Theory and Applications of Categories, Vol. 13, No. 7, 2004, pp. 114–128.

NORMED COMBINATORIAL HOMOLOGY AND NONCOMMUTATIVE TORI

Dedicated to Aurelio Carboni on the occasion of his sixtieth birthday

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Abstract. Cubical sets have a *directed homology,* studied in a previous paper and consisting of *preordered abelian groups,* with a positive cone generated by the structural cubes. By this additional information, cubical sets can provide a sort of 'noncommutative topology', agreeing with some results of noncommutative geometry but lacking the metric aspects of C^*-algebras. Here, we make such similarity stricter by introducing *normed* cubical sets and their *normed* directed homology, formed of normed preordered abelian groups. The normed cubical sets $\mathsf{N}C_\vartheta$ associated with 'irrational' rotations have thus the same classification up to isomorphism as the well-known irrational rotation C^*-algebras A_ϑ.

Introduction

This is a sequel of a previous work [7], referred to as Part I, which introduced a *directed* homology of cubical sets, consisting of *preordered* abelian groups. Some of its main results are linked with noncommutative geometry; here, we strengthen such links, by enriching cubical sets and their homology groups with a *norm.* (Subsection 1.2 of Part I will be cited as I.1.2.)

First, let us note that this homology norm can distinguish between metrically-different realisations of the same homotopy type, the one of the circle. Thus, applying the normed directed 1-homology group $\mathsf{N}{\uparrow}H_1$ to the standard normed directed circle $\mathsf{N}{\uparrow}\mathbf{S}^1$, where the length of a homology generator is 2π, we get $2\pi.{\uparrow}\mathbf{Z}$ as a *normed ordered subgroup* of the line. Similarly, the normed directed 1-torus $\mathsf{N}{\uparrow}\mathbf{T} = {\uparrow}\mathbf{R}/\mathbf{Z}$ gives the group of integers ${\uparrow}\mathbf{Z}$ with natural norm and order, since now the length of a homology generator is 1. Finally, the (naturally normed) singular cubical set of the punctured plane $\mathbf{R}^2 \setminus \{0\}$ assigns to the group $\mathbf{Z}$ the coarse preorder and the *zero* (semi)norm, making manifest the existence of (reversible) 1-cycles of arbitrarily small length (3.5).

These rather obvious aspects become of interest in a well-known situation where an ordinary topological approach fails. Let us recall that the group $G_\vartheta = \mathbf{Z}+\vartheta\mathbf{Z}$ (ϑ irrational)

Work supported by MIUR Research Projects.

Received by the editors 2003-07-18.

Published on 2004-12-05.

2000 Mathematics Subject Classification: 55U10, 81R60, 55Nxx.

Key words and phrases: Cubical sets, noncommutative C*-algebras, combinatorial homology, normed abelian groups.

is dense in the real line, and the orbit space $\mathbf{R}/G_\vartheta$ has a trivial topology, the coarse one. This trivial quotient, corresponding to the set of leaves of an irrational Kronecker foliation of the 2-torus, has been interpreted as a 'noncommutative space', the irrational rotation C^*-algebra A_ϑ, also called a noncommutative torus [4, 5, 11, 1]. As proved in [10, 11], K-theory gives precise classifications of these algebras, depending on ϑ: first, *up to strong Morita equivalence*, ϑ is determined up to the action of the group $\mathrm{PGL}(2, \mathbf{Z})$ (cf. 1.4); second, *up to isomorphism*, $A_\vartheta \cong A_\zeta$ if and only if $G_\vartheta = G_\zeta$ (as subsets of $\mathbf{R}$), if and only if $\zeta \in \pm\vartheta + \mathbf{Z}$ (cf. 1.4).

In Part I, we showed how that trivial quotient can be replaced with a naturally occurring cubical set, $C_\vartheta = {\uparrow}\mathbf{R}/G_\vartheta$, the quotient of the cubical set ${\uparrow}\mathbf{R}$ whose n-cubes are the continuous *order-preserving* mappings $\mathbf{I}^n \to \mathbf{R}$, under the action of the group G_ϑ. In fact, the *directed* 1-homology group ${\uparrow}H_1(C_\vartheta)$ is isomorphic to ${\uparrow}G_\vartheta$ as an ordered subgroup of $\mathbf{R}$ (Thm. I.4.8); it follows (Thm. I.4.9) that the classification of the cubical sets C_ϑ up to isomorphism coincides with that of the algebras A_ϑ up to strong Morita equivalence, recalled above. A comparison with the stricter classification of the latter *up to isomorphism* suggests that cubical sets provide a sort of 'noncommutative topology', without the metric character of noncommutative geometry.

Here, to account for this character, we enrich C_ϑ with a natural *normed* structure $\mathsf{N}C_\vartheta$, essentially produced by the length of (increasing) paths $\mathbf{I} \to \mathbf{R}$ (1.4). Now, *normed* directed 1-homology gives $\mathsf{N}{\uparrow}H_1(\mathsf{N}C_\vartheta) \cong {\uparrow}G_\vartheta$ as a normed ordered subgroup of $\mathbf{R}$ (Thm. 4.1). It follows easily that the normed cubical sets $\mathsf{N}C_\vartheta$ have precisely the same classification up to isomorphism as the C^*-algebras A_ϑ (Thm. 4.2).

We end this introduction with some technical remarks. Norms for sets (2.1), cubical sets (1.1) and abelian groups (3.1) will take values in $[0, \infty]$, so that these categories have all products (and some useful left adjoints); morphisms in these categories are always assumed to be (weakly) contracting, so that isomorphisms are isometrical. Moreover, in an abelian group, $\|x\| = 0$ will not imply $x = 0$: this assumption would annihilate useful information, as for the punctured plane recalled above.

Preorder of homology groups does not play a relevant role here, since the metric information is sufficient for our main goals; however, preorder is an independent aspect, which distinguishes - for instance - between $\mathsf{N}{\uparrow}\mathbf{S}^1$ and its non-directed version $\mathsf{N}\square\mathbf{S}^1$ (1.5, 3.5); this might be of use in other less obvious cases. It is also interesting to note that, in the present proofs, the arguments concerning norms are similar to the ones concerning preorders in Part I, if more complicated; this is likely related with the fact that preorder is a simplified, two-valued generalised metric (1.5).

References and motivation for directed algebraic topology can be found in [6]; for cubical sets in [7], including why we prefer to work with them rather than with simplicial sets.

Finally, some comments on notation will be of help. Structures enriched with some sort of direction, including preorder, are usually denoted by the prefix 'd' or $\uparrow$. Structures enriched with a norm may be distinguished by 'N'. We always omit the latter for normed abelian groups: thus, the group $\mathbf{Z}$ has the natural norm, inherited from the reals, whenever the context requires a norm and we are not specifying a different one; similarly for $\lambda\mathbf{Z}$ ($\lambda > 0$), G_ϑ and the (naturally) ordered groups $\lambda{\uparrow}\mathbf{Z}$, ${\uparrow}G_\vartheta$.

