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DERIVATIONS OF CATEGORICAL GROUPS

Dedicated to Aurelio Carboni on the occasion of his sixtieth birthday

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ABSTRACT. In this paper we introduce and study the categorical group of derivations, $\mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{A})$, from a categorical group $\mathbb{G}$ into a braided categorical group $(\mathbb{A}, c)$ equipped with a given coherent left action of $\mathbb{G}$. Categorical groups provide a 2-dimensional vision of groups and so this object is a sort of 0-cohomology at a higher level for categorical groups. We show that the functor $\mathbb{D}\mathrm{er}(-, \mathbb{A})$ is corepresentable by the semidirect product $\mathbb{A} \rtimes \mathbb{G}$ and that $\mathbb{D}\mathrm{er}(\mathbb{G}, -)$ preserves homotopy kernels. Well-known cohomology groups, and exact sequences relating these groups, in several different contexts are then obtained as examples of this general theory.

1. Introduction

In many algebraic structures such as groups, associative algebras and Lie algebras, the notion of derivation has played an important role in the study of extensions and cohomology [1, 19, 25]. The abelian group $\mathrm{Der}(G, A)$ of derivations (or crossed homomorphisms) from a group G into a G-module A appears in the lowest dimensions of the well-known long exact sequences, in both variables, in group cohomology and so this group is, perhaps up to a shift-dimension, the 0-cohomology group of G with coefficients in A, [1].

Categorical groups are monoidal groupoids in which each object is invertible, up to isomorphism, with respect to the tensor product [3, 21, 28, 29] and this kind of structures have been shown to be relevant in diverse fields such as ring theory, group cohomology and algebraic topology [6, 7, 9, 16, 22, 27, 29, 30, 31]. Categorical groups (sometimes equipped with a braiding or a symmetry) form a 2-category that can be seen as a 2-dimensional analogue of the category of (abelian) groups.

In this paper we are led, from the consideration of (braided or symmetric) categorical groups, to the notion of derivation in this setting. Thus, we introduce the categorical group of derivations $\mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{A})$ as a sort of 0-cohomology for categorical groups which is not in fact a group but a categorical group. The coefficients we take are $\mathbb{G}$-modules, that is, braided categorical groups $(\mathbb{A}, c)$ equipped with a given coherent left action of a categorical group $\mathbb{G}$ and so this cohomology is, in a weak sense, "abelian" cohomology for categorical groups.

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When the categorical group $\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A})$ is suitably specialized and we take π_0 and π_1, then both Eilenberg-Mac Lane's [13] (in low dimensions) and Dedecker's [12] cohomology groups appear as projections, on the category of (abelian) groups, of that categorical group. This kind of results also hold for suitably defined categorical groups of invariant objects and also of derivations module inner derivations (see [17]). Further, they should be valid in next dimensions in the sense that already studied cohomology groups with coefficients in braided or symmetric categorical groups [6, 8, 9, 30] should be, through π_0 and π_1, projections on the category of (abelian) groups of a cohomology that lives at this higher level of categorical groups.

The paper is organized as follows. Section 2 is devoted to fixing notations and recalling the main notions and results we use throughout the paper.

In Section 3 we introduce the categorical group of derivations and make clear the relevance of considering braided (non-symmetric) categorical groups as coefficients. We include several examples, that shall be useful to illustrate the theory developed, showing the relationship between this cohomology theory and many other more classic ones. Then we analyze the group categorical structure of the homomorphisms, in the comma category of categorical groups over a categorical group $\mathbb{G}$, with range the projection $\mathbb{A} \rtimes \mathbb{G} \xrightarrow{\mathbf{pr}} \mathbb{G}$, and we show that there is an isomorphism of categorical groups

$$\mathbb{D}\mathrm{er}_{\mathbf{T}}(\mathbb{H},\mathbb{A}) \cong \mathbb{H}\mathrm{om}_{(\mathcal{CG},\mathbb{G})}\left(\begin{array}{c} \mathbb{H} \\ \downarrow{\scriptstyle\mathbf{T}} \\ \mathbb{G} \end{array}, \begin{array}{c} \mathbb{A} \rtimes \mathbb{G} \\ \downarrow{\scriptstyle\mathbf{pr}} \\ \mathbb{G} \end{array} \right).$$

In particular, we obtain the universal property of the categorical group semidirect product $\mathbb{A} \rtimes \mathbb{G}$ and show that the categorical group $\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A})$ is corepresentable by $\mathbb{A} \rtimes \mathbb{G}$, two results that generalize the corresponding well-known results at the level of groups.

The above isomorphism reveals that the functor $\mathbb{D}\mathrm{er}(\mathbb{G},-)$ must have certain left-exactness properties. The notion of 2-exactness in the setting of categorical groups was introduced in [22, 31], and we show in Corollary 4.5 that there exists a 2-exact sequence

$$\mathbb{D}\mathrm{er}(\mathbb{G},K(\mathbf{T})) \xrightarrow{\mathbf{j}_*} \mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A}) \xrightarrow{\mathbf{T}_*} \mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{B})$$

associated to any homomorphism of $\mathbb{G}$-modules $\mathbf{T} : (\mathbb{A}, c) \to (\mathbb{B}, c)$ with kernel $K(\mathbf{T})$. Associated to such a 2-exact sequence there is then an induced exact sequence connecting the homotopy groups at dimensions 1 and 0 that specializes to well-known exact sequences both in group cohomology and Dedecker's cohomology.

2. Preliminaries

Let us start by recalling that, in a monoidal category $\mathbb{G} = (\mathbb{G}, \otimes, a, I, l, r)$, an object X is termed *invertible* if the functors $Y \mapsto X \otimes Y$ and $Y \mapsto Y \otimes X$ are equivalences (the natural isomorphisms of left and right unit say precisely that the unit object I is an invertible object). A (right) inverse for an invertible object X consists of an object X^* and an isomorphism $\gamma_X : X \otimes X^* \xrightarrow{\sim} I$. It is then possible to choose an isomorphism $\vartheta_X : X^* \otimes X \to I$ in such a way $(X, X^*, \gamma_X, \vartheta_X)$ is a duality in $\mathbb{G}$. In particular there are natural isomorphisms $X \cong (X^*)^*$, $(X \otimes Y)^* \cong Y^* \otimes X^*$ and we choose $I^* = I$.

A *categorical group* (see [2, 3, 21, 23, 24, 28, 29] for general background) is a monoidal groupoid $\mathbb{G} = (\mathbb{G}, \otimes, a, I, l, r)$ where every object is invertible. A categorical group is termed *strict* when the isomorphisms of associativity and left and right unit are identity arrows and the isomorphisms γ_X and ϑ_X can be chosen as identities.

A categorical group $\mathbb{G}$ is said to be a *braided categorical group* if it is also equipped with a family of natural isomorphisms $c = c_{X,Y} : X \otimes Y \to Y \otimes X$ (the braiding) that interacts with a, r and l such that, for any $X, Y, Z \in \mathbb{G}$, the following equalities hold:

$$(1 \otimes c_{X,Z}) \cdot a_{Y,X,Z} \cdot (c_{X,Y} \otimes 1) = a_{Y,Z,X} \cdot c_{X,Y\otimes Z} \cdot a_{X,Y,Z} \ ,$$

$$(c_{Y,X} \otimes 1) \cdot a^{-1}_{Y,X,Z} \cdot (1 \otimes c_{Z,X}) = a^{-1}_{X,Y,Z} \cdot c_{Y\otimes Z,X} \cdot a^{-1}_{Y,Z,X} \ .$$

A braided categorical group $(\mathbb{G}, c)$ is called a *symmetric categorical group* if the condition $c^2 = 1$ is satisfied.

If $\mathbb{G}$ and $\mathbb{H}$ are categorical groups, then a *homomorphism* $\mathbf{T} = (T, \mu) : \mathbb{G} \to \mathbb{H}$ consists of a functor $T : \mathbb{G} \to \mathbb{H}$ and a family of natural isomorphisms

$$\mu = \mu_{X,Y} : T(X \otimes Y) \longrightarrow T(X) \otimes T(Y) \ ,$$

such that the usual coherence condition holds (observe that a canonical isomorphism $\mu_0 : T(I) \longrightarrow I$ can be deduced from the family $\mu_{X,Y}$ and also, for any object $X \in \mathbb{G}$, there exists an isomorphism $\lambda_X : T(X^*) \longrightarrow T(X)^*$). If $\mathbb{G}$ and $\mathbb{H}$ are braided categorical groups, then compatibility of $\mu_{X,Y}$ with the braiding is also required.

Below we will denote by $\mathcal{C}G$ (resp. $\mathcal{B}CG$) the 2-category of (braided) categorical groups where the arrows are the homomorphisms $(T, \mu) : \mathbb{G} \to \mathbb{H}$ and the 2-cells (here called *morphisms*) from (T, μ) to (T', μ') are natural transformation $\epsilon : T \Rightarrow T'$ such that, for any objects $X, Y \in \mathbb{G}$, $(\epsilon_X \otimes \epsilon_Y) \cdot \mu_{X,Y} = \mu'_{X,Y} \cdot \epsilon_{X\otimes Y}$.

Recall that if $\mathbb{G} \in \mathcal{C}G$, then the set of connected components of $\mathbb{G}$, $\pi_0(\mathbb{G})$, has a group structure (which is abelian if $\mathbb{G} \in \mathcal{B}CG$) where the operation is given by $[X]\cdot[Y] = [X\otimes Y]$. Also, the abelian group $\pi_1(\mathbb{G}) = \mathrm{Aut}_{\mathbb{G}}(I)$ is associated to $\mathbb{G}$.

