Theory and Applications of Categories, Vol. 14, No. 17, 2005, pp. 399–423.

NOTES ON ENRICHED CATEGORIES WITH COLIMITS OF SOME CLASS

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ABSTRACT. The paper is in essence a survey of categories having ϕ-weighted colimits for all the weights ϕ in some class Φ. We introduce the class Φ^+ of Φ-*flat* weights which are those ψ for which ψ-colimits commute in the base $\mathcal{V}$ with limits having weights in Φ; and the class Φ^- of Φ-*atomic* weights, which are those ψ for which ψ-limits commute in the base $\mathcal{V}$ with colimits having weights in Φ. We show that both these classes are *saturated* (that is, what was called *closed* in the terminology of [AK88]). We prove that for the class $\mathcal{P}$ of *all* weights, the classes $\mathcal{P}^+$ and $\mathcal{P}^-$ both coincide with the class $\mathcal{Q}$ of *absolute* weights. For any class Φ and any category $\mathcal{A}$, we have the free Φ-cocompletion $\Phi(\mathcal{A})$ of $\mathcal{A}$; and we recognize $\mathcal{Q}(\mathcal{A})$ as the Cauchy-completion of $\mathcal{A}$. We study the equivalence between $(\mathcal{Q}(\mathcal{A}^{\mathrm{op}}))^{\mathrm{op}}$ and $\mathcal{Q}(\mathcal{A})$, which we exhibit as the restriction of the Isbell adjunction between $[\mathcal{A},\mathcal{V}]^{\mathrm{op}}$ and $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$ when $\mathcal{A}$ is small; and we give a new Morita theorem for any class Φ containing $\mathcal{Q}$. We end with the study of Φ-continuous weights and their relation to the Φ-flat weights.

1. Introduction

The present observations had their beginnings in an analysis of the results obtained by Borceux, Quintero and Rosický in their article [BQR98], which in turn followed on from that of Borceux and Quintero [BQ96]. These authors were concerned with extending to the enriched case the notion of accessible category and its properties, described for ordinary categories in the books [MP89] of Makkai and Paré and [AR94] of Adàmek and Rosický. They were led to discuss categories – now meaning $\mathcal{V}$-categories – with finite limits (in a suitable sense), or more generally with α-small limits, or with filtered colimits (in a suitable sense), and more generally with α-filtered colimits, or again with α-flat colimits, and to discuss the connexions between these classes of limits and of colimits. When we looked in detail at their work, we observed that many of the properties they discussed hold in fact for categories having colimits of *any* given class Φ, while others hold when Φ is the class of colimits commuting in the base category $\mathcal{V}$ with the limits of some class Ψ – such particular properties as finiteness or filteredness arising only as special cases of the *general* results. Approaching in this abstract way, not generalizations

The first author gratefully acknowledges the support of the Australian Research Council. The second author gratefully acknowledges the support of the British Engineering and Physical Sciences Research Council (grant GR/R63004/01).

Received by the editors 2005-09-02.

Transmitted by Ross Street. Published on 2005-12-03. Revised on 2006-04-29. Original version at www.tac.mta.ca/tac/volumes/14/17/14-17a.dvi..

2000 Mathematics Subject Classification: 18A35, 18C35, 18D20.

Key words and phrases: limits, colimits, flat, atomic, small presentable, Cauchy completion.

of accessible categories as such, but the study of categories with colimits (or limits) of some class, brings considerable notional simplifications.

Although our original positive results are limited in number, their value may be judged by the extra light they cast on several of the results in [BQR98]. To expound these results, it has seemed to us necessary to repeat some known facts so as to provide the proper context. The outcome is that we have produced a rather complete study of categories having colimits of a given class, which is to a large extent self-contained: a kind of survey paper containing a fair number of original results.

We begin by reviewing and completing some known material in the first sections: in Section 2 the general notions of weighted limits and colimits for enriched categories; in Section 3 the free Φ-cocompletion $\Phi(\mathcal{A})$ of a $\mathcal{V}$-category $\mathcal{A}$; and in Section 4 results on the recognition of categories of the form $\Phi(\mathcal{A})$.

Section 5 treats generally the commutation of limits and colimits in the base $\mathcal{V}$: it introduces classes of the form Φ^+ of Φ-*flat* weights – those weights whose colimits in $\mathcal{V}$ commute with Φ-weighted limits – and classes of the form Φ^- of Φ-*atomic* weights – those weights whose limits in $\mathcal{V}$ commute with Φ-weighted colimits. We show that each of these classes is saturated.

Section 6 focuses on the class $\mathcal{Q} = \mathcal{P}^-$ where $\mathcal{P}$ is the class of *all* (small) weights; this $\mathcal{Q}$ is the class of *small projective* or *atomic* weights, which is also, as Street showed in [Str83], the class of *absolute* weights. We show that $\mathcal{Q}$ is also the class $\mathcal{P}^+$ of $\mathcal{P}$-flat weights. We recall that a weight $\phi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ corresponds to a module $\overline{\phi} : \mathcal{I} \longrightarrow\!\!\!\!\!\circ\;\; \mathcal{K}$, while a weight $\psi : \mathcal{K} \to \mathcal{V}$ corresponds to a module $\underline{\psi} : \mathcal{K} \longrightarrow\!\!\!\!\!\circ\;\; \mathcal{I}$; and we recall that the relation between a left adjoint module $\overline{\phi}$ and its right adjoint $\underline{\psi}$ gives rise to an equivalence between $(\mathcal{Q}(\mathcal{K}^{\mathrm{op}}))^{\mathrm{op}}$ and $\mathcal{Q}(\mathcal{K})$, which is in fact the restriction to the small projectives of the *Isbell Adjunction* between $[\mathcal{K}, \mathcal{V}]^{\mathrm{op}}$ and $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$.

Section 7 studies the Cauchy-completion $\mathcal{Q}(\mathcal{A})$ for a general category $\mathcal{A}$ and gives an extension of the classical Morita theorem: for any class Φ containing $\mathcal{Q}$ we have $\Phi(\mathcal{A}) \simeq \Phi(\mathcal{B})$ if and only if $\mathcal{Q}(\mathcal{A}) \simeq \mathcal{Q}(\mathcal{B})$. (We use $\cong$ to denote isomorphism and $\simeq$ to denote equivalence.)

Finally we consider in section 8 the class of Φ-continuous functors $\mathcal{N}^{\mathrm{op}} \to \mathcal{V}$, where $\mathcal{N}$ is a small category admitting Φ-colimits; and we compare these with the Φ-flat functors. For $\mathcal{V} = \mathbf{Set}$, some special cases of the results here appeared in [ABLR02].

We have benefited greatly from discussions with Francis Borceux and with Ross Street, both of whom have contributed significantly to the improvement of our exposition; we thankfully acknowledge their help.

2. Revision of weighted limits and colimits

The necessary background knowledge about enriched categories is largely contained in [Kel82], augmented by [Kel82-2] and the Albert-Kelly article [AK88].

We deal with categories enriched in a symmetric monoidal closed category $\mathcal{V}$, supposing as usual that the ordinary category $\mathcal{V}_0$ underlying $\mathcal{V}$ is locally small, complete and cocomplete. (A set is *small* when its cardinal is less than a chosen inaccessible cardinal ∞, and a category is *locally small* when each of its hom-sets is small.) We henceforth use "category", "functor", and "natural transformation" to mean "$\mathcal{V}$-category", "$\mathcal{V}$-functor", and "$\mathcal{V}$-natural transformation", except when more precision is needed. We call a $\mathcal{V}$-category *small* when its set of *isomorphism classes* of objects is a small set; a $\mathcal{V}$-category that is not small is sometimes said to be *large*. $\mathcal{V}$-**CAT** is the 2-category of $\mathcal{V}$-categories, whereas $\mathcal{V}$-**Cat** is that of small $\mathcal{V}$-categories. **Set** is the category of small sets, **Cat** = **Set**-**Cat** is the 2-category of small categories, and **CAT** = **Set**-**CAT** is the 2-category of locally small categories.

A *weight* is a functor $\phi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ with domain $\mathcal{K}^{\mathrm{op}}$ *small*; weights were called *indexing-types* in [Kel82], [Kel82-2] and [AK88], where weighted limits were called *indexed limits*. (A functor with codomain $\mathcal{V}$ is often called a *presheaf*; so that a weight is a presheaf with a small domain.) Recall that the *ϕ-weighted limit* $\{\phi, T\}$ of a functor $T : \mathcal{K}^{\mathrm{op}} \to \mathcal{A}$ is defined representably by

2.1. $\mathcal{A}(a, \{\phi, T\}) \cong [\mathcal{K}^{\mathrm{op}}, \mathcal{V}](\phi, \mathcal{A}(a, T-))$,

while the *ϕ-weighted colimit* $\phi * S$ of $S : \mathcal{K} \to \mathcal{A}$ is defined dually by

2.2. $\mathcal{A}(\phi * S, a) \cong [\mathcal{K}^{\mathrm{op}}, \mathcal{V}](\phi, \mathcal{A}(S-, a))$,

so that $\phi * S$ is equally the ϕ-weighted limit of $S^{\mathrm{op}} : \mathcal{K}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$. Of course the limit $\{\phi, T\}$ consists not just of the object $\{\phi, T\}$ but also of the representation 2.1, or equally of the corresponding *counit* $\mu : \phi \to \mathcal{A}(\{\phi, T\}, T-)$; it is by *abus de langage* that we usually mention only $\{\phi, T\}$. When $\mathcal{V} = \mathbf{Set}$, we refind the classical (or "conical") limit of $T : \mathcal{K}^{\mathrm{op}} \to \mathcal{A}$ and the classical colimit of $S : \mathcal{K} \to \mathcal{A}$ as

2.3. $\lim T = \{\Delta 1, T\}$ and $\operatorname{colim} S = \Delta 1 * S$

where $\Delta 1 : \mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$ is the constant functor at the one point set 1. Recall too that the weighted limits and colimits can be calculated using the classical ones when $\mathcal{V} = \mathbf{Set}$: for then the presheaf $\phi : \mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$ gives the discrete op-fibration $d : \mathrm{el}(\phi) \to \mathcal{K}^{\mathrm{op}}$ where $\mathrm{el}(\phi)$ is the category of elements of ϕ, and now

2.4. $\{\phi, T\} = \lim\{\, \mathrm{el}(\phi) \xrightarrow{d} \mathcal{K}^{\mathrm{op}} \xrightarrow{T} \mathcal{A} \,\}$, 2.5. $\phi * S = \mathrm{colim}\{\ \mathrm{el}(\phi)^{\mathrm{op}} \xrightarrow{d^{\mathrm{op}}} \mathcal{K} \xrightarrow{S} \mathcal{A}\ \}$.

Recall finally that a functor $F : \mathcal{A} \to \mathcal{B}$ is said to *preserve the limit* $\{\phi, T\}$ as in 2.1 when $F(\{\phi, T\})$ is the limit of FT weighted by ϕ, with counit

$$\phi \xrightarrow{\mu} \mathcal{A}(\{\phi, T\}, T-) \xrightarrow{F} \mathcal{B}(F\{\phi, T\}, FT-)\ ;$$

and F is said to preserve the colimit $\phi * S$ as in 2.2 when F^{op} preserves $\{\phi, S^{\mathrm{op}}\}$.