1. A general overview

This is an outline of the main topics and results of the paper. The index α takes values 0, 1, also written $-, +$ (e.g. in superscripts).

1.1. Normed cubical sets. The theory of *normed* cubical sets will be briefly examined in Section 2, and their normed directed homology introduced in Section 3; for the moment, it will be sufficient to have a few definitions.

First, let us recall that a *cubical set* $X = ((X_n), (\partial_i^\alpha), (e_i))$ is a sequence of sets X_n ($n \geq 0$), together with mappings, called *faces* (∂_i^α) and *degeneracies* (e_i)

$$\partial_i^\alpha = \partial_{n,i}^\alpha \colon X_n \to X_{n-1}, \qquad e_i = e_{n,i} \colon X_{n-1} \to X_n \qquad (\alpha = \pm;\ i = 1, \ldots, n), \qquad (1)$$

satisfying the cubical relations

$$\begin{aligned} &\partial_i^\alpha.\partial_j^\beta = \partial_j^\beta.\partial_{i+1}^\alpha \quad (j \leq i), && e_j.e_i = e_{i+1}.e_j \quad (j \leq i), \\ &\partial_i^\alpha.e_j = e_j.\partial_{i-1}^\alpha \quad (j < i), \quad \textit{or}\ \ \mathrm{id} \quad (j = i), && \textit{or}\ \ e_{j-1}.\partial_i^\alpha \quad (j > i). \end{aligned} \qquad (2)$$

A *morphism* $f = (f_n) \colon X \to Y$ is a sequence of mappings $f_n \colon X_n \to Y_n$ commuting with faces and degeneracies. All this forms a category **Cub** which has all limits and colimits and is cartesian closed (it is a category of presheaves).

Now, a *normed cubical set* will be a cubical set X equipped with a sequence of 'norms' which annihilate on degenerate elements

$$\| - \| \colon X_n \to [0, +\infty], \qquad \|e_i(a)\| = 0 \qquad (\forall a \in X_n). \qquad (3)$$

We do *not* require any coherence condition for faces, nor any restriction on the norm of a point; for instance, a degenerate edge must have norm zero, but its vertices can have any norm. The category **NCub** of normed cubical sets has, for *morphisms*, the (weakly) contracting morphisms of cubical sets $f \colon X \to Y$, with $\|f_n(x)\| \leq \|x\|$, for all $x \in X_n$.

(The cubical sets we are to consider might be viewed as *c-sets* (I.1.9): such an object is a set K equipped with a sub-presheaf c_*K of the cubical set of components $\mathbf{Set}(I^n, K)$, such that K is covered by all distinguished cubes. And we could introduce *normed c-sets*, as c-sets with a norm on their structural presheaf; but this would be of limited utility, here.)

1.2. Elementary models. A *normal* cubical set has norm 1 on all *non-degenerate entries* (and 0 on the degenerate ones). All the 'elementary' cubical sets considered in I.1.5 will be equipped with this normal norm and denoted with the same symbols.

Thus, $\uparrow\mathbf{i} = \mathbf{2}$ will denote here the *normal directed elementary interval*, freely generated (as a normal cubical set) by a 1-cube u

$$0 \xrightarrow{u} 1 \qquad \partial_1^-(u) = 0, \;\; \partial_1^+(u) = 1, \qquad \|u\| = \|0\| = \|1\| = 1. \tag{4}$$

More generally, the *normed directed elementary n-cube* $\uparrow\mathbf{i}^n$ is the normal object generated by one n-cube, for $n \geq 0$. (It is a tensor power $\uparrow\mathbf{i}\otimes\ldots\otimes\uparrow\mathbf{i}$, cf. 27).

The normed directed elementary circle $\uparrow\mathbf{s}^1$ is the normal object generated by a 1-cube u with equal faces

$$\partial_1^-(u) = * = \partial_1^+(u), \qquad \|u\| = \| * \| = 1. \tag{5}$$

Similarly, the *normed directed elementary n-sphere* $\uparrow\mathbf{s}^n (n > 1)$ is the normal object generated by an n-cube u, all whose faces are totally degenerate (hence equal)

$$\partial_i^\alpha(u) = (e_1)^{n-1}(\partial_1^-)^n(u), \qquad \|u\| = \| * \| = 1 \qquad (\alpha = \pm;\; i = 1, \ldots, n), \tag{6}$$

while $\uparrow\mathbf{s}^0 = \mathbf{s}^0$ is the normal object generated by two vertices. The n-dimensional torus $\uparrow\mathbf{t}^n$ will be defined in 27, as a tensor power of $\uparrow\mathbf{s}^1$.

Finally, the normed ordered circle $\uparrow\mathbf{o}^1$ is the normal object generated by two edges with the same faces

$$v^- \overset{u'}{\underset{u''}{\rightrightarrows}} v^+ \qquad \partial_1^\alpha(u') = \partial_1^\alpha(u''), \qquad \|u'\| = \|u''\| = \|v^-\| = \|v^+\| = 1. \tag{7}$$

and more generally the *normed ordered sphere* $\uparrow\mathbf{o}^n$ is the normal object generated by two n-cubes u', u'' with the same boundary: $\partial_i^\alpha(u') = \partial_i^\alpha(u'')$.

For the links of these objects with suspension, pointed or not, see I.1.7 and I.5.2.

1.3. Normed directed circles. Since metric aspects are relevant in our treatment, we shall distinguish between the standard circle $\mathbf{S}^1$, equipped with the natural geodetic metric, and the standard 1-torus $\mathbf{T}$, with the metric induced by the line

$$\mathbf{S}^1 \cong \mathbf{R}/2\pi\mathbf{Z}, \qquad \mathbf{T} = \mathbf{R}/\mathbf{Z}, \tag{8}$$

so that a simple loop has, respectively, a length of 2π and 1.

To interpret such spaces as normed cubical sets, let us start from the directed line $\uparrow\mathbf{R}$, a cubical set produced by topology and natural order: its n-cubes are the continuous order-preserving mappings $a: \mathbf{I}^n \to \mathbf{R}$, with obvious faces and degeneracies - produced by the cofaces and codegeneracies of the standard cubes $\mathbf{I}^n$. (In Part I, this cubical set is rather viewed as the corresponding c-set; see the last remark of 1.1.)

The *normed directed line* $\mathsf{N}\uparrow\mathbf{R}$ will be this cubical set, with the following, obvious norm on the n-cube $a: \mathbf{I}^n \to \mathbf{R}$

$$n = 0\colon \|a\| = 1, \qquad n = 1\colon \|a\| = a(1) - a(0), \qquad n > 1\colon \|a\| = 0; \tag{9}$$

note that, in degree 1, a is an *increasing path* and $\|a\|$ is its length.