The *kernel* $(K(\mathbf{T}), \mathbf{j}, \epsilon)$ of a homomorphism $\mathbf{T} : \mathbb{G} \to \mathbb{H}$ was defined in [22, 31] and we now recall an explicit description of this universal object. $K(\mathbf{T})$ is the categorical group whose objects are pairs (X, u_X) where $X \in \mathbb{G}$ and $u_X : T(X) \to I$ is an arrow in $\mathbb{H}$; an arrow $f : (X, u_X) \to (Y, u_Y)$ is an arrow $f : X \to Y$ in $\mathbb{G}$ such that $u_X = u_Y \cdot T(f)$; the tensor product is given by $(X, u_X)\otimes(Y, u_Y) = (X\otimes Y, u_X \cdot u_Y)$, where $u_X \cdot u_Y : T(X\otimes Y) \to I$ is the composite $T(X \otimes Y) \xrightarrow{\mu_{X,Y}} T(X) \otimes T(Y) \xrightarrow{u_X\otimes u_Y} I \otimes I \xrightarrow{r_I = l_I} I$; the unit object is (I, μ_0) and the associativity and left-unit and right-unit constraints are given by $a_{X,Y,Z}$, l_X and r_X respectively; an inverse for any object (X, u_X) is given by $(X^*, (u^*_X)^{-1}\lambda_X)$, where X^* is an inverse for X. As for $\mathbf{j} : K(\mathbf{T}) \to \mathbb{G}$, it is the strict homomorphism sending f :

$(X, u_X) \to (Y, u_Y)$ to $f : X \to Y$. Finally, $\epsilon : \mathbf{Tj} \to \mathbf{0}$ is the morphism whose component at (X, u_X) is given by u_X. If $\mathbb{G}$ and $\mathbb{H}$ are braided (symmetric) categorical groups, then $K(\mathbf{T})$ is also a braided (symmetric) categorical group, where the braiding (symmetry) $c = c_{(X,u_X),(Y,u_Y)} : (X, u_X) \otimes (Y, u_Y) \longrightarrow (Y, u_Y) \otimes (X, u_X)$ is exactly $c_{X,Y}$, and $\mathbf{j}$ is a homomorphism of braided (symmetric) categorical groups. The categorical group $K(\mathbf{T})$ just described is a standard homotopy kernel and so it is determined, up to isomorphism, by the following strict universal property: given a homomorphism $\mathbf{F} : \mathbb{K} \to \mathbb{G}$ and a morphism $\tau : \mathbf{TF} \to \mathbf{0}$, there exists a unique homomorphism $\mathbf{F}' : \mathbb{K} \to K(\mathbf{T})$ such that $\mathbf{jF}' = \mathbf{F}$ and $\epsilon\mathbf{F}' = \tau$.

In [22], the following notion of exactness for homomorphisms of categorical groups was introduced. Let $\mathbb{K} \xrightarrow{\mathbf{F}} \mathbb{G} \xrightarrow{\mathbf{T}} \mathbb{H}$ be two homomorphisms and $\tau : \mathbf{TF} \to \mathbf{0}$ a morphism. From the universal property of the kernel of $\mathbf{T}$, there exists a homomorphism $\mathbf{F}'$ making the following diagram commutative:

$$\begin{tikzcd} & K(\mathbf{T}) \arrow[dr, "\mathbf{j}"] & \\ \mathbb{K} \arrow[ur, dashed, "\mathbf{F}'"] \arrow[rr, "\mathbf{F}"'] & & \mathbb{G} \end{tikzcd}$$

which is given, for any $X \in \mathbb{K}$, by $F'(X) = (F(X), \tau_X)$, for any arrow f in $\mathbb{K}$, by $F'(f) = F(f)$ and where, for any $X, Y \in \mathbb{K}$, $(\mu_{F'})_{X,Y} = (\mu_F)_{X,Y}$. Then, the triple $(\mathbf{F}, \tau, \mathbf{T})$ (or sometimes just the sequence $\mathbb{K} \xrightarrow{\mathbf{F}} \mathbb{G} \xrightarrow{\mathbf{T}} \mathbb{H}$ if τ is understood) is said to be *2-exact* if F' is full and essentially surjective. Note that if $(K(\mathbf{T}), \mathbf{j}, \epsilon)$ is the kernel of $\mathbf{T} : \mathbb{G} \to \mathbb{H}$, then the triple $(\mathbf{j}, \epsilon, \mathbf{T})$ is 2-exact and there exists (see [27]) an induced exact sequence of groups

$$\begin{tikzcd} 0 \arrow[r] & \pi_1(K(\mathbf{T})) \arrow[r, "\pi_1(\mathbf{j})"] & \pi_1(\mathbb{G}) \arrow[r, "\pi_1(\mathbf{T})"] & \pi_1(\mathbb{H}) \arrow[dll, "\delta"'] \\ & \pi_0(K(\mathbf{T})) \arrow[r, "\pi_0(\mathbf{j})"] & \pi_0(\mathbb{G}) \arrow[r, "\pi_0(\mathbf{T})"] & \pi_0(\mathbb{H}) \end{tikzcd} \tag{1}$$

where, for any $u \in \pi_1(\mathbb{H})$, $\delta(u)$ is the connected component of the object $(I, u{\cdot}\mu_0) \in K(\mathbf{T})$. Moreover, if the functor T is essentially surjective, the above exact sequence is right exact. In general, if $(\mathbf{F}, \tau, \mathbf{T})$ is a 2-exact sequence of categorical groups, then $\pi_0(\mathbb{K} \xrightarrow{\mathbf{F}} \mathbb{G} \xrightarrow{\mathbf{T}} \mathbb{H})$ and $\pi_1(\mathbb{K} \xrightarrow{\mathbf{F}} \mathbb{G} \xrightarrow{\mathbf{T}} \mathbb{H})$ are exact sequences of groups.

Let $\mathbb{G}$ be a fixed categorical group. A $\mathbb{G}$-*module* (see [9, 17]) is defined as a braided categorical group $(\mathbb{A}, c)$ together with a homomorphism (a $\mathbb{G}$-*action*) $(T, \mu) : \mathbb{G} \to \mathcal{E}q(\mathbb{A}, c)$ from $\mathbb{G}$ to the categorical group $\mathcal{E}q(\mathbb{A}, c)$ of the equivalences of the braided categorical group $(\mathbb{A}, c)$.

It is straightforward to see that giving a $\mathbb{G}$-module $(\mathbb{A}, c)$ is equivalent to giving a functor

$$ac : \mathbb{G} \times \mathbb{A} \longrightarrow \mathbb{A} \quad , \quad (X, A) \mapsto {}^{X}A \ , (f, u) \mapsto {}^{f}u \ ,$$

together with natural isomorphisms

$$\phi = \phi_{X,Y,A} : {}^{(X\otimes Y)}A \to {}^{X}({}^{Y}A) \; ; \; \phi_0 = \phi_{0,A} : {}^{I}A \to A \; ; \; \psi = \psi_{X,A,B} : {}^{X}(A \otimes B) \to {}^{X}A \otimes {}^{X}B \; ,$$

which satisfy certain coherence conditions [9, 14] (note that, for any $X \in \mathbb{G}$, an isomorphism $\psi_0 = \psi_{0,X} : {}^{X}I \longrightarrow I$ can be deduced from the family $\psi_{X,A,B}$).

If $(\mathbb{A}, c)$ and $(\mathbb{B}, c)$ are $\mathbb{G}$-modules, then a homomorphism of $\mathbb{G}$-modules from $(\mathbb{A}, c)$ to $(\mathbb{B}, c)$ is a homomorphism of braided categorical groups $\mathbf{T} = (T, \mu) : (\mathbb{A}, c) \longrightarrow (\mathbb{B}, c)$ that is equivariant in the sense that there exists a family of natural isomorphisms

$$\nu = \nu_{X,A} : T({}^{X}A) \longrightarrow {}^{X}T(A),$$

such that, for any objects $X, Y \in \mathbb{G}$ and $A, B \in \mathbb{A}$, the following conditions hold:

i) $\phi_{X,Y,T(A)} \cdot \nu_{X\otimes Y,A} = {}^{X}\nu_{Y,A} \cdot \nu_{X,{}^{Y}A} \cdot T(\phi_{X,Y,A})$,

ii) $\phi_{0,T(A)} \cdot \nu_{I,A} = T(\phi_{0,A})$,

iii) $(\nu_{X,A} \otimes \nu_{X,B}) \cdot \mu_{{}^{X}A,{}^{X}B} \cdot T(\psi_{X,A,B}) = \psi_{X,T(A),T(B)} \cdot {}^{X}\mu_{A,B} \cdot \nu_{X,A\otimes B}$,

iv) ${}^{X}T(c_{A,B}) \cdot \nu_{X,A\otimes B} = \nu_{X,B\otimes A} \cdot T({}^{X}c_{A,B})$.

Let us observe that, if $(\mathbf{T}, \nu) : (\mathbb{A}, c) \longrightarrow (\mathbb{B}, c)$ is a homomorphism of $\mathbb{G}$-modules and we consider the kernel $(K(\mathbf{T}), \mathbf{j})$ of the underlying homomorphism of braided categorical groups $\mathbf{T}$, then $K(\mathbf{T})$ is also a $\mathbb{G}$-module with action given by ${}^{X}(A, u) = ({}^{X}A, v)$ where $v = \psi_{0,X} \cdot {}^{X}u \cdot \nu_{X,A}$. The homomorphism $\mathbf{j}$ is equivariant, where the isomorphism $j({}^{X}(A, u)) \to {}^{X}j(A, u)$ is given by the identity $id_{{}^{X}A}$.

3. The categorical group of derivations

In the setting of categorical groups we establish the following notion of derivation.

3.1. Definition. *A derivation from a categorical group $\mathbb{G}$ into a $\mathbb{G}$-module $(\mathbb{A}, c)$ is a functor $D : \mathbb{G} \to \mathbb{A}$ together with a family of natural isomorphisms*

$$\beta = \beta_{X,Y} : D(X \otimes Y) \to D(X) \otimes {}^{X}D(Y), \quad X, Y \in \mathbb{G},$$

such that, for any objects $X, Y, Z \in \mathbb{G}$, the following coherence condition holds:

$$\begin{gathered}(1 \otimes \psi_{X,D(Y),{}^{Y}D(Z)}) \cdot (1 \otimes {}^{X}\beta_{Y,Z}) \cdot \beta_{X,Y\otimes Z} \cdot D(a_{X,Y,Z}) = \\ (1 \otimes (1 \otimes \phi_{X,Y,D(Z)})) \cdot a_{D(X),{}^{X}D(Y),{}^{X\otimes Y}D(Z)} \cdot (\beta_{X,Y} \otimes 1) \cdot \beta_{X\otimes Y,Z} \; .\end{gathered} \tag{2}$$

If (D, β) is a derivation, there exists an isomorphism $\bar\beta_0 : D(I) \to I$ determined uniquely by the two following equalities:

$$D(r_X) = r_{D(X)} \cdot (1 \otimes \psi_{0,X}) \cdot (1 \otimes {}^{X}\bar\beta_0) \cdot \beta_{X,I} \quad ; \quad D(l_X) = l_{D(X)} \cdot (1 \otimes \phi_{0,D(X)}) \cdot (\bar\beta_0 \otimes 1) \cdot \beta_{I,X} \; .$$

The derivation (D, β) is *normalized* if the isomorphism $\bar\beta_0$ is an identity.

