-\infty,+\infty],\geq)$, with tensor product $\otimes = +$ (where $-\infty + (+\infty) = +\infty$), as we show next.

Consider $X = \{n \,|\, n \in \mathbb{N}$, non-zero and even$\}$, $Z = \{-m \,|\, m \in \mathbb{N}$, non-zero and odd$\}$ and $Y = X \cup Z$. For

$$\begin{aligned} d_1 : X \times Y &\to [-\infty,+\infty] \\ (x,y) &\mapsto xy \end{aligned} \quad \text{and} \quad \begin{aligned} d_2 : Y \times Z &\to [-\infty,+\infty], \\ (y,z) &\mapsto yz \end{aligned}$$

and free ultrafilters $\mathfrak{x} \in UX$ and $\mathfrak{z} \in UZ$, we have

$$\begin{aligned} U(d_2 \cdot d_1)(\mathfrak{x}, \mathfrak{z}) &= \inf\{v \in \mathbf{V} \mid \forall A \in \mathfrak{x}\ \forall C \in \mathfrak{z}\ \exists x \in A\ \exists z \in C : (d_2 \cdot d_1)(x, z) \geq v\} \\ &= -\infty, \end{aligned}$$

since $(d_2 \cdot d_1)(x, z) = \inf_{y \in Y} y(x + z) = -\infty$. To calculate $(Ud_2 \cdot Ud_1)(\mathfrak{x}, \mathfrak{z})$, let $\mathfrak{y} \in UY$. If $X \in \mathfrak{y}$, then $Ud_1(\mathfrak{x}, \mathfrak{y}) = +\infty$ since every $A \in \mathfrak{x}$ is unlimited and every $B \in \mathfrak{y}$ has a positive element. If $X \notin \mathfrak{y}$, then $Z \in \mathfrak{y}$; hence $Ud_2(\mathfrak{y}, \mathfrak{z}) = +\infty$ since every $C \in \mathfrak{z}$ is unlimited and every $B \in \mathfrak{y}$ contains a negative element. Now

$$(Ud_2 \cdot Ud_1)(\mathfrak{x}, \mathfrak{z}) = \inf_{\mathfrak{y} \in UY} Ud_1(\mathfrak{x}, \mathfrak{y}) + Ud_2(\mathfrak{y}, \mathfrak{z}) = +\infty,$$

and therefore $U(d_2 \cdot d_1) \not\leq Ud_2 \cdot Ud_1$; that is U is not op-lax.

6.5. The filter monad. The filter monad (F, η, μ) in **Set**, with FX the set of filters of X, $Ff(\mathfrak{x}) = \{B \subseteq Y \mid f^{-1}(B) \in \mathfrak{x}\}$ for every $f : X \to Y$ and $\mathfrak{x} \in FX$, and η and μ defined as in the example above, satisfies (BC). Hence F can be extended into an endofunctor of **Rel**, that may be described by

$$\mathfrak{x}(Fr)\mathfrak{y} \ \Leftrightarrow\ r[\mathfrak{x}] \subseteq \mathfrak{y} \text{ and } r^\circ[\mathfrak{y}] \subseteq \mathfrak{x},$$

for every relation $r : X \nrightarrow Y$, $\mathfrak{x} \in FX$ and $\mathfrak{y} \in FY$. We observe that, contrarily to the case of the ultrafilter monad, in this situation we have to impose both conditions, $r[\mathfrak{x}] \subseteq \mathfrak{y}$ and $r^\circ[\mathfrak{y}] \subseteq \mathfrak{x}$, since each of them does not follow from the other. This was the reason why Pisani in [17] had to restrict the codomain in order to get a functor extension with the "non-symmetric" definition. Indeed, if we define $G : \mathbf{Rel} \to \mathbf{Rel}$, $\varepsilon : \mathrm{Id}_{\mathbf{Rel}} \to G$ and $\nu : GG \to G$ by $GX = FX$,

$$\begin{aligned} \mathfrak{x}(Gr)\mathfrak{y} &\ \Leftrightarrow\ r^\circ[\mathfrak{y}] \subseteq \mathfrak{x}, \\ x\,\varepsilon_X\,\mathfrak{x} &\ \Leftrightarrow\ \mathfrak{x} \subseteq \eta_X(x), \\ \mathfrak{X}\,\nu_X\,\mathfrak{x} &\ \Leftrightarrow\ \mathfrak{x} \subseteq \mu_X(\mathfrak{X}), \end{aligned}$$

we obtain a lax monad (G, ε, ν) in **Rel**.

6.6. The double powerset monad. Finally we present an example that shows that the Beck-Chevalley condition used throughout is not always satisfied. In the double powerset monad (D, η, μ) in **Set** induced by the adjunction

$$\mathbf{Set}^{\mathrm{op}} \underset{\mathbf{Set}(-,2)}{\overset{\mathbf{Set}(-,2)}{\rightleftarrows}} \mathbf{Set}$$

the functor D does not satisfy (BC) (see [13]). Indeed, it is easy to check that the D-image of the following pullback

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & \{0,1\} \arrow[d, "g"] \\ \{0,1\} \arrow[r, "f"] & \{0,1\}, \end{tikzcd}$$

where $f(0) = f(1) = 0$ and $g(0) = g(1) = 1$, does not satisfy (BC):

$$Df(\{\emptyset, \{0,1\}\}) = Dg(\{\emptyset, \{0,1\}\}) = P(\{0,1\}),$$

although there is no element on $D(\emptyset)$ mapped into $\{\emptyset, \{0,1\}\}$ by the pullback projections.

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