Theory and Applications of Categories, Vol. 14, No. 16, 2005, pp. 357–398.

ENLARGEMENTS OF CATEGORIES

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Abstract.

In order to apply nonstandard methods to modern algebraic geometry, as a first step in this paper we study the applications of nonstandard constructions to category theory. It turns out that many categorical properties are well behaved under enlargements.

Contents

1. Introduction

After Abraham Robinson's pioneering work on nonstandard analysis in the 1960s (see [Rob66]), nonstandard methods have been applied successfully in a wide area of mathematics, especially in stochastics, topology, functional analysis, mathematical physics and mathematical economy.

Nonstandard proofs of classical results are often conceptually more satisfying than the standard proofs; for example, existence of Haar measures on locally compact abelian groups can be proved elegantly by nonstandard methods (see [Par69], [Gor97] and [Ric76]).

Robinson himself also proved the usefulness of his methods for classical problems in algebraic geometry and number theory, namely the distribution of rational points on curves over number fields (see [Rob73] and [RR75]), infinite Galois theory, class field theory of infinite extensions (see [Rob67b] and [Rob69]) and the theory of Dedekind domains (see [Rob67a]).

This work has been written under hospitality of the Department of Pure Mathematics and Mathematical Statistics of the University of Cambridge and the "Sonderforschungsbereich geometrische Strukturen in der Mathematik" of the University of Mnster

Received by the editors 2004-08-20 and, in revised form, 2005-09-28.

Transmitted by Ieke Moerdijk. Published on 2005-10-31.

2000 Mathematics Subject Classification: 03H05,18A30,18E30,14A99.

There are plenty of reasons which could make nonstandard mathematics attractive for arithmetic geometry: If ${}^*\mathbb{Z}$ is an enlargement of $\mathbb{Z}$, then it contains infinite numbers and in particular infinite prime numbers. If P is such an infinite prime number, then the residue ring ${}^*\mathbb{Z}/P{}^*\mathbb{Z}$ is a field that behaves in many respects like a finite field, even though it is of characteristic zero and can — for suitable values of P — even contain an algebraic closure of $\mathbb{Q}$.

If ${}^*\mathbb{Q}$ denotes the field of fractions of ${}^*\mathbb{Z}$, then all p-adic fields $\mathbb{Q}_p$ (for standard primes p), as well as the field $\mathbb{R}$ and the ring of adeles $\mathbb{A}_{\mathbb{Q}}$, are all natural subquotients of ${}^*\mathbb{Q}$ (and the obvious analogues remain true if we replace $\mathbb{Q}$ with a number field K and ${}^*\mathbb{Q}$ with ${}^*K$). This seems to suggest that working with ${}^*\mathbb{Q}$ could lead to new insights into problems concerning local-global principles and adelic questions.

Limits, especially limits of "finite objects", play an important role in arithmetic geometry: Galois groups are limits of finite groups, the algebraic closure of a field k is the limit over finite extensions of k, Galois cohomology is the limit of the cohomology of the finite quotients of the Galois group, l-adic cohomology is the limit of étale cohomology with coefficients in the finite sheaves $\mathbb{Z}/l^n\mathbb{Z}$, and so on. This is another aspect that makes nonstandard methods attractive for arithmetic geometry, because those methods often make it possible to replace such a limit with a *finite object which behaves just like a finite object. This is exactly the point of view Robinson takes in his approach to infinite Galois theory which allows him to deduce infinite Galois theory from finite Galois theory. For more motivation for using nonstandard methods in arithmetic algebraic geometry we refer to [Fes03].

Modern arithmetic geometry makes heavy use of categorical and homological methods, and this unfortunately creates a slight problem with applying nonstandard methods directly: The nonstandard constructions are set-theoretical in nature and in the past have mostly been applied to sets (with structure) like $\mathbb{R}$ or $\mathbb{Q}$ or topological spaces and not to categories and (homological) functors. But in order to be able to talk about "*varieties over $\mathbb{Q}$" or "étale cohomology with coefficients in ${}^*\mathbb{Z}/P{}^*\mathbb{Z}$ for an infinite prime P", it seems necessary to apply the nonstandard construction to categories like the category of varieties over $\mathbb{Q}$ or the category of étale sheaves over a base scheme S.

Exactly for this reason, in this paper, we study "enlargements of categories", i.e. the application of the nonstandard construction to (small) categories and functors, to create the foundations needed to make use of the advantages of nonstandard methods described above in this abstract setup.

As a first application in a second paper, we will enlarge the fibred category of étale sheaves over schemes and then be able to define étale cohomology with coefficients in ${}^*\mathbb{Z}/P{}^*\mathbb{Z}$ for an infinite prime P or with coefficients in ${}^*\mathbb{Z}/l^{h}{}^*\mathbb{Z}$ for a finite prime l and an infinite natural number h. In the case of smooth and proper varieties over an algebraically closed field, the first choice of coefficients then leads to a new Weil-cohomology whereas the second allows a comparison with classical l-adic cohomology.

The plan of the paper is as follows: In the second chapter, we begin by introducing "$\hat{S}$-small" categories for a superstructure $\hat{S}$ (categories whose morphisms form a set which is an element of $\hat{S}$) and functors between these. Given an enlargement $\hat{S} \to \widehat{{}^*S}$, we describe how to "enlarge" $\hat{S}$-small categories and functors between them: We assign an $\widehat{{}^*S}$-small category ${}^*\mathcal{C}$ to every $\hat{S}$-small category $\mathcal{C}$ and a functor ${}^*F : {}^*\mathcal{C} \to {}^*\mathcal{D}$ to every functor $F : \mathcal{C} \to \mathcal{D}$ of $\hat{S}$-small categories. Then we show that most basic properties of given $\hat{S}$-small categories and functors between them, like the existence of certain finite limits or colimits, the representability of certain functors or the adjointness of a given pair of functors, carry over to the associated enlargements.

In the third chapter, we study the enlargements of filtered and cofiltered categories and the enlargements of filtered limits and colimits. Using the "saturation principle" of enlargements, we will be able to prove that all such limits in an $\hat{S}$-small category $\mathcal{C}$ are "dominated" (in a sense that will be made precise) by objects in ${}^*\mathcal{C}$.

In the fourth chapter, we will study $\hat{S}$-small additive and abelian categories (and additive functors between them), and their enlargements (which will be additive respectively abelian again), and we will give an interpretation of the enlargement of a category of R-modules (for a ring R that is an element of $\hat{S}$) as a category of *internal* ${}^*R$-modules.

The fifth chapter is devoted to derived functors between $\hat{S}$-small abelian categories. We will see that taking the enlargement is compatible with taking the i-th derived functor (again in a sense that will be made precise).

In the sixth chapter, we look at triangulated and derived $\hat{S}$-small categories and exact functors between them and study their enlargements. The enlargement of a triangulated category will be triangulated again, the enlargement of a derived functor will be exact, and we will prove several compatibilities between the various constructions.

In the seventh chapter, we add yet more structure and study $\hat{S}$-small fibred categories and $\hat{S}$-small additive, abelian and triangulated fibrations and their enlargements. This is important for most applications we have in mind.

Finally, we have added an appendix on basic definitions and properties of superstructures and enlargements for the convenience of the reader.

We would like to thank our referee whose suggestions led to several improvements of this paper.

2. Enlargements of categories

Let $\hat{S}$ be a superstructure and $* : \hat{S} \to \widehat{{}^*S}$ an enlargement. Even though a superstructure is not a universe in the sense of [SGA4I] (see A.2), we use the same terminology as in [SGA4I] by simply replacing "universe" by "superstructure" in all definitions. In particular, by an $\hat{S}$-small category $\mathcal{C}$, we mean a small category $\mathcal{C}$ whose set of morphisms is contained in $\hat{S}$, and we want to consider its image ${}^*\mathcal{C}$ in $\widehat{{}^*S}$. In this paragraph, we want to establish basic properties of ${}^*\mathcal{C}$: It is again a category (a $\widehat{{}^*S}$-small category to be precise), and it will inherit a lot of the properties of $\mathcal{C}$ (like having an initial object or final object).

Not surprisingly, the proofs will rely heavily on the transfer principle, and therefore it will be convenient to have short formal descriptions of the concepts involved. In particular, it will turn out to be easier to work with the following definition of an $\hat{S}$-small category instead of the one given above:

2.1. DEFINITION. *Let $\hat{S}$ be a superstructure. An $\hat{S}$-*small category *is a quadruple $\langle M, s, t, c\rangle$ with*

1. $M \in \hat{S} \setminus S$,
2. $s, t : M \to M$ *satisfying*
   (a) $ss = ts = s$,
   (b) $st = tt = t$,
3. $c \subseteq M \times M \times M$ *satisfying*
   (a) $\forall f, g, h \in M : (\langle f, g, h\rangle \in c) \Rightarrow (sf = tg) \wedge (tf = th) \wedge (sg = sh)$,
   (b) $\forall f, g \in M : (sf = tg) \Rightarrow (\exists! h \in M : \langle f, g, h\rangle \in c)$,
4. $\forall f \in M : \langle f, tf, f\rangle, \langle sf, f, f\rangle \in c$,
5. $\forall f_1, f_2, f_3, f_{12}, f_{23}, f_{123} \in M : \langle f_1, f_2, f_{12}\rangle, \langle f_{12}, f_3, f_{123}\rangle, \langle f_2, f_3, f_{23}\rangle \in c$
$$\Rightarrow \langle f_1, f_{23}, f_{123}\rangle \in c.$$

If $\mathcal{C} = \langle M, s, t, c\rangle$ is an $\hat{S}$-small category, we call $\mathrm{Mor}_{\mathcal{C}} := M$ *the* set of morphisms of $\mathcal{C}$.

2.2. REMARK. Given an $\hat{S}$-small category $\mathcal{C} = \langle M, s, t, c\rangle$ in the sense just defined, we can get an $\hat{S}$-small category in the usual sense of [SGA4I, I.1.0] as follows:

1. Define the set of *objects* of $\mathcal{C}$ as $\mathrm{Ob}(\mathcal{C}) := s(M)$ (note that also $\mathrm{Ob}(\mathcal{C}) = t(M)$ because of 2.1.2; note also that because of 2.1.4, elements of $s(M) = t(M)$ are units under composition, so objects correspond to their identity morphisms).
2. For objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, define $\mathrm{Mor}_{\mathcal{C}}(X, Y)$, the set of *morphisms from X to Y*, as $\{f \in M | sf = X \wedge tf = y\}$.
3. For an object $X \in \mathrm{Ob}(\mathcal{C})$ define $\mathrm{id}_X := X \overset{2.1.2}{\in} \mathrm{Mor}_{\mathcal{C}}(X, X)$.
4. Consider objects X, Y and Z of $\mathcal{C}$ and morphisms $f \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$ and $g \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$. According to 2.1.3, there is a unique morphism $h \in \mathrm{Mor}_{\mathcal{C}}(X, Z)$ such that $\langle f, g, h\rangle \in c$. Put $fg := f \circ g := h$.

If on the other hand we are given an $\hat{S}$-small category $\mathcal{D}$ in the sense of [SGA4I] and if we put

- $M := \bigsqcup_{X,Y \in \mathrm{Ob}(\mathcal{D})} \mathrm{Mor}_{\mathcal{D}}(X, Y)$,
- $sf := \mathrm{id}_X$ and $tf := \mathrm{id}_Y$ for $f \in \mathrm{Mor}_{\mathcal{D}}(X, Y)$ and
- $c := \big\{\langle f, g, h\rangle \,|\, \exists X, Y, Z \in \mathrm{Ob}(\mathcal{D}) : f \in \mathrm{Mor}_{\mathcal{D}}(Y, Z) \wedge g \in \mathrm{Mor}_{\mathcal{D}}(X, Y) \wedge f \circ g = h\big\}$,

then $\langle M, s, t, c\rangle$ is an $\hat{S}$-small category in the sense of 2.1.

2.3. Example.

1. Let $\mathcal{C}$ be a *small* category. Then A.3 shows that there is a superstructure $\hat{S}$ such that $\mathcal{C}$ is isomorphic to an $\hat{S}$-small category.

2. Let $\hat{S}$ be a superstructure. There is an $\hat{S}$-small category $\mathcal{C}$ that is equivalent to the category $\mathrm{Ens}^{\mathrm{fin}}$ of finite sets.

3. Let $\hat{S}$ be a superstructure. If $A \in \hat{S}$ is a ring, then there is an $\hat{S}$-small category $\mathcal{C}$ that is equivalent to $\mathrm{Mod}_A^{\mathrm{fin}}$, the category of finitely generated A-modules.

As a next step, we want to describe the notion of a *covariant functor* between $\hat{S}$-small categories that are given in terms of 2.1:

2.4. Definition. *Let $\hat{S}$ be a superstructure, and let $\mathcal{C} = \langle M, s, t, c\rangle$ and $\mathcal{D} = \langle M', s', t', c'\rangle$ be $\hat{S}$-small categories. A* (covariant) functor *$F : \mathcal{C} \to \mathcal{D}$ is a map $F : M \to M'$ satisfying*

- *$Fs = s'F$,*
- *$Ft = t'F$ and*
- *$\forall f, g, h \in M : \langle f, g, h\rangle \in c \Rightarrow \langle Ff, Fg, Fh\rangle \in c'$.*

2.5. Lemma. *Let $\hat{S}$ be a superstructure, and let $\mathcal{C}$ and $\mathcal{D}$ be $\hat{S}$-small categories. Then the following categories are also $\hat{S}$-small:*

1. *the opposite category $\mathcal{C}^\circ$ of $\mathcal{C}$,*

2. *the product category $\mathcal{C} \times \mathcal{D}$,*

3. *the category $\mathrm{Funct}\,(\mathcal{C}, \mathcal{D})$ of covariant functors from $\mathcal{C}$ to $\mathcal{D}$,*

4. *the category $\mathrm{Funct}^\circ(\mathcal{C}, \mathcal{D}) := \mathrm{Funct}\,(\mathcal{C}^\circ, \mathcal{D})$ of contravariant functors from $\mathcal{C}$ to $\mathcal{D}$.*

Proof. 1 and 2 are easy and 3 and 4 follows from the fact $M, M' \in \hat{S} \Rightarrow {M'}^M \in \hat{S}$. ∎

Now we can look at the effect of applying $*$ to an $\hat{S}$-small category $\mathcal{C}$:

2.6. Proposition. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement. and let $\mathcal{C}$ be an $\hat{S}$-small category. Then ${}^*\mathcal{C}$ is a $\widehat{{}^*S}$-small category with $\mathrm{Mor}_{{}^*\mathcal{C}} = {}^*\mathrm{Mor}_{\mathcal{C}}$ (so in particular all morphisms in ${}^*\mathcal{C}$ are internal).*

Proof. If $\mathcal{C} = \langle M, s, t, c\rangle$, then ${}^*\mathcal{C} = \langle {}^*M, {}^*s, {}^*t, {}^*c\rangle$, and an easy transfer immediately proves that this new quadruple satisfies all the conditions in 2.1. ∎

Before we go on, we need a technical lemma to compute the transfer of various formulas from $\hat{S}$ to $\widehat{{}^*S}$ (by $\varphi[X_1,\dots,X_n]$ we always mean a formula whose free variables are exactly $X_1,\dots,X_n$):

2.7. Lemma. *In the situation of 2.6, we have:*

1. $\varphi[X] \equiv \big[X \in \mathrm{Ob}(\mathcal{C})\big] \quad \Rightarrow \quad {}^*\varphi[X] \equiv \big[X \in \mathrm{Ob}({}^*\mathcal{C})\big]$,
2. $\varphi[f,X,Y] \equiv \big[f \in \mathrm{Mor}_{\mathcal{C}}(X,Y)\big] \quad \Rightarrow \quad {}^*\varphi[f,X,Y] \equiv \big[f \in \mathrm{Mor}_{{}^*\mathcal{C}}(X,Y)\big]$,
3. $\varphi[f,g,h] \equiv \big[f,\ g \text{ composable morphisms in } \mathcal{C} \text{ with } fg = h\big]$
   $\Rightarrow \quad {}^*\varphi[f,g,h] \equiv \big[f,\ g \text{ composable morphisms in } {}^*\mathcal{C} \text{ with } fg = h\big]$,
4. $\varphi[f] \equiv \big[f \text{ isomorphism in } \mathcal{C}\big] \quad \Rightarrow \quad {}^*\varphi[f] \equiv \big[f \text{ isomorphism in } {}^*\mathcal{C}\big]$,
5. $\varphi[f] \equiv \big[f \text{ monomorphism in } \mathcal{C}\big] \quad \Rightarrow \quad {}^*\varphi[f] \equiv \big[f \text{ monomorphism in } {}^*\mathcal{C}\big]$,
6. $\varphi[f] \equiv \big[f \text{ epimorphism in } \mathcal{C}\big] \quad \Rightarrow \quad {}^*\varphi[f] \equiv \big[f \text{ epimorphism in } {}^*\mathcal{C}\big]$,

Proof. The proofs are all simple. As an example we show 4.Let $\mathcal{C} = \langle M,s,t,c\rangle$.

$$\begin{aligned}{}^*\varphi[f] = {}^*\Big[&\exists g \in M : \exists X,Y \in \mathrm{Ob}(\mathcal{C}) : (\langle f,g,X\rangle \in c) \wedge (\langle g,f,Y\rangle \in c)\Big]\\ &\overset{1}{=} \Big[\exists g \in {}^*M : \exists X,Y \in \mathrm{Ob}({}^*\mathcal{C}) : (\langle f,g,X\rangle \in {}^*c) \wedge (\langle g,f,Y\rangle \in {}^*c))\Big],\end{aligned}$$

■

2.8. Corollary. *Let* $* : \hat{S} \to \widehat{{}^*S}$ *an enlargement, and let* $\mathcal{C}$ *and* $\mathcal{D}$ *be* $\hat{S}$*-small categories. Then we have:*

1. $\mathrm{Ob}({}^*\mathcal{C}) = {}^*\mathrm{Ob}(\mathcal{C})$, *and if* $m : \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{C}) \to \mathrm{Mor}_{\mathcal{C}}$ *denotes the map that sends* $\langle X,Y\rangle$ *to* $\mathrm{Mor}_{\mathcal{C}}(X,Y)$, *then* ${}^*m : \mathrm{Ob}({}^*\mathcal{C}) \times \mathrm{Ob}({}^*\mathcal{C}) \to \mathrm{Mor}_{{}^*\mathcal{C}}$ *maps* $\langle X,Y\rangle$ *to* $\mathrm{Mor}_{{}^*\mathcal{C}}(X,Y)$. *In particular we have* $\mathrm{Mor}_{{}^*\mathcal{C}}({}^*X,{}^*Y) = {}^*\mathrm{Mor}_{\mathcal{C}}(X,Y)$ *for* $X,Y \in \mathrm{Ob}(\mathcal{C})$.
2. $*$ *induces a covariant functor from* $\mathcal{C}$ *to* ${}^*\mathcal{C}$ *that takes isomorphisms (resp. monomorphisms, resp. epimorphisms) to isomorphisms (resp. monomorphisms, resp. epimorphisms).*
3. ${}^*(\mathcal{C}^\circ) = ({}^*\mathcal{C})^\circ$.
4. ${}^*(\mathcal{C} \times \mathcal{D}) = {}^*\mathcal{C} \times {}^*\mathcal{D}$.