Given two derivations $(D,\beta),(D',\beta'):\mathbb{G}\to\mathbb{A}$, a *morphism* from (D,β) to (D',β') consists of a natural transformation $\epsilon : D \Rightarrow D'$ such that, for any objects $X,Y\in\mathbb{G}$, the following condition holds:

$$(\epsilon_X \otimes {}^X\epsilon_Y)\cdot\beta_{X,Y} = \beta'_{X,Y}\cdot\epsilon_{X\otimes Y}\ . \tag{3}$$

The vertical composition of natural transformations determines a composition for morphisms between derivations so that we can consider the category $\mathrm{Der}(\mathbb{G},\mathbb{A})$ of derivations from $\mathbb{G}$ into $(\mathbb{A},c)$, which is actually a groupoid. It is a groupoid pointed by the trivial derivation (D_0,β_0), where $D_0:\mathbb{G}\to\mathbb{A}$ is the constant functor with value the unit object $I\in\mathbb{A}$ and, for any $X,Y\in\mathbb{G}$, $(\beta_0)_{X,Y} = (I \xrightarrow{r_I^{-1}} I\otimes I \xrightarrow{1\otimes\psi_{0,X}^{-1}} I\otimes {}^XI)$.

Even more, we have:

3.2. Theorem. *Let $\mathbb{G}$ be a categorical group and $(\mathbb{A},c)$ a $\mathbb{G}$-module. There is a tensor functor*

$$\otimes : \mathrm{Der}(\mathbb{G},\mathbb{A})\times\mathrm{Der}(\mathbb{G},\mathbb{A}) \longrightarrow \mathrm{Der}(\mathbb{G},\mathbb{A})$$

such that the groupoid $\mathrm{Der}(\mathbb{G},\mathbb{A})$ *becomes a categorical group,* $(\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A}),\otimes)$*, called the categorical group of derivations*

Proof. Given objects $(D,\beta),(D',\beta')\in\mathrm{Der}(\mathbb{G},\mathbb{A})$, we define

$$(D,\beta)\otimes(D',\beta') = (D\otimes D',\beta\otimes\beta')\,,$$

where $D\otimes D':\mathbb{G}\to\mathbb{A}$ is the functor given, for any $X\in\mathbb{G}$, by $(D\otimes D')(X) = D(X)\otimes D'(X)$ and, for any arrow f, by $(D\otimes D')(f) = D(f)\otimes D'(f)$, whereas $\beta\otimes\beta' = (\beta\otimes\beta')_{X,Y} : (D\otimes D')(X\otimes Y) \longrightarrow (D\otimes D')(X)\otimes {}^X(D\otimes D')(Y)$ is the family of natural isomorphisms determined, for each pair of objects $X,Y\in\mathbb{G}$, by the following equality:

$$(1\otimes\psi_{X,D(Y),D'(Y)})\cdot(\beta\otimes\beta')_{X,Y} = (1\otimes c_{X_{D(Y),D'(X)}}\otimes 1)\cdot(\beta_{X,Y}\otimes\beta'_{X,Y})\ .$$

To check that $\beta\otimes\beta'$ satisfies (2) follows, assuming that all canonical isomorphisms are identities, of the commutativity of the diagram

$$\begin{tikzcd}[column sep=small]
{}^XDY\otimes {}^{X\otimes Y}DZ\otimes D'X\otimes {}^XD'Y \arrow[rr, "1\otimes c_{{}^{X\otimes Y}DZ,\,D'X\otimes {}^XD'Y}"] \arrow[dr, "1\otimes c_{{}^{X\otimes Y}DZ,\,D'X}\otimes 1"] \arrow[dd, "c_{{}^XDY\otimes {}^{X\otimes Y}DZ,\,D'X}\otimes 1"'] \arrow[rr, phantom, "(I)" yshift=-30pt] & & {}^XDY\otimes D'X\otimes {}^XD'Y\otimes {}^{X\otimes Y}DZ \arrow[dd, "c_{{}^XDY,\,D'X}\otimes 1"] \\
\arrow[r, phantom, "(II)"] & {}^XDY\otimes D'X\otimes {}^{X\otimes Y}DZ\otimes {}^XD'Y \arrow[ur, "1\otimes c_{{}^{X\otimes Y}DZ,\,{}^XD'Y}"'] \arrow[dl, "c_{{}^XDY,\,D'X}\otimes 1"'] \arrow[r, phantom, "(III)"] & \\
D'X\otimes {}^XDY\otimes {}^{X\otimes Y}DZ\otimes {}^XD'Y \arrow[rr, "1\otimes c_{{}^{X\otimes Y}DZ,\,{}^XD'Y}"'] & & D'X\otimes {}^XDY\otimes {}^XD'Y\otimes {}^{X\otimes Y}DZ\ ,
\end{tikzcd}$$

which actually is commutative since regions (I) and (II) are commutative due to the coherence conditions of the braiding, and region (III) so is because the naturality of the isomorphisms $c_{X,Y}$. Thus $(D\otimes D',\beta\otimes\beta')$ is a derivation from $\mathbb{G}$ into $(\mathbb{A},c)$.

On the other hand, given morphisms between derivations, $\epsilon:(D,\beta)\Rightarrow(F,\alpha)$ and $\epsilon':(D',\beta')\Rightarrow(F',\alpha')$, we define $\epsilon\otimes\epsilon':(D\otimes D',\beta\otimes\beta')\Rightarrow(F\otimes F',\alpha\otimes\alpha')$ by $(\epsilon\otimes\epsilon')_X=\epsilon_X\otimes\epsilon'_X:D(X)\otimes D'(X)\to F(X)\otimes F'(X)$. Both the naturality of ϵ and ϵ' and the functoriality of $\otimes$ in $\mathbb{A}$ imply that $\epsilon\otimes\epsilon'$ is natural and to check that the corresponding equality (3) for $\epsilon\otimes\epsilon'$ holds is routine.

The above data define a categorical group

$$\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A})=(\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A}),\otimes,\bar{a},\bar{I},\bar{l},\bar{r}),$$

where $\bar{a}:((D,\beta)\otimes(D',\beta'))\otimes(D'',\beta'')\Rightarrow(D,\beta)\otimes((D',\beta')\otimes(D'',\beta''))$ is the morphism determined by the natural transformation $\bar{a}:(D\otimes D')\otimes D''\Rightarrow D\otimes(D'\otimes D'')$ given by $\bar{a}_X=a_{D(X),D'(X),D''(X)}$, $X\in\mathbb{G}$; the unit object $\bar{I}$ is the trivial derivation; the left unit constraint $\bar{l}=\bar{l}_{(D,\beta)}:(D_0,\beta_0)\otimes(D,\beta)\Rightarrow(D,\beta)$ is the morphism determined by the natural transformation $\bar{l}:D_0\otimes D\Rightarrow D$ given by $\bar{l}_X=l_{D(X)}$; the right unit constraint $\bar{r}=\bar{r}_{(D,\beta)}:(D,\beta)\otimes(D_0,\beta_0)\Rightarrow(D,\beta)$ is the morphism determined by the natural transformation $\bar{r}:D\otimes D_0\Rightarrow D$ given by $\bar{r}_X=r_{D(X)}$.

An inverse for any object $(D,\beta)\in\mathrm{Der}(\mathbb{G},\mathbb{A})$ is given by (D^*,α), where $D^*(X)=D(X)^*$ and $\alpha_{X,Y}:D(X\otimes Y)^*\to D(X)^*\otimes{}^{X}D(Y)^*$ is given by $\beta^*_{X,Y}$ and canonical isomorphisms.

∎

3.3. Remark. If $(\mathbb{A},c)$ is a symmetric categorical group, then $\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A})$ is a symmetric categorical group where the symmetry

$$\bar{c}=\bar{c}_{(D,\beta),(D',\beta')}:(D,\beta)\otimes(D',\beta')\Rightarrow(D',\beta')\otimes(D,\beta)$$

is the morphism determined by the natural transformation $\bar{c}:D\otimes D'\Rightarrow D'\otimes D$ whose component at X is $\bar{c}_X=c_{D(X),D'(X)}:D(X)\otimes D'(X)\to D'(X)\otimes D(X)$. However, if c is just a braiding in $\mathbb{A}$ (but not a symmetry), then $\bar{c}$ is not a braiding in $\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A})$ because, in such a case, $\bar{c}=\bar{c}_{(D,\beta),(D',\beta')}$ is not a morphism between derivations since (3) does not hold.

The above remark suggest that, if $(\mathbb{A},c)$ is just a braided (non-symmetric) categorical group, then the groupoid $\mathrm{Der}(\mathbb{G},\mathbb{A})$ has another different monoidal structure of that defined in Theorem 3.2. More precisely, we have:

3.4. Theorem. *Let $\mathbb{G}$ be a categorical group and $(\mathbb{A},c)$ a $\mathbb{G}$-module. Then the functor*

$$\begin{aligned}\mathrm{Der}(\mathbb{G},\mathbb{A})\times\mathrm{Der}(\mathbb{G},\mathbb{A})&\xrightarrow{\;\bar{\otimes}\;}\mathrm{Der}(\mathbb{G},\mathbb{A})\\ \big((D,\beta),(D',\beta')\big)&\longmapsto(D\bar{\otimes}D',\beta\bar{\otimes}\beta')\end{aligned}$$

where $(D\bar{\otimes}D')(X) = D'(X)\otimes D(X)$, $X\in\mathbb{G}$, *and*

$$(\beta\bar{\otimes}\beta')_{X,Y} : (D\bar{\otimes}D')(X\otimes Y) \to (D\bar{\otimes}D')(X)\otimes {}^{X}(D\bar{\otimes}D')(Y),$$

is the family of natural isomorphisms given, for any $X, Y \in \mathbb{G}$, *by*

$$(\beta\bar{\otimes}\beta')_{X,Y} = (1\otimes\psi^{-1}_{X,D'(Y),D(Y)})\cdot(1\otimes c^{-1}_{D(X),{}^{X}D'(Y)}\otimes 1)\cdot(\beta'_{X,Y}\otimes\beta_{X,Y})\ , \tag{4}$$

defines a monoidal structure such that $\mathrm{Der}(\mathbb{G},\mathbb{A})$ *becomes a categorical group denoted by* $(\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A}),\bar{\otimes})$. *Moreover, the family of natural transformations*

$$\bar{c}_{D,D'} : D\otimes D' \Rightarrow D\bar{\otimes}D'$$

given, for any $X\in\mathbb{G}$, *by*

$$(\bar{c}_{D,D'})_X = c_{D(X),D'(X)} : D(X)\otimes D'(X) \to D'(X)\otimes D(X),$$

is such that

$$(id, \bar{c}_{D,D'}) : (\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A}),\otimes) \to (\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A}),\bar{\otimes})$$

is a monoidal isomorphism.

PROOF. The proof that $(\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A}),\bar{\otimes})$ is a categorical group is similar to the one written down in Theorem 3.2.