We spoke above of a "class Φ of colimits" or a "class Ψ of limits"; but this is loose and rather dangerous language – the only thing that one can sensibly speak of is *a class Φ of weights*. Then a category $\mathcal{A}$ *admits Φ-limits*, or is *Φ-complete*, if $\mathcal{A}$ admits the limit $\{\phi, T\}$ for each weight $\phi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ in Φ and each $T : \mathcal{K}^{\mathrm{op}} \to \mathcal{A}$; while $\mathcal{A}$ *admits Φ-colimits*, or is *Φ-cocomplete*, when $\mathcal{A}$ admits the colimit $\phi * S$ for each $\phi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ in Φ and each $S : \mathcal{K} \to \mathcal{A}$ (and thus when $\mathcal{A}^{\mathrm{op}}$ is Φ-complete). Moreover a functor $\mathcal{A} \to \mathcal{B}$ between Φ-complete categories is said to be *Φ-continuous* when it preserves all Φ-limits, and one defines *Φ-cocontinuous* dually. We write Φ-**Conts** for the 2-category of Φ-complete categories, Φ-continuous functors, and all natural transformations – which is a (non full) sub-2-category of $\mathcal{V}$-**CAT**; and similarly Φ-**Cocts** for the 2-category of Φ-cocomplete categories, Φ-cocontinuous functors, and all natural transformations.

To give a class Φ of weights is to give, for each small $\mathcal{K}$, those $\phi \in \Phi$ with domain $\mathcal{K}^{\mathrm{op}}$; let us use as in [AK88] the notation

2.6. $\Phi[\mathcal{K}] = \{\phi \in \Phi \mid dom(\phi) = \mathcal{K}^{\mathrm{op}}\}$,

so that

2.7. $\Phi = \Sigma_{\mathcal{K}\ small}\Phi[\mathcal{K}]$.

In future, we look on $\Phi[\mathcal{K}]$ as a full subcategory of the functor category $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$ (which we may also call a *presheaf* category). The smallest class of weights is the empty class 0, and 0-**Conts** is just $\mathcal{V}$-**CAT**. The largest class of weights consists of *all* weights – that is, all presheaves with small domains – and we denote this class by $\mathcal{P}$; the 2-category $\mathcal{P}$-**Conts** is just the 2-category **Conts** of *complete* categories and *continuous* functors, and similarly $\mathcal{P}$-**Cocts** = **Cocts**.

There may well be different classes Φ and Ψ for which the sub-2-categories Φ-**Conts** and Ψ-**Conts** of $\mathcal{V}$-**CAT** coincide; which is equally to say that Φ-**Cocts** and Ψ-**Cocts** coincide. When $\mathcal{V}$ = **Set**, for instance, **Conts** = $\mathcal{P}$-**Conts** coincides with Φ-**Conts** where Φ consists of the weights for products and for equalizers. We define the *saturation* Φ^* of a class Φ of weights as follows: the weight ψ belongs to Φ^* when every Φ-complete category is also ψ-complete and every Φ-continuous functor is also ψ-continuous. Note that Φ^* was called in [AK88] the *closure* of Φ; we now prefer the term "saturation", since "closure" already has so many meanings. Clearly then, we have

2.8. Φ-**Conts** $=$ Ψ-**Conts** $\Leftrightarrow$ Φ-**Cocts** $=$ Ψ-**Cocts** $\Leftrightarrow$ $\Phi^* = \Psi^*$.

When $\mathcal{V} = \mathbf{Set}$, we can of course consider Φ-**Conts** where Φ consists of the $\Delta 1 : \mathcal{K}^{\text{op}} \to \mathbf{Set}$ for all $\mathcal{K}$ in some class $\mathcal{D}$ of small categories; and we might write $\mathcal{D}$-**Conts** for this 2-category Φ-**Conts** of $\mathcal{D}$-*complete* categories, $\mathcal{D}$-*continuous* functors, and all natural transformations. We underline the fact, however, that when $\mathcal{V} = \mathbf{Set}$, NOT every Φ-**Conts** is of the form $\mathcal{D}$-**Conts** for some $\mathcal{D}$ as above; a simple example of this situation is given in [AK88].

We spoke of $\mathcal{V}$-**CAT** as a 2-category, the category $\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A}, \mathcal{B})$ having as its objects the $\mathcal{V}$-functors $T : \mathcal{A} \to \mathcal{B}$ and as its arrows the $\mathcal{V}$-natural transformations $\alpha : T \to S : \mathcal{A} \to \mathcal{B}$. When $\mathcal{A}$ is small, however, we also have the $\mathcal{V}$-category $[\mathcal{A}, \mathcal{B}]$, whose underlying ordinary category $[\mathcal{A}, \mathcal{B}]_0$ is $\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A}, \mathcal{B})$; an example is of course the presheaf $\mathcal{V}$-category $[\mathcal{K}^{\text{op}}, \mathcal{V}]$ of 2.1.

When $\mathcal{A}$ is not small, $[\mathcal{A}, \mathcal{B}]$ may not exist as a $\mathcal{V}$-category, since the end $\int_a \mathcal{B}(Fa, Ga)$ giving the $\mathcal{V}$-valued hom $[\mathcal{A}, \mathcal{B}](F, G)$ may not exist in $\mathcal{V}$ for all $F, G : \mathcal{A} \to \mathcal{B}$. However it may exist for *some* pairs F, G, and then we can speak of $[\mathcal{A}, \mathcal{B}](F, G)$. This allows us the convenience of speaking of the limit $\{\phi, T\}$ of 2.1 or the colimit $\phi * S$ of 2.2 even when $\mathcal{K}$ is not small (so that ϕ is no longer a weight, in the sense of this article) : for instance, we say that $\phi * S$ *exists* if the right side of 2.2 *exists in* $\mathcal{V}$ *for each* a, *and is representable* as the left side of 2.2. In particular, we can speak, even when $\mathcal{A}$ is not small, of the possible existence of the left Kan extension $\mathrm{Lan}_K T$ of some $T : \mathcal{A} \to \mathcal{B}$ along some $K : \mathcal{A} \to \mathcal{C}$, recalling from Chapter 4 of [Kel82] that it is given by

2.9. $\mathrm{Lan}_K T(c) \cong \mathcal{C}(K-, c) * T$,

existing when the colimit on the right exists for each c.

3. Revision of the free Φ-cocompletion of a category and of saturated classes of weights

Another piece of background knowledge that we need to recall concerns the "left biadjoint" to the forgetful 2-functor $U_\Phi : \Phi\text{-}\mathbf{Cocts} \to \mathcal{V}\text{-}\mathbf{CAT}$. (Note that it is convenient to deal with colimits rather than limits.)

Recall from Section 4.8 of [Kel82] that a presheaf $F : \mathcal{A}^{\text{op}} \to \mathcal{V}$, where $\mathcal{A}$ need not be small, is said to be *accessible* if it is the left Kan extension of some $\phi : \mathcal{K}^{\text{op}} \to \mathcal{V}$ with $\mathcal{K}$ small along some $H^{\text{op}} : \mathcal{K}^{\text{op}} \to \mathcal{A}^{\text{op}}$; which is to say, by 2.9, that F has the form

3.1. $Fa \cong \mathcal{A}(a, H-) * \phi$,

which by (3.9) of [Kel82] may equally be written as

3.2. $Fa \cong \phi * \mathcal{A}(a, H-)$.

It is shown in Proposition 4.83 of [Kel82][1] that, whenever F is accessible, it is a left Kan extension as above for some H that is *fully faithful*; in other words, that F is the left Kan extension of its restriction to some small full subcategory of $\mathcal{A}^{\mathrm{op}}$. Whenever F is accessible, $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}](F, G)$ exists for each G, an easy calculation using the Yoneda isomorphism giving

3.3. $\int_a [Fa, Ga] \cong \int_a [\phi * \mathcal{A}(a, H-), Ga] \cong [\mathcal{K}^{\mathrm{op}}, \mathcal{V}](\phi, GH^{\mathrm{op}})$.

Accordingly, for any $\mathcal{A}$ there is a $\mathcal{V}$-category $\mathcal{PA}$ having as its objects the accessible presheaves, and with its $\mathcal{V}$-valued hom given by the usual formula $\int_a [Fa, Ga]$; it was first introduced by Lindner [Lin74]. Any presheaf $\mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is accessible if $\mathcal{A}$ is small, being the left Kan extension of itself along the identity, so that $\mathcal{PA}$ coincides with $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ for a small $\mathcal{A}$.

Every representable $\mathcal{A}(-, b)$ is accessible; we express it in the form 3.2 by taking $\phi = I : \mathcal{I}^{\mathrm{op}} \to \mathcal{V}$ and $H = b : \mathcal{I} \to \mathcal{A}$, where $\mathcal{I}$ is the unit $\mathcal{V}$-category and I is the unit for $\otimes$. Accordingly we have the fully-faithful *Yoneda embedding* $Y : \mathcal{A} \to \mathcal{PA}$ sending b to $\mathcal{A}(-, b)$, which we sometimes loosely treat as an inclusion. Now calculating 3.3 with $F = Yb$ gives at once the *Yoneda isomorphism*

3.4. $\mathcal{PA}(Yb, G) \cong Gb$.

By Proposition 5.34 of [Kel82] the category $\mathcal{PA}$ admits all small colimits, these being formed pointwise from those in $\mathcal{V}$. So the typical object F of $\mathcal{PA}$ as in 3.2 can be written as

3.5. $F \cong \phi * YH$,

this now being a colimit in $\mathcal{PA}$. We can see 3.5 as expressing the general accessible F as a small colimit in $\mathcal{PA}$ of representables.

Recall from [Kel82] p.154 that, given a class Φ of weights and a full subcategory $\mathcal{A}$ of a Φ-cocomplete category $\mathcal{B}$, the *closure of* $\mathcal{A}$ *in* $\mathcal{B}$ *under* Φ*-colimits* is the smallest full replete subcategory of $\mathcal{B}$ containing $\mathcal{A}$ and closed under the formation of Φ-colimits in $\mathcal{B}$ – namely the intersection of all such. For any class Φ of weights, and any category $\mathcal{A}$, we write $\Phi(\mathcal{A})$ for the closure of $\mathcal{A}$ in $\mathcal{PA}$ under Φ-colimits, with $Z : \mathcal{A} \to \Phi(\mathcal{A})$ and $W : \Phi(\mathcal{A}) \to \mathcal{PA}$ for the full inclusions, so that $Y : \mathcal{A} \to \mathcal{PA}$ is the composite WZ; note that W is Φ-cocontinuous. We now reproduce (the main point of) [Kel82] Theorem 5.35. The proof below is a little more direct than that given there, which referred back to earlier propositions. The result itself must be older still, at least for certain classes Φ.

[1]See also Proposition 3.16 below. Added 2006-04-29.