2.9. Corollary. *Let* $* : \hat{S} \to \widehat{{}^*S}$ *an enlargement, and let* $\mathcal{C}$ *be an* $\hat{S}$*-small category.*

1. *Let* $\mathcal{D}$ *be a (full) subcategory of* $\mathcal{C}$*. Then* $\mathcal{D}$ *is* $\hat{S}$*-small, and* ${}^*\mathcal{D}$ *is a* $\widehat{{}^*S}$*-small (full) subcategory of* ${}^*\mathcal{C}$*.*

2. *Let* X *be an object of* $\mathcal{C}$*, then* $\mathcal{C}/X$*, the category of objects over* X*, is* $\hat{S}$*-small, and* ${}^*(\mathcal{C}/X) = {}^*\mathcal{C}/{}^*X$*.*

3. *Let* $\mathcal{D}$ *be a subcategory of* $\mathcal{C}$*, and let* F *be a covariant (resp. contravariant) functor from* $\mathcal{C}$ *into another* $\hat{S}$*-small category* $\mathcal{C}'$*. Then* ${}^*(F|\mathcal{D}) = {}^*F|{}^*\mathcal{D} : {}^*\mathcal{D} \to {}^*\mathcal{C}'$*.*

4. *Let* X *be an object of* $\mathcal{C}$*, and let* $\mathcal{R}$ *be a sieve of* X *(see [SGA4I, I.4.1]), i.e. a full subcategory of* $\mathcal{C}/X$ *with the property that if* Y *is an object of* $\mathcal{C}/X$ *and if there exists a morphism from* Y *to an object of* $\mathcal{R}$ *in* $\mathcal{C}/X$*, then* Y *belongs to* $\mathcal{R}$*. Then* ${}^*\mathcal{R}$ *is a sieve of* ${}^*X$*.*

2.10. Corollary. *Let* $* : \hat{S} \to \widehat{{}^*S}$ *be an enlargement, let* $\mathcal{C}$ *be an* $\hat{S}$*-small category, and let* X *and* Y *be two objects of* $\mathcal{C}$*. Then* X *and* Y *are isomorphic if and only if* ${}^*X$ *and* ${}^*Y$ *are isomorphic in* ${}^*\mathcal{C}$*.*

Proof. We have

$$\begin{aligned}\big[X \cong Y\big] &\Leftrightarrow \big[\exists f \in \mathrm{Mor}_{\mathcal{C}}(X,Y) : f \text{ isomorphism}\big] \\ &\overset{\text{transfer, 2.7.4, 2.8.1}}{\Longleftrightarrow} \big[\exists f \in \mathrm{Mor}_{{}^*\mathcal{C}}({}^*X,{}^*Y) : f \text{ isomorphism}\big] \Leftrightarrow \big[{}^*X \cong {}^*Y\big].\end{aligned}$$

■

2.11. Definition. *Let* $\hat{S}$ *be a superstructure. A* finite $\hat{S}$*-small category is an* $\hat{S}$*-small category* $\mathcal{C}$ *whose set of morphisms* $\mathrm{Mor}_{\mathcal{C}}$ *is a finite set.*

For the next proposition, recall from 2.4 that if $* : \hat{S} \to \widehat{{}^*S}$ is an enlargement and if $\mathcal{C}$ and $\mathcal{D}$ are ${}^*S$-small categories, then a functor from $\mathcal{C}$ to $\mathcal{D}$ is a map between the sets $\mathrm{Mor}_{\mathcal{C}}$ and $\mathrm{Mor}_{\mathcal{D}}$, which are elements of $\widehat{{}^*S}$, so in particular such a functor is an element of the superstructure $\widehat{{}^*S}$. Therefore we can ask — as for any element of $\widehat{{}^*S}$ — whether a given functor is an *internal* element of $\widehat{{}^*S}$ (see A.8). Such functors, that are internal elements of $\widehat{{}^*S}$, we will call *internal functors*.

2.12. Proposition. *Let* $* : \hat{S} \to \widehat{{}^*S}$ *be an enlargement, and let* $\mathcal{C}$ *and* $\mathcal{D}$ *be* $\hat{S}$*-small categories.*

1. ${}^*\mathrm{Funct}\,(\mathcal{C},\mathcal{D})$ *(resp.* ${}^*\mathrm{Funct}^\circ(\mathcal{C},\mathcal{D})$*) is the subcategory of internal functors and internal morphisms of functors of* $\mathrm{Funct}\,({}^*\mathcal{C},{}^*\mathcal{D})$ *(resp.* $\mathrm{Funct}^\circ({}^*\mathcal{C},{}^*\mathcal{D})$*). In particular,* $*$ *maps covariant (resp. contravariant) functors from* $\mathcal{C}$ *to* $\mathcal{D}$ *to covariant (resp. contravariant) functors from* ${}^*\mathcal{C}$ *to* ${}^*\mathcal{D}$*.*

2. If $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor, then the following diagram of categories and functors commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "*"'] & \mathcal{D} \arrow[d, "*"] \\ {}^*\mathcal{C} \arrow[r, "{}^*F"'] & {}^*\mathcal{D}. \end{tikzcd}$$

3. If $\mathcal{C}$ is a finite *$\hat{S}$-small category, then*

$$ {}^*\mathrm{Funct}\,(\mathcal{C},\mathcal{D}) = \mathrm{Funct}\,({}^*\mathcal{C},{}^*\mathcal{D}) \qquad \textit{and} \qquad {}^*\mathrm{Funct}^\circ(\mathcal{C},\mathcal{D}) = \mathrm{Funct}^\circ({}^*\mathcal{C},{}^*\mathcal{D}).$$

Proof. Let $\mathcal{C} = \langle M, s, t, c\rangle$ and $\mathcal{D} = \langle M', s', t', c'\rangle$. We have the following statement in $\hat{S}$:

$$\mathrm{Ob}(\mathrm{Funct}\,(\mathcal{C},\mathcal{D})) = \{F : M \to M' \mid F \text{ subject to 2.4 concerning } \mathcal{C},\, \mathcal{D}\}.$$

Transfer of this gives us:

$$\begin{aligned} \mathrm{Ob}({}^*\mathrm{Funct}\,(\mathcal{C},\mathcal{D})) &\overset{2.8.1}{=} {}^*\mathrm{Ob}(\mathrm{Funct}\,(\mathcal{C},\mathcal{D})) \\ &= \{F : {}^*M \to {}^*M' \mid F \textit{ internal} \text{ and subject to 2.4 concerning } {}^*\mathcal{C},\, {}^*\mathcal{D}\}, \end{aligned}$$

i.e. the objects of ${}^*\mathrm{Funct}\,(\mathcal{C},\mathcal{D})$ are just the internal functors from ${}^*\mathcal{C}$ to ${}^*\mathcal{D}$. Now for the morphisms of functors, we have the following statement in $\hat{S}$:

$$\begin{aligned} &\forall F, G \in \mathrm{Ob}(\mathrm{Funct}\,(\mathcal{C},\mathcal{D})) \;:\; \mathrm{Mor}_{\mathrm{Funct}\,(\mathcal{C},\mathcal{D})}(F,G) \\ &\qquad = \{f : \mathrm{Ob}(\mathcal{C}) \to M' \mid f \text{ fulfills certain properties concerning } \mathcal{C} \text{ and } \mathcal{D}\}, \end{aligned}$$

and transferring this by using 2.7.2 and 2.8.1, we get

$$\begin{aligned} &\forall F, G \in \mathrm{Ob}({}^*\mathrm{Funct}\,(\mathcal{C},\mathcal{D})) \;:\; \mathrm{Mor}_{{}^*\mathrm{Funct}\,(\mathcal{C},\mathcal{D})}(F,G) \\ &\quad = \{f : \mathrm{Ob}({}^*\mathcal{C}) \to {}^*M' \mid f \textit{ internal}, \text{ fulfills certain properties concerning } {}^*\mathcal{C} \text{ and } {}^*\mathcal{D}\}, \end{aligned}$$

i.e. for internal functors F and G from ${}^*\mathcal{C}$ to ${}^*\mathcal{D}$, the morphisms between F and G in ${}^*\mathrm{Funct}\,(\mathcal{C},\mathcal{D})$ are precisely the internal morphisms of functors between F and G. This proves 1 for covariant functors; the proof for contravariant functors is analogous.

To prove 2, we see immediately that for an arbitrary morphism $f \in \mathrm{Mor}_{\mathcal{C}}$ we have

$${}^*F\big[*(f)\big] = {}^*F\big[{}^*f\big] \overset{A.10.8}{=} {}^*\big[F(f)\big] = *\big[F(f)\big].$$

To prove 3, according to 1 we have to show that all functors from ${}^*\mathcal{C}$ to ${}^*\mathcal{D}$ and all morphisms of functors from ${}^*\mathcal{C}$ to ${}^*\mathcal{D}$ are internal if $\mathcal{C}$ is finite. But in this case, $* : \mathcal{C} \to {}^*\mathcal{C}$ is a *bijection*, i.e. both functors from ${}^*\mathcal{C}$ to ${}^*\mathcal{D}$ and morphisms of functors from ${}^*\mathcal{C}$ to ${}^*\mathcal{D}$ are maps from a *finite* set to an internal set. But such maps are internal (see A.11.4). ■

2.13. Definition. *For a set of sets N, let* $(N\text{-}\mathbf{Ens})$ *denote the full subcategory of the category of sets consisting of those sets which are elements of N.*

2.14. Remark. Note that if $N \in \hat{S}$ is a set of sets in a superstructure $\hat{S}$, then $(N\text{-}\mathbf{Ens})$ is an $\hat{S}$-small category (because the set of morphisms in $(N\text{-}\mathbf{Ens})$ is then obviously an element of $\hat{S}$). Furthermore, if $\hat{S}$ is an arbitrary superstructure, then any full subcategory of the category of sets that is $\hat{S}$-small equals $(N\text{-}\mathbf{Ens})$ for a suitable $N \in \hat{S}$.

2.15. Proposition. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, and let $N \in \hat{S}$ be a set of sets. Then ${}^*(N\text{-}\mathbf{Ens})$ is the subcategory of $({}^*N\text{-}\mathbf{Ens})$ that consists of all the objects but whose morphisms are only the* internal *maps.*

Proof. Transfer of the statement $\mathrm{Ob}(N\text{-}\mathbf{Ens}) \overset{2.1}{=} \{\mathrm{id}_X | X \in N\} \cong N$ gives us $\mathrm{Ob}({}^*(N\text{-}\mathbf{Ens})) \cong {}^*N$, so the objects of ${}^*(N\text{-}\mathbf{Ens})$ are exactly the objects of $({}^*N\text{-}\mathbf{Ens})$. As for the morphisms, transfer of

$$\forall \mathrm{id}_X, \mathrm{id}_Y \in \mathrm{Ob}(N\text{-}\mathbf{Ens}) \, : \, \mathrm{Mor}_{(N\text{-}\mathbf{Ens})}(\mathrm{id}_X, \mathrm{id}_Y) = Y^X$$

gives the desired result. ∎

2.16. Corollary. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, let $N \in \hat{S}$ be a set of sets, and let $\mathcal{C}$ be an $\hat{S}$-small category with $M := \mathrm{Mor}_{\mathcal{C}} \subseteq N$.*

1. *Let $F : \mathcal{C}^\circ \to (N\text{-}\mathbf{Ens})$ be a functor (i.e. F is a presheaf on $\mathcal{C}$ with values in $(N\text{-}\mathbf{Ens})$). Then ${}^*F$ is a presheaf on ${}^*\mathcal{C}$ with values in $({}^*N\text{-}\mathbf{Ens})$.*
2. *If in particular we look at the presheaf $h_X := \mathrm{Mor}_{\mathcal{C}}(\_, X) : \mathcal{C}^\circ \to (M\text{-}\mathbf{Ens}) \subseteq (N\text{-}\mathbf{Ens})$ for an element $X \in \mathrm{Ob}(\mathcal{C})$, then ${}^*h_X = \mathrm{Mor}_{{}^*\mathcal{C}}(\_, {}^*X) =: h_{{}^*X}$.*
3. *Consider the statement*

$$\varphi[F, X] \equiv \Big[\big(F \in \mathrm{Ob}(\mathrm{Funct}^\circ(\mathcal{C}, (N\text{-}\mathbf{Ens})))\big) \wedge \big(X \in \mathrm{Ob}(\mathcal{C})\big) \wedge \big(F \textit{ is representable by } X\big)\Big]$$

(note that this makes sense because of $M \subseteq N$). Then

$${}^*\varphi[F, X] = \Big[F \textit{ is an internal presheaf on } {}^*\mathcal{C} \textit{ with values in } ({}^*N\text{-}\mathbf{Ens}) \textit{ which is internally representable by } X\Big].$$

(By "internally representable" we mean that there exists an object $Y \in \mathrm{Ob}({}^*\mathcal{C})$ and an internal *isomorphism of functors between F and $\mathrm{Mor}_{{}^*\mathcal{C}}(\_, Y)$.)*

4. *If a presheaf F on $\mathcal{C}$ with values in $(N\text{-}\mathbf{Ens})$ is representable by an object $X \in \mathrm{Ob}(\mathcal{C})$, then ${}^*F$ is representable by ${}^*X$.*
5. *If all presheaves on $\mathcal{C}$ with values in $({}^*N\text{-}\mathbf{Ens})$ are representable, then all* internal *presheaves on ${}^*\mathcal{C}$ with values in $({}^*N\text{-}\mathbf{Ens})$ are representable.*

PROOF. 1 is easy. 2 follows directly from 2.8.1.
To prove 3, let $h : \mathcal{C} \to \mathrm{Funct}^\circ(\mathcal{C}, (N\text{-}\mathbf{Ens})), X \mapsto h_X$ denote the canonical fully faithful embedding where — as in 2 — h_X denotes the functor $\mathrm{Mor}_{\mathcal{C}}(\_, X) : \mathcal{C}^\circ \to (N\text{-}\mathbf{Ens})$. Then

$$\varphi[F, X] = \Big[\big(F \in \mathrm{Ob}(\mathrm{Funct}^\circ(\mathcal{C}, (N\text{-}\mathbf{Ens})))\big) \wedge \big(X \in \mathrm{Ob}(\mathcal{C})\big) \\ \wedge \big(\exists f \in \mathrm{Mor}_{\mathrm{Funct}^\circ(\mathcal{C},(N\text{-}\mathbf{Ens}))}(F, h(X)) \;:\; f \text{ isomorphism in } \mathrm{Funct}^\circ(\mathcal{C}, (N\text{-}\mathbf{Ens}))\big)\Big].$$

We then get

$$^*\varphi[F, X] \overset{2.7.4,2.8.1}{=} \Big[\big(F \in \mathrm{Ob}(^*\mathrm{Funct}^\circ(\mathcal{C}, (N\text{-}\mathbf{Ens})))\big) \wedge \big(X \in \mathrm{Ob}(^*\mathcal{C})\big) \\ \wedge \big(\exists f \in \mathrm{Mor}_{^*\mathrm{Funct}^\circ(\mathcal{C},(N\text{-}\mathbf{Ens}))}(F, (^*h)(X)) \;:\; f \text{ isomorphism in } ^*\mathrm{Funct}^\circ(\mathcal{C}, (N\text{-}\mathbf{Ens}))\big)\Big]$$

and by 2.12.1

$$^*\varphi[F, X] = \Big[\big(F \text{ is an internal presheaf on } \mathcal{C} \text{ with values in } (^*N\text{-}\mathbf{Ens})\big) \\ \wedge \big(X \in \mathrm{Ob}(^*C)\big) \wedge \big(F \text{ and } (^*h)(X) \text{ are internally isomorphic}\big)\Big].$$

What is $(^*h)(X)$?

$$\forall X, Y \in \mathrm{Ob}(\mathcal{C}) \;:\; [h(X)](Y) = \mathrm{Mor}_{\mathcal{C}}(Y, X) \\ \overset{\text{transfer},2.7.2,2.8.1}{\Longleftrightarrow} \forall X, Y \in \mathrm{Ob}(^*\mathcal{C}) \;:\; [(^*h)(X)](Y) = \mathrm{Mor}_{^*\mathcal{C}}(Y, X) = h_X(Y).$$

So we get:

$$^*\varphi[F, X] = \Big[F \text{ is an internal presheaf on } \mathcal{C} \text{ with values in } (^*N\text{-}\mathbf{Ens}) \\ \text{which is internally representable by } X\Big],$$

which is 3.

4 and 5 follows from 3. ∎

2.17. COROLLARY. *Let $* : \hat{S} \to \widehat{^*S}$ be an enlargement, let $\mathcal{I}$ be a* finite *$\hat{S}$-small category, and let $\mathcal{C}$ be an $\hat{S}$-small category. If for all functors $G : \mathcal{I} \to \mathcal{C}$ the projective limit $\varprojlim G$ (resp. inductive limit $\varinjlim G$) exists in $\mathcal{C}$, then for all functors $G : \mathcal{I} \to {}^*\mathcal{C}$, $\varprojlim G$ (resp. $\varinjlim G$) exists in ${}^*\mathcal{C}$.*

PROOF. Follows from 2.12.3 and 2.16.3. ∎

2.18. Corollary. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, and let $\mathcal{C}$ be an $\hat{S}$-small category. Then if $\mathcal{C}$ has one of the following properties, then so has ${}^*\mathcal{C}$:*

1. *$\mathcal{C}$ has an initial object.*
2. *$\mathcal{C}$ has a final object.*
3. *$\mathcal{C}$ has a null object.*
4. *Arbitrary finite direct sums exist in $\mathcal{C}$.*
5. *Arbitrary finite direct products exist in $\mathcal{C}$.*
6. *Arbitrary finite fibred sums exist in $\mathcal{C}$.*
7. *Arbitrary finite fibred products exist in $\mathcal{C}$.*
8. *Difference cokernels of two arbitrary morphisms exist in $\mathcal{C}$.*
9. *Difference kernels of two arbitrary morphisms exist in $\mathcal{C}$.*

2.19. Proposition. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, let $\mathcal{C}$ and $\mathcal{D}$ be $\hat{S}$-small categories, and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two covariant functors. If F is left adjoint to G, then ${}^*F : {}^*\mathcal{C} \to {}^*\mathcal{D}$ is left adjoint to ${}^*G : {}^*\mathcal{D} \to {}^*\mathcal{C}$.*

Proof. First of all, according to 2.12.1, ${}^*F$ is a functor from ${}^*\mathcal{C}$ to ${}^*\mathcal{D}$, and ${}^*G$ is a functor from ${}^*\mathcal{D}$ to ${}^*\mathcal{C}$.