Now, for any $(D,\beta),(D',\beta')\in\mathrm{Der}(\mathbb{G},\mathbb{A})$, $\bar{c}_{D,D'} : D\otimes D' \Rightarrow D\bar{\otimes}D'$ is actually a morphism between derivations since condition (3) holds. In fact, the following diagram is commutative

$$\begin{tikzcd}
D(X\otimes Y)\otimes D'(X\otimes Y) \arrow[rr, "c_{D(X\otimes Y),D'(X\otimes Y)}"] \arrow[d, "\beta_{X,Y}\otimes\beta'_{X,Y}"'] \arrow[drr, phantom, "(I)"] & & D'(X\otimes Y)\otimes D(X\otimes Y) \arrow[d, "\beta'_{X,Y}\otimes\beta_{X,Y}"] \\
DX\otimes {}^{X}DY\otimes D'X\otimes {}^{X}D'Y \arrow[rr, "c_{DX\otimes {}^{X}DY,D'X\otimes {}^{X}D'Y}"] \arrow[dr, "c_{DX\otimes {}^{X}DY,D'X}\otimes 1"] \arrow[dd, "1\otimes c_{{}^{X}DY,D'X}"'] \arrow[drr, phantom, "(III)"] \arrow[ddr, phantom, "(II)"] & & D'X\otimes {}^{X}D'Y\otimes DX\otimes {}^{X}DY \\
& D'X\otimes DX\otimes {}^{X}DY\otimes {}^{X}D'Y \arrow[ur, "1\otimes c_{DX\otimes {}^{X}DY,{}^{X}D'Y}"] \arrow[dr, "1\otimes c_{{}^{X}DY,{}^{X}D'Y}"] \arrow[d, phantom, "(IV)"] & \\
DX\otimes D'X\otimes {}^{X}DY\otimes {}^{X}D'Y \arrow[ur, "c_{DX,D'X}\otimes 1"] \arrow[rr, "c_{DX\otimes D'X}\otimes c_{{}^{X}DY\otimes {}^{X}D'Y}"'] \arrow[d, "1\otimes\psi^{-1}_{X,DY,D'Y}"'] \arrow[drr, phantom, "(V)"] & & D'X\otimes DX\otimes {}^{X}D'Y\otimes {}^{X}DY \arrow[uu, "1\otimes c_{DX,{}^{X}D'Y}"'] \arrow[d, "1\otimes\psi^{-1}_{X,D'Y,DY}"] \arrow[uul, phantom, "(II)"] \\
DX\otimes D'X\otimes {}^{X}(DY\otimes D'Y) \arrow[rr, "c_{DX\otimes D'X}\otimes {}^{X}(c_{DY,D'Y})"'] & & D'X\otimes DX\otimes {}^{X}(D'Y\otimes DY),
\end{tikzcd}$$

because regions (I) and (IV) are clearly commutative and regions (II), (III) and (V) are commutative due to the coherence conditions of the braiding and the compatibility condition between the braiding and the canonical isomorphism $\psi = \psi_{X,A,B}$ of the action.

Moreover, $(id, \bar{c})$ is a monoidal isomorphism between $(\mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{A}), \otimes)$ and $(\mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{A}), \bar{\otimes})$ because the required coherence condition, expressed in the following diagram, holds. In fact, the diagram

$$\begin{tikzcd}
D(X)\otimes D'(X)\otimes D''(X) \arrow[rrrr, "c_{D(X)\otimes D'(X),D''(X)}"] \arrow[ddddd, "c_{D(X),D'(X)\otimes D''(X)}"'] \arrow[ddrr, "1\otimes c_{D'(X),D''(X)}"'] & & & & D''(X)\otimes D(X)\otimes D'(X) \arrow[ddddd, "1\otimes c_{D(X),D'(X)}"] \\
& & (I) & & \\
& & D(X)\otimes D''(X)\otimes D'(X) \arrow[uurr, "c_{D(X),D''(X)}\otimes 1"] \arrow[dddrr, "c_{D(X),D''(X)\otimes D'(X)}"] & (II) & \\
& (III) & & & \\
& & & & \\
D'(X)\otimes D''(X)\otimes D(X) \arrow[rrrr, "c_{D'(X),D''(X)}\otimes 1"'] & & & & D''(X)\otimes D'(X)\otimes D(X)
\end{tikzcd} ,$$

is commutative due to naturality (region (III)) and the coherence conditions of the braiding (regions (I) and (II)). ∎

3.5. Example.

i) *Groups as categorical groups: Derivations.*

If G is a group, the discrete category it defines, denoted by $G[0]$, is a strict categorical group where the tensor product is given by the group operation. When A is an abelian group, $A[0]$ is braided (even symmetric) due to the commutativity of A. It is easy to check that $\mathcal{E}q(A[0]) = \mathrm{Aut}(A)[0]$ and a $G[0]$-action on $A[0]$ is actually a G-action, in the usual sense, on A, that is, a G-module structure on A. In such a case $\mathbb{D}\mathrm{er}(G[0], A[0]) = \mathrm{Der}(G, A)[0]$, since an object of $\mathbb{D}\mathrm{er}(G[0], A[0])$ only consists of a map $d : G \to A$ such that, for any $x, y \in G$, $d(xy) = d(x)\,{}^{x}d(y)$ and all morphisms are identities.

If A is an abelian group, the category with only one object it defines, denoted by $A[1]$, is also a strict braided (even symmetric) categorical group where both the composition and the tensor product are given by the group operation.

The categorical group $\mathcal{E}q(A[1])$ can be described as follows: The objects are pairs $(a, f) \in A \times \mathrm{Aut}(A)$; the set of arrows from (a, f) to (b, g) is empty if $f \neq g$ and the one point set, $\{(a, b, f)\}$, if $f = g$; composition is given by $(a, b, f) \cdot (b, c, f) = (a, c, f)$; the tensor product on objects is $(a, f) \otimes (b, g) = (g(a) + b, gf)$, while on morphisms it is $(a, b, f) \otimes (c, d, g) = (g(a) + c, g(b) + d, gf)$.

If G is a group, it is not difficult to see that a $G[0]$-action on $A[1]$ is the same thing as a G-action on A together with an Eilenberg-Mac Lane 1-cochain of G with coefficients in the G-module A. Thus, we can consider the categorical group $\mathbb{D}\mathrm{er}(G[0], A[1])$. An object of this category consists of a map $\beta : G \times G \to A$, $(x, y) \mapsto \beta_{x,y}$, such that, for any $x, y, z \in G$, ${}^{x}\beta_{y,z} + \beta_{x,yz} = \beta_{xy,z} + \beta_{x,y}$, that is, an Eilenberg-Mac Lane 2-cocycle of G with coefficients in A. This 2-cocycle is normalized (i.e., $\beta_{x,1} = 0 = \beta_{1,x}$) if the derivation is normalized. A morphism between two objects β and β' consists of a map $d : G \to A$ such that, for any $x, y \in G$, $\beta'_{x,y} + d(xy) = d(x) + {}^{x}d(y) + \beta_{x,y}$, that is, a 1-cochain showing that β and β' are cohomologous. Thus, $\pi_0(\mathbb{D}\mathrm{er}(G[0], A[1])) = H^2(G, A)$ the 2-nd Eilenberg-Mac lane cohomology group of G with coefficients in A. On the other hand, since the unit object is the trivial 2-cocycle, $\pi_1(\mathbb{D}\mathrm{er}(G[0], A[1])) = \mathrm{Der}(G, A)$.

ii) *Categorical groups associated to crossed modules: Derivations.*

Recall that a crossed module of groups is a system $\mathcal{L} = (H, G, \varphi, \delta)$, where $\delta : H \to G$ is a group homomorphism and $\varphi : G \to \mathrm{Aut}(H)$ is an action (so that H is a G-group) for which the following conditions are satisfied:

$$\delta({}^{x}h) = x\delta(h)x^{-1} \quad , \quad {}^{\delta(h)}h' = hh'h^{-1} \ .$$

It is well-known that crossed modules of groups correspond to strict categorical groups. Given a crossed module $\mathcal{L}$, the corresponding strict categorical group $\mathbb{G}(\mathcal{L})$ can be described as follows: the objects are the elements of the group G; an arrow $h : x \to y$ is an element $h \in H$ with $x = \delta(h)y$; the composition is multiplication in H; the tensor product is given by

$$(x \xrightarrow{h} y) \otimes (x' \xrightarrow{h'} y') = (xx' \xrightarrow{h^{y}h'} yy') \ .$$

A crossed module $\mathcal{L}$ together with a map $\{-,-\} : G \times G \to H$ satisfying certain equalities (see [11]) is called a reduced 2-crossed module. Reduced 2-crossed modules $(\mathcal{L}, \{-,-\})$ (also called braided crossed modules in [4]) correspond to strict braided categorical groups $\mathbb{G}(\mathcal{L}, \{-,-\}) = (\mathbb{G}(\mathcal{L}), c)$ where the braiding $c = c_{x,y} : xy \to yx$ is given by $c_{x,y} = \{x, y\}$ (see [5, 15]).

If A is an abelian group, $A[1]$ is the strict braided categorical group associated to the braided crossed module $A \xrightarrow{0} 0$ (this is actually a stable crossed module [11] with the identity as symmetry). More generally, any homomorphism of abelian groups $\delta : A \to B$, together with the trivial action of B on A and the zero homomorphism $0 : B \times B \to A$, is a stable crossed module. The associated symmetric strict categorical group is precisely the kernel of the homomorphism of symmetric categorical groups $A[1] \to B[1]$. Note that, in this case, sequence (1) specializes to the exact sequence $0 \to \mathrm{Ker}(\delta) \to A \xrightarrow{\delta} B \to \mathrm{Coker}(\delta) \to 0$.

The actor crossed module, $\mathrm{Act}(\mathcal{L})$, of a crossed module $\mathcal{L} = (H, G, \varphi, \delta)$ was introduced in [26] and was shown to be the analogue of the automorphism group of a group. This crossed module $\mathrm{Act}(\mathcal{L})$ is precisely (see [3]) the crossed module associated to the categorical group $\mathcal{A}\mathrm{ut}(\mathbb{G}(\mathcal{L}))$ (where $\mathcal{A}\mathrm{ut}(\mathbb{A}, c)$ stands for the categorical subgroup of $\mathcal{E}q(\mathbb{A}, c)$ whose objects, called automorphisms, are the equivalences (T, μ) that are strict and where T is an isomorphism). It consists of the group morphism $\Delta : D(G, H) \to \mathrm{Aut}(\mathcal{L})$ [18], where $D(G, H)$ is the Whitehead group of regular derivations (that is, the units of the monoid $\mathrm{Der}(G, H)$), $\mathrm{Aut}(\mathcal{L})$ is the group of automorphisms of $\mathcal{L}$ (that is, pairs of group automorphisms $\phi_0 \in \mathrm{Aut}(G)$ and $\phi_1 \in \mathrm{Aut}(H)$ such that $\phi_0 \cdot \delta = \delta \cdot \phi_1$ and $\phi_1({}^{x}h) = {}^{\phi_0(x)}\phi_1(h))$, Δ is given by $\Delta(d) = (\sigma_d, \theta_d)$, where $\sigma_d(x) = \delta(d(x))x$ and $\theta_d(h) = d(\delta(h))h$, and the action of $\mathrm{Aut}(\mathcal{L})$ on $D(G, H)$ is given by ${}^{(\phi_0,\phi_1)}d = \phi_1 \cdot d \cdot \phi_0^{-1}$.