3.6. Proposition. *For any Φ-cocomplete category $\mathcal{B}$, composition with Z gives an equivalence of categories*

$$\Phi\text{-}\mathbf{Cocts}(\Phi(\mathcal{A}),\mathcal{B}) \to \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A},\mathcal{B})$$

with an equivalence inverse given by the left Kan extension along Z. Thus $\Phi(-)$ provides a left bi-adjoint to the forgetful 2-functor Φ-$\mathbf{Cocts} \to \mathcal{V}$-$\mathbf{CAT}$.

Proof. By 2.9, the left Kan extension $\mathrm{Lan}_Z G$ of $G : \mathcal{A} \to \mathcal{B}$ is given by $\mathrm{Lan}_Z G(F) = \Phi(\mathcal{A})(Z-,F) * G$, existing when this last colimit exists for each F in $\Phi(\mathcal{A})$. However $\Phi(\mathcal{A})(Z-,F) = \mathcal{P}\mathcal{A}(WZ-,WF) = \mathcal{P}\mathcal{A}(Y-,WF)$, which by Yoneda is isomorphic to WF. Consider the full subcategory of $\Phi(\mathcal{A})$ given by those F for which $WF * G$ *does* exist; it contains the representables by Yoneda, and it is closed under Φ-colimits: for these exist in $\Phi(\mathcal{A})$ and are preserved by W, while [Kel82] (3.23) gives $(\phi*S)*G \cong \phi*(S-*G)$, either side existing if the other does; so the subcategory in question is all of $\Phi(\mathcal{A})$.

What is more: $\mathrm{Lan}_Z G = W - *G : \Phi(\mathcal{A}) \to \mathcal{B}$ preserves Φ-colimits since W does so and colimits are cocontinuous in their weights (see [Kel82] (3.23) again). So one does indeed have a functor $\mathrm{Lan}_Z : \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A},\mathcal{B}) \to \Phi\text{-}\mathbf{Cocts}(\Phi(\mathcal{A}),\mathcal{B})$, while one has trivially the restriction functor $\Phi\text{-}\mathbf{Cocts}(\Phi(\mathcal{A}),\mathcal{B}) \to \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A},\mathcal{B})$ given by composition with Z. By [Kel82] (4.23), the canonical $G \to \mathrm{Lan}_Z(G)Z$ is invertible for all G since Z is fully faithful. Thus it remains to consider the canonical $\alpha : \mathrm{Lan}_Z(SZ) \to S$ for a Φ-cocontinuous $S : \Phi(\mathcal{A}) \to \mathcal{B}$. The F-component of α for $F \in \Phi(\mathcal{A})$ is the canonical $\alpha_F : WF * SZ \to SF$; and clearly the collection of those F for which α_F is invertible contains the representables and is closed under Φ-colimits: therefore it is the totality of $\Phi(\mathcal{A})$. ∎

3.7. Remarks. We may express this by saying that $\Phi(\mathcal{A})$ is the *free Φ-cocomplete category on $\mathcal{A}$*. As a particular case, $\mathcal{P}\mathcal{A}$ itself is the free cocomplete category on $\mathcal{A}$; in other words $\Phi(\mathcal{A}) = \mathcal{P}\mathcal{A}$ when Φ is the class of *all* weights – which is why (identifying $\mathcal{P}(\mathcal{A})$ with $\mathcal{P}\mathcal{A}$) we use $\mathcal{P}$ as the name for this class of all weights.

As shown in [Kel82], one can form $\Phi(\mathcal{A})$ by transfinite induction. Define successively full replete subcategories $\mathcal{A}_\alpha$ of $\mathcal{P}\mathcal{A}$ as α runs through the ordinals: $\mathcal{A}_0$, which is equivalent to $\mathcal{A}$, consists of the representables, now in the sense of those presheaves isomorphic to some $\mathcal{A}(-,a)$; then $\mathcal{A}_{\alpha+1}$ consists of $\mathcal{A}_\alpha$ together with all Φ-colimits in $\mathcal{P}\mathcal{A}$ of diagrams in $\mathcal{A}_\alpha$; and for a limit ordinal α we set $\mathcal{A}_\alpha = \bigcup_{\beta<\alpha}\mathcal{A}_\beta$. This sequence stabilizes if, as we suppose, there exist arbitrarily large inaccessible cardinals: for we have $\Phi(\mathcal{A}) = \Phi_\alpha(\mathcal{A})$ when α is the smallest regular cardinal greater than $card(ob(\mathcal{K}))$ for all small $\mathcal{K}$ with $\Phi[\mathcal{K}]$ non-empty. It follows that $\Phi(\mathcal{A})$ *is a small category when $\mathcal{A}$ and Φ are small.* In a number of important cases, one has $\Phi(\mathcal{A}) = \mathcal{A}_1$ in the notation above; it is so when $\Phi = \mathcal{P}$ since by 3.5 every accessible F is a small colimit of representables, and in the case $\mathcal{V} = \mathbf{Set}$ it is so by [Kel82] Theorem 5.37 when Φ consists of the weights for finite conical colimits. However there is no special value in this condition, which (as we shall see in Proposition 3.15 below) always holds for a small $\mathcal{A}$ when the class Φ is *saturated.*

An explicit description of the saturation Φ^* of a class Φ of weights was given by Albert and Kelly in [AK88], in the following terms:

3.8. Proposition. *The weight $\psi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ lies in the saturation Φ^* of the class Φ if and only if the object ψ of $\mathcal{P}\mathcal{K} = [\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$ lies in the full subcategory $\Phi(\mathcal{K})$ of $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$.*

There is another useful way of putting this. When $\mathcal{K}$ is small, both $\Phi[\mathcal{K}]$ and $\Phi(\mathcal{K})$ make sense for any class Φ; and in fact we have

3.9. $\Phi[\mathcal{K}] \subset \Phi(\mathcal{K})$,

since for $\phi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ the Yoneda isomorphism

3.10. $\phi \cong \phi * Y$

exhibits ϕ as an object of $\Phi(\mathcal{K})$ when $\phi \in \Phi$. We can write Proposition 3.8 as

3.11. $\Phi^*[\mathcal{K}] = \Phi(\mathcal{K})$,

so that Φ is a saturated class precisely when

3.12. $\Phi[\mathcal{K}] = \Phi(\mathcal{K})$

for each small $\mathcal{K}$. In other words the class Φ is saturated precisely when, for each small $\mathcal{K}$, *the full subcategory $\Phi[\mathcal{K}]$ of $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$ contains the representables $\mathcal{K}(-, k)$ and is closed in $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$ under Φ-colimits.*

3.13. Example. Consider the case when $\mathcal{V}$ is locally finitely presentable as a closed category in the sense of [Kel82-2], and Φ is the class of finite weights as described there; this includes the case where $\mathcal{V} = \mathbf{Set}$ and Φ is the set of weights for the classical finite colimits. Then $\Phi^*[\mathcal{K}] = \Phi^*(\mathcal{K})$ is the closure of $\mathcal{K}$ in $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$ under finite colimits, which by [Kel82-2] Theorem 7.2 is the full subcategory of $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$ given by the finitely presentable objects.

It follows of course from the definitions of $\Phi(\mathcal{A})$ and of Φ^* that

3.14. $\Phi^*(\mathcal{A}) = \Phi(\mathcal{A})$

for any $\mathcal{A}$. We cannot write 3.12 when $\mathcal{K}$ is replaced by a non-small $\mathcal{A}$, since then $\Phi[\mathcal{A}]$ has no meaning; but a partial replacement for it is provided by the following, which was Proposition 7.4 in [AK88]:

3.15. Proposition. *If the presheaf $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ lies in $\Phi(\mathcal{A})$ for some saturated class Φ, then F is a Φ-colimit in $\mathcal{P}A$ of representables; that is, $F \cong \phi * YH$ for some $\phi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ in Φ and some $H : \mathcal{K} \to \mathcal{A}$. Since Φ-colimits are formed in $\Phi(\mathcal{A})$ as in $\mathcal{P}\mathcal{A}$, F is equally the colimit $\phi * ZH$ in $\Phi(\mathcal{A})$.*

In other words an F in $\Phi(\mathcal{A})$ has the form 3.5 with ϕ in Φ. Equally, this asserts that $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is the left Kan extension of $\phi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ along $H^{\mathrm{op}} : \mathcal{K}^{\mathrm{op}} \to \mathcal{A}^{\mathrm{op}}$. In fact, we can take H here to be fully faithful, as was shown [Kel82] Proposition 4.83 for the case $\Phi = \mathcal{P}$:

3.16. Proposition. *For a saturated class Φ, any F in $\Phi(\mathcal{A})$ is of the form* $\mathrm{Lan}_{H^{\mathrm{op}}}\phi$ *for some $\phi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ in Φ and some fully faithful $H : \mathcal{K} \to \mathcal{A}$.*

Proof. We already have that $F \cong \mathrm{Lan}_{T^{\mathrm{op}}}\psi$ for some $\psi : \mathcal{L}^{\mathrm{op}} \to \mathcal{V}$ in Φ and some $T : \mathcal{L} \to \mathcal{A}$. Let T factorize as $T = HP$ where $H : \mathcal{K} \to \mathcal{A}$ is fully faithful and $P : \mathcal{L} \to \mathcal{K}$ is bijective on objects. Then $\mathcal{K}$ is small since $\mathcal{L}$ is small. Now $F \cong \mathrm{Lan}_{T^{\mathrm{op}}}\psi \cong \mathrm{Lan}_{H^{\mathrm{op}}}\phi$, where $\phi = \mathrm{Lan}_{P^{\mathrm{op}}}\psi$. However $\phi = \mathrm{Lan}_{P^{\mathrm{op}}}\psi \cong \psi * YP$, which, as a Φ-colimit of representables, lies in $\Phi(\mathcal{K})$, and hence in $\Phi[\mathcal{K}]$. ∎

It may be useful to understand extreme special cases of one's notation. First observe that the saturation 0^* of the empty class 0 consists precisely of the representables – that is, $0^*[\mathcal{K}] = 0^*(\mathcal{K})$ consists of the isomorphs of the various $\mathcal{K}(-,k) : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$. Another extreme case involves the empty $\mathcal{V}$-category 0 with no objects. Of course $\mathcal{P}0 = [0^{\mathrm{op}}, \mathcal{V}]$ is the terminal category 1; its unique object is the unique functor $! : 0^{\mathrm{op}} \to \mathcal{V}$ and $1(!,!)$ is the terminal object 1 of $\mathcal{V}$. (This differs in general from the *unit $\mathcal{V}$-category $\mathcal{I}$*, with one object $*$ but with $\mathcal{I}(*,*)=I$.) So for any class Φ, we have $\Phi[0] = 0$ if $! : 0^{\mathrm{op}} \to \mathcal{V}$ is not in Φ, and $\Phi[0] = [0^{\mathrm{op}}, \mathcal{V}] = 1$ otherwise. Now $\Phi(0)$ is the closure of 0 in $\mathcal{P}0$ under Φ-colimits, and any diagram $T : \mathcal{K} \to 0$ has $\mathcal{K} = 0$, so that $\Phi(0) = 0$ if $! \notin \Phi$ and otherwise $\Phi(0)$ contains $! * Y = !$, giving $\Phi(0) = 1$. So in fact $\Phi(0) = \Phi[0]$, being 0 or 1. Both are possible for a saturated Φ; for $\mathcal{P}0 = 1$, while the Albert-Kelly theorem (Proposition 3.8 above) gives $0^*[0] = 0(0) = 0[0] = 0$.