Now, the groups $\mathbf{Z}$ and $2\pi\mathbf{Z}$ act on the line, by translations, as well as on $\mathsf{N}{\uparrow}\mathbf{R}$. The quotient cubical sets are, by definition, the *normed directed circle* $\mathsf{N}{\uparrow}\mathbf{S}^1$ and *normed directed 1-torus* $\mathsf{N}{\uparrow}\mathbf{T}$

$$\mathsf{N}{\uparrow}\mathbf{S}^1 = (\mathsf{N}{\uparrow}\mathbf{R})/(2\pi\mathbf{Z}), \qquad \mathsf{N}{\uparrow}\mathbf{T} = (\mathsf{N}{\uparrow}\mathbf{R})/\mathbf{Z}; \tag{10}$$

the quotient norm is obviously $\|[a]\| = \|a\|$, each action being isometrical. (General quotient norms will be dealt with in 2.1.)

1.4. Normed cubical sets of irrational rotation. Consider now, on the real line, the action (by translations) of the additive subgroup $G_\vartheta = \mathbf{Z}+\vartheta\mathbf{Z}$, where ϑ is an irrational real number. G_ϑ is dense in $\mathbf{R}$ and the orbit space $\mathbf{R}/G_\vartheta = \mathbf{T}/\vartheta\mathbf{Z}$ is topologically trivial: an uncountable set with the coarse topology.

In Part I, we have seen the advantage of replacing this useless quotient with the cubical set

$$C_\vartheta = {\uparrow}\mathbf{R}/G_\vartheta, \tag{11}$$

where ${\uparrow}\mathbf{R}$ is the cubical set recalled above (1.3), with the obvious action of the group G_ϑ. Indeed, the homology of the cubical set C_ϑ is not trivial, but coincides with the homology of the group $G_\vartheta \cong \mathbf{Z}^2$, whence with the homology of the torus $\mathbf{T}^2$. Moreover, *directed* homology gives further relevant information: ${\uparrow}H_1(C_\vartheta) \cong {\uparrow}G_\vartheta$ as an *ordered* group (the natural order of real numbers), by Theorem I.4.8; it follows (I.4.9) that $C_\vartheta \cong C_\zeta$ if and only if ϑ and ζ are in the same orbit of the action of the group $\mathrm{PGL}(2,\mathbf{Z})$: in other words, each of them can be obtained from the other applying, finitely many times, the transformations $R(t) = t-1$ and $T^k(t) = t+k$ (on $\mathbf{R}\setminus\mathbf{Q}$; for $k \in \mathbf{Z}$, or $k = \pm 1$). This is precisely the classification of the rotation C^*-algebra A_ϑ up to strong Morita equivalence ([11], Thm. 4), recalled in I.4.1.

But these algebras have a stricter classification *up to isomorphism*, coming from the metric information contained in C^*-algebras: essentially, the fact that the traces of the projections of $K_0(A_\vartheta)$ form the subset $G_\vartheta \cap [0,1] \subset \mathbf{R}$ (a result of Pimsner, Voiculescu and Rieffel, see [10] and [11], Thm. 1.2), as recalled in I.4.1. It follows easily that $A_\vartheta \cong A_\zeta$ if and only if $G_\vartheta = G_\zeta$ (as subsets of $\mathbf{R}$), if and only if $\zeta \in \pm\vartheta + \mathbf{Z}$ (as in the proof of 4.2).

Here, we will obtain similar results enriching the cubical set C_ϑ with a norm. First, let us replace the cubical set ${\uparrow}\mathbf{R}$ with the normed cubical set $\mathsf{N}{\uparrow}\mathbf{R}$ (9). Again, the group $G_\vartheta = \mathbf{Z}+\vartheta\mathbf{Z}$ acts isometrically on it, and the quotient has an obvious norm

$$\mathsf{N}C_\vartheta = (\mathsf{N}{\uparrow}\mathbf{R})/G_\vartheta, \qquad \|[a]\| = \|a\|. \tag{12}$$

$\mathsf{N}C_\vartheta$ will be called an *irrational rotation normed cubical set.* Our main results here (Thms 4.1, 4.2) will prove that the normed homology $\mathsf{N}H_1(\mathsf{N}{\uparrow}\mathbf{R}/G_\vartheta)$ is isomorphic to G_ϑ, as a *normed subgroup of the line*, and deduce that the classification of the normed cubical sets NCJ, up to isomorphism, is *the same* as the one of the rotation algebras AJ, recalled above.

More generally, similar results hold for $G_\vartheta = \sum_j \vartheta_j\mathbf{Z} \subset \mathbf{R}$, where $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$ is an n-tuple of real numbers linearly independent on the rationals (4.3).

1.5. Metric spaces. The last two subsections suggest that it would be useful, starting from a *metric* space X, to define a norm $\mathsf{N}\square X$ on the singular cubical set $\square X$. Here, we only sketch a beginning of this program, sufficient for normed homology in degree 0 and 1 (in higher degrees, Riemannian manifolds and smooth cubes might be more convenient).

We shall use a generalised, non-symmetric notion of metric space adequate for directed algebraic topology (cf. [6]), and natural within the theory of enriched categories (cf. [9]). Thus, a *directed metric space* or *d-metric space*, is a set X equipped with a *d-metric* $\delta\colon X\times X \to [0,\infty]$, satisfying the axioms

$$\delta(x,x) = 0, \qquad\qquad \delta(x,y) + \delta(y,z) \ \geq\ \delta(x,z). \tag{13}$$

(If the value ∞ is forbidden, such a function is usually called a *quasi-pseudo-metric*, cf. [8].) A *symmetric* d-metric (satisfying $\delta(x,y) = \delta(y,x)$) will be called here a (generalised) *metric*; it is the same as an *écart* in Bourbaki [2].

d**Mtr** will denote the category of such d-metric spaces, with *d-contractions* $f\colon X \to Y$ $(\delta(x,x') \geq \delta(f(x), f(x')))$. Limits and colimits exist and are calculated as in **Set**; products have the l_∞ d-metric and equalisers the restricted one, while sums have the obvious d-metric and coequalisers have the d-metric induced on the quotient ([6], 4.7).

Now, it is clear how we should define the norm of a cube $a\colon \mathbf{I}^n \to X$ in degrees 0, 1

$$\begin{array}{l} n = 0\colon\ \|a\| = 1, \\ n = 1\colon\ \|a\| = \sup_p \sum_i \delta(a(t_{i-1}), a(t_i)), \qquad t_i = i/p \quad (i = 0,1,\ldots,p); \end{array} \tag{14}$$

moreover, if X is a 1-dimensional manifold, we are done: all higher cubes can be given norm 0.