Now, if $(\mathcal{L}, \{-,-\})$ is a braided crossed module, we can consider the crossed module $\mathrm{Act}(\mathcal{L}, \{-,-\})$ defined, as above, by the group homomorphism $\Delta : D(G,H) \to \mathrm{Aut}(\mathcal{L}, \{-,-\})$, where this last group is the group of automorphisms of $(\mathcal{L}, \{-,-\})$, that is, automorphisms (ϕ_0, ϕ_1) of $\mathcal{L}$ such that $\phi_1(\{x,y\}) = \{\phi_0(x), \phi_0(y)\}$. Note that, for any $d \in D(G,H)$, $\theta_d(\{x,y\}) = \{\sigma_d(x), \sigma_d(y)\}$ and so $\Delta(d) \in \mathrm{Aut}(\mathcal{L}, \{-,-\})$. Thus, $\mathrm{Act}(\mathcal{L}, \{-,-\})$ is precisely the crossed module associated to the categorical group $\mathcal{A}\mathrm{ut}(\mathbb{G}(\mathcal{L}, \{-,-\}))$.

Using the actor of a crossed module, in [26] the notion of action of a crossed module $\mathcal{L}$ on another one $\mathcal{A}$ is defined as a morphism of crossed modules $\mathcal{L} \to \mathrm{Act}(\mathcal{A})$. In the same way, if $(\mathcal{A}, \{-,-\})$ is a braided crossed module, we can consider the crossed module $\mathrm{Act}(\mathcal{A}, \{-,-\})$ and define the notion of action of $\mathcal{L}$ on $(\mathcal{A}, \{-,-\})$. Now, since $\mathrm{Act}(\mathcal{A}, \{-,-\})$ is precisely the crossed module associated to the categorical group $\mathcal{A}\mathrm{ut}(\mathbb{G}(\mathcal{A}, \{-,-\}))$, then, by considering the inclusion $\mathcal{A}\mathrm{ut}(\mathbb{G}(\mathcal{A}, \{-,-\})) \hookrightarrow \mathcal{E}q(\mathbb{G}(\mathcal{A}, \{-,-\}))$, any action of a crossed module $\mathcal{L}$ on a braided crossed module $(\mathcal{A}, \{-,-\})$ determines an action of $\mathbb{G}(\mathcal{L})$ on the braided categorical group $\mathbb{G}(\mathcal{A}, \{-,-\}) = (\mathbb{G}(\mathcal{A}), c)$.

Thus, in such a case of having an action of a crossed module of groups $\mathcal{L} = (H \xrightarrow{\delta} G)$ on a braided crossed module of groups $\mathcal{A} = (L \xrightarrow{\rho} M, \{-,-\})$, we can consider the categorical group $\mathbb{D}\mathrm{er}(\mathcal{L}, \mathcal{A}) = \mathbb{D}\mathrm{er}(\mathbb{G}(\mathcal{L}), \mathbb{G}(\mathcal{A}))$ whose objects are triplets of maps $d : G \to M$, $f : H \to L$ and $l : G^2 \to L$, $(x,y) \mapsto l_{x,y}$, such that: i) $d(xy) = \rho(l_{x,y}) d(x)\, {}^{x}d(y)$; ii) f is a group morphism; iii) $l_{x,yz}\, {}^{d(x)}({}^{x}l_{y,z}) = l_{xy,z} l_{x,y}$; iv) $\rho(f(h)) d(x) = d(\delta(h)x)$. Note that this last condition implies that $\rho \cdot f = d \cdot \delta$. A morphism from (d,f,l) to (d',f',l') is a map $\epsilon : G \to L$, $x \mapsto \epsilon_x$, satisfying that: i) $d(x) = \rho(\epsilon_x)\, d'(x)$; ii) $l_{x,y}\, \epsilon_x\, {}^{d'(x)}({}^{x}\epsilon_y) = \epsilon_{xy}\, l'_{x,y}$; iii) $f(h)\epsilon_x = \epsilon_{\delta(h)x} f'(h)$.

In the particular case that $\mathcal{L} = (0 \xrightarrow{0} G)$ and the action is the trivial one, an object of $\mathbb{D}\mathrm{er}(\mathcal{L}, \mathcal{A})$ is precisely a Dedecker 2-cocycle [12], that is normalized if the derivation is also, and a morphism is precisely an equivalence [12] between the corresponding Dedecker 2-cocycles. Thus, in this case, $\pi_0(\mathbb{D}\mathrm{er}(\mathcal{L}, \mathcal{A})) = H^2(G, (\mathcal{A}, \{-,-\}))$ the 2-th non-abelian cohomology set (group in this case)[5] of G with coefficients in the reduced 2-crossed module $(\mathcal{A}, \{-,-\})$. The product in this group is given by $[(d,l)][(d',l')] = [(d,l) \otimes (d',l')] = [(d \otimes d', \bar{l})]$, where $(d \otimes d')(x) = d(x) d'(x)$ and $\bar{l}_{x,y} = l_{x,y}\, {}^{d(x)d(y)}l'_{x,y}\, {}^{d(x)}\{d(y), d'(x)\}$ (c.f. [5, Propositions 2.1 and 2.4]).

Moreover, when $\mathcal{L} = (0 \xrightarrow{0} G)$ then, for any action of $\mathcal{L}$ on $\mathcal{A}$, $\pi_1(\mathbb{D}\mathrm{er}(\mathcal{L}, \mathcal{A})) = \mathrm{Der}(G, \mathrm{Ker}(\rho))$ (note that G acts on L and then on $\mathrm{Ker}(\rho)$, so that $\mathrm{Ker}(\rho)$ is a G-module).

iii) *Semidirect product of categorical groups: Relative derivations.*

Let $\mathbb{G}$ be a categorical group and $(\mathbb{A}, c)$ a $\mathbb{G}$-module. The *semidirect product* of $(\mathbb{A}, c)$ by $\mathbb{G}$ (c.f. [14]) is the categorical group, denoted by $\mathbb{A} \rtimes \mathbb{G}$, whose underlying groupoid is the cartesian product $\mathbb{A} \times \mathbb{G}$. The tensor product is given by

$$(A, X) \otimes (B, Y) = (A \otimes {}^{X}B, X \otimes Y) \quad , \quad (u, f) \otimes (v, g) = (u \otimes {}^{f}v, f \otimes g).$$

The unit object is the pair (I, I) and the associativity and left and right unit constraints are obtained from those of $\mathbb{A}$ and $\mathbb{G}$ and the canonical morphisms defining the action.

Note that if G is a group and A a G-module, then $A[0] \rtimes G[0] = (A \rtimes G)[0]$, whereas $A[1] \rtimes G[0] = \mathbb{G}(\mathcal{L})$, where $\mathcal{L}$ is the crossed module given by the zero homomorphism $A \stackrel{0}{\to} G$ and the action of G on A.

If $(\mathbb{A}, c)$ is a $\mathbb{G}$-module, it is clear that, for any homomorphism $\mathbf{T} = (T, \mu) : \mathbb{H} \to \mathbb{G}$, $(\mathbb{A}, c)$ is an $\mathbb{H}$-module via the composite $\mathbb{H} \xrightarrow{\mathbf{T}} \mathbb{G} \xrightarrow{ac} \mathcal{E}q(\mathbb{A}, c)$ and we will denote by $\mathbb{D}\mathrm{er}_{\mathbf{T}}(\mathbb{H}, \mathbb{A})$ the corresponding categorical group of derivations. In particular, by considering the semidirect product $\mathbb{A} \rtimes \mathbb{G}$ and the canonical projection $\mathbf{pr} : \mathbb{A} \rtimes \mathbb{G} \to \mathbb{G}$, the other projection $\partial : \mathbb{A} \rtimes \mathbb{G} \longrightarrow \mathbb{A}$, $(A, X) \mapsto A$, defines a derivation $(\partial, id) \in \mathbb{D}\mathrm{er}_{\mathbf{pr}}(\mathbb{A} \rtimes \mathbb{G}, \mathbb{A})$.

iv) *Derivations into the Picard categorical group of a commutative ring.*

If S is a commutative ring, there is a symmetric categorical group (c.f. [30, 10]) $\mathcal{P}\mathrm{ic}(S) = (\mathcal{P}\mathrm{ic}(S), \otimes_S, a, S, l, r)$ where $\mathcal{P}\mathrm{ic}(S)$ is the category of invertible S-modules (i.e., finitely generated projective S-modules of constant rank 1), $\otimes_S$ is the tensor product of S-modules, the unit object is the S-module S, the associativity and unit constraints are the usual ones for the tensor product of modules,

$$(P \otimes_S Q) \otimes_S T \xrightarrow[\sim]{a_{P,Q,T}} P \otimes_S (Q \otimes_S T) \ , \ P \otimes_S S \xrightarrow[\sim]{r_P} P \xleftarrow[\sim]{l_P} S \otimes_S P \ ,$$

an inverse for any object P is given by the dual module $P^* = \mathrm{Hom}_S(P, S)$ and the symmetry $c_{P,Q} : P \otimes_S Q \to Q \otimes_S P$ is the usual isomorphism. Note that $\pi_0(\mathcal{P}\mathrm{ic}(S)) = \mathrm{Pic}(S)$, the Picard group of S, and for any invertible S-module P, $\mathrm{Aut}_{\mathcal{P}\mathrm{ic}(S)}(P) \cong U(S)$ the group of units of S. In particular, $\pi_1(\mathcal{P}\mathrm{ic}(S)) \cong U(S)$.

Let G be a group operating on a commutative ring S by ring automorphisms. Then, $\mathcal{P}\mathrm{ic}(S)$ is a $G[0]$-module with action $G[0] \times \mathcal{P}\mathrm{ic}(S) \longrightarrow \mathcal{P}\mathrm{ic}(S)$, $(\sigma, P) \mapsto {}^{\sigma}P$, where ${}^{\sigma}P$ is the same abelian group as P with action from S by $s \cdot x = \sigma^{-1}(s)x$, $s \in S$, $x \in P$. Note that this action is strict in the sense that the natural isomorphisms $\phi_{\sigma,\tau,P} : {}^{\sigma\tau}P \to {}^{\sigma}({}^{\tau}P)$, $\phi_{0,P} : {}^{1}P \to P$ and $\psi_{\sigma,P,Q} : {}^{\sigma}(P \otimes_S Q) \to {}^{\sigma}P \otimes_S {}^{\sigma}Q$ are identities.