Before ending this section, we recall a result characterizing Φ-cocomplete categories, along with a short proof. This was Proposition 4.5 in [AK88].

3.17. Proposition. *For any class Φ of weights, a category $\mathcal{A}$ admits Φ-colimits if and only if the fully faithful embedding $Z : \mathcal{A} \to \Phi(\mathcal{A})$ admits a left adjoint; that is, if and only if the full subcategory $\mathcal{A}$ given by the representables is reflective in $\Phi(\mathcal{A})$.*

Proof. If $\mathcal{A}$ is reflective, it admits Φ-colimits because $\Phi(\mathcal{A})$ does so. Suppose conversely that $\mathcal{A}$ admits Φ-colimits, and write $\mathcal{B}$ for the full subcategory of $\mathcal{PA}$ given by those objects admitting a reflection into $\mathcal{A}$; then $\mathcal{B}$ contains $\mathcal{A}$ and $\mathcal{B}$ is closed in $\mathcal{PA}$ under Φ-colimits since $\mathcal{A}$ admits these; so that $\mathcal{B}$ contains $\Phi(\mathcal{A})$, as desired. ∎

4. Recognition theorems

We recall from Proposition 5.62 of [Kel82] a result characterizing categories of the form $\Phi(\mathcal{A})$ – or more precisely functors of the form $Z : \mathcal{A} \to \Phi(\mathcal{A})$. At the same time, we give a direct proof; for the proof in [Kel82] refers back to earlier results in that book.

We begin with a piece of notation: for a category $\mathcal{A}$ and a class Φ of weights, we write $\mathcal{A}_\Phi$ for the full subcategory of $\mathcal{A}$ given by those $a \in \mathcal{A}$ for which the representable $\mathcal{A}(a,-) : \mathcal{A} \to \mathcal{V}$ preserves all Φ-colimits (That is, all Φ-colimits that *exist in* $\mathcal{A}$). There is no agreed name for $\mathcal{A}_\Phi$; the objects of $\mathcal{A}_\Phi$ are usually called *finitely presentable* when the Φ-colimits are the classical filtered colimits; while when Φ is the class $\mathcal{P}$ of all weights, the objects of $\mathcal{A}_\Phi$ were called *small projectives* in [Kel82], but have also been called *atoms* by some authors. Let us use the name Φ-*atoms* for the objects of $\mathcal{A}_\Phi$. When $\mathcal{A}$ admits Φ-colimits and hence Φ^*-colimits, it follows from the definition of $\mathcal{A}_\Phi$ that

4.1. $\mathcal{A}_{\Phi^*} = \mathcal{A}_\Phi$.

For a functor $G : \mathcal{A} \to \mathcal{B}$, we get for each $b \in \mathcal{B}$ the presheaf $\mathcal{B}(G-, b) : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$. If this is accessible for every b, we have a functor $\tilde{G} : \mathcal{B} \to \mathcal{P}\mathcal{A}$ in the notation of [Kel82]. Note that $\tilde{G}G \cong Y : \mathcal{A}^{\mathrm{op}} \to \mathcal{P}\mathcal{A}$ when G is fully faithful.

The following is the characterization result of Proposition 5.62 of [Kel82] with a slightly expanded form of its statement.

4.2. Proposition. *In order that $G : \mathcal{A} \to \mathcal{B}$ be equivalent to the free Φ-cocompletion $Z : \mathcal{A} \to \Phi(\mathcal{A})$ of $\mathcal{A}$ for a class Φ of weights, the following conditions are necessary and sufficient:*
(i) G is fully faithful (allowing us to treat $\mathcal{A}$ henceforth as a full subcategory of $\mathcal{B}$);
(ii) $\mathcal{B}$ is Φ-cocomplete;
(iii) the closure of $\mathcal{A}$ in $\mathcal{B}$ under Φ-colimits is $\mathcal{B}$ itself;
(iv) $\mathcal{A}$ is contained in the full subcategory $\mathcal{B}_\Phi$ of $\mathcal{B}$.
When these conditions hold, each functor $\mathcal{B}(G-, b) : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is accessible and in fact lies in the full subcategory $\Phi(\mathcal{A})$ of $\mathcal{P}\mathcal{A}$. Thus we have a functor $\tilde{G} : \mathcal{B} \to \mathcal{P}\mathcal{A}$ given by $\tilde{G}(b) = \mathcal{B}(G-, b)$, and this factorizes as WK where W is, as before, the inclusion from $\Phi(\mathcal{A})$ to $\mathcal{P}\mathcal{A}$. The functor $K : \mathcal{B} \to \Phi(\mathcal{A})$ here is an equivalence, an equivalence inverse being given by $\mathrm{Lan}_Z G : \Phi(\mathcal{A}) \to \mathcal{B}$, which by Proposition 3.6 is the unique Φ-cocontinuous extension of G to $\Phi(\mathcal{A})$.

Proof. The necessity of the first two conditions is clear. That of the third results from the fact that the inclusion $W : \Phi(\mathcal{A}) \to \mathcal{P}\mathcal{A}$ preserves Φ-colimits by definition. For that of the fourth condition, the point is that $\Phi(\mathcal{A})(Za, -) \cong \mathcal{P}\mathcal{A}(Ya, W-)$ preserves Φ-colimits: for W does so, while $\mathcal{P}\mathcal{A}(Ya, -) : \mathcal{P}\mathcal{A} \to \mathcal{V}$ preserves all small colimits, being isomorphic by Yoneda to the evaluation E_a.

We turn now to the proof of sufficiency. First, to see that each $\mathcal{B}(G-, b)$ lies in the full replete subcategory $\Phi(\mathcal{A})$ of $\mathcal{P}\mathcal{A}$, consider the full subcategory of $\mathcal{B}$ given by those b for which this is so; this contains $\mathcal{A}$ since $\mathcal{B}(G-, Ga) \cong Ya$ because G is fully faithful, and it is closed in $\mathcal{B}$ under Φ-colimits by (iv), since $\Phi(\mathcal{A})$ is closed under these in $\mathcal{P}\mathcal{A}$; so it is all of $\mathcal{B}$.

Thus we have indeed a functor $K : \mathcal{B} \to \Phi(\mathcal{A})$, sending b to $\mathcal{B}(G-, b)$. We next show that K or equivalently $\tilde{G} = WK : \mathcal{B} \to \mathcal{P}\mathcal{A}$ is fully faithful. Consider the full subcategory of $\mathcal{B}$ given by those b for which the map $\tilde{G}_{b,c} : \mathcal{B}(b, c) \to \mathcal{P}\mathcal{A}(\tilde{G}(b), \tilde{G}(c))$ is invertible for all c. We observe that it contains $\mathcal{A}$ since G is fully faithful, and that it is closed under Φ-colimits since $\mathcal{A} \subset \mathcal{B}_\Phi$. Thus it is all of $\mathcal{B}$.

It remains to show that K and $S = \mathrm{Lan}_Z G : \Phi(\mathcal{A}) \to \mathcal{B}$ are equivalence-inverses. Recall that $\tilde{G}G \cong Y$ since G is fully faithful. Also recall from Proposition 3.6 that S is the essentially unique Φ-cocontinuous functor with $SZ \cong G$. So $WKSZ \cong \tilde{G}G \cong Y \cong WZ$, giving $KSZ \cong Z$ since W is fully faithful, and then giving $KS \cong 1$ by Proposition 3.6 since KS and 1 are Φ-cocontinuous, K being so because $\mathcal{A} \subset \mathcal{B}_\Phi$. Finally $KS \cong 1$ gives $KSK \cong K$; whence $SK \cong 1$ since K (as we saw) is fully faithful. ∎

Proposition 4.2 is of particular interest in the case of a small $\mathcal{A}$. We may cast the result for a small $\mathcal{A}$ in the form:

4.3. Proposition. *For a class Φ of weights, the following properties of a category $\mathcal{B}$ are equivalent:*
(i) For some small $\mathcal{K}$, there is an equivalence $\mathcal{B} \simeq \Phi(\mathcal{K})$;
(ii) $\mathcal{B}$ is Φ-cocomplete and has a small full subcategory $\mathcal{A} \subset \mathcal{B}_\Phi$ such that every object of $\mathcal{B}$ is a Φ^*-colimit of a diagram in $\mathcal{A}$;
(iii) $\mathcal{B}$ is Φ-cocomplete and has a small full sub-category $\mathcal{A} \subset \mathcal{B}_\Phi$ such that the closure of $\mathcal{A}$ in $\mathcal{B}$ under Φ-colimits is $\mathcal{B}$ itself.
Under the hypothesis (iii) – and so a fortiori under (ii) – if $G : \mathcal{A} \to \mathcal{B}$ denotes the inclusion, the functor $\tilde{G} : \mathcal{B} \to \mathcal{P}\mathcal{A}$ is fully faithful, with $\Phi(\mathcal{A})$ for its replete image.

4.4. Remarks. (a) When Φ is the class $\mathcal{P}$ of all weights, we get a characterization here of the functor category $\mathcal{P}\mathcal{K} = [\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$ for a small $\mathcal{K}$; note that it differs from the characterization given in [Kel82] Theorem 5.26, which replaces the condition that $\mathcal{B}$ be the colimit closure of $\mathcal{A}$ by the condition that $\mathcal{A}$ be strongly generating in $\mathcal{B}$; but these conditions are very similar in strength by [Kel82] Proposition 3.40.
(b) Theorem 5.3 of [BQR98] is the special case where $\mathcal{V}$ is locally finitely presentable as a closed category in the sense of [Kel82-2] and Φ is the saturated class of α-flat presheaves. (See Section 5 below.)

Let us mention the following consequence of Proposition 4.3.

4.5. Proposition. *For a small $\mathcal{K}$ and a saturated class Φ, let $\mathcal{A}$ be a full reflective subcategory of $\Phi(\mathcal{K})$ that is closed in $\Phi(\mathcal{K})$ under Φ-colimits. Then $\mathcal{A}$ is equivalent to $\Phi(\mathcal{L})$ for a small $\mathcal{L}$.*

Proof. Write $J : \mathcal{A} \to \Phi(\mathcal{K})$ for the inclusion, with $R : \Phi(\mathcal{K}) \to \mathcal{A}$ for its left adjoint, and regard $Z : \mathcal{K} \to \Phi(\mathcal{K})$ as an *inclusion* of the representables in $\Phi(\mathcal{K})$. The objects RZk of $\mathcal{A}$ with $k \in \mathcal{K}$ constitute a full subcategory $\mathcal{L}$ of $\mathcal{A}$. By hypothesis, $\mathcal{A}$ admits Φ-colimits and J preserves these. The subcategory $\mathcal{L}$ lies in $\mathcal{A}_\Phi$, because $\mathcal{A}(RZk, -) \cong \Phi(\mathcal{K})(Zk, J-)$ preserves Φ-colimits since both J and $\Phi(\mathcal{K})(Zk, -)$ (being the evaluation at k) do so. Finally every object a of $\mathcal{A}$ is a Φ-colimit of a diagram taking its values in $\mathcal{L}$; for $Ja \in \Phi(\mathcal{K})$ is a Φ-colimit $Ja * Z$, and R preserves this colimit, so that $a \cong RJa \cong Ja * RZ$, where the diagram $RZ : \mathcal{K} \to \mathcal{A}$ takes its values in $\mathcal{L}$. ∎

5. Limits and colimits commuting in $\mathcal{V}$

The new observations to which we now turn begin with the general study of the commutativity in $\mathcal{V}$ of limits and colimits. For a pair of weights $\psi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ and $\phi : \mathcal{L}^{\mathrm{op}} \to \mathcal{V}$, to say that

5.1. $\phi * - : [\mathcal{L}, \mathcal{V}] \to \mathcal{V}$ preserves ψ-limits

is equally to say that

5.2. $\{\psi, -\} : [\mathcal{K}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{V}$ preserves ϕ-colimits,

because each in fact asserts the invertibility, for every functor $S : \mathcal{K}^{\mathrm{op}} \otimes \mathcal{L} \to \mathcal{V}$, of the canonical comparison morphism

5.3. $\phi? * \{\psi-, S(-, ?)\} \to \{\psi-, \phi? * S(-, ?)\}$.