This way, the standard circle $\mathbf{S}^1$, with the geodetic metric, produces a normed cubical set $\mathsf{N}\square\mathbf{S}^1$ which is an extension of the normed directed circle $\mathsf{N}{\uparrow}\mathbf{S}^1$ (10); similarly, the 1-torus $\mathsf{N}\square\mathbf{T} = \mathsf{N}\square(\mathbf{R}/\mathbf{Z}) = (\mathsf{N}\square\mathbf{R})/\mathbf{Z}$ is an extension of the directed version $\mathsf{N}{\uparrow}\mathbf{T}$. On the other hand, the punctured plane $\mathbf{R}^2 \setminus \{0\}$ (with the euclidean metric) gets a norm on $\square_1(\mathbf{R}^2 \setminus \{0\})$ with arbitrarily small loops. Normed homology will distinguish all such 'spaces' (3.5).

One should notice that the functor $\mathsf{N}\square_1\colon \mathrm{d}\mathbf{Mtr} \to \mathsf{N}\mathbf{Set}$ does not preserve quotients: for instance, $\mathbf{R}/G_\vartheta$ has a trivial metric, 0 everywhere, while the norm of $(\mathsf{N}\square_1\mathbf{R})/G_\vartheta$ is not trivial.

Finally, it is interesting to note that a preorder amounts to a d-metric with values in $\{0,\infty\}$, setting $x \leq y$ when $\delta(x,y) = 0$. This can explain why various arguments for the norm, in the sequel, are a sort of enriched version of the corresponding arguments for preordering, in Part I.

2. Normed cubical sets

Normed sets and normed cubical sets are equipped with a sort of extended 'seminorm'.

2.1. Normed sets. As motivated in the Introduction, our norms - for sets, cubical sets or abelian groups - will always take values in the commutative ordered semiring $[0, \infty]$, an extension of $\mathbf{R}^+$ where ∞ acts in the obvious way, except perhaps in one case, $0.\infty = 0$ (if consistent with the product of cardinals)

$$a + \infty = \infty, \qquad 0.\infty = 0, \qquad b.\infty = \infty \quad (b > 0). \tag{15}$$

A *normed* set will be a set $X = (X, \| - \|)$ equipped with a *norm*, consisting of an arbitrary mapping

$$\| - \| \colon X \to [0, \infty]. \tag{16}$$

A (weak) *contraction* $f \colon X \to Y$ has $\|f(x)\| \leq \|x\|$, for all $x \in X$. NSet will denote the category of these *normed sets and contractions*; an isomorphism is thus a bijective isometry: $\|f(x)\| = \|x\|$, for all x. This category has all limits and colimits, constructed as in **Set** and equipped with a suitable norm (strictly determined).

Thus, a product $\prod X_i$ and a sum $\sum X_i$ (where $X_i = (X_i, \| - \|_i)$ have the following norms

$$\begin{array}{cc} \|(x_i)\| = \sup_i \|x_i\|_i & ((x_i) \in \prod X_i), \\ \|(x, i)\| = \|x\|_i & (x \in X_i), \end{array} \tag{17}$$

while a normed subset has the restricted norm, and a quotient $X/\!\sim$ has the induced one

$$\|\xi\| = \inf\{\|x\| \ \mid x \in \xi\} \qquad (\xi \in X/\!\sim). \tag{18}$$

Plainly, infinite products exist because we are allowing an infinite norm. By the same reason, the forgetful functor $| - | \colon \mathsf{NSet} \to \mathbf{Set}$ has a left adjoint $\mathsf{N}_\infty S$, which equips the set S with the *discrete* norm, always ∞. The right adjoint $\mathsf{N}_0 S$ has the *codiscrete*, or coarse norm, always zero. On the other hand, the *unit-ball* functor B_1

$$B_1 \colon \mathsf{NSet} \to \mathbf{Set}, \qquad B_1(X) = \{x \in X \mid \|x\| \leq 1\}, \tag{19}$$

has a left adjoint $\mathsf{N}_1 S$, which equips the set S with the constant norm at 1. The set S *will often be identified* with $\mathsf{N}_1 S$, called the *associated normal normed set*.

2.2. Tensor products. We have seen that, in the cartesian product $X \times Y$ of normed sets, $\|(x, y)\| = \|x\| \vee \|y\|$.

But NSet has a closed symmetric monoidal structure, with tensor product $X \otimes Y$ given by the cartesian product $|X| \times |Y|$ of the underlying sets, with a different norm on a pair $x \otimes y$ (written thus to avoid confusion with the cartesian product)

$$\|x \otimes y\| = \|x\|.\|y\|. \tag{20}$$

The identity of the tensor product is the singleton $\{*\} = \mathsf{N}_1\{*\}$, with $\| * \| = 1$; note that the representable functor produced by it is (isomorphic to) B_1, which acquires thus a privileged status

$$\mathsf{NSet}(\{*\}, -) \colon \mathsf{NSet} \to \mathbf{Set}, \qquad \mathsf{NSet}(\{*\}, X) = B_1(X), \tag{21}$$

and can be viewed as the 'true' forgetful functor to **Set**, even if not faithful.

The internal hom is the set of all mappings, equipped with the *Lipschitz norm*, i.e. the least Lipschitz constant of a mapping (possibly ∞, again)

$$\begin{gathered} Lip_\infty(Y,Z) = \mathbf{Set}(|Y|,|Z|), \\ \|f\| = \inf\{L \in [0,\infty] \ \mid \ \|f(y)\| \le L.\|y\|, \ \forall\, y \in Y\}. \end{gathered} \tag{22}$$

In fact, the usual bijection $\mathbf{Set}(X\times Y,Z) = \mathbf{Set}(X,\mathbf{Set}(Y,Z))$ which identifies $f\colon X\times Y \to Z$ with $g\colon X \to \mathbf{Set}(Y,Z)$ under the condition $f(x,y) = g(x)(y)$, provides two isometries

$$\begin{gathered} Lip_\infty(X\otimes Y,Z) = Lip_\infty(X, Lip_\infty(Y,Z)), \qquad\qquad \|f\| \le L \iff \|g\| \le L, \\ \mathsf{N}\mathbf{Set}(X\otimes Y,Z) = \mathsf{N}\mathbf{Set}(X, Lip_\infty(Y,Z)). \end{gathered} \tag{23}$$

And of course, the unit-ball functor B_1, applied to the normed set of all mappings, gives back the contracting ones

$$B_1(Lip_\infty(Y,Z)) = \mathsf{N}\mathbf{Set}(Y,Z), \tag{24}$$

as it happens in the well-known case of Banach spaces, in the interplay between bounded linear maps and linear contractions (cf. [12]).