Thus, in the above conditions, $\mathcal{P}\mathrm{ic}(S)$ is a $G[0]$-module and we can consider the (symmetric) categorical group $\mathbb{D}\mathrm{er}(G[0], \mathcal{P}\mathrm{ic}(S))$. An object of this categorical group consists of a family of invertible S-modules $\{P_\sigma\}_{\sigma \in G}$ together with S-isomorphisms $\beta_{\sigma,\tau} : P_{\sigma\tau} \longrightarrow P_\sigma \otimes {}^{\sigma}P_\tau$ such that, for any $\sigma, \tau, \gamma \in G$, $(1 \otimes {}^{\sigma}\beta_{\tau,\gamma}) \cdot \beta_{\sigma,\tau\gamma} = (\beta_{\sigma,\tau} \otimes 1) \cdot \beta_{\sigma\tau,\gamma}$. Given objects $(P_\sigma, \beta_{\sigma,\tau})$ and $(Q_\sigma, \alpha_{\sigma,\tau})$, a morphism between them consists of a family of S-isomorphisms $\epsilon_\sigma : P_\sigma \to Q_\sigma$ such that $(\epsilon_\sigma \otimes {}^{\sigma}\epsilon_\tau) \cdot \beta_{\sigma,\tau} = \alpha_{\sigma,\tau} \cdot \epsilon_{\sigma\tau}$. In the case where G is finite and S a Galois extension of $R = {}^{G}S$, there is a surjective homomorphism $\pi_0(\mathbb{D}\mathrm{er}(G[0], \mathcal{P}\mathrm{ic}(S))) \to \mathrm{Br}(S/R)$, the Brauer group of S/R-Azumaya algebras and, in fact,

$$\pi_0(\mathbb{D}\mathrm{er}(G[0], \mathcal{P}\mathrm{ic}(S))) = Z^2(S, G) \cong \widehat{\mathrm{Br}}(S/R)$$

where the latter denotes the abelian group of S-isomorphism classes of S/R-Azumaya algebras (c.f. [20]).

4. Derivations and semidirect product. Exactness.

Let $\mathbb{G}$ be a categorical group and let us consider the comma 2-category, $(\mathcal{C}G, \mathbb{G})$, of categorical groups over $\mathbb{G}$, whose objects are the homomorphisms of categorical groups $(T, \mu) : \mathbb{H} \to \mathbb{G}$ and whose arrows are the commutative triangles of homomorphisms

$$\begin{tikzcd} \mathbb{H} \arrow[rr, "\mathbf{F}=(F,\eta)"] \arrow[dr, "\mathbf{T}=(T,\mu)"'] & & \mathbb{H}' \arrow[dl, "\mathbf{T}'=(T',\mu')"] \\ & \mathbb{G}\,. & \end{tikzcd}$$

A 2-cell from $\mathbf{F_1}$ to $\mathbf{F_2}$ is a morphism $\epsilon : \mathbf{F_1} \Rightarrow \mathbf{F_2}$ between these homomorphisms such that $T'(\epsilon_H) = id_{T(H)}$ for any $H \in \mathbb{H}$. Then, for any $\mathbb{G}$-module $\mathbb{A}$, we have a category, $\mathbb{H}\mathrm{om}_{(\mathcal{C}G,\mathbb{G})}\left(\begin{array}{cc} \mathbb{H} & \mathbb{A}\rtimes\mathbb{G} \\ \downarrow\mathbf{T}\,, & \downarrow\mathbf{pr} \\ \mathbb{G} & \mathbb{G} \end{array}\right)$, whose set of objects is $\mathrm{Hom}_{(\mathcal{C}G,\mathbb{G})}\left(\begin{array}{cc} \mathbb{H} & \mathbb{A}\rtimes\mathbb{G} \\ \downarrow\mathbf{T}\,, & \downarrow\mathbf{pr} \\ \mathbb{G} & \mathbb{G} \end{array}\right)$ and whose arrows, from $\mathbf{F_1}$ to $\mathbf{F_2}$, are the 2-cells between them. This category is actually a groupoid and, even more, we have:

4.1. Proposition. *There is a tensor functor*

$$\mathbb{H}\mathrm{om}_{(\mathcal{CG},\mathbb{G})}\left(\begin{array}{cc} \mathbb{H} & \mathbb{A}\rtimes\mathbb{G} \\ \downarrow\mathbf{T}, & \downarrow\mathbf{pr} \\ \mathbb{G} & \mathbb{G} \end{array}\right) \times \mathbb{H}\mathrm{om}_{(\mathcal{CG},\mathbb{G})}\left(\begin{array}{cc} \mathbb{H} & \mathbb{A}\rtimes\mathbb{G} \\ \downarrow\mathbf{T}, & \downarrow\mathbf{pr} \\ \mathbb{G} & \mathbb{G} \end{array}\right) \xrightarrow{\otimes} \mathbb{H}\mathrm{om}_{(\mathcal{CG},\mathbb{G})}\left(\begin{array}{cc} \mathbb{H} & \mathbb{A}\rtimes\mathbb{G} \\ \downarrow\mathbf{T}, & \downarrow\mathbf{pr} \\ \mathbb{G} & \mathbb{G} \end{array}\right)$$

such that the groupoid $\mathbb{H}\mathrm{om}_{(\mathcal{C}G,\mathbb{G})}\left(\begin{array}{cc} \mathbb{H} & \mathbb{A}\rtimes\mathbb{G} \\ \downarrow\mathbf{T}\,, & \downarrow\mathbf{pr} \\ \mathbb{G} & \mathbb{G} \end{array}\right)$ *becomes a categorical group.*

Proof. If $(F_1, \mu_1), (F_2, \mu_2) \in \mathbb{H}\mathrm{om}_{(\mathcal{C}G,\mathbb{G})}\left(\begin{array}{cc} \mathbb{H} & \mathbb{A}\rtimes\mathbb{G} \\ \downarrow\mathbf{T}\,, & \downarrow\mathbf{pr} \\ \mathbb{G} & \mathbb{G} \end{array}\right)$ we define $(F_1, \mu_1) \otimes (F_2, \mu_2) = (F_1 \otimes F_2, \bar{\mu})$, where $F_1 \otimes F_2 : \mathbb{H} \to \mathbb{A} \rtimes \mathbb{G}$ is the functor given by $(F_1 \otimes F_2)(H) = (\partial F_1(H) \otimes \partial F_2(H), T(H))$ (see Example 3.5 iii)) and, for any $H, G \in \mathbb{H}$, $\bar{\mu} = \bar{\mu}_{H,G} : (F_1 \otimes F_2)(H \otimes G) \longrightarrow (F_1 \otimes F_2)(H) \otimes (F_1 \otimes F_2)(G)$ is the family of natural isomorphisms such that $\partial(\bar{\mu}_{H,G}) = (1 \otimes \psi^{-1}_{T(H),\partial F_1(G),\partial F_2(G)}) \cdot (1 \otimes c_{H_{(\partial F_1(G)),\partial F_2(H)}} \otimes 1) \cdot (\partial(\mu_{1_{H,G}}) \otimes \partial(\mu_{2_{H,G}}))$ and $pr(\bar{\mu}_{H,G}) = \mu_{H,G}$. To check that $\bar{\mu}$ satisfies the required coherence condition is straightforward.

On the other hand, given morphisms $\epsilon : (F_1, \mu_1) \Rightarrow (F_2, \mu_2)$ and $\epsilon' : (F_1', \mu_1') \Rightarrow (F_2', \mu_2')$, $\epsilon \otimes \epsilon' : (F_1 \otimes F_1', \bar{\mu}_1) \Rightarrow (F_2 \otimes F_2', \bar{\mu}_2)$ is given by $(\epsilon \otimes \epsilon')_H = (\partial(\epsilon_H) \otimes \partial(\epsilon'_H), id_{T(H)}) : (\partial F_1(H) \otimes \partial F_1'(H), T(H)) \to (\partial F_2(H) \otimes \partial F_2'(H), T(H))$. It is routine to check that $\epsilon \otimes \epsilon'$ is natural and satisfies the appropriate coherence condition.

The above data define a categorical group

$$\mathbb{H}\mathrm{om}_{(\mathcal{C}G,\mathbb{G})}\left(\begin{array}{cc}\mathbb{H} & \mathbb{A}\rtimes\mathbb{G}\\ \Big\downarrow\mathbf{T}\ , & \Big\downarrow\mathbf{pr}\\ \mathbb{G} & \mathbb{G}\end{array}\right)=\left(\mathbb{H}\mathrm{om}_{(\mathcal{C}G,\mathbb{G})}\left(\begin{array}{cc}\mathbb{H} & \mathbb{A}\rtimes\mathbb{G}\\ \Big\downarrow\mathbf{T}\ , & \Big\downarrow\mathbf{pr}\\ \mathbb{G} & \mathbb{G}\end{array}\right),\otimes,\bar{a},\bar{I},\bar{l},\bar{r}\right)$$

where $\bar{a}:((F_1,\mu_1)\otimes(F_2,\mu_2))\otimes(F_3,\mu_3)\Rightarrow(F_1,\mu_1)\otimes((F_2,\mu_2)\otimes(F_3,\mu_3))$ is the morphism determined by the natural transformation given by $\bar{a}_H=a_{\partial F_1(H),\partial F_2(H),\partial F_3(H)}$, $H\in\mathbb{H}$; the unit object $\bar{I}$ is the homomorphism $\mathbf{F}_0=(F_0,\eta_0):\mathbb{H}\to\mathbb{A}\rtimes\mathbb{G}$, where $F_0(H)=(I,T(H))$ and $(\eta_0)_{H,G}:(I,T(H\otimes G))\to(I\otimes{}^{H}I,T(H)\otimes T(G))$ is given by $(\eta_0)_{H,G}=((1\otimes\psi_{0,H}^{-1})\cdot r_I^{-1},\mu_{H,G})$; the left-unit constraint $\bar{l}=\bar{l}_{\mathbf{F}}:\mathbf{F}_0\otimes\mathbf{F}\Rightarrow\mathbf{F}$ is the morphism determined by the natural transformation $\bar{l}:F_0\otimes F\Rightarrow F$ given by $\bar{l}_H=(l_{\partial F(H)},id_{T(H)})$; the right-unit constraint $\bar{r}=\bar{r}_{\mathbf{F}}:\mathbf{F}\otimes\mathbf{F}_0\Rightarrow\mathbf{F}$ is the morphism determined by the natural transformation $\bar{r}:F\otimes F_0\Rightarrow F$ given by $\bar{r}_H=(r_{\partial F(H)},id_{T(H)})$.