Now choose a set $N \in \hat{S}$ that contains the (disjoint) union of the set of morphisms in $\mathcal{C}$ and the set of morphisms in $\mathcal{D}$, and consider the following two covariant functors $\alpha, \beta : \mathcal{C}^\circ \times \mathcal{D} \to (N\text{-}\mathbf{Ens})$:

$$\begin{tikzcd}
\mathcal{C}^\circ \times \mathcal{D} \arrow[rr, shift left] \arrow[rr, shift right] & & (N\text{-}\mathbf{Ens}) \\
 & & \mathrm{Mor}_{\mathcal{D}}(FX, Y) \\
(X, Y) \arrow[urr, mapsto, "\alpha"] \arrow[drr, mapsto, "\beta"'] & & \\
 & & \mathrm{Mor}_{\mathcal{C}}(X, GY).
\end{tikzcd}$$

Saying that F is left adjoint to G is equivalent to saying that α and β are isomorphic in the category $\mathrm{Funct}\,(\mathcal{C}^\circ \times \mathcal{D}, (N\text{-}\mathbf{Ens}))$ which is $\hat{S}$-small because of 2.5.1, 2 and 3. Because of 2.10, this is equivalent to ${}^*\alpha$ an ${}^*\beta$ being isomorphic in ${}^*\mathrm{Funct}\,(\mathcal{C}^\circ \times \mathcal{D}, (N\text{-}\mathbf{Ens}))$. Now we can apply 2.8.3 and 4, 2.12.1, and 2.15 to see that this is a subcategory of $\mathrm{Funct}\,(({}^*\mathcal{C})^\circ \times {}^*\mathcal{D}, ({}^*N\text{-}\mathbf{Ens}))$, i.e. if F is left adjoint to G, then ${}^*\alpha$ and ${}^*\beta$ are isomorphic in this latter category, and it is clear by transfer and by 2.8.1 that ${}^*\alpha$ and ${}^*\beta$ are given as follows:

$$\begin{tikzcd}({}^*\mathcal{C})^\circ \times {}^*\mathcal{D} \arrow[r, shift left] \arrow[r, shift right] & ({}^*N\text{-}\textsf{Ens})\end{tikzcd}$$

$$\begin{tikzcd}[column sep=large] & \mathrm{Mor}_{{}^*\mathcal{D}}(({}^*F)X, Y) \\ (X,Y) \arrow[ur, mapsto, "{}^*\alpha"] \arrow[dr, mapsto, "{}^*\beta"'] & \\ & \mathrm{Mor}_{{}^*\mathcal{C}}(X, ({}^*G)Y),\end{tikzcd}$$

i.e. the fact that ${}^*\alpha$ and ${}^*\beta$ are isomorphic is equivalent to ${}^*F$ being left adjoint to ${}^*G$. ■

3. Limits and enlargements

So far, so good! — But until now, the only property of enlargements we have used is the transfer principle, and therefore we have not revealed any properties of enlarged categories ${}^*\mathcal{C}$ that the original category $\mathcal{C}$ has not already had. This will change in the next proposition.

Remember that a category $\mathcal{I}$ is called *pseudo-cofiltered* if the following two conditions are satisfied (compare [SGA4I, I.2.7]):

- For any pair of morphisms $\varphi_1 : i_1 \to i$ and $\varphi_2 : i_2 \to i$, there are morphisms $\psi_1 : j \to i_1$ and $\psi_2 : j \to i_2$ in $\mathcal{I}$ such that $\varphi_1 \circ \psi_1 = \varphi_2 \circ \psi_2$: $\begin{tikzcd}[row sep=small] & i_1 \arrow[dr, "\varphi_1"] & \\ j \arrow[ur, dotted, "\exists\psi_1"] \arrow[dr, dotted, "\exists\psi_2"'] & & i \\ & i_2 \arrow[ur, "\varphi_2"'] & \end{tikzcd}$

- For any pair of morphisms $\varphi_1, \varphi_2 : i_1 \to i_2$ in $\mathcal{I}$, there is a morphism $\psi : j \to i_1$ in $\mathcal{I}$ such that $\varphi_1 \circ \psi = \varphi_2 \circ \psi$: $\begin{tikzcd} j \arrow[r, dotted, "\exists\psi"] & i_1 \arrow[r, shift left, "\varphi_1"] \arrow[r, shift right, "\varphi_2"'] & i_2\end{tikzcd}$

$\mathcal{I}$ is called *cofiltered* if it is pseudo-cofiltered, not empty and connected, where in this case being connected is equivalent to the condition that for any pair of objects $i_1, i_2 \in \mathrm{Ob}(\mathcal{I})$, there is a pair of morphisms $\psi_1 : j \to i_1$ and $\psi_2 : j \to i_2$: $\begin{tikzcd}[row sep=small] & i_1 \\ j \arrow[ur, dotted, "\exists\psi_1"] \arrow[dr, dotted, "\exists\psi_2"'] & \\ & i_2\end{tikzcd}$

A category $\mathcal{I}$ is called *pseudo-filtered*, if $\mathcal{I}^\circ$ is pseudo-cofiltered, and $\mathcal{I}$ is called *filtered* if $\mathcal{I}^\circ$ is cofiltered.

3.1. Proposition. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, and let $\mathcal{I}$ be an $\hat{S}$-small category.*

1. *If $\mathcal{I}$ is cofiltered, then there exists an object $i_{-\infty} \in \mathrm{Ob}({}^*\mathcal{I})$, together with a family of morphisms $\{p_i : i_{-\infty} \to {}^*i\}_{i \in \mathrm{Ob}(\mathcal{I})}$ with the property that for all morphisms $i_1 \xrightarrow{\varphi} i_2$*

in $\mathcal{I}$, the following triangle of morphisms in ${}^*\mathcal{I}$ commutes:

$$\begin{tikzcd} & {}^*i_1 \arrow[dd, "{}^*\varphi"] \\ i_{-\infty} \arrow[ur, "p_{i_1}"] \arrow[dr, "p_{i_2}"'] & \\ & {}^*i_2 \end{tikzcd}$$

2. *If $\mathcal{I}$ is filtered, then there exists an object $i_\infty \in \mathrm{Ob}({}^*\mathcal{I})$, together with a family of morphisms $\{\iota_i : {}^*i \to i_\infty\}_{i \in \mathrm{Ob}(\mathcal{I})}$ with the property that for all morphisms $i_1 \xrightarrow{\varphi} i_2$ in $\mathcal{I}$, the following triangle of morphisms in ${}^*\mathcal{I}$ commutes:*

$$\begin{tikzcd} i_1 \arrow[dr, "\iota_{i_1}"] \arrow[dd, "{}^*\varphi"'] & \\ & i_\infty \\ i_2 \arrow[ur, "\iota_{i_2}"'] & \end{tikzcd}$$

Proof. We only give a proof for 1, the proof for 2 follows immediately by looking at the opposite category $\mathcal{I}^\circ$. Because a filtered category has all cones of finite subdiagramms (comp. [Bor94][Lemma 2.13.2]) we have the following statement:

> Let $\mathcal{J}$ be a finite subsystem of $\mathcal{I}$. By this we mean a selection of finitely many objects of $\mathcal{I}$ and of finitely many morphisms between those objects (note that $\mathcal{J}$ in general will not be a subcategory of $\mathcal{I}$). Then there exists an object $i_\mathcal{J}$ of $\mathcal{I}$ and a family of morphisms $p_j^\mathcal{J} : i_\mathcal{J} \to j$ for every object j of $\mathcal{I}$ that is in $\mathcal{J}$ such that for any morphism $j_1 \xrightarrow{\varphi} j_2$ contained in $\mathcal{J}$, the following triangle of morphisms in $\mathcal{I}$ commutes: (1)
>
> $$\begin{tikzcd} & j_1 \arrow[dd, "\varphi"] \\ i_\mathcal{J} \arrow[ur, "p_{j_1}^\mathcal{J}"] \arrow[dr, "p_{j_2}^\mathcal{J}"'] & \\ & j_2 \end{tikzcd}$$

Let $M \in \hat{S}$ denote the set of all morphisms in $\mathcal{I}$, and let F denote the set of all finite subsystems of $\mathcal{I}$ — we certainly have $F \in \hat{S}$. For $\mathcal{J} \in F$, define a set $U_\mathcal{J}$ as follows:

$$U_\mathcal{J} = \Big\{ \langle i, p \rangle \,\Big|\, i \in \mathrm{Ob}(\mathcal{I}), p \in M^{\mathrm{Ob}(\mathcal{I})}, \forall j \in \{\text{objects in } \mathcal{J}\} : \\ p(j) \in \mathrm{Mor}_\mathcal{I}(i, j), \forall j_1 \xrightarrow{\varphi} j_2 \in \{\text{morphisms in } \mathcal{J}\} : p(j_2) = \varphi \circ p(j_1) \Big\}$$

Statement (1) shows two things:

- For all $\mathcal{J} \in F$, the set $U_\mathcal{J}$ is not empty (note that for objects i not contained in $\mathcal{J}$, $p(i)$ can be chosen arbitrarily in M, we can for example set $p(i) := \mathrm{id}_i$).
- For $\mathcal{J}_1, \ldots, \mathcal{J}_n \in F$, the intersection $U_{\mathcal{J}_1} \cap \ldots \cap U_{\mathcal{J}_n}$ is not empty (apply (1) to $\mathcal{J} := \mathcal{J}_1 \cup \ldots \cup \mathcal{J}_n$).

According to the saturation principle A.8.4, these imply that the intersection $\bigcap_{\mathcal{J} \in F} {}^*U_\mathcal{J}$ is not empty, i.e. we find a pair $\langle i_{-\infty}, p \rangle$, consisting of an object $i_{-\infty} \in \mathrm{Ob}({}^*\mathcal{I})$ and an internal map $p : \mathrm{Ob}({}^*\mathcal{I}) \to {}^*M$ in this intersection. For an object i of $\mathcal{I}$, there certainly is a $\mathcal{J} \in F$ containing i, and because $\langle i_{-\infty}, p \rangle$ is in ${}^*U_\mathcal{J}$, we conclude that $p({}^*i)$ is a morphism from $i_{-\infty}$ to ${}^*i$ and can thus set $p_i := p({}^*i)$.

If $i_1 \xrightarrow{\varphi} i_2$ is any morphism in $\mathcal{I}$, we find a $\mathcal{J} \in F$ that contains i_1, i_2 and φ, and the fact that $\langle i_{-\infty}, p \rangle$ is an element of ${}^*U_\mathcal{J}$ implies that the associated triangle commutes. This concludes the proof of the proposition. ∎

3.2. Example. (Construction of algebraic closures)
Let k be a field. To construct an algebraic closure of k, it suffices to construct an extension K of k such that K contains all finite algebraic extensions of k, because then taking the algebraic closure of k in K yields an algebraic closure of k.

To construct such a field K, choose a set of sets U such that for every finite algebraic extension L of k, there is a bijection from the set which underlies L to an element of U.

Consider the category $\mathcal{C}$ whose objects are pairs $\langle L, s\rangle$ with L an element of U and s a field-structure on L turning L into a finite algebraic extension of k, and whose morphisms are defined as

$$\mathrm{Mor}_{\mathcal{C}}(\langle L_1, s_1\rangle, \langle L_2, s_2\rangle) := \begin{cases} \{*\} & \text{if there is a } k\text{-embedding } L_1 \hookrightarrow L_2, \\ \emptyset & \text{otherwise.} \end{cases}$$

We then can find a base set S such that $\mathcal{C}$ is (isomorphic to) an $\hat{S}$-small category and such that U is an element of $\hat{S} \setminus S$ (compare 2.3.1).

It is easy to see that $\mathcal{C}$ is *filtered*, so that we find an object i_∞ of ${}^*\mathcal{C}$ and morphisms $\iota_L : {}^*L \to i_\infty$ for every object L of $\mathcal{C}$ as in 3.1.2.

By transfer, i_∞ defines a pair $\langle K, s\rangle$ where K is a set (and an element of ${}^*U$) and s is a field-structure on K. The existence of the ι_L and transfer show that we have k-embeddings $L \hookrightarrow K$ for all finite algebraic extensions L of k, and we are done.

Note that in order to prove the existence of enlargements (compare A.9), methods similar to the construction of algebraic closures in elementary algebra are used; therefore it is not really surprising that we are able to give a short proof for the existence of algebraic closures as soon as we have the powerful tool of enlargements at our disposal.

3.3. Corollary. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, let $\mathcal{I}$ be an $\hat{S}$-small category, and let $\varphi[X]$ be a formula in $\hat{S}$.*

1. *Assume that $\mathcal{I}$ is cofiltered, let $i_{-\infty}$ and $\{p_i\}$ be as in 3.1.1, and assume that for all morphisms $\psi : i \to j$ in $\mathcal{I}$, statement $\varphi[i]$ implies $\varphi[j]$. Then*
$$\Big(\forall i \in \mathrm{Ob}(\mathcal{I}) : \varphi[i]\Big) \quad \Longleftrightarrow \quad {}^*\varphi[i_{-\infty}].$$
2. *Assume that $\mathcal{I}$ is filtered, let i_∞ and $\{\iota_i\}$ be as in 3.1.2, and assume that for all morphisms $\psi : i \to j$ in $\mathcal{I}$, statement $\varphi[j]$ implies $\varphi[i]$. Then*
$$\Big(\forall i \in \mathrm{Ob}(\mathcal{I}) : \varphi[i]\Big) \quad \Longleftrightarrow \quad {}^*\varphi[i_\infty].$$

Proof. We only have to prove 1, because 2 follows from this by looking at $\mathcal{I}^\circ$. If $\varphi[i]$ is true for all $i \in \mathrm{Ob}(\mathcal{I})$, then by transfer and by 2.8.1, ${}^*\varphi[i]$ is true for all $i \in \mathrm{Ob}({}^*\mathcal{I})$, so it is in particular true for $i := i_{-\infty} \in \mathrm{Ob}({}^*\mathcal{I})$. On the other hand, by assumption, the following statement is true in $\hat{S}$:

$$\forall j, k \in \mathrm{Ob}(\mathcal{I}) : \Big[\mathrm{Mor}_{\mathcal{I}}(j,k) \neq \emptyset\Big] \Rightarrow \Big[\varphi[j] \Rightarrow \varphi[k]\Big],$$

and transfer of this gives us

$$\forall j, k \in \mathrm{Ob}({}^*\mathcal{I}) : \Big[\mathrm{Mor}_{{}^*\mathcal{I}}(j,k) \neq \emptyset\Big] \Rightarrow \Big[{}^*\varphi[j] \Rightarrow {}^*\varphi[k]\Big].$$

If we specialize this to $j := i_{-\infty}$ and $k := {}^*i$ for an $i \in \mathrm{Ob}(\mathcal{I})$ and note that $\mathrm{Mor}_{{}^*\mathcal{I}}(i_{-\infty}, {}^*i)$ is not empty (because it contains p_i), we get $\big[{}^*\varphi[i_{-\infty}] \Rightarrow {}^*\varphi[{}^*i]\big]$. But by transfer, the statements ${}^*\varphi[{}^*i]$ and $\varphi[i]$ are equivalent, and this completes the proof. ∎

3.4. Example. Let X be a topological space, let $U \subseteq X$ be an open subspace, and let $\mathcal{F}$ be a presheaf on X.

Recall that a *covering* of U is a family $\mathcal{U} = (U_j)_{j\in\mathcal{J}}$ of open subspaces of U with $\bigcup_{j\in\mathcal{J}} U_j = U$, and a *refinement* of $\mathcal{U}$ is a covering $\mathcal{V} = (V_k)_{k\in\mathcal{K}}$ of U, such that there exists a map $f : \mathcal{K} \to \mathcal{J}$ satisfying $V_k \subseteq U_{f(k)}$ for all $k \in \mathcal{K}$. Consider the following partially ordered set $\mathcal{I}$: An element of $\mathcal{I}$ is represented by a covering of U, and two coverings define the same element if each is a refinement of the other. The partial order is defined by

$$\mathcal{V} \leq \mathcal{U} :\Leftrightarrow \mathcal{V} \text{ is a refinement of } \mathcal{U}.$$

If $\mathcal{U}$ and $\mathcal{V}$ are coverings of U, then $\mathcal{U} \cap \mathcal{V} := (U_j \cap V_k)_{(j,k)\in\mathcal{J}\times\mathcal{K}}$ is obviously a covering of U and a refinement of both $\mathcal{U}$ and $\mathcal{V}$, which shows that $\mathcal{I}$, considered as a category, is cofiltered.

For a covering $\mathcal{U}$ of U, recall the *sheaf condition for $\mathcal{F}$ with respect to $\mathcal{U}$*, which states that the following sequence is exact:

$$\mathcal{F}(U) \to \prod_{j\in\mathcal{J}} \mathcal{F}(U_j) \rightrightarrows \prod_{i,j\in\mathcal{J}} \mathcal{F}(U_i \cap U_j).$$

It is easy to see that if $\mathcal{V}$ is a refinement of $\mathcal{U}$ and if $\mathcal{F}$ satisfies the sheaf condition with respect to $\mathcal{V}$, then it also satisfies the sheaf condition with respect to $\mathcal{U}$. This in particular implies that if $\mathcal{U}$ and $\mathcal{V}$ are coverings of U that define the same element of $\mathcal{I}$, then $\mathcal{F}$ satisfies the sheaf condition with respect to $\mathcal{U}$ iff it satisfies the sheaf condition with respect to $\mathcal{V}$. It therefore makes sense to say that $\mathcal{F}$ satisfies the sheaf condition with respect to an element of $\mathcal{I}$.

Now let $\hat{S}$ be a superstructure that contains the small category $\mathcal{I}$, and let $\varphi[X]$ be the following formula in $\hat{S}$:

$$\varphi[X] \equiv \Big[X \text{ is an element of } \mathcal{I}, \text{ and } \mathcal{F} \text{ satisfies the sheaf condition with respect to } X\Big].$$

By proposition 3.1.1, we find an object $i_{-\infty}$ of ${}^*\mathcal{I}$, represented by a “hyper covering” $\mathcal{U} = (U_j)_{j\in\mathcal{J}}$ of ${}^*U$, where $\mathcal{U}$ is an internal family of *open subsets of ${}^*U$ which covers ${}^*U$ and refines all standard coverings of $\mathcal{U}$.