2.3. Normed cubical sets. We have already defined such objects and their category, $\mathsf{N}\mathbf{Cub}$ (1.1).

Recall that *normal* cubical sets have norm 1 on all non-degenerate entries (1.2). As for normed sets (19), they are produced by an obvious functor $\mathsf{N}_1\colon \mathbf{Cub} \to \mathsf{N}\mathbf{Cub}$. But, here, its right adjoint $B_1\colon \mathsf{N}\mathbf{Cub} \to \mathbf{Cub}$ selects only those entries which have norm ≤ 1 *together* with all their iterated faces, of any order

$$\mathsf{N}_1 : \mathbf{Cub} \ \leftrightarrows \ \mathsf{N}\mathbf{Cub} : B_1, \qquad\qquad \mathsf{N}_1 \dashv B_1. \tag{25}$$

Since limits and colimits in **Cub** are constructed componentwise in **Set**, the same holds for $\mathsf{N}\mathbf{Cub}$, with - on each component - the norm resulting from the (co)limit, as in 2.1.

Similarly, the (non symmetric!) tensor product of **Cub** ([3]; I.1.4) can be lifted to $\mathsf{N}\mathbf{Cub}$

$$(X\otimes Y)_n = (\textstyle\sum_{p+q=n} |X_p|\times|Y_q|)/\sim_n, \qquad\qquad \|x\otimes y\| = \|x\|.\|y\|, \tag{26}$$

since the equivalence relation only identifies pairs whose norm is 0 and, moreover, a degenerate tensor always has a degenerate factor. The identity of the tensor product is the singleton $\{*\} = \mathsf{N}_1\{*\}$, with $\|*\| = 1$. Normal cubical sets are closed under tensor product.

Thus, the normed directed elementary n-cube $\uparrow\!\mathbf{i}^n$ (1.2) is a tensor power of $\uparrow\!\mathbf{i}$, and the *normal directed elementary n-torus* $\uparrow\!\mathbf{t}^n$ is defined as a tensor power of the normal directed elementary circle

$$\uparrow\!\mathbf{i}^n = \uparrow\!\mathbf{i}\otimes\ldots\otimes\uparrow\!\mathbf{i} = (\uparrow\!\mathbf{i})^{\otimes n}, \qquad\qquad \uparrow\!\mathbf{t}^n = \uparrow\!\mathbf{s}^1\otimes\ldots\otimes\uparrow\!\mathbf{s}^1 = (\uparrow\!\mathbf{s}^1)^{\otimes n} \tag{27}$$

2.4. Connected components. For a cubical set X, the *homotopy normed set* $\pi_0(X)$ is defined as a quotient of normed sets (18)

$$\pi_0(X) = X_0/\simeq, \qquad \|[x]\| = \inf\{\|y\| \mid y \simeq x\}, \tag{28}$$

where the equivalence relation $\simeq$ (*connection*) is generated by being vertices of a common edge. The *connected component* of X at an equivalence class $[x] \in \pi_0(X)$ is the normed cubical subset formed by all cubes of X whose vertices lie in $[x]$; X is always the sum of its connected components. If X is not empty, we say that it is connected if it has one connected component, or equivalently if $\pi_0(X)$ is a singleton.

The forgetful functor $(-)_0\colon \mathsf{N}\mathbf{Cub} \to \mathsf{N}\mathbf{Set}$ has a left adjoint, the *discrete* normed cubical set on a normed set

$$D\colon \mathsf{N}\mathbf{Set} \to \mathsf{N}\mathbf{Cub}, \tag{29}$$

where components are constant, $(DS)_n = S$ ($n \in \mathbf{N}$), faces and degeneracies are identities, the norms of vertices are unchanged (and the norms in higher degree are zero). Then, the functor $\pi_0\colon \mathsf{N}\mathbf{Cub} \to \mathsf{N}\mathbf{Set}$ is left adjoint to D.

3. Normed directed homology

Directed homology of cubical sets, studied in Part I, is enriched with norms.

3.1. Normed abelian groups. Normed directed homology will take values in normed preordered abelian groups, a 'metric' version of the category d$\mathbf{Ab}$ of preordered abelian groups used in Part I.

Here, a normed abelian group L is equipped with a norm $\|\lambda\| \in [0, \infty]$ such that

$$\|0\| = 0, \qquad \|-\lambda\| = \|\lambda\|, \qquad \|\lambda + \mu\| \leq \|\lambda\| + \|\mu\|. \tag{30}$$

Note that, for $n \in \mathbf{N}$, we only have $\|n.\lambda\| \leq n.\|\lambda\|$ (requiring equality would make quotients difficult to handle).

For a *normed preordered abelian group* $\uparrow L$, no coherence conditions between preorder and norm are required. In the category Nd$\mathbf{Ab}$ of such objects, a *morphism* is a contracting homomorphism ($\|f(\lambda)\| \leq \|\lambda\|$) which respects preorder. But also the purely algebraic homomorphisms of the underlying abelian groups will intervene, denoted by dashed arrows, $\dashrightarrow$.

Nd$\mathbf{Ab}$ has all limits and colimits, computed as in $\mathbf{Ab}$ and equipped with a suitable norm (as in 2.1) and preorder (as in d$\mathbf{Ab}$). The tensor product $\uparrow L \otimes \uparrow M$ of d$\mathbf{Ab}$ (with positive cone generated by the tensors of positive elements, I.2.2) can be lifted to Nd$\mathbf{Ab}$, with a norm

$$\|\xi\| = \inf\{\textstyle\sum_i \|\lambda_i\|.\|\mu_i\| \mid \xi = \sum_i \lambda_i \otimes \mu_i \in \uparrow L \otimes \uparrow M\}, \tag{31}$$

which solves the universal problem for preorder-preserving bi-homomorphisms $\varphi\colon \uparrow L \times \uparrow M \to \uparrow N$ such that $\|\varphi(\lambda, \mu)\| \leq \|\lambda\|.\|\mu\|$. This makes a closed symmetric monoidal structure: the internal hom $\uparrow\mathrm{Hom}(\uparrow M, \uparrow N)$ is the abelian group of *all* homomorphisms of the underlying abelian groups, with the positive cone of preorder preserving homomorphisms (as in I.2.2) and the Lipschitz norm (22).

The unit of the tensor product is the ordered group of integers $\uparrow\mathbf{Z}$ with the natural norm, $|k|$. Again, the representable functor $\mathsf{Nd}\mathbf{Ab}(\uparrow\mathbf{Z}, -)$, applied to the internal Hom, gives back the set of morphisms

$$\mathsf{Nd}\mathbf{Ab}(\uparrow\mathbf{Z}, \uparrow L) = B_1(L^+), \qquad B_1(\mathrm{Hom}^+(\uparrow M, \uparrow N)) = \mathsf{Nd}\mathbf{Ab}(\uparrow M, \uparrow N). \tag{32}$$

The forgetful functor $\mathsf{Nd}\mathbf{Ab} = \mathrm{d}\mathbf{Ab}$ has a left adjoint $\mathsf{N}_\infty\uparrow L$ and right adjoint $\mathsf{N}_0\uparrow L$, respectively giving to a preordered abelian group $\uparrow L$ its discrete ∞-norm ($\|\lambda\| = \infty$ for $\lambda \neq 0$) or the coarse one ($\|\lambda\| = 0$).