An inverse for an object (F_1,μ_1) is given by (F_1^*,η) where, for any $H\in\mathbb{H}$, $F_1^*(H)=((\partial F_1(H))^*,T(H))$ and $\eta_{H,G}:F_1^*(H\otimes G)\to F_1^*(H)\otimes F_1^*(G)$ is given by $\mu_{1_{H,G}}$ and canonical isomorphisms. ■

The above construction gives a contravariant functor

$$\mathbb{H}\mathrm{om}_{(\mathcal{C}G,\mathbb{G})}\left(\begin{array}{cc} & \mathbb{A}\rtimes\mathbb{G}\\ -\ , & \Big\downarrow\mathbf{pr}\\ & \mathbb{G}\end{array}\right):(\mathcal{C}G,\mathbb{G})\to\mathcal{C}G\,.$$

We also have, using Example 3.5 iii), another contravariant functor

$$\mathbb{D}\mathrm{er}(-,\mathbb{A}):(\mathcal{C}G,\mathbb{G})\longrightarrow\mathcal{C}G$$

that is given, on objects, by $\mathbb{D}\mathrm{er}(\mathbf{T},\mathbb{A})=\mathbb{D}\mathrm{er}_{\mathbf{T}}(\mathbb{H},\mathbb{A})$. For any arrow $\mathbf{F}=(F,\eta):\mathbf{T}\to\mathbf{T}'$, $\mathbb{D}\mathrm{er}(\mathbf{F},\mathbb{A}):\mathbb{D}\mathrm{er}_{\mathbf{T}'}(\mathbb{H}',\mathbb{A})\longrightarrow\mathbb{D}\mathrm{er}_{\mathbf{T}}(\mathbb{H},\mathbb{A})$ is the homomorphism of categorical groups $(\widehat{T},\widehat{\mu})$, where $\widehat{T}((D,\beta))=(DF,\widehat{\beta})$ with $\widehat{\beta}_{X,Y}=\beta_{F(X),F(Y)}\cdot D(\eta_{X,Y})$ and $\widehat{\mu}=\widehat{\mu}_{(D,\beta),(D',\beta')}$ is the morphism determined by the identity $id:(D\otimes D')F\Rightarrow DF\otimes D'F$.

The following theorem shows that these two functors are naturally isomorphic.

4.2. Theorem. *For any $\mathbb{G}$-module $(\mathbb{A},c)$ and any object $\mathbb{H}\xrightarrow{\mathbf{T}=(T,\mu)}\mathbb{G}\in(\mathcal{C}G,\mathbb{G})$, there exists an isomorphism of categorical groups*

$$\mathbb{D}\mathrm{er}_{\mathbf{T}}(\mathbb{H},\mathbb{A})\cong\mathbb{H}\mathrm{om}_{(\mathcal{C}G,\mathbb{G})}\left(\begin{array}{cc}\mathbb{H} & \mathbb{A}\rtimes\mathbb{G}\\ \Big\downarrow\mathbf{T}\ , & \Big\downarrow\mathbf{pr}\\ \mathbb{G} & \mathbb{G}\end{array}\right)$$

given by $(D,\beta)\mapsto(F,\eta)$, where $F(H)=(D(H),T(H))$ and, for any $H,G\in\mathbb{H}$, $\eta_{H,G}=(\beta_{H,G},\mu_{H,G})$.

Proof. We shall define homomorphisms of categorical groups

$$\mathbb{D}\mathrm{er}_{\mathbf{T}}(\mathbb{H},\mathbb{A}) \xrightarrow{(\kappa,\theta)} \mathbb{H}\mathrm{om}_{(\mathcal{CG},\mathbb{G})}\begin{pmatrix} \mathbb{H} & \mathbb{A}\rtimes\mathbb{G} \\ \downarrow\mathbf{T}\,, & \downarrow\mathbf{pr} \\ \mathbb{G} & \mathbb{G} \end{pmatrix} \;,\quad \mathbb{H}\mathrm{om}_{(\mathcal{CG},\mathbb{G})}\begin{pmatrix} \mathbb{H} & \mathbb{A}\rtimes\mathbb{G} \\ \downarrow\mathbf{T}\,, & \downarrow\mathbf{pr} \\ \mathbb{G} & \mathbb{G} \end{pmatrix} \xrightarrow{(\Sigma,\zeta)} \mathbb{D}\mathrm{er}_{\mathbf{T}}(\mathbb{H},\mathbb{A})$$

such that $(\kappa,\theta)\cdot(\Sigma,\zeta) = id$ and $(\Sigma,\zeta)\cdot(\kappa,\theta) = id$.

For any object $(D,\beta) \in \mathbb{D}\mathrm{er}_{\mathbf{T}}(\mathbb{H},\mathbb{A})$ we define $\kappa((D,\beta)) = (F,\eta)$, where $F(H) = (D(H),T(H))$, $F(f) = (D(f),T(f))$ and $\eta_{H,G} : F(H\otimes G) \to F(H)\otimes F(G)$ is given by $\eta_{H,G} = (\beta_{H,G},\mu_{H,G})$. For any morphism $\epsilon : (D,\beta) \Rightarrow (D',\beta')$, $\kappa(\epsilon) : (F,\eta) \Rightarrow (F',\eta')$ is the morphism determined by the natural transformation given by $\kappa(\epsilon)_H = (\epsilon_H, id_{T(H)})$. Moreover, since $\kappa((D,\beta)\otimes(D',\beta')) = \kappa((D,\beta))\otimes\kappa((D',\beta'))$, we take $\theta_{(D,\beta),(D',\beta')} = id$, that is, $(\theta_{(D,\beta),(D',\beta')})_H = (id_{D(H)\otimes D'(H)}, id_{T(H)})$.

Now, for any object $(F,\eta) \in \mathbb{H}\mathrm{om}_{(\mathcal{CG},\mathbb{G})}\begin{pmatrix} \mathbb{H} & \mathbb{A}\rtimes\mathbb{G} \\ \downarrow\mathbf{T}\,, & \downarrow\mathbf{pr} \\ \mathbb{G} & \mathbb{G} \end{pmatrix}$, we define $\Sigma((F,\eta)) = (D,\beta)$, where $D(H) = \partial F(H)$, $D(f) = \partial F(f)$ and $\beta_{H,G} : D(H\otimes G) \to D(H)\otimes {}^{H}D(G)$ is given by $\beta_{H,G} = \partial(\eta_{H,G})$. For any morphism $\tau : (F,\eta) \Rightarrow (F',\eta')$, $\Sigma(\tau) : (D,\beta) \Rightarrow (D',\beta')$ is the morphism determined by the natural transformation given by $\Sigma(\tau)_H = \partial(\tau_H)$. Moreover, it is easy to see that $\Sigma((F,\eta)\otimes(F',\eta')) = \Sigma((F,\eta)\otimes\Sigma((F',\eta'))$ and so we take $\zeta_{(F,\eta),(F',\eta')} = id$, that is, $(\zeta_{(F,\eta),(F',\eta')})_H = id_{\partial F(H)\otimes\partial F'(H)}$.

Finally it is straightforward to check that (κ,θ) and (Σ,ζ) are inverse to each other. ∎

4.3. Corollary. (Universal property of the semidirect product) *Let $\mathbb{G}$ be a categorical group and let $(\mathbb{A},c)$ be a $\mathbb{G}$-module. Then, for any homomorphism $\mathbf{T} : \mathbb{H} \to \mathbb{G}$ and any $\mathbf{T}$-derivation $\mathbf{D} : \mathbb{H} \to \mathbb{A}$, there exists an unique homomorphism, up to isomorphism, $\mathbf{F} : \mathbb{H} \longrightarrow \mathbb{A}\rtimes\mathbb{G}$, such that the following diagram is commutative:*

$$\begin{tikzcd} & \mathbb{H} \arrow[dl, "\mathbf{D}"'] \arrow[d, "\mathbf{F}"] \arrow[dr, "\mathbf{T}"] & \\ \mathbb{A} & \mathbb{A}\rtimes\mathbb{G} \arrow[l, "\partial"'] \arrow[r, "\mathbf{pr}"] & \mathbb{G} . \end{tikzcd}$$

By taking $\mathbb{H} = \mathbb{G}$ and $\mathbf{T} = id_{\mathbb{G}}$, we obtain:

4.4. Corollary. *For any $\mathbb{G}$-module $(\mathbb{A},c)$, there exists an isomorphism of categorical groups*

$$\mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A}) \cong \mathbb{H}\mathrm{om}_{(\mathcal{CG},\mathbb{G})}\begin{pmatrix} \mathbb{G} & \mathbb{A}\rtimes\mathbb{G} \\ \Vert\;, & \downarrow\mathbf{pr} \\ \mathbb{G} & \mathbb{G} \end{pmatrix} .$$

If $(\mathbf{T} = (T,\mu),\delta) : (\mathbb{A},c) \longrightarrow (\mathbb{B},c)$ is a homomorphism of $\mathbb{G}$-modules, there is an induced homomorphism of categorical groups

$$\mathbf{T}_* = (T_*,\bar{\mu}) : \mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{A}) \longrightarrow \mathbb{D}\mathrm{er}(\mathbb{G},\mathbb{B})$$

where $T_*(D,\beta) = (TD, \bar{\beta})$ with $\bar{\beta}_{X,Y} = (1 \otimes \delta_{X,D(Y)}) \cdot \mu_{D(X),\, {}^{X}D(Y)} \cdot T(\beta_{X,Y})$ and $T_*(\epsilon) = T\epsilon$, for any morphism $\epsilon : (D,\beta) \Rightarrow (D',\beta')$. On the other hand,

$$\bar{\mu} = \bar{\mu}_{(D,\beta),(D',\beta')} : T_*((D,\beta) \otimes (D',\beta')) \Rightarrow T_*(D,\beta) \otimes T_*(D',\beta')$$

is the morphism determined by the natural transformation $T(D \otimes D') \Rightarrow TD \otimes TD'$ with component at X given by $\mu_{D(X),D'(X)}$.