When these statements are true for every such S, we say that *ϕ-colimits commute with ψ-limits in $\mathcal{V}$*. For classes Φ and Ψ of weights, if 5.1 (or equivalently 5.2) holds for all $\phi \in \Phi$ and all $\psi \in \Psi$, we say that *Φ-colimits commute with Ψ-limits in $\mathcal{V}$*. For any class Ψ of weights we may consider the class Ψ^+ of all weights ϕ for which ϕ-colimits commute with Ψ-limits in $\mathcal{V}$; and for any class Φ of weights we may consider the class Φ^- of all weights ψ for which Φ-colimits commute with ψ-limits in $\mathcal{V}$. We have here of course a Galois connection, with $\Phi \subset \Psi^+$ if and only if $\Psi \subset \Phi^-$. Note that $[\mathcal{L}, \mathcal{V}]$ and $\mathcal{V}$ in 5.1 admit all (small) limits; so that by the definition above of the saturation Ψ^* of a class Ψ of weights, if $\phi * -$ preserves all Ψ-limits, it also preserves Ψ^*-limits. From this and a dual argument, one concludes that:

5.4. Proposition. *For any classes Φ and Ψ of weights, the classes Φ^- and Ψ^+ are saturated; so that $\Psi^{+\,*} = \Psi^+$ and $\Phi^{-\,*} = \Phi^-$. Moreover $\Psi^+ = \Psi^{*\,+}$ and $\Phi^- = \Phi^{*\,-}$.*

When Ψ consists of the weights for finite limits (in the usual sense for ordinary categories, or in the sense of [Kel82-2] when $\mathcal{V}$ is locally finitely presentable as a closed category), it has been customary to call the elements of Ψ^+ the *flat* weights, as they are those ϕ having $\phi * - : [\mathcal{L}, \mathcal{V}] \to \mathcal{V}$ left exact. (Note the corresponding use of "α-flat" in Definition 4.1 of [BQR98].) Accordingly for a general Ψ we call the elements of Ψ^+ the *Ψ-flat* weights.

Recall that the limit functor $\{\psi, -\} : [\mathcal{K}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{V}$ of 5.2 is just the representable functor $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}](\psi, -)$. Accordingly $\psi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ lies in Φ^- for a given class Φ if and only if it lies in the subcategory $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}]_\Phi$:

5.5. $\Phi^-[\mathcal{K}] = \Phi^-(\mathcal{K}) = [\mathcal{K}^{\mathrm{op}}, \mathcal{V}]_\Phi$.

In other words, the elements ψ of $\Phi^-[\mathcal{K}]$ are the *Φ-atoms* of $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$; we also call them the *Φ-atomic* weights. When Φ is the class $\mathcal{P}$ of all weights, the elements of $\mathcal{P}^-$ are also called the *small projective* weights.

Part of the saturatedness of Φ^- – namely the closedness of $\Phi^-[\mathcal{K}]$ in $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$ under Φ^--colimits – is the special case for $\mathcal{A} = [\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$ of the following more general result:

5.6. Proposition. *For any class Φ of weights and any category $\mathcal{A}$, the full subcategory $\mathcal{A}_\Phi$ of $\mathcal{A}$ is closed in $\mathcal{A}$ under any Φ^--colimits that exist in $\mathcal{A}$.*

PROOF. Let the colimit $\psi * S$ exist, where $\psi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ lies in Φ^- and $S : \mathcal{K} \to \mathcal{A}$ takes its values in $\mathcal{A}_\Phi$. Then by definition

$$\mathcal{A}(\psi * S, a) \cong [\mathcal{K}^{\mathrm{op}}, \mathcal{V}](\psi, \mathcal{A}(S-, a)).$$

Since each $\mathcal{A}(Sk, -)$ preserves Φ-colimits, and since $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}](\psi, -)$ preserves Φ-colimits by 5.5, it follows that $\mathcal{A}(\psi * S, -)$ preserves Φ-colimits: that is to say $\psi * S \in \mathcal{A}_\Phi$. ■

5.7. EXAMPLE. When $\mathcal{V} = \mathbf{Set}$, let Ψ be the class of weights for (classical conical) finite limits: that is, the set of all $\Delta 1 : \mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$ with $\mathcal{K}$ finite. Then Ψ^+ consists of those $\phi : \mathcal{L}^{\mathrm{op}} \to \mathbf{Set}$ with $\phi * - : [\mathcal{L}, \mathbf{Set}] \to \mathbf{Set}$ left exact; that is, the *flat* presheaves $\phi : \mathcal{L}^{\mathrm{op}} \to \mathbf{Set}$. As is well known, these are those presheaves ϕ for which $(\mathrm{el}(\phi))^{\mathrm{op}}$ is filtered. Since $\phi * S$ for $S : \mathcal{L} \to \mathcal{A}$ is given as in 2.5 by $\mathrm{colim}\{\ \mathrm{el}(\phi)^{\mathrm{op}} \xrightarrow{d^{\mathrm{op}}} \mathcal{L} \xrightarrow{S} \mathcal{A}\ \}$, a functor $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set}$ is Ψ^+-cocontinuous if and only if it preserves filtered colimits; that is, if and only if it is finitary. By 5.5, therefore, Ψ^{+-} consists of those $\psi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ for which $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\psi, -)$ preserved filtered colimits; that is, those ψ that are finitely presentable in $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$. It follows from 3.13 that Ψ^{+-} coincides in this case with Ψ^*.

5.8. EXAMPLE. With $\mathcal{V} = \mathbf{Set}$ again, let Ψ consist of the single object $0^{\mathrm{op}} \to \mathbf{Set}$, where 0 is the empty category: so a Ψ-limit is a terminal object. Now $\phi : \mathcal{L}^{\mathrm{op}} \to \mathbf{Set}$ lies in Ψ^+ whenever $\phi * - : [\mathcal{L}, \mathbf{Set}] \to \mathbf{Set}$ preserves the terminal object; which is to say that $\phi * \Delta 1 \cong 1$, or equally that $\mathrm{colim}(\phi) \cong 1$, or again that $\mathrm{el}(\phi)$ is connected. So the presheaf $\psi : \mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$ lies in Ψ^{+-} just when $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}](\psi, -)$ preserves connected (conical) colimits. This time Ψ^{+-} strictly includes Ψ^*. For $\Psi^*(\mathcal{K})$, being the closure of the representables in $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ under Ψ-colimits, consists of the representables together with the initial object $\Delta 0 : \mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$. When $\mathcal{K}$ has one object, being given by the monoid $\{1, e\}$ with $e^2 = e$, the subcategory $\mathcal{Q}(\mathcal{K})$ of $[\mathcal{K}^{\mathrm{op}}, \mathbf{Set}]$ given by the Cauchy completion of $\mathcal{K}$ has, by Section 5.8 of [Kel82], two objects, the representable object $*$ and the equalizer E of the two maps $1, e : * \to *$, which splits the idempotent e; and E is not $\Delta 0$ since there is an arrow from $*$ to E because $ee = e$. Now $\mathcal{Q} = \mathcal{P}^-$ by Section 6 below, and $\mathcal{P}^- \subset \Psi^{+-}$ because $\Psi^+ \subset \mathcal{P}$. So in this case, there are objects of $\Psi^{+-}(\mathcal{K})$ which are not contained in $\Psi^*(\mathcal{K})$, and Ψ^* is properly contained in Ψ^{+-}.

5.9. REMARK. When $\mathcal{V} = \mathbf{Set}$, it is well known (see for example Theorem 5.38 of [Kel82]) that the flat weights $\mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ are precisely the filtered conical colimits of representables, and hence constitute the closure of $\mathcal{K}$ in $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$ under filtered conical colimits. This is false for a general $\mathcal{V}$ that is locally finitely presentable as a closed category; if [BQR98] seems to suggest otherwise, it is only because those authors *define* "filtered colimit" to mean "colimit weighted by a flat weight".

6. The class $\mathcal{Q}$ of small projective weights

This section is devoted to the study of the saturated class $\mathcal{Q} = \mathcal{P}^-$ of *small projective* weights. So for a small $\mathcal{K}$, 5.5 gives

6.1. $\mathcal{Q}[\mathcal{K}] = \mathcal{Q}(\mathcal{K}) = [\mathcal{K}^{\mathrm{op}}, \mathcal{V}]_{\mathcal{P}}$,

consisting of those $\phi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ for which $\{\phi, -\} = [\mathcal{K}^{\mathrm{op}}, \mathcal{V}](\phi, -) : [\mathcal{K}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{V}$ preserves all small colimits. We shall establish the following alternative characterizations of $\mathcal{Q}$. First from Proposition 6.14 below:

6.2. $\phi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ lies in $\mathcal{Q}$ if and only the corresponding module $\mathcal{I} \multimapdotinv \mathcal{K}$ is a left adjoint.

Proposition 6.20 below gives:

6.3. $\mathcal{Q}$ is the class $\mathcal{P}^+$ of $\mathcal{P}$-flat weights.

Finally, as Street showed in [Str83],

6.4. $\mathcal{Q}$ is the class of *absolute* weights.

Moreover there is an adjunction $L \dashv R : [\mathcal{K}, \mathcal{V}]^{\mathrm{op}} \to [\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$, due in the case $\mathcal{V} = \mathbf{Set}$ to Isbell, which restricts to an equivalence $(\mathcal{Q}(\mathcal{K}^{\mathrm{op}}))^{\mathrm{op}} \simeq \mathcal{Q}(\mathcal{K})$ between the full subcategories of small projectives in $[\mathcal{K}, \mathcal{V}]$ and in $[\mathcal{K}^{\mathrm{op}}, \mathcal{V}]$. In terms of modules, this equivalence sends a right adjoint module $\mathcal{K} \multimapdotinv \mathcal{I}$ to its left adjoint $\mathcal{I} \multimapdotinv \mathcal{K}$.