The forgetful functor $\mathsf{Nd}\mathbf{Ab} = \mathsf{N}\mathbf{Set}$ has a left adjoint, associating to a normed set S the *free normed ordered abelian group* $\uparrow\mathbf{Z}S$, which is the free abelian group generated by the underlying set, equipped with the obvious norm

$$\| \textstyle\sum_x k_x.x\| = \sum_x |k_x|.\|x\|, \tag{33}$$

($(k_x)_{x\in S}$ is a quasi-null family of integers) and with the order whose positive cone is the monoid $\mathbf{N}S$ of positive combinations, with $k_x \in \mathbf{N}$.

3.2. Chain complexes. We shall also use the category $\mathsf{Nd}\mathrm{C}_*\mathbf{Ab}$ of *normed directed chain complexes*: their components are normed preordered abelian groups, differentials are not assumed to respect norms or preorders, *but chain morphisms are*: they must be *contracting* and preorder-preserving. It is again an additive category with all limits and colimits.

The *normed directed homology* of such a complex $\uparrow C_*$ is a sequence of normed preordered abelian groups, consisting of the ordinary homology subquotients

$$\mathsf{N}\uparrow H_n \colon \mathsf{Nd}\mathrm{C}_*\mathbf{Ab} \to \mathsf{Nd}\mathbf{Ab}, \qquad \mathsf{N}\uparrow H_n(\uparrow C_*) = \mathrm{Ker}\partial_n/\mathrm{Im}\partial_{n+1}, \tag{34}$$

with the induced norm and preorder. Similarly, we have the category of normed directed cochain complexes $\mathsf{Nd}\mathrm{C}_*\mathbf{Ab}$ and its cohomology.

When we want to forget about preorder, we take out the prefixes d, $\uparrow$. Thus, $\mathsf{N}\mathbf{Ab}$ denotes the category of *normed abelian groups* (and contracting homomorphisms), while $NC * Ab$ stands for *normed chain complexes*; their *normed homology* will be written as

$$\mathsf{N}H_n \colon \mathsf{N}\mathrm{C}_*\mathbf{Ab} \to \mathsf{N}\mathbf{Ab}. \tag{35}$$

3.3. Normed directed homology. The normed cubical set X determines a chain complex of free normed ordered abelian groups (3.2)

$$\begin{aligned} &\mathsf{N}\uparrow C_n(X) = (\uparrow\mathbf{Z}X_n)/(\uparrow\mathbf{Z}(\mathrm{Deg}_n X)) = \uparrow\mathbf{Z}\overline{X}_n && (\overline{X}_n = X_n \setminus \mathrm{Deg}_n X), \\ &\partial_n(\hat{x}) = \textstyle\sum_{i,\alpha} (-1)^{i+\alpha} (\partial_i^\alpha x)\hat{} && (x \in X_n). \end{aligned} \tag{36}$$

As usual, $\hat{x}$ is the class of the n-cube x up to degenerate cubes. Note that all degenerate chains have norm 0, whence all representatives of $\hat{x}$ have the same norm in $\uparrow\mathbf{Z}X_n$: this justifies the identification of the quotient with $\uparrow\mathbf{Z}\overline{X}_n$, from the 'metric' point of view. Therefore (as in Part I), we shall generally write the equivalence class $\hat{x}$ as x, identifying all degenerate cubes with 0. (For these classes and their chain complex, we shall avoid the usual term 'normalised', which might give rise to confusion with norms.)

Also here (cf. I.2.1), *the positive cone and the norm are not respected by the differential* $\partial_n\colon \mathsf{N}{\uparrow}C_n(X) \dashrightarrow \mathsf{N}{\uparrow}C_{n-1}(X)$, which is just a homomorphism of the underlying abelian groups, as stressed by the dashed arrow. On the other hand, a morphism of normed cubical sets $f\colon X \to Y$ induces a sequence of *morphisms* $\mathsf{N}{\uparrow}C_n(X) \to \mathsf{N}{\uparrow}C_n(Y)$, which do preserve preorder and respect norms. We have defined a covariant functor

$$\mathsf{N}{\uparrow}C_*\colon \mathsf{N}\mathbf{Cub} \to \mathsf{Nd}\mathrm{C}_*\mathbf{Ab}, \tag{37}$$

with values in the category $\mathsf{Nd}\mathrm{C}_*\mathbf{Ab}$ of normed directed chain complexes of abelian groups (3.2). This produces the *normed directed homology* of a cubical set, as a sequence of normed preordered abelian groups

$$\mathsf{N}{\uparrow}H_n\colon \mathsf{N}\mathbf{Cub} \to \mathsf{Nd}\mathbf{Ab}, \qquad \mathsf{N}{\uparrow}H_n(X) = \mathsf{N}{\uparrow}H_n(\mathsf{N}{\uparrow}C_*X), \tag{38}$$

given by the ordinary homology subquotient, with the induced preorder and norm. When we forget preorder, the normed chain and homology functors will be written as $\mathsf{N}C_*X$ and $\mathsf{N}H_*X$.

Extending I.2.1 with the introduction of norms, we can consider *normed directed* combinatorial (co)homology of cubical sets, *with coefficients in a normed preordered abelian group* ${\uparrow}L$, starting from the normed directed chain complexes (cf. 3.1)

$$\mathsf{N}{\uparrow}C_*(X;{\uparrow}L) = \mathsf{N}{\uparrow}C_*(X) \otimes {\uparrow}L, \qquad \mathsf{N}{\uparrow}C_*(X;{\uparrow}L) = \mathrm{Hom}(\mathsf{N}{\uparrow}C_*(X), {\uparrow}L). \tag{39}$$

Below, we only consider $\mathsf{N}{\uparrow}H_n(X) = \mathsf{N}{\uparrow}H_n(X;{\uparrow}\mathbf{Z})$.

3.4. Elementary computations. This is an extension of I.2.3 to metric aspects.

Plainly, the homology of a sum $X = \sum X_i$ of normed cubical sets is a direct sum $\mathsf{N}{\uparrow}H_nX = \bigoplus_i \mathsf{N}{\uparrow}H_nX_i$ of *normed preordered* abelian groups. It follows that, for every cubical set X

$$\mathsf{N}{\uparrow}H_0(X) = {\uparrow}\mathbf{Z}.\pi_0X, \qquad \|[x]\| = \inf\{\|y\| \mid y \simeq x\}, \tag{40}$$

is the free normed ordered abelian group generated by the homotopy *normed* set π_0X (2.4, with the norm recalled above, for generators).

$$\mathsf{N}{\uparrow}H_0({\uparrow}\mathbf{s}^n) = \mathsf{N}{\uparrow}H_n({\uparrow}\mathbf{s}^n) = {\uparrow}\mathbf{Z} \qquad (n > 0), \tag{41}$$

with the natural norm, since an n-cycle ku (notation of 1.2) has norm $|k|$. On the other hand

$$\mathsf{N}{\uparrow}H_n({\uparrow}\mathbf{o}^n) = 2.{\uparrow}_d\mathbf{Z}, \tag{42}$$

with the natural norm (and discrete order). In fact, an n-chain $hu' + ku''$ (notation of 1.2) is a cycle when $h + k = 0$, with norm $2|h|$.