Recalling that the kernel of a homomorphism of $\mathbb{G}$-modules is also a $\mathbb{G}$-module and using the isomorphism in the above Corollary we have (c.f. [27, Proposition 3.1.2]):

4.5. Corollary. *Let $\mathbb{G}$ be a categorical group and $(\mathbf{T},\delta) : (\mathbb{A}, c) \to (\mathbb{B}, c)$ a homomorphism of $\mathbb{G}$-modules with kernel $(K(\mathbf{T}), \mathbf{j}, \epsilon)$. Then, the categorical group $\mathbb{D}\mathrm{er}(\mathbb{G}, K(\mathbf{T}))$ is isomorphic to the kernel of the induced homomorphism*

$$\mathbf{T}_* = (T_*, \bar{\mu}) : \mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{A}) \longrightarrow \mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{B}).$$

In particular, the sequence

$$\mathbb{D}\mathrm{er}(\mathbb{G}, K(\mathbf{T})) \xrightarrow{\mathbf{j}_*} \mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{A}) \xrightarrow{\mathbf{T}_*} \mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{B})$$

is 2-exact and there is an induced exact sequence of groups

$$\begin{array}{l} 0 \longrightarrow \pi_1(\mathbb{D}\mathrm{er}(\mathbb{G}, K(\mathbf{T}))) \xrightarrow{\pi_1(\mathbf{j})_*} \pi_1(\mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{A})) \xrightarrow{\pi_1(\mathbf{T}_*)} \pi_1(\mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{B})) \xrightarrow{\delta} \\ \pi_0(\mathbb{D}\mathrm{er}(\mathbb{G}, K(\mathbf{T}))) \xrightarrow{\pi_0(\mathbf{j}_*)} \pi_0(\mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{A})) \xrightarrow{\pi_0(\mathbf{T}_*)} \pi_0(\mathbb{D}\mathrm{er}(\mathbb{G}, \mathbb{B})). \end{array}$$

4.6. Example. As an example of the sequences in the above corollary, let us consider

$$\phi = (\phi_1, \phi_0) : \mathcal{A} = (L \xrightarrow{\rho} M, \{-,-\}) \longrightarrow \mathcal{B} = (L'' \xrightarrow{\rho''} M'', \{-,-\}),$$

a surjective morphism of reduced 2-crossed modules of groups (i.e., a morphism ϕ with $\phi_0 : M \twoheadrightarrow M''$ and $\phi_1 : L \twoheadrightarrow L''$ epimorphisms). Let G be a group and suppose that the crossed module $\mathcal{L} = (0 \to G)$ acts on $(\mathcal{A}, \{-,-\})$ and $(\mathcal{B}, \{-,-\})$ (see Example 3.5 ii)) in such a way that ϕ preserves the action. If $L' = \mathrm{Ker}(\phi_1)$ and $M' = \mathrm{Ker}(\phi_0)$, let $\mathcal{F} = (L' \xrightarrow{\rho'} M', \{-,-\})$ be the 2-reduced crossed module fiber of ϕ (where M' acts on L' by restriction of the action of M on L and $\{-,-\} : M' \times M' \to L'$ is also induced by restriction). Then $\mathcal{L}$ also acts on $(\mathcal{F}, \{-,-\})$ and it is easy to check that $\mathbb{G}(\mathcal{F}, \{-,-\})$ is equivalent to the kernel of the induced homomorphism of $(\mathbb{G}(\mathcal{L}) = G[0])$-modules $\mathbb{G}(\mathcal{A}, \{-,-\}) \longrightarrow \mathbb{G}(\mathcal{B}, \{-,-\})$. Thus, according to Corollary 4.5, there is a 2-exact sequence

$$\mathbb{D}\mathrm{er}(G[0], \mathbb{G}(\mathcal{F}, \{-,-\})) \longrightarrow \mathbb{D}\mathrm{er}(G[0], \mathbb{G}(\mathcal{A}, \{-,-\})) \longrightarrow \mathbb{D}\mathrm{er}(G[0], \mathbb{G}(\mathcal{B}, \{-,-\}))$$

and therefore, when the action is the trivial one, there is a group exact sequence (see Example 3.5 ii))

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Hom}(G,\mathrm{Ker}(\rho')) \arrow[r] & \mathrm{Hom}(G,\mathrm{Ker}(\rho)) \arrow[r] & \mathrm{Hom}(G,\mathrm{Ker}(\rho'')) \arrow[dll] \\ & H^2(G,(\mathcal{F},\{-,-\})) \arrow[r] & H^2(G,(\mathcal{A},\{-,-\})) \arrow[r] & H^2(G,(\mathcal{B},\{-,-\})). \end{tikzcd}$$

In the particular case that $\mathcal{A} = (A \to 0)$ and $\mathcal{B} = (A'' \to 0)$, where both A and A'' are G-modules and $\phi : A \twoheadrightarrow A''$ is an epimorphism of G-modules, the fiber crossed module is $\mathcal{F} = (A' \to 0)$, where $A' = \mathrm{Ker}(\phi)$. Then, the above sequence specializes to the well-known exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Der}(G,A') \arrow[r] & \mathrm{Der}(G,A) \arrow[r] & \mathrm{Der}(G,A'') \arrow[dll] \\ & H^2(G,A') \arrow[r] & H^2(G,A) \arrow[r] & H^2(G,A''). \end{tikzcd}$$

Note that in this last case of $\mathcal{A}$ and $\mathcal{B}$ being crossed modules associated to G-modules A and A'', but $\phi : A \to A''$ being any morphism of G-modules, the kernel $K(\phi)$ of the induced homomorphism of categorical groups $\phi : A[1] \to A''[1]$ is (see Example 3.5 ii)) the strict symmetric categorical group associated to the crossed module defined by ϕ and the trivial action of A'' on A. Then the 2-exact sequence

$$\mathbb{D}\mathrm{er}(G[0],K(\phi)) \longrightarrow \mathbb{D}\mathrm{er}(G[0],A[1]) \longrightarrow \mathbb{D}\mathrm{er}(G[0],A''[1])$$

induces the exact sequence of groups

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Der}(G,A') \arrow[r] & \mathrm{Der}(G,A) \arrow[r] & \mathrm{Der}(G,A'') \arrow[dll] \\ & H^2(E(\phi_*)) \arrow[r] & H^2(G,A) \arrow[r] & H^2(G,A'') \end{tikzcd}$$

since it is plain to see that $\pi_1(\mathbb{D}\mathrm{er}(G[0],K(\phi))) = \mathrm{Der}(G,A')$ and $\pi_0(\mathbb{D}\mathrm{er}(G[0],K(\phi))) = H^2(E(\phi_*))$, the 2^{nd} cohomology group of the mapping cone $E(\phi_*)$ (see [25]) of the cochain transformation $\phi_* : \mathrm{Hom}_G(B_\bullet, A) \to \mathrm{Hom}_G(B_\bullet, A'')$, where $B_\bullet$ is a free resolution of the trivial G-module $\mathbb{Z}$.

References

[1] M. Barr, J. Beck, Seminar on triples and categorical homology theory, Springer Lecture Notes in Math. 80, 1969.

[2] F. Borceux, Handbook of categorical algebra 1,2, Cambridge University Press, 1994.

[3] L. Breen, Théorie de Schreier supérieure, Ann. Scient. Éc. Norm. Sup. 4^e série, 25 (1992), 465-514.

[4] R. Brown, N.D. Gilbert, Algebraic models of 3-types and automorphisms structure for crossed modules, Proc. London Math. Soc. 59(3), (1989) 51-73.

[5] M. Bullejos, A.M. Cegarra, A 3-dimensional non-abelian cohomology with applications to homotopy classification of continuous maps, Canadian Journal of Mathematics 43(2) (1991), 265-296.

[6] M. Bullejos, P. Carrasco, A.M. Cegarra, Cohomology with coefficients in symmetric cat-groups. An extension of Eilenberg-MacLane's classification theorem, Math. Proc. Camb. Phil. Soc. 114 (1993), 163-189.

[7] P. Carrasco, A.M. Cegarra, (Braided) tensor structures on homotopy groupoids and nerves of (braided) categorical groups, Communications in Algebra 24(13) (1996), 3995-4058.

[8] P. Carrasco, A.M. Cegarra, Schreier theory for central extensions of categorical groups, Communications in Algebra 24(13) (1996), 4059-4112.

[9] P. Carrasco, A.R. Garzón, J.G. Miranda, Schreier theory for singular extensions of categorical groups and homotopy classification, Communications in Algebra 28(5) (2000), 2585-2613.

[10] A.M. Cegarra, L. Fernández, Cohomology of cofibred categorical groups, J. Pure Appl. Algebra 143 (1999), 107-154.

[11] D. Conduché, Modules croisés généralisés de longeur 2, J. Pure Appl. Algebra 34 (1984), 155-178.

[12] P. Dedecker, Cohomologie non-abélienne, Séminaire de l'Institut Mathématique Lille, 1965.

[13] S. Eilenberg, S. Mac Lane, Cohomology theory in Abstract group I, Ann. of Math. 48 (1947), 51-78.

[14] A.R. Garzón, H. Inassaridze, Semidirect products of categorical groups. Obstruction theory, Homology, Homotopy and its applications 3 (6) (2001), 111-138.

[15] A.R. Garzón, J.G. Miranda, Homotopy theory for (braided) cat-groups, Cahiers Top. et Géom. Diff. Cat. XXXVIII-2 (1997), 99-139.

[16] A.R. Garzón, J.G. Miranda, A. del Río, Tensor structures on homotopy groupoids of topological spaces, Intern. Math. Journal 2(5) (2002), 407-431.

[17] A.R. Garzón, A. del Río, Low-dimensional cohomology for categorical groups, Cahiers Top. et Géom. Diff. Cat. XLIV-4 (2003), 247-280.

[18] N.D. Gilbert, Derivations, automorphisms and crossed modules, Communications in Algebra 18(8) (1990), 2703-2734.

[19] K.W. Gruenberg, Cohomological topics in group theory. Springer Lecture Notes in Math. 143, 1970.

[20] A. Hattory, On the groups $H^n(S, G)$ and the Brauer group of commutative rings, Sci. Papers College Gen. Ed. Univ. Tokyo 28 (1978), 1-20.

[21] A. Joyal, R. Street, Braided tensor categories, Advances in Math. 82(1) (1991), 20-78.

[22] S. Kasangian, E.M. Vitale, Factorization systems for symmetric cat-groups, Theory and Applications of Categories 7(5) (2000), 47-70.

[23] G.M. Kelly, On Mac Lane's conditions for coherence of natural associativities, commutativities, etc. J. of Algebra 1 (1964), 397-402.

[24] S. Mac Lane, Natural associativity and commutativity, Rice University Studies 49 (1963), 28-46.

[25] S. Mac Lane, Homology, Springer 1963.

[26] K.J. Norrie, Actions and automorphisms of crossed modules, Bull. Soc. Math. France 118 (1990), 129-146.

[27] A. Rousseau, Bicatégories monoidales et extensions de gr-catégories, Homology, Homotopy and its applications 5 (1) (2003), 437-547.

[28] N. Saavedra, Catégories tannakiennes, Springer Lecture Notes in Math. 265, 1972.

[29] H.X. Sinh, Gr-catégories, Université Paris VII, Thèse de doctorat, 1975.

[30] K. H. Ulbrich, Group Cohomology for Picard Categories, J. of Algebra 91 (1984), 464-498.

[31] E.M. Vitale, A Picard-Brauer exact sequence of categorical groups, J. Pure Applied Algebra 175 (2002), 383-408.

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