Applying corollary 3.3.1 to this situation, we get that $\mathcal{F}$ satisfies the sheaf condition with respect to *every covering* iff ${}^*\mathcal{F}$ satisfies the *sheaf condition for the *one hyper covering* $\mathcal{U}$. This latter condition is obviously satisfied iff the following two conditions are both satisfied:

1. If $s, t \in {}^*(\mathcal{F}(U)) = ({}^*\mathcal{F})(U)$ satisfy $s|U_j = t|U_j$ for all $j \in \mathcal{J}$, then $s = t$.
2. If $(s_j)_{j\in\mathcal{J}}$ is an internal family with $s_j \in ({}^*\mathcal{F})(U_j)$, satisfying $s_j|U_j \cap U_{j'} = s_{j'}|U_j \cap U_{j'}$ for all $j, j' \in \mathcal{J}$, then there exists an $s \in {}^*(\mathcal{F}(U))$ with $s_j = s|U_j$ for all $j \in \mathcal{J}$.

3.5. Corollary. *Let* $* : \hat{S} \to \widehat{{}^*S}$ *be an enlargement, let* $\mathcal{I}$ *be an* $\hat{S}$*-small category, let* $N \in \hat{S}$ *be a set of sets, and let* $F : \mathcal{I} \to (N\text{-}\mathbf{Ens})$ *be a covariant functor.*

1. *Assume that* $\mathcal{I}$ *is cofiltered, and let* $i_{-\infty}$ *be as in 3.1.1. Then we have a canonical injection* $\mu : \varprojlim F \hookrightarrow {}^*F(i_{-\infty})$*, and for every object* $i \in \mathrm{Ob}(\mathcal{I})$*, we have a commutative diagram*

$$\begin{tikzcd} & {}^*F(i_{-\infty}) \arrow[dr, "{}^*F(p_i)"] & \\ \varprojlim F \arrow[ur, hook, dashed, "\mu"] \arrow[rr, "(x_i)_{i\in\mathrm{Ob}(\mathcal{I})} \mapsto {}^*x_i"'] & & {}^*F({}^*i) \end{tikzcd} \tag{2}$$

2. *Assume that* $\mathcal{I}$ *is filtered, and let* i_∞ *and* $\{\iota_i\}$ *be as in 3.1.2. Then we have a canonical injection* $\nu : \varinjlim F \hookrightarrow {}^*F(i_\infty)$.

Proof. In the situation of 1, let x be an element of $\varprojlim F$, given by a family $(x_i)_{i\in\mathrm{Ob}(\mathcal{I})}$ of elements $x_i \in F(i)$ such that for all morphisms $\varphi : i \to j$ in $\mathcal{I}$, the element x_i is mapped to x_j under $F(\varphi)$. If we consider each x_i as a morphism $\{*\} \to F(i)$ in $(N\text{-}\mathbf{Ens})$, we consequently can consider x as a map $x : \mathrm{Ob}(\mathcal{I}) \to \mathrm{Mor}_{(N\text{-}\mathbf{Ens})}$ with the property

$$\Big[\forall i \in \mathrm{Ob}(\mathcal{I}) : x(i) \in \mathrm{Mor}_{(N\text{-}\mathbf{Ens})}(\{*\}, Fi)\Big] \\ \wedge \Big[\forall i, j \in \mathrm{Ob}(\mathcal{I}) : \forall \varphi \in \mathrm{Mor}_{\mathcal{I}}(i, j) : x(j) = F\varphi \circ x(i)\Big].$$

Then by transfer, ${}^*x$ is a map from $\mathrm{Ob}({}^*\mathcal{I})$ to $\mathrm{Mor}_{{}^*(N\text{-}\mathbf{Ens})} \overset{2.15}{\hookrightarrow} \mathrm{Mor}_{({}^*N\text{-}\mathbf{Ens})}$ satisfying

$$\Big[\forall i \in \mathrm{Ob}({}^*\mathcal{I}) : x(i) \in \mathrm{Mor}_{({}^*N\text{-}\mathbf{Ens})}(\{*\}, {}^*F(i))\Big] \\ \wedge \Big[\forall i, j \in \mathrm{Ob}({}^*\mathcal{I}) : \forall \varphi \in \mathrm{Mor}_{{}^*\mathcal{I}}(i, j) : x(j) = {}^*F(\varphi) \circ x(i)\Big],$$

i.e. by setting $\tilde{x}_i := \big[{}^*x(i)\big](*) \in {}^*F(i)$ for all objects $i \in \mathrm{Ob}({}^*I)$, we get a compatible system $(\tilde{x}_i)_{i\in\mathrm{Ob}({}^*I)}$, and we define $\mu(x)$ as $\tilde{x}_{i_{-\infty}}$.

To see that the so defined map μ is injective, let $y = (y_i)$ be another element of P with the property $\mu(x) = \mu(y)$. We have to show that $x_i = y_i$ for all objects i of $\mathcal{I}$. Look at the formula $\varphi[X] \equiv \big[x_X = y_X\big]$ in $\hat{S}$. We have to show that the statement $\varphi[i]$ is true for all $i \in \mathrm{Ob}(\mathcal{I})$. If $\varphi : i \to j$ is a morphism in $\mathcal{I}$, then $\varphi[i]$ implies $\varphi[j]$, so we can apply 3.3.1, i.e. we have to show that ${}^*\varphi[i_{-\infty}]$ is true in $\widehat{{}^*S}$. But obviously ${}^*\varphi[X] = \big[\tilde{x}_X = \tilde{y}_X\big]$, so ${}^*\varphi[i_{-\infty}]$ is true because of $\mu(x) = \mu(y)$. So we see that μ indeed is injective.

In the situation of 2, let x be an element of $\varinjlim F$, represented by an element x_j of $F(j)$ for an $j \in \mathrm{Ob}(\mathcal{I})$. Put $\nu(x) := \iota_j({}^*x_j)$. To see that this is well defined, it suffices to show that for a morphism $\varphi : j \to k$ in $\mathcal{I}$ we have $\iota_j({}^*x_j) = \iota_k\bigl[{}^*(\varphi(x_j))\bigr]$, but this is clear by definition of the $\{\iota_i\}$. So we get a well defined map $\nu : \varinjlim F \to F(i_\infty)$.

To show that ν is injective, let y be another element of $\varinjlim F$, represented by $y_k \in F(k)$, with $x \neq y$. We have to show $\nu(x) \neq \nu(y)$. Look at the formula

$$\varphi[X] \equiv \Bigl[\forall \varphi \in \mathrm{Mor}_{\mathcal{I}}(j, X) : \forall \psi \in \mathrm{Mor}_{\mathcal{I}}(k, X) : \varphi(x_j) \neq \psi(y_k)\Bigr].$$

Because we assume $x \neq y$, we know that $\varphi[i]$ is true for all $i \in \mathrm{Ob}(\mathcal{I})$. Then by transfer ${}^*\varphi[i]$ must be true for all $i \in \mathrm{Ob}({}^*\mathcal{I})$, so that in particular ${}^*\varphi[i_\infty]$ is true in $\widehat{{}^*S}$ (to see this, we could have also applied 3.3.2). Now

$$\varphi[i_\infty] = \Bigl[\forall \varphi \in \mathrm{Mor}_{\mathcal{I}}({}^*j, i_\infty) : \forall \psi \in \mathrm{Mor}_{\mathcal{I}}({}^*k, i_\infty) : \varphi({}^*x_j) \neq \psi({}^*y_k)\Bigr],$$

and if we specialize this to $\varphi := \iota_j$ and $\psi := \iota_k$, we see that in particular $\nu(x) \neq \nu(y)$ is true, i.e. ν is indeed injective. ∎

Let $\mathcal{I}$ be a cofiltered category, let $\mathcal{C}$ be any category, and let $G : \mathcal{I} \to \mathcal{C}$ be a covariant functor. Remember that by definition, the projective limit $\varprojlim G$ is the presheaf $X \mapsto \varprojlim_{i \in \mathrm{Ob}(\mathcal{I})} \mathrm{Mor}_{\mathcal{C}}(X, Gi)$ on $\mathcal{C}$, and that we say that "the projective limit exists in $\mathcal{C}$" if $\varprojlim G$ is representable by an object P of $\mathcal{C}$.

Assume for a moment that $\mathcal{I}$ has an initial object i_0. In that case, $\varprojlim G$ obviously exists in $\mathcal{C}$, and it is represented by Gi_0, i.e. we have a commutative diagram of presheaves

$$\begin{tikzcd} \varprojlim G \arrow[r, "\sim"] \arrow[d, hook, "\prod(\mathrm{proj}_i)"'] & h_{Gi_0} \arrow[d, hook, "\prod G(i_0 \to i)"] \\ \prod_{i \in \mathrm{Ob}(\mathcal{I})} h_{Gi} \arrow[r, equal] & \prod_{i \in \mathrm{Ob}(\mathcal{I})} h_{Gi}. \end{tikzcd}$$

Now, in general $\mathcal{I}$ will not have an initial object, but we have seen in 3.1.1 that in the category ${}^*\mathcal{I}$, there is an object $i_{-\infty}$ which is "nearly as good as an initial object for $\mathcal{I}$", and we could hope that in the following commuting diagram of presheaves

$$\begin{tikzcd} \varprojlim G \arrow[r] \arrow[d, hook, "\prod(\mathrm{proj}_i)"'] & \tilde{G} := \bigl[X \mapsto \mathrm{Mor}_{{}^*\mathcal{C}}(X, {}^*Gi_{-\infty})\bigr] \arrow[d, hook, "\prod p_i"] \\ \prod_{i \in \mathrm{Ob}(\mathcal{I})} h_{Gi} \arrow[r, "*"'] & \prod_{i \in \mathrm{Ob}(\mathcal{I})} \bigl[X \mapsto \mathrm{Mor}_{{}^*\mathcal{C}}({}^*X, {}^*Gi)\bigr] \end{tikzcd}$$

the morphism $\varprojlim G \to \tilde{G}$ is an isomorphism. Unfortunately, there are two problems with this: First, for an object X of $\mathcal{C}$, an object i of $\mathcal{I}$ and a section $\varphi \in \tilde{G}(X)$, $p_{i*}(\varphi)$ must be of the form ${}^*\varphi_i$ for a suitable $\varphi_i \in \mathrm{Mor}_{\mathcal{C}}(X, Gi)$, and there is no reason why this should be the case.

Second, remember that the pair $\langle i_{-\infty}, (p_i)_i\rangle$ is in general not uniquely determined. For example, for any morphism $p : i'_{-\infty} \to i_{-\infty}$ in ${}^*\mathcal{I}$, the pair $\langle i'_{-\infty}, (p_i \circ p)_i\rangle$ has the same property.

Now let $\tilde{G}'$ be the presheaf $\left[X \mapsto \mathrm{Mor}_{{}^*\mathcal{C}}(X, {}^*Gi'_{-\infty})\right]$, and for an object X of $\mathcal{C}$, let s and t be two distinct sections in $\tilde{G}'(X)$. If p is not a monomorphism, it can happen that s and t become equal in $\tilde{G}(X)$, and this shows that $\prod p_{i*}$ in general will not be a monomorphism.

To solve the first problem, we will define a subpresheaf $\tilde{G}^{\mathrm{fin}}$ of $\tilde{G}$ whose sections are mapped to something of the right form under $\prod p_{i*}$. To solve the second problem, we will introduce an equivalence relation $\sim$ on $\tilde{G}^{\mathrm{fin}}$ such that two section of $\tilde{G}$ that are mapped to the same section under $\prod p_{i*}$ will be equivalent.

Having done that, we will be able to prove $\varprojlim G \cong \tilde{G}^{\mathrm{fin}}/{\sim}$.

3.6. Corollary. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, and let $G : \mathcal{I} \to \mathcal{C}$ be a covariant functor of $\hat{S}$-small categories.*

1. *Assume that $\mathcal{I}$ is cofiltered, and let $i_{-\infty}$ and $\{p_i\}$ be as in 3.1.1. Define the presheaf $\tilde{G}$ on $\mathcal{C}$ by $\tilde{G}(X) := \mathrm{Mor}_{{}^*\mathcal{C}}({}^*X, ({}^*G)(i_{-\infty}))$ for $X \in \mathrm{Ob}(\mathcal{C})$, define the subpresheaf $\tilde{G}^{\mathrm{fin}}$ of $\tilde{G}$ by*

$$\tilde{G}^{\mathrm{fin}}(X) := \left\{ \varphi \in \tilde{G}(X) \,\middle|\, \forall i \in \mathrm{Ob}(\mathcal{I}) : \exists \varphi_i \in \mathrm{Mor}_{\mathcal{C}}(X, G(i)) : ({}^*G)(p_i) \circ \varphi = {}^*\varphi_i \right\}$$

for $X \in \mathrm{Ob}(\mathcal{C})$, and define an equivalence relation $\sim$ on $\tilde{G}^{\mathrm{fin}}$ as follows:

$$\forall X \in \mathrm{Ob}(\mathcal{C}) : \forall \varphi, \psi \in \tilde{G}^{\mathrm{fin}}(X) : \varphi \sim \psi :\Leftrightarrow \forall i \in \mathrm{Ob}(\mathcal{I}) : ({}^*G)(p_i) \circ \varphi = ({}^*G)(p_i) \circ \psi.$$

Then we have a canonical monomorphism $\varprojlim G \hookrightarrow \tilde{G}$ of presheaves on $\mathcal{C}$ that induces an isomorphism $\varprojlim G \xrightarrow{\sim} \tilde{G}^{\mathrm{fin}}/{\sim}$. If in addition $\mathrm{Mor}_{\mathcal{C}}(X, G(i))$ *is* finite *for all $X \in \mathrm{Ob}(\mathcal{C})$ and all $i \in \mathrm{Ob}(\mathcal{I})$, then $\tilde{G}^{\mathrm{fin}} = \tilde{G}$.*

The following diagram of presheaves on $\mathcal{C}$ commutes:

$$\begin{tikzcd} \varprojlim G \arrow[r, "\sim"] \arrow[d, hook, "\prod(\mathrm{proj}_i)"'] & \tilde{G}^{\mathrm{fin}}/{\sim} \arrow[d, hook, "\prod p_i"] \\ \prod_{i \in \mathrm{Ob}(\mathcal{I})} h_{Gi} \arrow[r, "*"'] & \prod_{i \in \mathrm{Ob}(\mathcal{I})} \left[X \mapsto \mathrm{Mor}_{{}^*\mathcal{C}}({}^*X, {}^*Gi)\right]. \end{tikzcd} \tag{3}$$

2. *Assume that $\mathcal{I}$ is filtered, and let i_∞ and $\{\iota_i\}$ be as in 3.1.2. Define the presheaf $\tilde{G}$ on $\mathcal{C}^\circ$ by $\tilde{G}(X) := \mathrm{Mor}_{{}^*\mathcal{C}}(({}^*G)(i_\infty), {}^*X)$ for $X \in \mathrm{Ob}(\mathcal{C})$, define the subpresheaf $\tilde{G}^{\mathrm{fin}}$ of $\tilde{G}$ by*

$$\tilde{G}^{\mathrm{fin}}(X) := \left\{ \varphi \in \tilde{G}(X) \,\middle|\, \forall i \in \mathrm{Ob}(\mathcal{I}) : \exists \varphi_i \in \mathrm{Mor}_{\mathcal{C}}(G(i), X) : \varphi \circ ({}^*G)(\iota_i) = {}^*\varphi_i \right\}$$

for $X \in \mathrm{Ob}(\mathcal{C})$*, and define an equivalence relation* $\sim$ *on* $\tilde{G}^{\mathrm{fin}}$ *as follows:*

$$\forall X \in \mathrm{Ob}(\mathcal{C}) : \forall \varphi, \psi \in \tilde{G}^{\mathrm{fin}}(X) : \varphi \sim \psi :\Leftrightarrow \forall i \in \mathrm{Ob}(\mathcal{I}) : \varphi \circ ({}^*G)(\iota_i) = \psi \circ ({}^*G)(\iota_i).$$

Then we have a canonical monomorphism $\varinjlim G \hookrightarrow \tilde{G}$ *of presheaves on* $\mathcal{C}^\circ$ *that induces an isomorphism* $\varinjlim G \xrightarrow{\sim} \tilde{G}^{\mathrm{fin}}/{\sim}$*. If in addition* $\mathrm{Mor}_{\mathcal{C}}(G(i), X)$ *is* finite *for all* $X \in \mathrm{Ob}(\mathcal{C})$ *and all* $i \in \mathrm{Ob}(\mathcal{I})$*, then* $\tilde{G}^{\mathrm{fin}} = \tilde{G}$*.*

For every $i \in \mathrm{Ob}(\mathcal{I})$*, the following diagram of presheaves on* $\mathcal{C}^\circ$ *commutes:*

$$\begin{tikzcd}
\varinjlim G \arrow[rr, "\sim"] \arrow[d, hook, "\prod(\mathrm{proj}_i)"'] & & \tilde{G}^{\mathrm{fin}}/{\sim} \arrow[d, hook, "\prod \iota_i"] \\
\prod_{i \in \mathrm{Ob}(\mathcal{I})} h_{Gi} \arrow[r, "*"'] & \prod_{i \in \mathrm{Ob}(\mathcal{I})} \big[X \mapsto \mathrm{Mor}_{{}^*\mathcal{C}}({}^*Gi, {}^*X)\big]. &
\end{tikzcd} \tag{4}$$

Proof. We only prove 1, because 2 follows from 1 by considering $\mathcal{I}^\circ$ and $\mathcal{C}^\circ$. Let X be an object of $\mathcal{C}$. The first claim is equivalent to the statement that for every object $X \in \mathrm{Ob}(\mathcal{C})$ there is a canonical injection

$$\alpha_X : \varprojlim \mathrm{Mor}_{\mathcal{C}}(X, G(i)) \hookrightarrow \mathrm{Mor}_{{}^*\mathcal{C}}({}^*X, ({}^*G)(i_{-\infty}))$$

which is functorial in X. To see this, apply 3.5.1 to $N := \mathrm{Mor}_{\mathcal{C}}$ and $F : \mathcal{I} \to (N\text{-}\mathbf{Ens})$, $i \mapsto \mathrm{Mor}_{\mathcal{C}}(X, Gi)$ (the functorality is clear by construction). We thus get a monomorphism $\varprojlim G \hookrightarrow \tilde{G}$ of presheaves on $\mathcal{C}$ which we call α.

We have already proved that the image of this monomorphism is contained in $\tilde{G}^{\mathrm{fin}}$ — this is just (2), so that we get an induced map $\varprojlim G \xrightarrow{\bar{\alpha}} \tilde{G}^{\mathrm{fin}}/{\sim}$, and (2) obviously also shows that $\bar{\alpha}$ is injective. Furthermore, it is clear by construction that (3) indeed is a well-defined, commuting diagram of presheaves on $\mathcal{C}$.