3.5. Normed homology of circles. The normed directed 1-homology group of the normed directed circle $\mathsf{N}{\uparrow}\mathbf{S}^1$ and 1-torus $\mathsf{N}{\uparrow}\mathbf{T}$ (1.3) are easy to compute, taking into account the length of the standard generating 1-cycle, a simple loop. Thus

$$\mathsf{N}{\uparrow}H_1(\mathsf{N}{\uparrow}\mathbf{S}^1) = 2\pi.{\uparrow}\mathbf{Z}, \qquad \mathsf{N}{\uparrow}H_1(\mathsf{N}{\uparrow}\mathbf{T}) = {\uparrow}\mathbf{Z}, \tag{43}$$

with the natural norm and order. The corresponding non-directed versions $\mathsf{N}\mathbf{S}^1, \mathsf{N}\mathbf{T}$ (1.5) yield the same norm and the coarse preorder (since both generators of the group can be realised as 1-cycles)

$$\mathsf{N}{\uparrow}H_1(\mathsf{N}{\uparrow}\mathbf{S}^1) = 2\pi.{\uparrow}_c\mathbf{Z}, \qquad \mathsf{N}{\uparrow}H_1(\mathsf{N}{\uparrow}\mathbf{T}) = {\uparrow}_c\mathbf{Z}. \tag{44}$$

Finally, the punctured plane $\mathbf{R}^2 \setminus \{0\}$ (with the euclidean metric) gets the coarse preorder and the zero 'norm', since the homology generator contains arbitrarily small cycles

$$\mathsf{N}{\uparrow}H_1(\mathsf{N}\square(\mathbf{R}^2 \setminus \{0\})) = \mathsf{N}_0{\uparrow}_c\mathbf{Z}. \tag{45}$$

(Of course, in all these cases, $\mathsf{N}{\uparrow}H_0$ is the normed ordered abelian group ${\uparrow}\mathbf{Z}$.)

3.6. Theorem. [Tensor products] *Given two normed cubical sets X, Y, there is a natural isomorphism and a monomorphism*

$$\begin{gathered}\mathsf{N}{\uparrow}C_*(X \otimes Y) = \mathsf{N}{\uparrow}C_*(X) \otimes \mathsf{N}{\uparrow}C_*(Y),\\ \mathsf{N}{\uparrow}H_*(X) \otimes \mathsf{N}{\uparrow}H_*(Y) \rightarrowtail \mathsf{N}{\uparrow}H_*(X \otimes Y).\end{gathered} \tag{46}$$

Proof. Recall, from the proof of I.2.7, that we can identify the preordered abelian groups

$${\uparrow}C_n(X \otimes Y) = \textstyle\bigoplus_{p+q=n} {\uparrow}C_p(X) \otimes {\uparrow}C_q(Y), \tag{47}$$

respecting their canonical positive bases, i.e. the sum of the sets $\overline{X}_p \times \overline{Y}_q$, for $p + q = n$. This identification preserves the norm of the tensors of the basis, $\|x \otimes y\| = \|x\|.\|y\|$, and - by the Künneth formula - induces a monomorphism in homology. ■

3.7. Elementary cubical tori. Also because of the previous theorem, the normed directed homology of the normed elementary torus ${\uparrow}\mathbf{t}^n = ({\uparrow}\mathbf{s}^1)^{\otimes n}$ is expressed as in I.2.9

$$\mathsf{N}{\uparrow}H_i({\uparrow}\mathbf{t}^n) = {\uparrow}\mathbf{Z}^{\binom{n}{i}} \qquad (0 \leq i \leq n), \tag{48}$$

but now ${\uparrow}\mathbf{Z}$ is the *normed* ordered abelian group of integers.

4. Normed rotation structures corresponding to noncommutative tori

We compute the normed homology of the normed cubical sets $\mathsf{N}C_\vartheta$.

4.1. THEOREM. *For any irrational number J, the normed homology groups of* $\mathsf{N}C_\vartheta = (\mathsf{N}{\uparrow}\mathbf{R})/G_\vartheta$ *(1.4) are as follows*

$$\begin{gathered}\mathsf{N}H_1(\mathsf{N}C_\vartheta) = G_\vartheta = \mathbf{Z} + \vartheta\mathbf{Z} \subset \mathbf{R},\\ \mathsf{N}H_0(\mathsf{N}C_\vartheta) = \mathbf{Z} \subset \mathbf{R}, \qquad \mathsf{N}H_2(\mathsf{N}C_\vartheta) = \mathsf{N}_0\mathbf{Z},\end{gathered} \tag{49}$$

(with the norm induced by the reals in degrees 0 and 1; and null in degree 2). The first isomorphism above has a simple description on the positive cone $G_\vartheta \cap \mathbf{R}^+$

$$\begin{array}{lll}\varphi\colon G_\vartheta \to \mathsf{N}H_1(\mathsf{N}{\uparrow}\mathbf{R}/G_\vartheta), & \varphi(\rho) = [pa_\rho] & (\rho \in G_\vartheta \cap \mathbf{R}^+),\\ a_\rho\colon \mathbf{I} \to \mathbf{R}, & a_\rho(t) = \rho t, & \end{array} \tag{50}$$

where $p\colon \mathbf{R} \to \mathbf{R}/G_\vartheta$ *is the canonical projection. (The preorder of these homology groups has been determined in I.4.8; see 1.4.)*

PROOF. The algebraic part of the statement is already known from Part I: the homology of $\mathsf{N}{\uparrow}\mathbf{R}/G_\vartheta$ is, algebraically, as claimed in (1) and the mapping φ in (50) is an algebraic isomorphism (I.4.8). Moreover, the norm on $\mathsf{N}H_0$ is plain (3.4.1), while the one on $\mathsf{N}H2$ comes from the fact that all 2-chains have norm zero.

The rest of the proof, concerning the norm of $\mathsf{N}H_1$, is a non-obvious enrichment of the one of I.4.8 concerning preorder (cf. the last remark in (1.5); part of the complication comes from the fact that, here, we cannot reduce the argument to positive chains.