Recall that by a module $\mathcal{A} \multimapdotinv \mathcal{B}$ is meant a functor $\mathcal{B}^{\mathrm{op}} \otimes \mathcal{A} \to \mathcal{V}$ with $\mathcal{A}$ and $\mathcal{B}$ small, and that modules with their usual composition and 2-cells form a bicategory $\mathcal{V}$-$\mathbf{Mod}$. Recall further that each functor $T : \mathcal{A} \to \mathcal{B}$ gives rise to modules $T_* : \mathcal{A} \multimapdotinv \mathcal{B}$ and $T^* : \mathcal{B} \multimapdotinv \mathcal{A}$, where

6.5. $T_*(b, a) = \mathcal{B}(b, Ta)$ and $T^*(a, b) = \mathcal{B}(Ta, b)$,

and that T_* is left adjoint to T^* in $\mathcal{V}$-$\mathbf{Mod}$. Recall finally that the bicategory $\mathcal{V}$-$\mathbf{Mod}$ is *closed*, admitting all *right liftings* and all *right extensions* as follows: given modules $f : \mathcal{A} \multimapdotinv \mathcal{B}$, $g : \mathcal{C} \multimapdotinv \mathcal{A}$ and $h : \mathcal{C} \multimapdotinv \mathcal{B}$, we have the right lifting $\{\!|\, f, h \,|\!\} : \mathcal{C} \multimapdotinv \mathcal{A}$ of h through f and the right extension $[\![g, h]\!] : \mathcal{A} \multimapdotinv \mathcal{B}$ of h along g, given by:

6.6. $\{\!|\, f, h \,|\!\}(a, c) = \int_b [f(b, a), h(b, c)]$

and

6.7. $[\![g, h]\!](b, a) = \int_c [g(a, c), h(b, c)]$,

satisfying the universal properties

6.8. $\mathcal{V}\text{-}\mathbf{Mod}(\mathcal{A}, \mathcal{B})(f, [\![g, h]\!]) \cong \mathcal{V}\text{-}\mathbf{Mod}(\mathcal{C}, \mathcal{B})(fg, h) \cong \mathcal{V}\text{-}\mathbf{Mod}(\mathcal{C}, \mathcal{A})(g, \{\!|\, f, h \,|\!\})$.

The second isomorphism corresponds by Yoneda to a morphism $\epsilon : f\{\!|\, f, h \,|\!\} \to h : \mathcal{C} \to \mathcal{B}$ which is said, in the language of [StWa78], to *exhibit* $\{\!|\, f, h \,|\!\}$ *as the right lifting of* h *through* f. Such a lifting $\{\!|\, f, h \,|\!\}$ is *respected* by a $k : \mathcal{D} \multimapdotinv \mathcal{C}$ when the 2-cell ϵk exhibits $\{\!|\, f, h \,|\!\}k$ as the right lifting $\{\!|\, f, hk \,|\!\}$ of hk through f, and the lifting $\{\!|\, f, h \,|\!\}$ is *absolute* when it is respected by every such arrow k.

As in any closed bicategory, we have the following characterization of left adjoints:

6.9. Proposition. *In $\mathcal{V}$-**Mod**, the following statements are equivalent:*
(i) $f : \mathcal{A} \relbar\!\!\circ\!\!\rightarrow \mathcal{B}$ has a right adjoint;
(ii) for all $h : \mathcal{C} \relbar\!\!\circ\!\!\rightarrow \mathcal{B}$, the right lifting $\{\!| f, h |\!\}$ of h through f is absolute;
(iii) the right lifting $\{\!| f, 1 |\!\}$ of $1 : \mathcal{B} \relbar\!\!\circ\!\!\rightarrow \mathcal{B}$ through f is respected by f.
When these are satisfied, the right adjoint f^* of f is the right lifting $\{\!| f, 1 |\!\}$ of 1 through f; moreover the right lifting $\{\!| f, h |\!\}$ of (ii) above is given by f^*h.

There exists of course a dual characterization of right adjoints, in terms of right extensions. Thus letting h in 6.8 be $1_{\mathcal{B}} : \mathcal{B} \relbar\!\!\circ\!\!\rightarrow \mathcal{B}$ yields an adjunction

6.10. $\{\!| -, 1 |\!\} \dashv [\![-, 1]\!] : \mathcal{V}\text{-}\mathbf{Mod}(\mathcal{B}, \mathcal{A})^{\mathrm{op}} \to \mathcal{V}\text{-}\mathbf{Mod}(\mathcal{A}, \mathcal{B}),$

which restricts to an equivalence between the right adjoints $\mathcal{B} \relbar\!\!\circ\!\!\rightarrow \mathcal{A}$ and the left adjoints $\mathcal{A} \relbar\!\!\circ\!\!\rightarrow \mathcal{B}$, for $\{\!| -, 1 |\!\}$ sends a left adjoint to its right adjoint, while $[\![-, 1]\!]$ sends a right adjoint to its left adjoint.

We now translate Proposition 6.9 into the language of functors. Consider again morphisms $f : \mathcal{A} \relbar\!\!\circ\!\!\rightarrow \mathcal{B}$, $h : \mathcal{C} \relbar\!\!\circ\!\!\rightarrow \mathcal{B}$ and $k : \mathcal{D} \relbar\!\!\circ\!\!\rightarrow \mathcal{C}$ in $\mathcal{V}$-**Mod**. These correspond respectively to functors $F : \mathcal{A} \to [\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$, $H : \mathcal{C} \to [\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$, and $K : \mathcal{D} \to [\mathcal{C}^{\mathrm{op}}, \mathcal{V}]$; let us also write $H' : \mathcal{B}^{\mathrm{op}} \to [\mathcal{C}, \mathcal{V}]$ for the other functor corresponding to h. One checks straightforwardly that

6.11. k respects the right lifting $\{\!| f, h |\!\}$ of h through f

is equivalent to

6.12. for all a in $\mathcal{A}$ and all d in $\mathcal{D}$, $Kd * - : [\mathcal{C}, \mathcal{V}] \to \mathcal{V}$ preserves the limit $\{Fa, H'\}$,

which, by the equivalence of 5.1 and 5.2, is further equivalent to

6.13. for all a in $\mathcal{A}$ and all d in $\mathcal{D}$, the colimit $Kd{*}H$ is preserved by $\{Fa, -\} : [\mathcal{B}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{V}$.

Our particular interest is in the case $\mathcal{A} = \mathcal{I}$ of the above: to give a module $f : \mathcal{I} \relbar\!\!\circ\!\!\rightarrow \mathcal{B}$ is to give a presheaf $\phi : \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$, and we write $\overline{\phi}$ for f. Equally to give a module $g : \mathcal{B} \relbar\!\!\circ\!\!\rightarrow \mathcal{I}$ is to give a presheaf $\psi : \mathcal{B} \to \mathcal{V}$, and we write $\underline{\psi}$ for g. Now 6.9 gives the following proposition, in which the assertion (ii) is the direct translation of the fact that for any module $h : \mathcal{C} \relbar\!\!\circ\!\!\rightarrow \mathcal{B}$ the right lifting $\{\!| f, h |\!\}$ is respected by any module $\mathcal{I} \relbar\!\!\circ\!\!\rightarrow \mathcal{C}$.

6.14. Proposition. *Given a weight $\phi : \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$, the following conditions are equivalent:*
(i) $\overline{\phi}$ has a right adjoint $\underline{\psi}$;
(ii) the representable functor $\{\phi, -\} = [\mathcal{B}^{\mathrm{op}}, \mathcal{V}](\phi, -) : [\mathcal{B}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{V}$ is cocontinuous, that is, ϕ is a small projective;
(iii) the representable functor $\{\phi, -\} : [\mathcal{B}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{V}$ preserves the colimit $\phi * Y$ of $Y : \mathcal{B} \to [\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ weighted by $\phi : \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$.
When these are satisfied, a right adjoint $\underline{\psi}$ of $\overline{\phi}$ is given by taking

6.15. $\quad \psi = [\mathcal{B}^{\mathrm{op}}, \mathcal{V}](\phi, Y-).$

Dually, $\underline{\psi}$ has a left adjoint if and only ψ is a small projective in $[\mathcal{B}, \mathcal{V}]$, and then a left adjoint $\overline{\phi}$ of $\underline{\psi}$ is given by

6.16. $\quad \phi = [\mathcal{B}, \mathcal{V}](\psi, Y'-),$

where Y' is the Yoneda embedding $\mathcal{B}^{\mathrm{op}} \to [\mathcal{B}, \mathcal{V}]$.

Recall that every functor $G : \mathcal{B} \to \mathcal{C}$ where $\mathcal{B}$ is small and $\mathcal{C}$ is cocomplete has the essentially unique cocontinuous extension $\mathrm{Lan}_Y G = - * G : [\mathcal{B}^{\mathrm{op}}, \mathcal{V}] \to \mathcal{C}$ along the Yoneda embedding $Y : \mathcal{B} \to [\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$, and $- * G$ has in fact the right adjoint $\tilde{G} : \mathcal{C} \to [\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ given by $\tilde{G}(c) = \mathcal{C}(G-, c)$. Moreover $\tilde{G}G$ is isomorphic to Y when G is fully faithful. Applying this when G is the Yoneda embedding $Y'^{\mathrm{op}} : \mathcal{B} \to [\mathcal{B}, \mathcal{V}]^{\mathrm{op}}$, we get a commutative diagram

$$\begin{tikzcd}
{[\mathcal{B}, \mathcal{V}]^{\mathrm{op}}} \arrow[rr, shift left, "R"] & & {[\mathcal{B}^{\mathrm{op}}, \mathcal{V}]} \arrow[ll, shift left, "L"] \\
& \mathcal{B} \arrow[ul, "Y'^{\mathrm{op}}"] \arrow[ur, "Y"'] &
\end{tikzcd}$$

with L left adjoint to R; an easy calculation gives

6.17. $\; L(\phi) = [\mathcal{B}^{\mathrm{op}}, \mathcal{V}](\phi, Y-)$ and $R(\psi) = [\mathcal{B}, \mathcal{V}](\psi, Y'-)$.