It remains to be seen that $\bar{\alpha}$ is surjective. So let $p : {}^*X \to ({}^*G)(i_{-\infty})$ be an element of $\tilde{G}^{\mathrm{fin}}(X)$. For $i \in \mathrm{Ob}(\mathcal{I})$, by definition of $\tilde{G}^{\mathrm{fin}}$, there exists a $\varphi_i : X \to G(i)$ such that $({}^*G)(p_i) \circ p = {}^*\varphi_i$. If $i \xrightarrow{\psi} j$ is a morphism in $\mathcal{I}$, then by definition of the p_i we have

$$\begin{aligned}
& \psi \circ p_i = p_j \\
\Rightarrow \quad & ({}^*G)({}^*\psi) \circ ({}^*G)(p_i) = ({}^*G)(p_j) \\
\Rightarrow \quad & ({}^*G)({}^*\psi) \circ \big[({}^*G)(p_i) \circ p\big] = \big[({}^*G)(p_j) \circ p\big] \\
\Rightarrow \quad & ({}^*G)({}^*\psi)({}^*\varphi_i) = {}^*\varphi_j \\
\Leftrightarrow \quad & G(\psi)(\varphi_i) = \varphi_j
\end{aligned}$$

which proves that $\varphi := \{\varphi_i\}_{i \in \mathrm{Ob}(\mathcal{I})}$ defines an element of $(\varprojlim G)(X)$. By construction we have $\alpha_X(\varphi) \sim p$, i.e. φ is a preimage of p under $\bar{\alpha}$. This shows that $\bar{\alpha}$ indeed is an isomorphism of presheaves on $\mathcal{C}$.

Finally, let us assume that $\mathrm{Mor}_{\mathcal{C}}(X, G(i))$ is finite for all $X \in \mathrm{Ob}(\mathcal{C})$ and $i \in \mathrm{Ob}(\mathcal{I})$. Then $*$ induces bijections $\mathrm{Mor}_{\mathcal{C}}(X, G(i)) \xrightarrow{\sim} \mathrm{Mor}_{{}^*\mathcal{C}}({}^*X, ({}^*G)({}^*i))$ because of A.10.2, i.e. all elements on the right hand side are of the form ${}^*\varphi$ for a suitable φ, so that all sections of $\tilde{G}$ belong to $\tilde{G}^{\mathrm{fin}}$. ∎

3.7. Example.

1. Let $\mathcal{C}$ be the category of finite rings (whose elements are contained in the set $\mathbb{Z}$, say), let $\mathcal{I}$ be the category of the ordered set $\mathbb{N}_+$ (ordered by $\geq$), let $\hat{S}$ be a superstructure that contains $\mathcal{I}$ and $\mathcal{C}$ as $\hat{S}$-small categories, let p be a prime number, let $G : \mathcal{I} \to \mathcal{C}$ be the functor $n \mapsto \mathbb{Z}/p^n\mathbb{Z}$, and let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement.

   Then $\mathcal{I}$ is cofiltered, and we can apply 3.6.1 to $\varprojlim G$: $\mathcal{I}$ is the category of the ordered set ${}^*\mathbb{N}$, as $i_{-\infty}$, we can choose any $h \in {}^*\mathbb{N} \setminus \mathbb{N}$, the category ${}^*\mathcal{C}$ is the category of *finite rings (with internal ring-structure, whose elements are contained in ${}^*\mathbb{Z}$) and internal ring-homomorphisms as morphisms, and ${}^*G$ is the functor that sends h to ${}^*\mathbb{Z}/p^{h*}\mathbb{Z}$. The corollary then tells us (because all the sets of morphisms in $\mathcal{C}$ are finite) that to give a compatible system of ring-homomorphisms from a finite Ring R into every $\mathbb{Z}/p^n\mathbb{Z}$ is the same as giving a ring-homomorphism (which is automatically internal because R is finite) from R into the *finite ring ${}^*\mathbb{Z}/p^{h*}\mathbb{Z}$ where two such ring-homomorphisms give the same compatible system if and only if their compositions with all projections ${}^*\mathbb{Z}/p^{h*}\mathbb{Z} \twoheadrightarrow \mathbb{Z}/p^n\mathbb{Z}$ coincide. This is equivalent to the existence of a ring-isomorphism $\mathbb{Z}_p \xrightarrow{\sim} \left({}^*\mathbb{Z}/p^{h*}\mathbb{Z}\right)/I$ where I is the (external) ideal generated by $\{p^k | k \in {}^*\mathbb{N} \setminus \mathbb{N}\}$.

2. Let $\mathcal{C}$ be a small category of abelian groups that contains all $\mathbb{Z}/n\mathbb{Z}$ for $n \in \mathbb{N}_+$ and $\mathbb{Q}/\mathbb{Z}$ as objects, let $\mathcal{I}$ be the category of the ordered set $\mathbb{N}_+$, (now ordered by $|$), let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement where $\hat{S}$ contains $\mathcal{C}$ and $\mathcal{I}$ as $\hat{S}$-small categories, and let $G : \mathcal{I} \to \mathcal{C}$ be the functor that sends $n \in \mathbb{N}_+$ to $\mathbb{Z}/n\mathbb{Z}$ and a morphism $m \mid n$ in $\mathcal{I}$ to $\mathbb{Z}/m\mathbb{Z} \xrightarrow{1 \mapsto n/m} \mathbb{Z}/n\mathbb{Z}$.

   Then $\mathcal{I}$ is filtered, and we can apply 3.6.2 to $\varinjlim G$: The category ${}^*\mathcal{I}$ is the category of the ordered set ${}^*\mathbb{N}_+$, and if $h \in {}^*\mathbb{N}_+ \setminus \mathbb{N}$ is any infinite natural number, then as i_∞ we can take the number $(h!)$. The category ${}^*\mathcal{C}$ is a category of abelian groups (with internal group-structure and with internal group homomorphisms as morphisms), and ${}^*G$ is the functor that sends a $k \in {}^*\mathbb{N}_+$ to ${}^*\mathbb{Z}/k{}^*\mathbb{Z}$. The corollary then tells us that to give a morphism from $\varinjlim G$ to an abelian group A in $\mathcal{C}$, we have to give an (internal) group homomorphism φ from ${}^*\mathbb{Z}/h!{}^*\mathbb{Z}$ to ${}^*A$, such that for all $n \in \mathbb{N}_+$, the composition $\psi : {}^*(\mathbb{Z}/n\mathbb{Z}) \xrightarrow{1 \mapsto h!/n} {}^*\mathbb{Z}/h!{}^*\mathbb{Z} \xrightarrow{\varphi} {}^*A$ is of the form ${}^*\varphi_n$ for a $\varphi : \mathbb{Z}/n\mathbb{Z} \to A$. As this condition is obviously equivalent to the condition that $\psi(1) \in A$, we see that a morphism from $\varinjlim G$ to A is given by a morphism $\varphi : {}^*\mathbb{Z}/h!{}^*\mathbb{Z} \to {}^*A$ such that for all $n \in \mathbb{N}_+$, we have $\varphi(h!/n) \in A \subseteq {}^*A$.

   Two such morphisms φ_1 and φ_2 correspond to the same morphism $\varinjlim G \to A$ if and only if their compositions with all inclusions $\mathbb{Z}/n\mathbb{Z} \hookrightarrow {}^*\mathbb{Z}/h!{}^*\mathbb{Z}$ coincide, i.e. if and only if $\varphi_1(h!/n) = \varphi_2(h!/n) \in A$ for all $n \in \mathbb{N}_+$.

   We have the isomorphism $\alpha : \varinjlim G \xrightarrow{\sim} \mathbb{Q}/\mathbb{Z}$, defined by the morphisms α_n :

$\mathbb{Z}/n\mathbb{Z} \xrightarrow{1\mapsto 1/n} \mathbb{Q}/\mathbb{Z}$ for $n \in \mathbb{N}_+$. To which morphisms $\varphi : {}^*\mathbb{Z}/h!{}^*\mathbb{Z} \to {}^*\mathbb{Q}/\mathbb{Z}$ does α correspond? — By transfer, φ is uniquely determined by $x := \varphi(1)$, and by (4), for every $n \in \mathbb{N}_+$, the following diagram in ${}^*\mathcal{C}$ must commute:

$$\begin{tikzcd} {}^*\mathbb{Z}/h!{}^*\mathbb{Z} \arrow[r, "1\mapsto x", "\varphi"'] & {}^*(\mathbb{Q}/\mathbb{Z}) \\ \mathbb{Z}/n\mathbb{Z} \arrow[u, "1\mapsto h!/n"] \arrow[ur, "\alpha_n"'] & \end{tikzcd}$$

This shows that we must have $1/n = \alpha_n(1) = (h!/n) \cdot x \in {}^*(\mathbb{Q}/\mathbb{Z})$, i.e. $x = (1 + mn)/h!$ for an $m \in {}^*\mathbb{Z}$. Because this holds for every $n \in \mathbb{N}_+$, we get $x = (1 + N)/h!$ with $N \in {}^*\mathbb{Z}$ a number divisible by all $n \in \mathbb{N}_+$.

4. Enlargements of additive and abelian categories

Next, we want to have a look at $\hat{S}$-small categories with more structure, namely at *additive* and *abelian* categories. We start by giving a formal definition:

4.1. Definition. *Let $\hat{S}$ be a superstructure.*

1. *An* additive $\hat{S}$-small category *is a pair $\langle \mathcal{A}, P\rangle$, consisting of an $\hat{S}$-small category $\mathcal{A} = \langle M, s, t, c\rangle$ and a set $P \subseteq M \times M \times M$, subject to the following conditions:*

   (a) *For all objects A, B of $\mathcal{A}$, the intersection $\big[\mathrm{Mor}_{\mathcal{A}}(A, B) \times \mathrm{Mor}_{\mathcal{A}}(A, B) \times \mathrm{Mor}_{\mathcal{A}}(A, B)\big] \cap P$ is a map $P_{A,B} : \mathrm{Mor}_{\mathcal{A}}(A, B) \times \mathrm{Mor}_{\mathcal{A}}(A, B) \to \mathrm{Mor}_{\mathcal{A}}(A, B)$ that endows $\mathrm{Mor}_{\mathcal{A}}(A, B)$ with the structure of an abelian group.*

   (b) *For all $A, B, C \in \mathrm{Ob}\mathcal{A}$, the map $\mathrm{Mor}_{\mathcal{A}}(B, C) \times \mathrm{Mor}_{\mathcal{A}}(A, B) \xrightarrow{\circ} \mathrm{Mor}_{\mathcal{A}}(A, C)$ given by c is bilinear with respect to the group-structures defined in (1).*

   (c) *$\mathcal{A}$ has a zero object.*

   (d) *$\mathcal{A}$ has arbitrary finite sums.*

2. *An additive $\hat{S}$-small category $\langle \mathcal{A}, P\rangle$ is called* abelian *if it is abelian in the usual sense, i.e. if all morphisms in $\mathcal{A}$ have a kernel and a cokernel and if for each morphism f, the canonical map $\mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism.*

4.2. Remark.

1. For a superstructure $\hat{S}$, an additive $\hat{S}$-small category is just an additive category in the usual sense whose underlying category is $\hat{S}$-small.

2. Let $\mathcal{A}$ be a small, additive category. Then there is a base set S and an $\hat{S}$-small additive category which is isomorphic to $\mathcal{A}$ (compare 2.3.1).

4.3. Remark. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. Then F is exact if and only if F maps $\ker(f)$ to $\ker(Ff)$ for every morphism f in $\mathcal{A}$ and if F maps epimorphisms to epimorphisms:

It is clear that the condition is necessary. If on the other hand $A \xrightarrow{f} B \xrightarrow{g} C$ is exact in $\mathcal{A}$, then this is equivalent to the existence of a factorization

$$\begin{tikzcd} & A \arrow[d, dashed, "\bar{f}"'] \arrow[dr, "f"] & & \\ 0 \arrow[r] & \ker(g) \arrow[r] & B \arrow[r, "g"] & C \end{tikzcd}$$

with an epimorphism $\bar{f}$. Applying F to this diagram and using $F\ker(g) = \ker(Fg)$ and the fact that $F\bar{f}$ is an epimorphism then shows that $FA \xrightarrow{Ff} FB \xrightarrow{Fg} FC$ is exact in $\mathcal{B}$, so the condition is also sufficient.

4.4. Proposition. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, and let $\langle \mathcal{A}, P\rangle$ be an additive $\hat{S}$-small category. Then:*

1. *$\langle {}^*\mathcal{A}, {}^*P\rangle$ is an additive $\widehat{{}^*S}$-small category, and the functor $*$ is additive.*
2. *If $\varphi[X, Y, Z, f, g]$ is the formula $\left[X \xrightarrow{f} Y \text{ is a morphism in } \mathcal{A} \text{ with kernel } Z \xrightarrow{g} X\right]$ in $\hat{S}$, then ${}^*\varphi[X, Y, Z, f, g]$ is the formula $\left[X \xrightarrow{f} Y \text{ is a morphism in } {}^*\mathcal{A} \text{ with kernel } Z \xrightarrow{g} X\right]$ in $\widehat{{}^*S}$, and the analogous statement is true for "cokernel", "image" or "coimage" instead of "kernel".*
3. *If in addition $\langle \mathcal{A}, P\rangle$ is abelian, then so is $\langle {}^*\mathcal{A}, {}^*P\rangle$, and the functor $*$ is exact.*

Proof. By easy transfer, we see that $\langle {}^*\mathcal{A}, {}^*P\rangle$ satisfies conditions (1) and (2), and 2.18 shows that it also satisfies conditions (3) and (4), so it is indeed an additive category.

By 2.16.4, we know that $*$ maps the zero object of $\mathcal{A}$ to the zero object of ${}^*\mathcal{A}$ and the sum $X \oplus Y$ of two objects $X, Y \in \mathcal{A}$ to the sum of ${}^*X$ and ${}^*Y$ in ${}^*\mathcal{A}$ — this shows that $*$ is an *additive* functor and therefore completes the proof of 1.

To prove 2, we only have to formalize φ:

$$\begin{aligned} \varphi[X, Y, Z, f, g] = \Big[&\big(X, Y, Z \in \mathrm{Ob}(\mathcal{A}) \wedge f \in \mathrm{Mor}_{\mathcal{A}}(X, Y) \wedge g \in \mathrm{Mor}_{\mathcal{A}}(Z, X)\big) \\ &\wedge \big(\forall A \in \mathrm{Ob}(\mathcal{A}) : \forall h \in \mathrm{Mor}_{\mathcal{A}}(A, X) : [fh = 0] \Rightarrow [\exists! h' \in \mathrm{Mor}_{\mathcal{A}}(A, Z) : gh' = h]\big)\Big]. \end{aligned}$$

It is then clear (again by using 2.8.1) that ${}^*\varphi$ is what we claim in 2.

Now let us assume that $\langle \mathcal{A}, P\rangle$ is abelian. First, in ${}^*\mathcal{A}$ kernels and cokernels of arbitrary morphisms exist because of 2.18. To prove that coimage and image of every morphism in

${}^*\mathcal{A}$ are canonically isomorphic, we first write down the corresponding statement in $\mathcal{A}$:

$$\begin{aligned}&\forall A, B, C, D \in \mathrm{Ob}(\mathcal{A}) : \forall f \in \mathrm{Mor}_{\mathcal{A}}(A, B) :\\ &\qquad\qquad \forall \varphi \in \mathrm{Mor}_{\mathcal{A}}(A, C) : \forall g \in \mathrm{Mor}_{\mathcal{A}}(C, D) : \forall \psi \in \mathrm{Mor}_{\mathcal{A}}(D, B) :\\ &\Big[\big(A \xrightarrow{\varphi} C \text{ is the coimage of } f\big) \wedge \big(D \xrightarrow{\psi} B \text{ is the image of } f\big) \wedge \big(\psi g \varphi = f\big)\Big]\\ &\qquad\qquad\qquad\qquad\qquad\qquad \Rightarrow g \text{ is an isomorphism.}\end{aligned}$$

To see that the transfer of this statement is exactly what we want in ${}^*\mathcal{A}$, we can apply 2 for "image" and "coimage". This proves that ${}^*\mathcal{A}$ is indeed an abelian category.

Finally, $*$ maps kernels to kernels because of 2.16.4 and epimorphisms to epimorphisms because of 2.7.6, so 4.3 shows that $*$ in this case is exact. ■

From now on, when talking about an additive or abelian $\hat{S}$-small category $\langle \mathcal{A}, P\rangle$, we will simply denote it by $\mathcal{A}$ and drop P from the notation.