First, the mapping φ is certainly contracting, because on the positive elements $\rho \in G_\vartheta \cap \mathbf{R}^+$ we have $\|[pa_\rho]\| \le \|[a_\rho]\| = \rho$ (and G_ϑ is totally ordered). We have to prove that it is also expansive, $\|\varphi(\rho)\| \ge |\rho|$. To simplify the argument, a 1-chain z of ${\uparrow}\mathbf{R}$ which projects to a cycle $p_\sharp(z)$ in ${\uparrow}\mathbf{R}/G_\vartheta$, or to a boundary, will be called a *pre-cycle* or a *pre-boundary*, respectively. (Note that, since $p_\sharp$ is surjective, the homology of ${\uparrow}\mathbf{R}/G_\vartheta$ is isomorphic to the quotient of pre-cycles modulo pre-boundaries.)

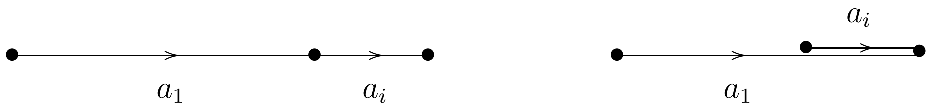
Let $z = \sum_i \lambda_i a_i$ be a pre-cycle. Assuming that all a_i's *are different*, the norm $\|z\|$ in $\mathsf{N}C_*(\mathsf{N}{\uparrow}\mathbf{R})$ and the *weight* $|z|$ (introduced for the proof) are expressed as follows:

$$\|z\| = \textstyle\sum_i |\lambda_i|.\|a_i\|, \qquad |z| = \sum_i |\lambda_i|. \tag{51}$$

We will prove that z is equivalent, modulo pre-boundaries, to a pre-cycle $\pm\, a_\rho$ with lesser norm (and lesser weight). Since each homology class in $\mathsf{N}H_1(\mathsf{N}{\uparrow}\mathbf{R}/G_\vartheta)$ has precisely one representative of type $\pm\, pa_\rho$, the latter reaches the minimal norm in its homology class. Thus, $\|\varphi(\rho)\| \ge |\rho|$.

Let $z = z' + z''$, putting in z' all the summands $\lambda_i a_i$ which are pre-cycles themselves, and replace any such a_i, up to pre-boundaries, with a_{ρ_i}, where $\rho_i = \partial^+ a_i - \partial^- a_i \in G_\vartheta^+$; norm and weight can only decrease, because of possible coincidences of ρ_i's. If $z'' = 0$ we are done, otherwise $z'' = z - z'$ is still a pre-cycle; let us act on it. Reorder its paths a_i so that a_1 has a minimal $|\lambda_1|$ (> 0); since $\partial^+ a_1$ has to annihilate in $\partial p_\sharp(z')$, there is some index $i > 1$ such that:

- either a_i has a coefficient λ_i of the same sign as λ_1, and $\partial^+ a_1 - \partial^- a_i \in G_\vartheta$; by a G_ϑ-translation of a_i (leaving pa_i unaffected), we can assume that $\partial^+ a_1 = \partial^- a_i$ (as in the left diagram below) and then replace (modulo boundaries) $\lambda_1 a_1 + \lambda_i a_i$ with $\lambda_1 \hat{a}_1 + (\lambda_i - \lambda_1) a_i$ where $\hat{a}_1 = a_1 * a_i$ is the concatenation; norm can only decrease while weight is strictly less: at most the previous one minus $|\lambda_1|$;



(52)

- or a_i has a coefficient of opposite sign and $\partial^+ a_1 - \partial^+ a_i \in G_\vartheta$. Again, we can assume that $\partial^+ a_1 = \partial^+ a_i$ (as in the right diagram above). Then, replace (modulo boundaries) $\lambda_1 a_1 + \lambda_i a_i$ with $\lambda_1 \hat{a}_1 + (\lambda_i + \lambda_1) a_i$ where $\hat{a}_1$ is any increasing path from $\min(\partial^- a_1, \partial^- a_i)$ to $\max(\partial^- a_1, \partial^- a_i)$; norm and weight behave as above.

Continuing this way, the procedure ends in a finite number of steps, because weight strictly decreases; this means that, modulo pre-boundaries, we have changed z into an integral combination of pre-cycles of the required form, $z' = \sum_i \lambda_i a_{\rho_i}$, with $\|z'\| \leq \|z\|$.

Now, we can replace $2a_\rho$ with $a_{2\rho}$ and $\lambda.a_\rho$ with $\pm\, a_{|\lambda|\rho}$ (modulo pre-boundaries): we get a pre-cycle $z'' = \sum_i \lambda_i a_{\rho_i}$ with the same norm and $\lambda_i = \pm 1$. Then, operating with G_ϑ-translation and concatenation, we get a pre-cycle $z''' = a_{\rho'} - a_{\rho''}$ with the same norm, $\|z'''\| = \rho' + \rho'' \leq \|z\|$. Finally, we replace the latter with $\pm\, a_\rho$, with norm $\rho = |\rho' - \rho''| \leq \|z\|$. ∎

4.2. Theorem. *The normed c-sets $\mathsf{N}{\uparrow}\mathbf{R}/G_\vartheta$ and $\mathsf{N}{\uparrow}\mathbf{R}/G_\zeta$ are (isometrically) isomorphic if and only if $G_\vartheta = G_\zeta$ as subsets of $\mathbf{R}$, if and only if $\zeta \in \pm\,\vartheta + \mathbf{Z}$.*

Proof. By Theorem 4.1, if our normed c-sets are isomorphic, also their normed groups $\mathbf{N}H_1$ are, and $G_\vartheta \cong G_\zeta$ (isometrically). Since the values of the norm $\| - \|\colon G_\vartheta \to \mathbf{R}$ form the set $G_\vartheta \cap \mathbf{R}^+$, it follows that G_ϑ coincides with G_ζ. Finally, if this is the case, then $\vartheta = a + b\zeta$ and $\zeta = c + d\vartheta$, whence $\vartheta = a + bc + bd\vartheta$ and $d = \pm 1$. ∎

4.3. An extension. Extending the previous case (and enriching I.4.4b), take an n-tuple of real numbers $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$, linearly independent on the rationals, and consider the normed additive subgroup $G_\vartheta = \sum_j \vartheta_j \mathbf{Z} \subset \mathbf{R}$, acting freely and isometrically on the line. (The previous case corresponds to the pair $(1, \vartheta)$.)

Again, the normed cubical set $\mathsf{N}{\uparrow}\mathbf{R}/G_\vartheta$ has a normed directed homology, isomorphic to the normed ordered abelian group ${\uparrow}G_\vartheta$

$$\mathsf{N}{\uparrow}H_1(\mathsf{N}{\uparrow}\mathbf{R}/G_\vartheta) = {\uparrow}G_\vartheta = {\uparrow}\textstyle\sum_j \vartheta_j \mathbf{Z}) \qquad\qquad (G_\vartheta^+ = G_\vartheta \cap \mathbf{R}^+). \qquad (53)$$

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