This adjunction, which we shall call the *Isbell adjunction*, is in fact the case $\mathcal{A} = \mathcal{I}$ of the adjunction 6.10. Moreover when $\phi \in [\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ is a small projective, it follows from 6.15 that $L(\phi) = \psi$ where the module $\underline{\psi}$ is the right adjoint of the module $\overline{\phi}$; so that in fact ψ too is a small projective. Dually, when $\psi \in [\mathcal{B}, \mathcal{V}]$ is a small projective, it follows from 6.16 that $R(\psi) = \phi$ where $\overline{\phi}$ is the left adjoint of $\underline{\psi}$; with ϕ too a small projective. In other words the adjunction $L \dashv R$ restricts to an equivalence at the level of small projectives, which we may write as

6.18. $\; (\mathcal{Q}(\mathcal{B}^{\mathrm{op}}))^{\mathrm{op}} \simeq \mathcal{Q}(\mathcal{B}).$

If $\phi \in [\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ and $\psi \in [\mathcal{B}, \mathcal{V}]$ are small projectives which correspond in this equivalence, the functor $[\mathcal{B}, \mathcal{V}](\psi, -) : [\mathcal{B}, \mathcal{V}] \to \mathcal{V}$, being cocontinuous, has the form $- * \theta$ where θ is its composite with $Y' : \mathcal{B}^{\mathrm{op}} \to [\mathcal{B}, \mathcal{V}]$. However this composite is ϕ by 6.16, and we can write $- * \phi$ as $\phi * -$; so we have

6.19. $\; [\mathcal{B}, \mathcal{V}](\psi, -) \cong \phi * - : [\mathcal{B}, \mathcal{V}] \to \mathcal{V}.$

In terms of modules, this is just the observation that a right extension along a right adjoint is given by composition with its left adjoint – since for a ϕ and a ψ as above we have an adjunction $\overline{\phi} \dashv \underline{\psi} : \mathcal{B} \,{-\!\!\circ\!\!\!\to}\, \mathcal{I}$. This leads to another characterization of the small projectives:

6.20. Proposition. *For a weight* $\phi : \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$*, the following conditions are equivalent:*
(i) ϕ *is a small projective;*
(ii) $\phi * - : [\mathcal{B}, \mathcal{V}] \to \mathcal{V}$ *is representable;*
(iii) $\phi * - : [\mathcal{B}, \mathcal{V}] \to \mathcal{V}$ *is continuous;*
(iv) $\phi * - : [\mathcal{B}, \mathcal{V}] \to \mathcal{V}$ *preserves the limit* $\{\phi, Y'\}$ *of* $Y' : \mathcal{B}^{\mathrm{op}} \to [\mathcal{B}, \mathcal{V}]$ *weighted by* ϕ*.*

Proof. $(i) \Rightarrow (ii)$ by 6.19. It is trivial that $(ii) \Rightarrow (iii) \Rightarrow (iv)$. By the equivalence of 5.1 and 5.2, (iv) is equivalent to the preservation by $\{\phi, -\}$ of the colimit $\phi * Y$; and this is equivalent to (i) by Proposition 6.14. ∎

6.21. Remark. The assertion (iii) of the proposition above may be expressed by saying that $\mathcal{Q}$ is the class $\mathcal{P}^+$ of $\mathcal{P}$-flat weights.

There is a further characterization of the weights in $\mathcal{Q}$, due to Street. A weight $\phi : \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ is said to be *absolute* if each limit $\{\phi, T\}$, where $T : \mathcal{B}^{\mathrm{op}} \to \mathcal{C}$ say, is preserved by *every* functor $P : \mathcal{C} \to \mathcal{D}$; or equally if each colimit $\psi * S$, where $S : \mathcal{B} \to \mathcal{C}$, is preserved by every functor $P : \mathcal{C} \to \mathcal{D}$. Street showed the following, in a context wider than ours, in [Str83]; we give a proof (in our context) for completeness:

6.22. Theorem. *A weight* $\phi : \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ *is absolute precisely when it is a small projective in* $[\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$*.*

Proof. One direction is clear: to say that $\phi : \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ is a small projective is, by the equivalence of 5.1 and 5.2, to say that, for each $T : \mathcal{B}^{\mathrm{op}} \to [\mathcal{A}, \mathcal{V}]$ with $\mathcal{A}$ small and for each weight $\psi : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$, the limit $\{\phi, T\}$ is preserved by the functor $\psi * - : [\mathcal{A}, \mathcal{V}] \to \mathcal{V}$; so that each absolute $\phi : \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ is certainly a small projective in $[\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$.

As a preliminary to the proof of the converse, recall that the defining property of the colimit $\phi * S$ for $S : \mathcal{B} \to \mathcal{C}$ is an isomorphism

6.23. $\quad \mathcal{C}(\phi * S, c) \cong [\mathcal{B}^{\mathrm{op}}, \mathcal{V}](\phi, \mathcal{C}(S-, c)).$

However $\mathcal{C}(Sb, c) = S^*(b, c)$; and then if $\phi : \mathcal{B}^{\mathrm{op}} \to \mathcal{V}$ corresponds to the module $\overline{\phi} : \mathcal{I} \longrightarrow\!\!\!\!\circ\;\; \mathcal{B}$, the right side of 6.23 is $\{\!\!\{ \overline{\phi}, S^* \}\!\!\}(*, c)$, where $*$ denotes the unique object of $\mathcal{I}$. Finally the object $\phi * S$ of $\mathcal{C}$ corresponds to a functor $\phi * S : \mathcal{I} \to \mathcal{C}$ and hence to a module $(\phi * S)^* : \mathcal{C} \longrightarrow\!\!\!\!\circ\;\; \mathcal{I}$ with $(\phi * S)^*(*, c) = \mathcal{C}(\phi * S, c)$; so that the defining equation 6.23 of $\phi * S$ may be written as

6.24. $\quad (\phi * S)^* \cong \{\!\!\{ \overline{\phi}, S^* \}\!\!\}$

which is just to say that the lifting of S^* through $\overline{\phi}$ is given by $(\phi * S)^*$.

To ask $P : \mathcal{C} \to \mathcal{D}$ to preserve the colimit $\phi * S$ is to ask the invertibility of the canonical comparison $\phi * (PS) \to P(\phi * S)$ or equally of the canonical comparison $(\phi * S)^* P^* \to (\phi * PS)^*$. By 6.24 this may be written in the form

6.25. $\{\!\!\{ \overline{\phi}, S^* \}\!\!\} P^* \to \{\!\!\{ \overline{\phi}, S^* P^* \}\!\!\}$;

so that P preserves $\phi * S$ exactly when P^* respects the right lifting $\{\!\!\{ \overline{\phi}, S^* \}\!\!\}$.

We now complete the proof of the converse, showing a small projective weight ϕ to be absolute. Supposing $\phi * S$ to exist for $S : \mathcal{B} \to \mathcal{C}$, we are to show that the right lifting $\{\!\!\{ \overline{\phi}, S^* \}\!\!\}$ of 6.24 is respected by P^* for every $P : \mathcal{C} \to \mathcal{D}$. But this is certainly the case since, $\overline{\phi}$ being a left adjoint by Proposition 6.14, the lifting in question is absolute by Proposition 6.9. ∎

7. Cauchy completion and the Morita theorems

For any category $\mathcal{A}$, the inclusion $J : \mathcal{A} \to \mathcal{Q}(\mathcal{A})$ expresses $Q(\mathcal{A})$ as the free Q-cocomplete category on $\mathcal{A}$, which by Theorem 6.22 is the free cocompletion of $\mathcal{A}$ under absolute colimits. It is determined by the universal property 3.6, which here, because every functor preserves *absolute* colimits, becomes:

7.1. $\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{Q}(\mathcal{A}), \mathcal{B}) \simeq \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A}, \mathcal{B})$ for any $\mathcal{B}$ with absolute colimits.

Proposition 4.5 here takes the following stronger form:

7.2. Proposition. *The inclusion $J : \mathcal{A} \to \mathcal{Q}(\mathcal{A})$ is an equivalence if and and only if $\mathcal{A}$ admits all absolute colimits.*

Proof. The "only if" part is trivial. If $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{Q}$-cocomplete, we have $\mathcal{Q}\text{-}\mathbf{Cocts}(\mathcal{A}, \mathcal{B}) = \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A}, \mathcal{B}) \simeq \mathcal{Q}\text{-}\mathbf{Cocts}(\mathcal{Q}(\mathcal{A}), \mathcal{B})$, whence it follows that $J : \mathcal{A} \to \mathcal{Q}(\mathcal{A})$ is an equivalence. ∎

7.3. Proposition. *For a small $\mathcal{B}$, let $\phi \in [\mathcal{B}^{\mathrm{op}}, \mathcal{V}]$ and $\psi \in [\mathcal{B}, \mathcal{V}]$ be small projective weights related by the equivalence 6.18. Then for any category $\mathcal{A}$ and any functor $F : \mathcal{B} \to \mathcal{A}$, we have an isomorphism $\{\psi, F\} \cong \phi * F$, either side existing if the other does. Accordingly, $\mathcal{A}$ admits absolute limits if and only if it admits absolute colimits.*

Proof. Let $\{\psi, F\}$ exist; as the ψ-weighted limit of F in $\mathcal{A}$, it is also the ψ-weighted colimit of F^{op} in $\mathcal{A}^{\mathrm{op}}$. Since ψ-weighted colimits are absolute by Theorem 6.22, the canonical $\psi * \mathcal{A}(F-, a) \to \mathcal{A}(\{\psi, F\}, a)$ is invertible; but $\psi * \mathcal{A}(F-, a)$ is isomorphic by 6.19 to $[\mathcal{B}^{\mathrm{op}}, \mathcal{V}](\phi, \mathcal{A}(F-, a))$, exhibiting $\{\psi, F\}$ as the colimit $\phi * F$. ∎

The equivalence 6.18 above was for small categories $\mathcal{B}$; it admits the following extension to arbitrary categories:

7.4. Proposition. *For any category $\mathcal{A}$, we have an equivalence $(\mathcal{Q}(\mathcal{A}^{\mathrm{op}}))^{\mathrm{op}} \simeq \mathcal{Q}(\mathcal{A})$.*

PROOF. Let $\mathcal{B}$ admit absolute colimits; so $\mathcal{B}^{\mathrm{op}}$ too admits absolute colimits, by Proposition 7.3. Then for any $\mathcal{A}$, we have equivalences

$$\begin{aligned}\mathcal{Q}\text{-}\mathbf{Cocts}(\mathcal{Q}(\mathcal{A}),\mathcal{B}) &\simeq \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A},\mathcal{B})\\ &\cong (\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A}^{\mathrm{op}},\mathcal{B}^{\mathrm{op}}))^{\mathrm{op}}\\ &\simeq (\mathcal{V}\text{-}\mathbf{CAT}(\mathcal{Q}(\mathcal{A}^{\mathrm{op}}),\mathcal{B}^{\mathrm{op}}))^{\mathrm{op}}\\ &\cong \mathcal{V}\text{-}\mathbf{CAT}((\mathcal{Q}(\mathcal{A}^{\mathrm{op}}))^{\mathrm{op}},\mathcal{B})\\ &= \mathcal{Q}\text{-}\mathbf{Cocts}((\mathcal{Q}(\mathcal{A}^{\mathrm{op}}))^{\mathrm{op}},\mathcal{B}).\end{aligned}$$

The desired equivalence $(\mathcal{Q}(\mathcal{A}^{\mathrm{op}}))^{\mathrm{op}} \simeq \mathcal{Q}(\mathcal{A})$ follows. ∎

A category $\mathcal{A}$ which admits absolute colimits, and hence absolute limits, is said to be *Cauchy-complete*; and $\mathcal{Q}(\mathcal{A})$ is called the *Cauchy-completion* of $\mathcal{A}$; this concept was introduced by Lawvere in [Law73]. For a general class Φ of weights, the free completion of $\mathcal{A}$ under Φ-*limits* is of course $(\Phi(\mathcal{A}^{\mathrm{op}}))^{\mathrm{op}}$; so by Proposition 7.4, $\mathcal{Q}(\mathcal{A})$ is also the completion of $\mathcal{A}$ under *absolute limits*.