4.5. Proposition. *Let* $* : \hat{S} \to \widehat{{}^*S}$ *be an enlargement, let* $\mathcal{A}$ *and* $\mathcal{B}$ *be* $\hat{S}$*-small additive categories, and let* $F : \mathcal{A} \to \mathcal{B}$ *be a functor.*

1. *If* F *is additive, then* ${}^*F : {}^*\mathcal{A} \to {}^*\mathcal{B}$ *is also additive.*
2. *If* $\mathcal{A}$ *and* $\mathcal{B}$ *are abelian and if* F *is left exact (resp. right exact, resp. exact), then* ${}^*F$ *is also left exact (resp. right exact, resp. exact).*

Proof. To prove 1, we have to show that ${}^*F$ maps the zero object to the zero object and direct sums to direct sums:

$$F(0_{\mathcal{A}}) = 0_{\mathcal{B}} \quad \overset{\text{transfer}}{\Longrightarrow} \quad ({}^*F)(\underbrace{{}^*0_{\mathcal{A}}}_{=0_{{}^*\mathcal{A}}}) = \underbrace{{}^*0_{\mathcal{B}}}_{=0_{{}^*\mathcal{B}}},$$

and

$$\begin{aligned}&\forall X, Y \in \mathrm{Ob}(\mathcal{A}) : F(X \oplus Y) = F(X) \oplus F(Y)\\ &\qquad \overset{2.16.3,\text{transfer}}{\Longrightarrow} \quad \forall X, Y \in \mathrm{Ob}({}^*\mathcal{A}) : ({}^*F)(X \oplus Y) = ({}^*F)(X) \oplus ({}^*F)(Y).\end{aligned}$$

Now let us assume that $\mathcal{A}$ and $\mathcal{B}$ are abelian and that F is left exact. We have to show that ${}^*F$ then also is left exact, i.e. that ${}^*F$ maps kernels to kernels. But this follows, using 2.8.1 and 2.16.4, by transfer of

$$\forall A, B \in \mathrm{Ob}(\mathcal{A}) : \forall f \in \mathrm{Mor}_{\mathcal{A}}(A, B) : F\ker(f) = \ker(Ff).$$

■

4.6. Lemma. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, and let $\mathcal{A}$ and $\mathcal{B}$ be $\hat{S}$-small categories.*

1. *If $\mathcal{A}$ and $\mathcal{B}$ are additive, then the formula*
$$\varphi[F] \equiv \Big[F \textit{ is an additive functor from } \mathcal{A} \textit{ to } \mathcal{B}\Big]$$
becomes
$${}^*\varphi[F] \equiv \Big[F \textit{ is an additive functor from } {}^*\mathcal{A} \textit{ to } {}^*\mathcal{B}\Big].$$

2. *If $\mathcal{A}$ and $\mathcal{B}$ are abelian, then the formula*
$$\varphi[F] \equiv \Big[F \textit{ is an exact functor from } \mathcal{A} \textit{ to } \mathcal{B}\Big]$$
becomes
$${}^*\varphi[F] \equiv \Big[F \textit{ is an exact functor from } {}^*\mathcal{A} \textit{ to } {}^*\mathcal{B}\Big].$$

Proof. To prove 1, we can formalize φ as follows:

$$\varphi[F] = \Big[F \in \mathrm{Ob}(\mathrm{Funct}\,(\mathcal{A},\mathcal{B}))\Big] \wedge \Big[F(0_{\mathcal{A}}) = 0_{\mathcal{B}}\Big] \\ \wedge \Big[\forall A, B \in \mathrm{Ob}(\mathcal{A}) : F(A \oplus B) = F(A) \oplus F(B)\Big],$$

and then the claim follows by transfer and using 2.8.1 and 2.16.3. To prove 2, we use 4.3 and write

$$\varphi[F] = \Big[F \text{ is an additive functor from } \mathcal{A} \text{ to } \mathcal{B}\Big] \\ \wedge \Big[F \text{ maps kernels to kernels and epimorphisms to epimorphisms}\Big],$$

and if we use 1, 2.7.6, 2.8.1 and 2.16.4, the claim again follows by transfer. ∎

4.7. Proposition. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, let R be a ring whose underlying set is an element of $\hat{S}$, let $T \in \hat{S}$ be a set of sets, and let $(R\text{-}\underline{\mathrm{Mod}})_T$ be the full subcategory of the category of R-modules whose objects have underlying sets in T — this is obviously an $\hat{S}$-small category in the sense of 2.2.*

1. *The ring structure on R induces a canonical internal ring structure on ${}^*R$.*
2. *Let M be an object of $(R\text{-}\underline{\mathrm{Mod}})_T$. Then the R-module structure on M induces a canonical internal ${}^*R$-module structure on ${}^*M$.*
3. *The enlargement of $(R\text{-}\underline{\mathrm{Mod}})_T$ is (in the sense of 2.2) the $\hat{S}$-small category of internal ${}^*R$-modules with underlying sets in ${}^*T$ and with internal, ${}^*R$-linear maps as morphisms.*

PROOF. The ring structure of R is given by two maps $R \times R \to R$ (addition and multiplication), subject to certain conditions. Likewise, the R-module structure on M is given by two maps $M \times M \to M$ (addition) and $R \times M \to M$ (scalar multiplication), again subject to certain conditions. It is clear by transfer that the enlargements of these maps define an internal ring structure on ${}^*R$ and an internal ${}^*R$-module structure on ${}^*M$, thus proving 1 and 2. From this by transfer, 3 follows easily, keeping in mind that the enlargement of the set of maps between two sets (with certain conditions) is the set of *internal* maps between the enlargements of the two sets (with the induced conditions) — compare A.10.10. ■

4.8. REMARK. Proposition 4.7 has obvious analogues for *subcategories* of $(S\text{-}\underline{\mathrm{Mod}})_T$ like the full subcategories of *finitely generated* or *finite* R-modules. In particular, for the case $R = \mathbb{Z}$, we can apply this remark to the category $\mathrm{Ab}^{\mathrm{fin}}$ of finite abelian groups.

5. Enlargements and derived functors

5.1. LEMMA. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, let $\mathcal{A}$ be an $\hat{S}$-small abelian category, and consider the formula $\varphi[X] \equiv \Big[X \text{ is an injective object of } \mathcal{A}\Big]$ in $\hat{S}$. Then*

$$^*\varphi[X] = \Big[X \textit{ is an injective object of } {}^*\mathcal{A}\Big].$$

PROOF. An object X of $\mathcal{A}$ is injective iff for every monomorphism $f : A \hookrightarrow B$ and every morphism $g : A \to X$ in $\mathcal{A}$, there exists a morphism $h : B \to X$ in $\mathcal{A}$ so that $h \circ f = g$:

$$\begin{tikzcd} & & X \\ 0 \arrow[r] & A \arrow[ur, "g"] \arrow[r, "f"'] & B. \arrow[u, dotted, "h"'] \end{tikzcd}$$

Therefore we have

$$\begin{aligned} \varphi[X] = \Big(X \in \mathrm{Ob}(\mathcal{A})\Big) \wedge \Big(\forall A, B \in \mathrm{Ob}(\mathcal{A}) : \forall f \in \mathrm{Mor}_{\mathcal{A}}(A, B) : \\ (f \text{ monomorphism}) \Longrightarrow \forall g \in \mathrm{Mor}_{\mathcal{A}}(A, X) : \exists h \in \mathrm{Mor}_{\mathcal{A}}(B, X) : h \circ f = g\Big). \end{aligned}$$

With this description, the lemma follows easily from 2.7.5 and 2.8. ■

5.2. COROLLARY. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, and let $\mathcal{A}$ be an $\hat{S}$-small abelian category. Then $*$ maps injective (resp. projective) objects of $\mathcal{A}$ to injective (resp. projective) objects of ${}^*\mathcal{A}$, and if $\mathcal{A}$ has enough injectives (resp. projectives), then so has ${}^*\mathcal{A}$.*

PROOF. The corollary follows easily from 5.1: If I is an injective object of $\mathcal{A}$, then $\varphi[I]$ is true, so that by transfer ${}^*\varphi[{}^*I]$ is also true, i.e. ${}^*I$ is an injective object of ${}^*\mathcal{A}$. And if $\mathcal{A}$ has enough injectives, then the following statement is true:

$$\forall A \in \mathrm{Ob}(\mathcal{A}) : \exists I \in \mathrm{Ob}(\mathcal{A}) : \exists f \in \mathrm{Mor}_{\mathcal{A}}(A, I) : \Big[f \text{ monomorphism}\Big] \wedge \varphi[I].$$

Then by transfer and because of 2.7.5 and 2.8.1, it follows that also ${}^*\mathcal{A}$ has enough injectives. — The proof for projectives is analogous. ∎

5.3. COROLLARY. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, and let $\mathcal{A}$ and $\mathcal{B}$ be $\hat{S}$-small abelian category, and let $F : \mathcal{A} \to \mathcal{B}$ be a functor that maps injectives to injectives (resp. projectives to projectives). Then ${}^*F : {}^*\mathcal{A} \to {}^*\mathcal{B}$ also maps injectives to injectives (resp. projectives to projectives).*

PROOF. For injectives, this follows immediately from 5.1 by transfer of the statement

$$\forall A \in \mathrm{Ob}(\mathcal{A}) : \big[A \text{ injective}\big] \Rightarrow \big[FA \text{ injective}\big],$$

and for projectives, it is analogous. ∎

5.4. COROLLARY. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement, and let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor between $\hat{S}$-small abelian categories.*

1. *If $\mathcal{A}$ has enough injectives and if F is left exact, then for all $i \geq 0$, the right derived functors $\mathrm{R}^i F : \mathcal{A} \to \mathcal{B}$ and $\mathrm{R}^i({}^*F) : {}^*\mathcal{A} \to {}^*\mathcal{B}$ exist, and the following diagram of functors commutes:*

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\quad * \quad} & {}^*\mathcal{A} & = = = & {}^*\mathcal{A} \\ {\scriptstyle \mathrm{R}^iF}\downarrow & & \downarrow{\scriptstyle {}^*(\mathrm{R}^iF)} & & \downarrow{\scriptstyle \mathrm{R}^i({}^*F)} \\ \mathcal{B} & \xrightarrow[\quad * \quad]{} & {}^*\mathcal{B} & = = = & {}^*\mathcal{B} \end{array}$$

2. *If $\mathcal{A}$ has enough projectives and if F is right exact, then for all $i \geq 0$ the left derived functors $\mathrm{L}^i F : \mathcal{A} \to \mathcal{B}$ and $\mathrm{L}^i({}^*F) : {}^*\mathcal{A} \to {}^*\mathcal{B}$ exist, and the following diagram of functors commutes:*

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\quad * \quad} & {}^*\mathcal{A} & = = = & {}^*\mathcal{A} \\ {\scriptstyle \mathrm{L}^iF}\downarrow & & \downarrow{\scriptstyle {}^*(\mathrm{L}^iF)} & & \downarrow{\scriptstyle \mathrm{L}^i({}^*F)} \\ \mathcal{B} & \xrightarrow[\quad * \quad]{} & {}^*\mathcal{B} & = = = & {}^*\mathcal{B} \end{array}$$

PROOF. We only give a proof for 1, the proof for 2 is analogous. It is well known that the right derived functors of a left exact functor exist if the source category has enough injectives. Now $\mathcal{A}$ has enough injectives by assumption, so the $\mathrm{R}^i F$ exist, and ${}^*F$ is left exact by 4.5.2 and ${}^*\mathcal{A}$ has enough injectives because of 5.2, so the $\mathrm{R}^i({}^*F)$ also exist.

The left square commutes because of 2.12.2. To show that the right square commutes, look at the following statement in $\hat{S}$:

$$\forall A \in \mathrm{Ob}(\mathcal{A}) : \forall(A \xrightarrow{\varepsilon} I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} \dots \xrightarrow{\delta^i} I^{i+1}) \text{ injective resolution of } A \text{ in } \mathcal{A} :$$
$$\mathrm{R}^i F A = \mathrm{H}\left(I^{i-1} \xrightarrow{\delta^{i-1}} I^i \xrightarrow{\delta^i} I^{i+1}\right). \quad (5)$$

What is the transfer of this statement? To answer this question, we first note that the transfer of

$$\varphi[X,Y,Z,f,g] \equiv \Big[X \xrightarrow{f} Y \xrightarrow{g} Z \text{ is an exact sequence in } \mathcal{A}\Big]$$

is

$$^*\varphi[X,Y,Z,f,g] = \Big[X \xrightarrow{f} Y \xrightarrow{g} Z \text{ is an exact sequence in } {}^*\mathcal{A}\Big].$$

(To see this, we write the exactness in terms of equalities between kernels and cokernels etc. and then use 2.16.3.) From 5.1 we know that the transfer of the statement $\varphi[X] \equiv \Big[X \text{ is injective in } \mathcal{A}\Big]$ is $^*\varphi[X] = \Big[X \text{ is injective in } {}^*\mathcal{A}\Big]$, and combining these two results with 4.4.2 we get as the transfer of (5):

$$\forall A \in \mathrm{Ob}({}^*\mathcal{A}) : \forall (A \xrightarrow{\varepsilon} I^0 \xrightarrow{\delta^0} I^1 \xrightarrow{\delta^1} \dots \xrightarrow{\delta^i} I^{i+1}) \text{ injective resolution of } A \text{ in } {}^*\mathcal{A} :$$
$$^*(\mathrm{R}^i F)A = \underbrace{\mathrm{H}\left(I^{i-1} \xrightarrow{\delta^{i-1}} I^i \xrightarrow{\delta^i} I^{i+1}\right)}_{=\mathrm{R}^i({}^*F)A}.$$

This proves that the right square also commutes. ∎

5.5. Remark. In most cases it is not a restriction to assume that an abelian category is small and has enough injectives, at least if one assumes the existence of arbitrary large inaccessible cardinals:

Let $\mathcal{A}$ be a Grothendieck category with a generator $U \in \mathcal{A}$. For an object A in $\mathcal{A}$, we define the cardinality of A by $|A| := |\{A' \subset A\}|$. Let κ be an inaccessible (i.e. regular and limit) cardinal number with $\kappa > |U|$ and $\kappa > |Hom(U,U)|$ and let $\mathcal{A}^{<\kappa}$ be the full subcategory of $\mathcal{A}$ of objects $A \in \mathcal{A}$ with $|A| < \kappa$. Further let S be a set with $|S| > \kappa$.

5.6. Lemma.

1. *The category $\mathcal{A}^{<\kappa}$ is a small abelian category with enough injective objects, and the inclusion functor $i : \mathcal{A}^{<\kappa} \to \mathcal{A}$ is exact and respects injective objects.*
2. *The category $\mathcal{A}^{<\kappa}$ is an $\hat{S}$-small category*

Proof Sketch of proof. Each object of $\mathcal{A}^{<\kappa}$ is a quotient of the object $\sum_\kappa U$. Therefore $\mathcal{A}^{<\kappa}$ is small. If $0 \to A' \to A \to A'' \to 0$ is a short exact sequence in $\mathcal{A}$, than the inequalities $|A'| \le |A|$ and $|A''| \le |A|$ hold. We further have $|A \times B| \le 2^{\alpha^{\max(|A|,|B|)}}$. So the category $\mathcal{A}^{<\kappa}$ is an abelian subcategory of $\mathcal{A}$, and the inclusion functor is exact. The concrete construction of injective resolutions in [Gro57] and the choice of κ show that $\mathcal{A}^{<\kappa}$ has enough injectives. Because U and all subobjects of U are in $\mathcal{A}^{<\kappa}$, the inclusion functor i respects injectives (compare [Gro57, Lemma 1 in 1.10]). ∎

6. Enlargement of triangulated and derived categories

In this section we want to investigate what happens if we enlarge triangulated and derived categories. From now on, the proofs will be a little bit shorter, and we will not give all the details.
Let $\mathcal{T}$ be an additive category with an automorphism $\Sigma : \mathcal{T} \to \mathcal{T}$.

- A triangle in $\mathcal{T}$ is a triple (f, g, h) of morphisms in $\mathcal{T}$ of the form $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} \Sigma A$ with $A, B, C \in \mathcal{T}$.
- The category $\mathcal{T}$ together with the automorphism Σ and a class of triangles Δ is called a triangulated category if it fulfills certain axioms (compare for example [Wei94]).

Now let $\hat{S}$ be a superstructure, $(\mathcal{T}, \Sigma, \Delta)$ a triangulated category with $\mathcal{T}$ $\hat{S}$–small, and $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement. In particular this means that the set Δ is also $\hat{S}$–small. We get

6.1. Proposition. *The triple* $({}^*\mathcal{T}, {}^*\Sigma, {}^*\Delta)$ *is again a triangulated category.*

Proof. Follows easily from the transfer principle. ∎

We also get

6.2. Proposition. *If* $F : \mathcal{T} \to \mathcal{T}'$ *is a functor of triangulated categories then* ${}^*F : {}^*\mathcal{T} \to {}^*\mathcal{T}'$ *is also a functor of triangulated categories.*

Proof. Follows again easily from the transfer principle. ∎

Now let us look at triangulated subcategories.

6.3. Definition. *A triangulated subcategory* $\mathcal{D}$ *of a triangulated category* $\mathcal{T}$ *is called* thick *iff for all objects* $x, y \in \mathrm{Ob}(\mathcal{T})$*, we have the implication* $x \oplus y \in \mathrm{Ob}(\mathcal{D}) \Rightarrow x \in \mathrm{Ob}(\mathcal{D})$.

We have already seen that $*$ is exact. The next statement reflects this on a triangulated level.

6.4. Proposition. *Let* $\mathcal{T}$ *be an* $\hat{S}$*–small triangulated category and* $\mathcal{D} \subset \mathcal{T}$ *be a thick subcategory. Then* ${}^*\mathcal{D} \subset {}^*\mathcal{T}$ *is again thick and we have a canonical isomorphism*

$$ {}^*\mathcal{T}/{}^*\mathcal{D} \xrightarrow{\sim} {}^*(\mathcal{T}/\mathcal{D}) $$

of triangulated categories.

Proof. That ${}^*\mathcal{D} \subset {}^*\mathcal{T}$ is again a thick subcategory follows from the definition, the transfer principle, and corollary 2.16.2. The canonical functor $\mathcal{T} \to \mathcal{T}/\mathcal{D}$ induces a functor ${}^*\mathcal{T} \to {}^*(\mathcal{T}/\mathcal{D})$ and by transfer the subcategory ${}^*\mathcal{D}$ is the full subcategory of objects which are mapped to zero. So we get a functor ${}^*\mathcal{T}/{}^*\mathcal{D} \to {}^*(\mathcal{T}/\mathcal{D})$. Now we have the following general and well known fact about triangulated categories:

6.5. Lemma. *Let $\mathcal{T}$ and $\mathcal{T}'$ be triangulated categories and $F : \mathcal{T} \to \mathcal{T}'$ be a functor of triangulated categories with*

1. *F is essentially surjective on objects*
2. *for all $A', B' \in \mathrm{Ob}(\mathcal{T}')$ and $f \in \mathrm{Mor}_{\mathcal{T}'}(A', B')$ there are $A, B \in \mathrm{Ob}(\mathcal{T})$, $f \in \mathrm{Mor}_{\mathcal{T}}(A, B)$ and isomorphism $g_A : F(A) \to A'$ and $g_B : F(B) \to B'$ such that the diagram*

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "g_A"] & F(B) \arrow[d, "g_B"] \\ A' \arrow[r, "f"] & B \end{tikzcd}$$

is commutative.

Then the canonical functor $\mathcal{T}/\ker(F) \to \mathcal{T}'$ is an isomorphism of triangulated categories.

By the transfer principle and 2.6 both properties are fulfilled for the functor ${}^*\mathcal{T} \to {}^*(\mathcal{T}/\mathcal{D})$ and so the proposition follows. ∎

Now we want to consider the derived category of an abelian category. The derived category can be obtained in three steps. So let $\mathcal{A}$ be an abelian category. First we look at the category $\mathrm{Kom}\,(\mathcal{A})$ of complexes in the abelian category. Then we identify morphisms of complexes which are homotopic and get the homotopy–category of complexes $\mathcal{K}(\mathcal{A})$. This is a triangulated category and there we divide out the thick subcategory of all complexes which are quasi–isomorphic to the zero complex to get the derived category $\mathcal{D}(\mathcal{A})$. Now we look step by step what happens under enlargements. For that let $\mathcal{A}$ be an $\hat{S}$–small abelian category. Let $\mathcal{Z}$ be the category which has as objects $\mathbb{Z}$ and for each $i \leq j$ exactly one morphism from i to j. For a functor from $\mathcal{Z}$ to a category we denote by d_i the image of the unique morphism from i to $i+1$. The category $\mathrm{Kom}\,(\mathcal{A})$ of complexes in $\mathcal{A}$ is the category of functors from $\mathcal{Z}$ to $\mathcal{A}$ with the property that for all $i \in \mathcal{Z}$ the equality $d_{j+1} \circ d_j = 0$ holds. Therefore ${}^*\mathrm{Kom}\,(\mathcal{A})$ is the category of internal functors from ${}^*\mathcal{Z}$ to ${}^*\mathcal{A}$ with $d_{j+1} \circ d_j = 0$ for all $i \in {}^*\mathcal{Z}$. We call the objects of this category *$*$–complexes*. The category ${}^*\mathcal{Z}$ has as objects ${}^*\mathbb{Z}$ and again for each $i \leq j$ exactly one morphism from i to j. Because $\mathbb{Z}$ lies inside of ${}^*\mathbb{Z}$ we get a restriction functor from ${}^*\mathrm{Kom}\,(\mathcal{A})$ to $\mathrm{Kom}\,({}^*\mathcal{A})$. By the transfer principle we get ${}^*\mathcal{K}(\mathcal{A})$ out of ${}^*\mathrm{Kom}\,(\mathcal{A})$ if we identify morphisms which are internal homotopic. Because internal homotopies in ${}^*\mathrm{Kom}\,(\mathcal{A})$ become homotopies in $\mathrm{Kom}\,({}^*\mathcal{A})$ this functor induces a functor from ${}^*\mathcal{K}(\mathcal{A})$ to $\mathcal{K}({}^*\mathcal{A})$. With 6.4 one can see easily that we get a functor

$$\mathrm{res}_{\mathcal{A}} : {}^*\mathcal{D}(\mathcal{A}) \to \mathcal{D}({}^*\mathcal{A}).$$

For an abelian category $\mathcal{A}$ we denote by $\mathcal{D}^b(\mathcal{A})$ the full triangulated subcategory of $\mathcal{D}(\mathcal{A})$ of all complexes $A^\bullet \in \mathcal{D}(\mathcal{A})$ such that there is an $i \in \mathbb{N}$ with $\forall j \in \mathbb{Z} : |j| > i \Rightarrow h^i(A^\bullet) \simeq 0$. The full subcategory of $\mathrm{Kom}\,(\mathcal{A})$ of all complexes $A^\bullet \in \mathrm{Kom}\,(\mathcal{A})$ with $A^j \simeq 0$ for all $j \in \mathbb{Z}$ with $|j| > i$ for a specific $i \in \mathbb{N}$ is denoted by $\mathrm{Kom}^b\,(\mathcal{A})$.

6.6. Remark. It is not possible to define the functor $\mathrm{res}_{\mathcal{A}}$ for the bounded derived category, i.e. a functor ${}^*\mathcal{D}^b(\mathcal{A}) \to \mathcal{D}^b({}^*\mathcal{A})$, because the $*$–complexes in ${}^*\mathcal{D}^b(\mathcal{A})$ are not bounded by a standard natural number.

By transfer we get for each $i \in {}^*\mathbb{Z}$ a functor

$$\bullet^i : {}^*\mathrm{Kom}\,(\mathcal{A}) \to {}^*\mathcal{A}$$

and

$${}^*h^i : {}^*\mathcal{D}(\mathcal{A}) \to {}^*\mathcal{A}.$$

We summarize this in the following

6.7. Proposition. *Let $\mathcal{A}$ be an $\hat{S}$–small abelian category. Then for each $i \in \mathbb{Z}$, we have the following commutative diagram:*

$$\begin{tikzcd}
& {}^*\mathcal{A} \arrow[rr, "{}^*i"] \arrow[dd, equal] & & {}^*\mathcal{D}(\mathcal{A}) \arrow[rr, "{}^*h^i"] \arrow[dd, "\mathrm{res}_{\mathcal{A}}"] & & {}^*\mathcal{A} \arrow[dd, equal] \\
\mathcal{A} \arrow[ur, "*"] \arrow[dr, "*"'] \arrow[rr, "i"] & & \mathcal{D}(\mathcal{A}) \arrow[ur, "*"] \arrow[dr, "R*"] \arrow[rr, "h^i"] & & \mathcal{A} \arrow[ur, "*"] \arrow[dr, "*"] & \\
& {}^*\mathcal{A} \arrow[rr, "i"'] & & \mathcal{D}({}^*\mathcal{A}) \arrow[rr, "h^i"] & & {}^*\mathcal{A}
\end{tikzcd}$$

Here $R*$ denotes the right derived functor of the exact functor $* : \mathcal{A} \to {}^*\mathcal{A}$.

Proof. This follows immediately from the constructions. ∎

In general we do not have a non-trivial functor from $\mathcal{D}({}^*\mathcal{A})$ to ${}^*\mathcal{D}(\mathcal{A})$. But we do have one if we restrict ourselves to bounded complexes. Then we even get more. Namely we can identify $\mathcal{D}^b({}^*\mathcal{A})$ with a full but external subcategory of ${}^*\mathcal{D}(\mathcal{A})$.

6.8. Proposition. *Let $\mathcal{A}$ be an $\hat{S}$–small abelian category. There is a canonical fully faithful functor*

$$\mathcal{D}^b({}^*\mathcal{A}) \to {}^*\mathcal{D}(\mathcal{A})$$

which identifies $\mathcal{D}^b({}^*\mathcal{A})$ with the (external!) triangulated subcategory ${}^*\mathcal{D}^{fb}(\mathcal{A})$ of all $*$–complexes $A \in {}^*\mathcal{D}(\mathcal{A})$ such that there is an $i \in \mathbb{N}$ with

$$\forall j \in {}^*\mathbb{Z} : |j| > i \Rightarrow h^j(A) \simeq 0 \textit{ in } {}^*\mathcal{A}.$$

The functor $\mathrm{res}_{\mathcal{A}} : {}^*\mathcal{D}(\mathcal{A}) \to \mathcal{D}({}^*\mathcal{A})$ restricts to a functor ${}^*\mathcal{D}^{fb}(\mathcal{A}) \to \mathcal{D}^b({}^*\mathcal{A})$ which is an inverse of the above functor.

Proof. An object in $\mathcal{D}^b({}^*\mathcal{A})$ is represented by a complex $A^\bullet \in \mathrm{Kom}\,({}^*\mathcal{A})$ with $A^j \simeq 0$ for all $j \in \mathbb{Z}$ with $|j| > i$ for a certain $i \in \mathbb{Z}$. Because there are only finitely many $j \in \mathbb{Z}$ with $A^j \neq 0$, we can continue $A^\bullet$ to an internal functor from ${}^*\mathcal{Z}$ to ${}^*\mathcal{A}$ (compare A.11). The same is true for homotopies and quasi-isomorphisms. So the statement follows from this. ∎

6.9. PROPOSITION. *Let $\mathcal{A}$ and $\mathcal{B}$ be $\hat{S}$–small abelian categories, and we assume that $\mathcal{A}$ has enough injective objects. Further let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor. Then the functors in the following diagram exist and make it commutative.*

$$\begin{tikzcd} \mathcal{D}^+(\mathcal{A}) \arrow[r, "RF"] \arrow[d, "R*"'] & \mathcal{D}^+(\mathcal{B}) \arrow[d, "R*"'] \\ \mathcal{D}^+({}^*\mathcal{A}) \arrow[r, "R^*F"] & \mathcal{D}^+({}^*\mathcal{B}) \end{tikzcd}$$

PROOF. This can be proven in the same way as proposition 5.4. ∎

6.10. PROPOSITION. *Let $\mathcal{A}$ and $\mathcal{B}$ be $\hat{S}$–small abelian categories, and we assume that $\mathcal{A}$ has enough injective objects. Further let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of finite cohomological dimension. The functor ${}^*RF$ restricts to a functor*

$${}^*\mathcal{D}^{fb}(\mathcal{A}) \to {}^*\mathcal{D}^{fb}(\mathcal{B})$$

and we get a commutative diagram

$$\begin{tikzcd} {}^*\mathcal{D}^{fb}(\mathcal{A}) \arrow[r, "{}^*RF_{|{}^*\mathcal{D}^{fb}(\mathcal{A})}"] \arrow[d, "\mathrm{res}_{\mathcal{A}}"', "\wr"] & {}^*\mathcal{D}^{fb}(\mathcal{B}) \arrow[d, "\mathrm{res}_{\mathcal{B}}"', "\wr"] \\ \mathcal{D}^b({}^*\mathcal{A}) \arrow[r, "R^*F"] & \mathcal{D}^b({}^*\mathcal{B}) \end{tikzcd}$$

PROOF. We denote by ${}^*\mathrm{Kom}^{fb}(\mathcal{A})$ the category of all $*$–complexes $A^\bullet \in {}^*\mathrm{Kom}\,(\mathcal{A})$ such that there is an $i \in \mathbb{N}$ with

$$\forall j \in {}^*\mathbb{Z} : |j| > i \Rightarrow A^i \simeq 0 \text{ in } {}^*\mathcal{A}.$$

By transfer we have for each $*$–complex $A^\bullet \in {}^*\mathrm{Kom}^{fb}(\mathcal{A})$ a resolution in ${}^*\mathrm{Kom}^{fb}(\mathcal{A})$ $r : A^\bullet \to I^\bullet_{A^\bullet}$ with $I^\bullet_{A^\bullet} \in {}^*\mathrm{Kom}^{fb}(\mathcal{A})$ such that for all $i \in \mathbb{N}$ the object $I^i_{A^\bullet}$ is ${}^*F$–acyclic and ${}^*RF = {}^*F(I^\bullet_{A^\bullet})$. We see that $\mathrm{res}_{\mathcal{A}}(I^\bullet_{A^\bullet})$ has also acyclic components and so $R^*F(\mathrm{res}_{\mathcal{A}}(I^\bullet_{A^\bullet})) = {}^*F(\mathrm{res}_{\mathcal{A}}(I^\bullet_{A^\bullet})$ and the proposition follows. ∎

6.11. REMARK. It is easy to see that if $\mathcal{A}$ and $\mathcal{B}$ are $\hat{S}$-small abelian categories, if $\mathcal{A}$ has enough *projective* objects, and if $F : \mathcal{A} \to \mathcal{B}$ is a *right* exact functor (of finite cohomological dimension), the obvious statements which are analogous to 6.9 and 6.10 hold.

7. Enlargements of fibred categories

For each ring in the superstructure $\hat{S}$ we have the $\hat{S}$–small category of finitely generated modules over the ring. This gives us also for each standard ring (i.e. of the form ${}^*R$ for a ring R in $\hat{S}$) an internal category of modules in $\widehat{{}^*S}$. But we also want to have a category of modules for an internal but nonstandard ring. Furthermore we would like to have functorial properties with respect to morphisms between the rings. For this we show in this section that the notion of fibred categories behaves well under enlargements. How this solves the above problem we will see in example 7.10.

For the theory of fibred categories we refer to [SGA1, expos 6] and for additive, abelian, triangulated and derived fibrations to [SGA4III, expos 17].
First we recall the notations and definitions of fibred categories relevant for us.

7.1. Definition.

1. *Let $p : \mathcal{F} \to \mathcal{E}$ be a functor, and let S be an object in $\mathcal{E}$. The category $\mathcal{F}_S$ with*

   $\mathrm{Ob}(\mathcal{F}_S) := \{X \in \mathcal{F} | p(X) = S\}$ *and* $\mathrm{Mor}_{\mathcal{F}_S}(X, Y) := \{f \in \mathrm{Mor}_{\mathcal{F}}(X, Y) | p(f) = \mathrm{id}_S\}$

   is called the fibre of $\mathcal{F}$ in S.
2. *Let $p : \mathcal{F} \to \mathcal{E}$ be a functor, and let $\alpha : X \to Y$ be a morphism in $\mathcal{F}$. Then α is called* cartesian *if for all $X' \in \mathcal{F}$ and for all morphisms $u : X' \to Y$ such that there is a factorisation $p(u) = p(\alpha) \circ \beta$ there is a unique $\overline{u} \in \mathrm{Mor}_{\mathcal{F}}(X', X)$ with $u = \alpha \circ \overline{u}$ and $p(\overline{u}) = \beta$.*
3. *A functor $p : \mathcal{F} \to \mathcal{E}$ is called a* fibration *if for every morphism $\alpha : S' \to S$ in $\mathcal{E}$ and for every $Y \in \mathcal{F}_S$, there is a cartesian morphism $f : X \to Y$ with $p(f) = \alpha$. The category $\mathcal{F}$ is also called a category* fibred over $\mathcal{E}$.

7.2. Remark. The definition of a cartesian morphism differs slightly from [SGA1, VI.5.1] but compare [SGA1, VI.6.11].

Now let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement.

7.3. Proposition. *Let $p : \mathcal{F} \to \mathcal{E}$ be a fibration of $\hat{S}$–small categories. Then ${}^*p : {}^*\mathcal{F} \to {}^*\mathcal{E}$ is also a fibration and for all $S \in \mathcal{E}$ we have ${}^*(\mathcal{F}_S) = {}^*\mathcal{F}_{{}^*S}$.*

Proof. The last statement follows immediately from the definition. Now let $\mathcal{F} = \langle M, s_M, t_M, c_M \rangle$ and $\mathcal{E} = \langle N, s_N, t_N, c_N \rangle$. Then the functor is just a map $p : M \to N$. For a morphism in $\mathcal{F}$ to be cartesian we have the formula

$$\varphi[f] \equiv f \in M \wedge \forall X' \in \mathcal{F}\ \forall u \in \mathrm{Mor}_{\mathcal{F}}(X', t_M(f)) : \exists \beta \in N : p(u) = p(f) \circ \beta \Rightarrow \\ \exists! \overline{u} \in \mathrm{Mor}_{\mathcal{F}}(X', s_M(f)) : u = f \circ \overline{u} \wedge p(\overline{u}) = \beta$$

Now we have

$${}^*\varphi[f] \equiv f \in {}^*M \wedge \forall X' \in {}^*\mathcal{F}\ \forall u \in \mathrm{Mor}_{{}^*\mathcal{F}}(X', {}^*t_M(f)) : \exists \beta \in {}^*N : {}^*p(u) = {}^*p(f) \circ \beta \Rightarrow \\ \exists! \overline{u} \in \mathrm{Mor}_{{}^*\mathcal{F}}(X', {}^*s_M(f)) : u = f \circ \overline{u} \wedge {}^*p(\overline{u}) = \beta$$

But this means just that f is a cartesian morphism in ${}^*\mathcal{F}$. Now the theorem follows by applying the transfer principle. ∎

If all fibres of a fibration have a certain formal property, then the fibres of the enlargement of the fibration have the same property. We don't want to work this out in detail but we just want to give one example.

7.4. PROPOSITION. *Let $p : \mathcal{F} \to \mathcal{E}$ be a fibration of $\hat{S}$–small categories and we assume that for all $S \in \mathcal{E}$ the category $\mathcal{F}_S$ has fibre products. Then we also have for all $S \in {}^*\mathcal{E}$ fibre products in ${}^*\mathcal{F}_S$.*

PROOF. This can be proven in the same way as in corollary 2.17 ■

Now we want to add some more structures to the category $\mathcal{F}$. For that we recall some definitions from [SGA4III]. For a functor $p : \mathcal{A} \to \mathcal{B}$, a morphism f in $\mathcal{B}$ and objects $X, Y \in \mathcal{A}$ we define $\mathrm{Hom}_f(X, Y) := \{g \in \mathrm{Hom}_{\mathcal{A}}(X, Y) | p(g) = f\}$.

7.5. DEFINITION. *An* additive category over a category $\mathcal{B}$ *is a category $\mathcal{A}$ with a functor $p : \mathcal{A} \to \mathcal{B}$ and with abelian group structures on all* $\mathrm{Hom}_f(X, Y)$ *for all morphisms f in $\mathcal{B}$ and all objects $X, Y \in \mathcal{A}$ such that:*

1. *The composition in $\mathcal{A}$ is biadditive.*
2. *The fibres of the functor become additive categories.*

7.6. DEFINITION. *An* abelian category over a category $\mathcal{B}$ *is an additive category $F : \mathcal{A} \to \mathcal{B}$ such that the fibres are abelian categories and such that for all $f : x \to y$ in $\mathcal{B}$, the bifunctor*

$$\mathrm{Hom}_f(X, Y) : \mathcal{A}_x^{op} \times \mathcal{A}_y \to \mathrm{Ab}$$

is left exact in both X and Y.

7.7. DEFINITION. *A* triangulated category over a category $\mathcal{B}$ *is an additive category $F : \mathcal{A} \to \mathcal{B}$ over $\mathcal{B}$ together with a functor $T : \mathcal{A} \to \mathcal{A}$ and a class of triangles Δ_x in $\mathcal{A}_x$ for all $x \in \mathcal{B}$ such that:*

1. *T is an additive B–automorphism of $\mathcal{A}$.*
2. *For all $x \in \mathcal{B}$ the triple $(\mathcal{A}_x, T_x, \Delta_x)$ is a triangulated category.*
3. *For all $x, y \in \mathcal{B}$ and triangles $(X, Y, Z, u, v, w) \in \Delta_x, (X', Y', Z', u', v', w') \in \Delta_y$ and all commutative diagrams*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "j"] & Z \arrow[r, "w"] & T_x X \arrow[d, "f"] \\ X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & T_y X' \end{tikzcd}$$

   there is an $h : Z \to Z'$ such that this will become a morphism of triangles.

It is straightforward to enlarge such categories. We omit a precise statement and just state the following

7.8. Proposition. *The notion of small additive (abelian/triangulated) category over another small category behaves well under enlargement.*

Moreover we get

7.9. Proposition. *Let $p : \mathcal{A} \to \mathcal{B}$ be an abelian fibration of $\hat{S}$–small categories such that for all $B \in \mathcal{B}$ the abelian category $\mathcal{A}_B$ has enough injective/projective objects. Then for all $B \in {}^*\mathcal{B}$ the abelian category ${}^*\mathcal{A}_B$ has enough injective/projective objects.*

Proof. This can be proved in the same way as corollary 5.2. ∎

7.10. Example. Let $\mathcal{E}$ be the category of finite rings in $\hat{S}$ from example 3.7. Now let $\mathcal{F}$ be the category of pairs (R, M) where R is a finite Ring and M is a finitely generated R–module. A morphism from an object (R, M) to an object (R', M') is by definition a pair (φ, f) where φ is just a ring homomorphism $\varphi : R \to R'$ and f is just an R–linear module homomorphism from M to M'_R. Here M'_R denotes the module which is as abelian group the same as M' and gets its R–module structure via $\varphi : R \to R'$. This gives a category with the obvious composition. We have the natural functor $\mathcal{F} \to \mathcal{E}$ which maps the pair (R, M) to R and a morphism (φ, f) to φ. One can easily check that this gives an abelian fibred category over the category of finite rings. Now let ${}^*\mathcal{F} \to {}^*\mathcal{E}$ be the corresponding fibration in $\widehat{{}^*S}$. If R is an object in ${}^*\mathcal{E}$ we call the fibre ${}^*\mathcal{F}_R$ the category of *–finitely generated R–modules. The category ${}^*\mathcal{E}$ can easily be identified with the category of *–finite internal rings in $\hat{S}$. Furthermore ${}^*\mathcal{F}_R$ can be identified with a category of specific internal R–modules in $\widehat{{}^*S}$.

7.11. Remark. A motivation for the previous definitions is the following proposition from [SGA4III, XVII: Prop. 2.2.13]

7.12. Proposition. *There is an equivalence between*

1. *the 2-category of pseudo–functors from a category $\mathcal{B}^{op}$ to the category of additive (abelian/triangulated) categories with additive (left exact/triangulated) functors*
2. *the 2–category of additive (abelian/triangulated) fibred categories over $\mathcal{B}$.*

The notion of a 2–category also behaves well under enlargement. Now let (Cat) be a $\hat{S}$–small 2–category of small categories in $\hat{S}$. Then one can get by the transfer principle from the previous proposition the following

7.13. Proposition. *Let $\mathcal{B}$ be a $\hat{S}$–small category. There is an equivalence between*

1. *the 2-category of internal pseudo–functors from a category ${}^*\mathcal{B}^{op}$ to the category of additive (abelian/triangulated) categories in ${}^*(Cat)$ with additive (left exact/triangulated) functors.*
2. *the 2–category of additive (abelian/triangulated) fibred internal categories over ${}^*\mathcal{B}$.*

Finally we want to have a fibred version of proposition 6.8 and 6.10. For details on derived fibrations of an abelian fibration and cofibrations we refer again to [SGA4III, expose 17].

7.14. PROPOSITION. *Let $\mathcal{A} \to \mathcal{B}$ be an abelian cofibration which is bounded derivable, and furthermore we assume that there is an $n_0 \in \mathbb{N}$ such that for all morphisms $f : X \to Y$ in $\mathcal{B}$ the cohomological dimension of the left exact functor $f_* : \mathcal{A}_X \to \mathcal{A}_Y$ is less or equal to n_0. Then ${}^*\mathcal{A} \to {}^*\mathcal{B}$ is also an abelian cofibration which is bounded derivable and for all $f : X \to Y$ in ${}^*\mathcal{B}$ the cohomological dimension of the left exact functor $f_* : {}^*\mathcal{A}_X \to {}^*\mathcal{A}_Y$ is again less or equal to n_0. Further we define ${}^*\mathcal{D}^{fb}(\mathcal{A}/\mathcal{B})$ as the full subcategory of ${}^*\mathcal{D}(\mathcal{A}/\mathcal{B})$ which consists of those *complexes which are bounded by a standard natural number. Then ${}^*\mathcal{D}^{fb}(\mathcal{A}/\mathcal{B})$ defines also a triangulated cofibration over ${}^*\mathcal{B}$ and we have an canonical isomorphism of triangulated fibrations*

$${}^*\mathcal{D}^{fb}(\mathcal{A}/\mathcal{B}) \xrightarrow{\sim} \mathcal{D}^b({}^*\mathcal{A}/{}^*\mathcal{B}).$$

PROOF. By the transfer principle it follows that ${}^*\mathcal{A}/{}^*\mathcal{B}$ is again derivable and the statement about the cohomological dimension. The last property shows that ${}^*\mathcal{D}^{fb}(\mathcal{A}/\mathcal{B})$ is also cofibred over ${}^*\mathcal{B}$. The last statement of the proposition can be shown in a similar way as in 6.8 and 6.10. ∎

A. Superstructures and Enlargements

In this appendix we give a short introduction to enlargements and list (without proofs) some of their basic properties. Details can be found in [LR94] and [LW00].
For a set M, let $\mathcal{P}(M)$ denote the power set of M, i.e. the set of all subsets of M.

A.1. DEFINITION. *(Superstructure)*
Let S be an infinite set whose elements are no sets. Such a set we call a base set*, and its elements we call* base elements*. We define $\hat{S}$, the* superstructure over *S, as follows:*

$$\hat{S} := \bigcup_{n=0}^{\infty} S_n \qquad \text{where } S_0 := S \text{ and } \forall n \geq 1 : S_n := S_{n-1} \cup \mathcal{P}(S_{n-1}).$$

A.2. REMARK. Recall from [SGA4I, I.0] that a *universe* is a set U satisfying the following conditions:

1. $y \in x \in U \Rightarrow y \in U$,
2. $x, y \in U \Rightarrow \{x, y\} \in U$,
3. if $x \in U$ is a set, then $\mathcal{P}(x) \in U$,
4. if $(x_i)_{i \in I \in U}$ is a family with $x_i \in U$ for all i, then $\bigcup_{i \in I} x_i \in U$.

Every superstructure satisfies axioms 1, 2 and 3 of a universe, but not 4: Let $\hat{S}$ be a superstructure.

1. Let $y \in x \in \hat{S}$. Because x is a set and the base elements are no sets, there exists a uniquely determined $n \in \mathbb{N}_+$ such that $x \in S_n \setminus S_{n-1}$, i.e. $x \in \mathcal{P}(S_{n-1})$. So x is a subset of S_{n-1} and y an element of $S_{n-1} \subseteq \hat{S}$.

2. Let $x, y \in \hat{S}$. Then there is a $n \in \mathbb{N}_0$ such that $x, y \in S_n$, i.e. $\{x, y\} \in \mathcal{P}(S_n) \subseteq S_{n+1} \subseteq \hat{S}$.

3. Let $x \in \hat{S}$ be a set. Then as above, $x \subseteq S_n$ for a suitable $n \in \mathbb{N}_0$, and it follows $\mathcal{P}(x) \subseteq \mathcal{P}(S_n) \subseteq S_{n+1}$, i.e. $\mathcal{P}(x) \in S_{n+2} \subseteq \hat{S}$.

4. To see that axiom 4 does no hold, let I be a countable infinite subset of S, choose a bijection $\varphi : I \to \mathbb{N}_0$ and put $x_i := S_{\varphi(i)} \in \hat{S}$. (Note that for all n, the set S_n is an element of $\mathcal{P}(S_n) \subseteq S_{n+1} \subseteq \hat{S}$.) Then $\bigcup_{i \in I} x_i = \hat{S}$, which is *not* an element of $\hat{S}$.

Note that even though superstructures fail to be universes, axiom A.2.4 holds under the additional assumption that all sets x_i of the family are contained in a fixed set $U \in \hat{S}$, and in praxis all families considered will be of this sort. In addition to that, we have the following

A.3. Proposition. *Let U be a universe, and let $\mathcal{C}$ be an U-small category. Then there exists a base set S such that $\mathcal{C}$ is equivalent to an $\hat{S}$-small category.*

Proof. Let $M \in U$ denote the set of morphisms of $\mathcal{C}$. Choose a base set S whose cardinality is greater than that of M, and choose an injection $M \hookrightarrow S$. Then the construction given after 2.1 obviously produces an $\hat{S}$-small category that is isomorphic to $\mathcal{C}$. ∎

In the superstructure $\hat{S}$ to a base set S, we will find most of the mathematical objects of interest related to S: First of all, for sets $M, N \in \hat{S}$, the product set $M \times N$ is again an element of $\hat{S}$ when we identify an ordered pair $\langle a, b\rangle$ for $a \in M$, $b \in N$ with the set $\{a, \{a, b\}\}$, and for sets $M_1, \ldots, M_n \in \hat{S}$, the product set $M_1 \times \ldots \times M_n := (M_1 \times \ldots \times M_{n-1}) \times M_n$ is an element of $\hat{S}$. Therefore, relations between two sets $M, N \in \hat{S}$ and in particular functions from M to N are again elements of $\hat{S}$.

For example, if S contains the set of real numbers $\mathbb{R}$, then $\hat{S}$ will contain the sets $\mathbb{R}^n$ for $n \in \mathbb{N}_+$ as well as functions between subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, the set of continuous functions between such sets or the set of differentiable functions and so on.

A.4. Definition. *Let $\hat{S}$ be a superstructure, and let $f, x \in \hat{S}$. Then we define the element $(f\angle x)$ of $\hat{S}$ as follows: if f is a set that contains $\langle x, y\rangle$ for exactly one $y \in \hat{S}$, then $(f\angle x) := y$, otherwise $(f\angle x) := \emptyset$.*

(So if $f : A \to B$ is a function with $A, B \in \hat{S}$ and if $x \in A$, then $(f\angle x) = f(x)$.)

Now we want to define what *terms*, *formulas* and *statements* in a given superstructure are:

A.5. DEFINITION. *Let $\hat{S}$ be a superstructure.*

1. *A* term in $\hat{S}$ *is the following*

   (a) *An element of $\hat{S}$ is a term, a so called* constant *(note that $\hat{S}$ itself is not an element of $\hat{S}$ and therefore not a constant).*

   (b) *A* variable, *i.e. a symbol that is not an element of $\hat{S}$, is a term; usually we will denote variables by x, y, x_1, x_2, X, Y and so on.*

   (c) *If s and t are terms, then $\langle s, t\rangle$ and $(s\angle t)$ are terms.*

   (d) *A sequence of symbols is a term if and only if it can be built by (1), (2) and (3) in finitely many steps.*

   Note that terms that do not contain variables are elements of $\hat{S}$.

2. *A* formula in $\hat{S}$ *is the following:*

   (a) *If s and t are terms, then $s = t$ and $s \in t$ are formulas.*

   (b) *If ψ is a formula, then $\neg\psi$ is a formula.*

   (c) *If ψ and χ are formulas, then $(\psi \wedge \chi)$ is a formula.*

   (d) *If x is a variable, t is a term in which x does not appear and ψ is a formula, then $(\forall x \in t)\psi$ is a formula.*

   (e) *A sequence of symbols is a formula if and only if it can be built by (1), (2), (3) and (4) in finitely many steps.*

3. *Let φ be a formula in $\hat{S}$, and let X be a variable that occurs at position j of φ (remember that a formula is a sequence of symbols). We say that this occurrence of X is* bound *if position j is part of a subsequence of φ of the form $(\forall X \in t)\psi$ for a term t and a formula ψ. Otherwise, we call the occurrence of X* free.

4. *Let φ be a formula in $\hat{S}$. We say that a variable X is a* free variable of φ, *if there is a free occurrence of X at a position of φ. If $X_1, \ldots, X_n$ are the free variables of φ, we often write $\varphi[X_1, \ldots, X_n]$ instead of φ.*

5. *If $\varphi[X_1, \ldots, X_n]$ is a formula in $\hat{S}$ with free variables $X_1, \ldots, X_n$ and if $\tau_1, \ldots, \tau_n \in \hat{S}$ are constants, then $\varphi[\tau_1, \ldots, \tau_n]$ is the statement in $\hat{S}$ that we get when we replace any* free *occurrence of X_i in φ by τ_i.*

6. *A* statement in $\hat{S}$ *is a formula that has no free variables.*

A.6. Proposition. *Let φ be a statement in a superstructure $\hat{S}$.*

1. *If φ contains none of the logical symbols $\neg$, $\wedge$ or $\forall$, then φ is either of the form $s = t$ or of the form $s \in t$ with terms $s, t \in \hat{S}$ that contain no variables and are therefore elements of $\hat{S}$.*
2. *If φ contains at least one of the symbols $\neg$, $\wedge$ or $\forall$, then it is of one and only one of the following forms:*
   (a) *$\varphi = \neg\psi$ with a statement ψ,*
   (b) *$\varphi = (\psi_1 \wedge \psi_2)$ with statements ψ_1 and ψ_2.*
   (c) *$\varphi = (\forall X \in t)\psi$ with a term t without variables and a formula ψ which can only contain X as a free variable.*

A.7. Definition. *Let φ be a statement in a superstructure $\hat{S}$. We will call φ* true *or* valid *under the following conditions, depending on the structure of φ in the sense of A.6:*

1. *If φ is of the form $s = t$ or $s \in t$ with terms s and t that are elements of $\hat{S}$, then φ is* true *iff s equals t in the first case and iff s is an element of t in the second case.*
2. *If $\varphi = \neg\psi$ for a statement ψ, then φ is* true *iff ψ is not true.*
3. *If $\varphi = (\psi_1 \wedge \psi_2)$ for statements ψ_1 and ψ_2, then φ is* true *iff ψ_1 and ψ_2 both are true.*
4. *If $\varphi = (\forall X \in t)\psi$ for a term t which is an element of $\hat{S}$ and a formula ψ which can only contain X as a free variable, then we distinguish between the following three cases:*
   (a) *If t is a set and if X is a free variable of ψ, then φ is* true *iff the statement $\psi[\tau]$ is true for all $\tau \in t$.*
   (b) *If t is a set and if ψ is a statement, then φ is* true *iff ψ is true.*
   (c) *If t is no set, then φ is* true.

A.8. Definition. *(Enlargement)*
Let $* : \hat{S} \to \hat{W}$ be a map between superstructures. For $\tau \in \hat{S}$ we denote the image of τ under $*$ by ${}^*\tau$, and for a formula φ in $\hat{S}$, we define ${}^*\varphi$ to be the formula in $\hat{W}$ that we get when we replace any constant τ occurring in φ by ${}^*\tau$.
We call $*$ an enlargement *if the following conditions hold:*

1. *${}^*S = W$. (Because of this property, we will often write $* : \hat{S} \to \widehat{{}^*S}$.)*

2. $\forall s \in S : {}^*s = s$.

3. *(*transfer principle*)*
   If φ is a statement in $\hat{S}$, then φ is true iff ${}^*\varphi$ is true.

4. *(*saturation principle*)*
   Put $\mathcal{I} := \bigcup_{A \in \hat{S} \setminus S} {}^*A \subseteq \hat{W}$, let I be a nonempty set whose cardinality is not bigger than that of $\hat{S}$, and let $\{U_i\}_{i \in I}$ be a family of nonempty sets $U_i \in \mathcal{I}$ with the property that for all finite *subsets $J \subseteq I$, the intersection $\bigcap_{j \in J} U_j$ is nonempty. Then $\bigcap_{i \in I} U_i \neq \emptyset$.*

If $*$ is an enlargement, we call the elements of $\mathcal{I}$ the internal elements *of $\hat{W}$.*

A.9. Theorem. *For any base set S, there exists an enlargement $* : \hat{S} \to \widehat{{}^*S}$.*

An easy corollary of the transfer principle is that $*$ is injective: If σ and τ are elements of $\hat{S}$ and φ denotes the statement $\big[\sigma = \tau\big]$, then ${}^*\varphi \equiv \big[{}^*\sigma = {}^*\tau\big]$, i.e. ${}^*\sigma = {}^*\tau$ implies $\sigma = \tau$ by transfer.

From now on, in building formulas, we will as well use the symbols $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\exists$ and $\exists!$ as obvious abbreviations for the corresponding constructions involving only the symbols $\neg$, $\wedge$ and $\forall$. We will also often write $f(x)$ instead of $(f\angle x)$ and $\forall X \in t : \psi$ instead of $(\forall X \in t)\psi$ to make formulas more readable.

A.10. Proposition. *Let $* : \hat{S} \to \widehat{{}^*S}$ be an enlargement.*

1. *If $M \in \hat{S}$ is a set, then ${}^*M$ is also a set.*

2. *If $A \in \hat{S}$ is a set, then $\{{}^*a | a \in A\} \subseteq {}^*A$ with equality if and only if A is* finite*. In particular, if $A \subset S$ is finite, then ${}^*A = A$.*

3. ${}^*\emptyset = \emptyset$.

4. *If $A, B \in \hat{S}$ are sets, then*
$$ {}^*(A \cup B) = {}^*A \cup {}^*B,\ {}^*(A \cap B) = {}^*A \cap {}^*B,\ {}^*(A \setminus B) = {}^*A \setminus {}^*B,\ {}^*(A \times B) = {}^*A \times {}^*B. $$

5. *If $A_1, \dots, A_n \in \hat{S}$ are sets and $a_1 \in A_1, \dots, a_n \in A_n$, then*
$$ {}^*\langle a_1, \dots, a_n \rangle = \langle {}^*a_1, \dots, {}^*a_n \rangle. $$

6. *Let $A \in \hat{S}$ be a set and $\varphi[X_1, \dots, X_n]$ be a formula in $\hat{S}$. Then*
$$ {}^*\big\{\langle a_1, \dots, a_n \rangle \in A^n \big| \varphi[a_1, \dots, a_n] \text{ is true}\big\} = \big\{\langle a_1, \dots, a_n \rangle \in ({}^*A)^n \big| {}^*\varphi[a_1, \dots, a_n] \text{ is true}\big\}. $$

7. *If $A \in \hat{S}$ is a set, then*

$$^*[\mathcal{P}(A)] = \left\{B \in \mathcal{P}(^*A) \middle| B \text{ internal}\right\}.$$

8. *For $f, x \in \hat{S}$, we have $^*(f\angle x) = (^*f\angle ^*x)$, and if $A, B \in \hat{S}$ are sets and $f : A \to B$ is a function, then $^*f$ is a function from $^*A$ to $^*B$.*

9. *If $A, B, C \in \hat{S}$ are sets and $g : A \to B$ and $f : B \to C$ are maps, then $^*(f \circ g) = {}^*f \circ {}^*g$.*

10. *If $A, B \in \hat{S}$, then*

$$^*\left[B^A\right] = \left\{f : {}^*A \to {}^*B \middle| f \text{ internal}\right\}$$

(here B^A denotes as usual the set of maps from A to B).

11. *If $A \in \hat{S}$ is an ordered set (resp. partial ordered set, totally ordered set, group, abelian group, ring, commutative ring, commutative ring with unit, field, totally ordered field), then $^*A$ also is an ordered set (resp. partial ordered set, ...).*

A.11. Proposition. *Let $* : \hat{S} \to \widehat{^*S}$ be an enlargement.*

1. *Let $A, B \in \widehat{^*S}$ be* internal *sets, i.e. sets which are internal elements of $\widehat{^*S}$. Then $A \cup B$, $A \cap B$, $A \setminus B$ and $A \times B$ are also internal sets.*

2. *Let $B \in \widehat{^*S}$ be an* internal *set, and let $\varphi[X_1, \dots, X_n]$ be an* internal *formula in $\widehat{^*S}$, i.e. a formula in which all constants are internal elements of $\widehat{^*S}$. Then the set*

$$\left\{\langle b_1, \dots, b_n\rangle \in B^n \middle| \varphi[b_1, \dots, b_n] \text{ is true}\right\}$$

is internal.

3. *If $a \in A \in \widehat{^*S}$ and A is an internal set, then a is internal.*

4. *If $A \in \hat{S} \setminus S$ is a* finite *set, and $B \in \widehat{^*S}$ is an* internal *set, then any map from $^*A = A$ to B is internal.*

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