7.5. PROPOSITION. *For any class Φ of weights and any category $\mathcal{A}$, the category $\Phi(\mathcal{A})_\Phi$ is included in $\mathcal{Q}(\mathcal{A})$. If the class Φ contains $\mathcal{Q}$, we have an equality $\Phi(\mathcal{A})_\Phi = \mathcal{Q}(\mathcal{A})$.*

PROOF. By 3.14 and 4.1, we may as well assume Φ to be saturated.[2] We begin by proving the first assertion in the case of a small $\mathcal{A}$. Let us denote the inclusions again by $\mathcal{A} \xrightarrow{Z} \Phi(\mathcal{A}) \xrightarrow{W} [\mathcal{A}^{\mathrm{op}},\mathcal{V}]$ with $WZ = Y$. For $\phi \in \Phi(\mathcal{A})_\Phi$, the representable functor $\Phi(\mathcal{A})(\phi,-) : \Phi(\mathcal{A}) \to \mathcal{V}$ preserves Φ-colimits; in particular, it preserves the colimit $\phi * Z \cong \phi$ in $\Phi(\mathcal{A})$, which is a Φ-colimit since $\phi \in \Phi(\mathcal{A}) = \Phi[\mathcal{A}] \subset \Phi$.[3] Since W preserves Φ-colimits, we have $W(\phi * Z) \cong \phi * WZ = \phi * Y$ in $[\mathcal{A}^{\mathrm{op}},\mathcal{V}]$. The composite of $[\mathcal{A}^{\mathrm{op}},\mathcal{V}](\phi,-) = [\mathcal{A}^{\mathrm{op}},\mathcal{V}](W\phi,-)$ with W is $[\mathcal{A}^{\mathrm{op}},\mathcal{V}](W\phi,W-)$, which is isomorphic to $\Phi(\mathcal{A})(\phi,-)$ since W is fully faithful. Since both W and this composite preserve the colimit $\phi * Z$, it follows that $[\mathcal{A}^{\mathrm{op}},\mathcal{V}](\phi,-)$ preserves the colimit $W(\phi * Z) \cong \phi * Y$. Accordingly, ϕ is a small projective by 6.14.

We now prove the first statement for an arbitrary category $\mathcal{A}$. By Proposition 3.16, any $F \in \Phi(\mathcal{A})$ is of the form $\mathrm{Lan}_{H^{\mathrm{op}}}\phi$ for some fully faithful $H : \mathcal{K} \to \mathcal{A}$ with $\mathcal{K}$ small and some $\phi \in \Phi$. Because H is fully faithful, $\mathrm{Lan}_{H^{\mathrm{op}}} : [\mathcal{K}^{\mathrm{op}},\mathcal{V}] \to \mathcal{P}\mathcal{A}$ is also fully faithful; moreover, as a left adjoint, it preserves all colimits. For $\psi \in \Phi(\mathcal{K})$, its image $\mathrm{Lan}_{H^{\mathrm{op}}}\psi = \psi * YH$, as a Φ-colimit of representables, lies in $\Phi(\mathcal{A})$; so that $\mathrm{Lan}_{H^{\mathrm{op}}}$ restricts to a functor $L : \Phi(\mathcal{K}) \to \Phi(\mathcal{A})$. This functor, like $\mathrm{Lan}_{H^{\mathrm{op}}}$, is fully faithful, and it preserves Φ-colimits, since these are formed in $\Phi(\mathcal{K})$ as in $[\mathcal{K}^{\mathrm{op}},\mathcal{V}]$ and in $\Phi(\mathcal{A})$ as in $\mathcal{P}\mathcal{A}$. Since L is fully faithful, we have an isomorphism $\Phi(\mathcal{K})(\phi,-) \cong \Phi(\mathcal{A})(L(\phi),L-) = \Phi(\mathcal{A})(F,L-)$. If F belongs to $\Phi(\mathcal{A})_\Phi$ then, Φ-colimits are also preserved by $\Phi(\mathcal{A})(F,-)$, and hence by $\Phi(\mathcal{A})(F,L-)$. Thus Φ-colimits are preserved by $\Phi(\mathcal{K})(\phi,-)$, so that ϕ belongs to $\mathcal{Q}$ by the first part of the proof. So $F = \phi * YH$, as a $\mathcal{Q}$-colimit of representables, lies in $\mathcal{Q}(\mathcal{A})$.

[2] Sentence added 2006-04-29.
[3] Phrase added 2006-04-29.

Suppose now that $\mathcal{Q} \subset \Phi$. Since $\Phi \subset \mathcal{P}$, we have $\mathcal{P}^- \subset \Phi^-$, or $\mathcal{Q} \subset \Phi^-$. By Proposition 5.6, $\Phi(\mathcal{A})_\Phi$ is closed in $\Phi(\mathcal{A})$ under Φ^--colimits, and hence under $\mathcal{Q}$-colimits. Since $\mathcal{Q} \subset \Phi$ however, $\mathcal{Q}$-colimits are preserved by the inclusion $\Phi(\mathcal{A}) \to \mathcal{P}\mathcal{A}$. Thus $\Phi(\mathcal{A})_\Phi$ is also closed under $\mathcal{Q}$-colimits in $\mathcal{P}\mathcal{A}$; and since it contains the representables, it contains $\mathcal{Q}(\mathcal{A})$. ∎

For any class Φ containing $\mathcal{Q}$, for any $\mathcal{A}$ and, for any Φ-cocomplete $\mathcal{B}$, we have

$$\Phi\text{-}\mathbf{Cocts}(\Phi(\mathcal{A}), \mathcal{B}) \simeq \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{A}, \mathcal{B}) \simeq \mathcal{V}\text{-}\mathbf{CAT}(\mathcal{Q}(\mathcal{A}), \mathcal{B}) \simeq \Phi\text{-}\mathbf{Cocts}(\Phi(\mathcal{Q}(\mathcal{A})), \mathcal{B}),$$

so we have an equivalence,

7.6. $\Phi(\mathcal{A}) \simeq \Phi(\mathcal{Q}(\mathcal{A}))$.

The case $\Phi = \mathcal{P}$ of the following proposition is the principal classic Morita theorem:

7.7. Proposition. *Let Φ be a class of weights containing $\mathcal{Q}$. Then for any categories $\mathcal{A}$ and $\mathcal{B}$, we have $\Phi(\mathcal{A}) \simeq \Phi(\mathcal{B})$ if and only if $\mathcal{Q}(\mathcal{A}) \simeq \mathcal{Q}(\mathcal{B})$.*

Proof. If $\mathcal{Q}(\mathcal{A}) \simeq \mathcal{Q}(\mathcal{B})$ we get $\Phi(\mathcal{A}) \simeq \Phi(\mathcal{B})$ by 7.6. If $\Phi(\mathcal{A}) \simeq \Phi(\mathcal{B})$ then $\mathcal{Q}(\mathcal{A}) \simeq \mathcal{Q}(\mathcal{B})$ by Proposition 7.5 ∎

In the circumstances of this proposition, the categories $\mathcal{A}$ and $\mathcal{B}$ are said to be *Morita equivalent*.

8. Φ-continuous presheaves

We turn now to the study of categories of the form $\Phi\text{-}\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$, where $\mathcal{N}$ is a small Φ-cocomplete category and $\Phi\text{-}\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ denotes the full subcategory of $[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ determined by the Φ-continuous functors $\mathcal{N}^{\mathrm{op}} \to \mathcal{V}$. Since Φ-**Conts** equals Φ^*-**Conts** by the definition of Φ^*, we may as well suppose that Φ is saturated. Since the representables are certainly Φ-continuous, the Yoneda embedding factorizes through the inclusion $J : \Phi\text{-}\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}] \to [\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$, as say $Y = JK$; then since J is fully faithful, it follows from Yoneda that J is isomorphic (in the notation of Section 4) to $\tilde{K}$. For the case where $\mathcal{V} = \mathbf{Set}$ and Φ is a small set of weights of the form $\Delta 1 : \mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$, some of the results below appear in [ABLR02].

Since Φ-limits commute in $\mathcal{V}$ with all limits and with Φ^+-colimits, and since such limits and colimits are formed pointwise in $[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$, we have:

8.1. Proposition. *For any small Φ-cocomplete $\mathcal{N}$, the category $\Phi\text{-}\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ is closed in $[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ under all limits and under Φ^+-colimits. As a consequence, $\Phi^+(\mathcal{N}) \subset \Phi\text{-}\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$.*

In other words, each Φ-flat weight $\mathcal{N}^{\mathrm{op}} \to \mathcal{V}$ is Φ-continuous; we shall later give conditions for the converse to hold. That it does not hold in general is shown by the following example, which was Example 2.3 (vii) of [ABLR02]:

8.2. Example. With $\mathcal{V} = \mathbf{Set}$, let Φ be the saturated class of weights for which a Φ-cocomplete category is one with pushouts, and let $\mathcal{N}^{\mathrm{op}}$ be the one-object category given by a non-trivial group G, so that $[\mathcal{N}^{\mathrm{op}}, \mathbf{Set}]$ is the category of G-sets. Then $\mathcal{N}^{\mathrm{op}}$ has pullbacks, and $\Delta 1 : \mathcal{N}^{\mathrm{op}} \to \mathbf{Set}$ preserves pullbacks. Yet $\Delta 1$ is not Φ-flat: for 2.3 gives $\Delta 1 * - = \mathrm{colim}$, which by (3.35) of [Kel82] sends a presheaf to the set of connected components of its set of elements, and thus sends a G-set X to the set of its orbits. Now the G-sets $G \to 1 \leftarrow G$ have a pullback given by $G \leftarrow G \times G \to G$, and this pullback is not preserved by the passage to the sets of orbits.

Recall from [Kel82] that many important base-categories $\mathcal{V}$ are *locally bounded*, and that Theorem 6.11 of that work gives:

8.3. Proposition. *Whenever the base category $\mathcal{V}$ is locally bounded, Φ-$\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ is reflective in $[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$, for any class Φ of weights, and any small Φ-cocomplete $\mathcal{N}$.*

Sometimes however – as under certain hypotheses to be introduced below – we can infer the reflectiveness of Φ-$\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ more easily, without using the general theorem above, which involves a transfinite induction. Moreover additional hypotheses may imply special properties of the reflexion.

An important property of Φ-$\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ is the following; this is well known, one generalization of it being Theorem 5.56 in [Kel82].

8.4. Lemma. *For any functor $G : \mathcal{N} \to \mathcal{B}$ where $\mathcal{N}$ is Φ-cocomplete, the corresponding functor $\tilde{G} : \mathcal{B} \to [\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ takes its values in Φ-$\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ if and only if G preserves Φ-colimits.*

Proof. Consider a Φ-colimit $\phi * T$ in $\mathcal{N}$, where $\phi : \mathcal{L}^{\mathrm{op}} \to \mathcal{V}$ lies in Φ and where $T : \mathcal{L} \to \mathcal{N}$. To say that G preserves this colimit is to say that $G(\phi * T)$ (with the appropriate unit) is the colimit $\phi * GT$ in $\mathcal{B}$, which is also to say that, for each $b \in \mathcal{B}$, the object $\mathcal{B}(\phi * GT, b)$ (with the appropriate counit) is the limit $\{\phi, \mathcal{B}(GT-, b)\}$ in $\mathcal{V}$; this, in turn, is to say that each $\tilde{G}b : \mathcal{N}^{\mathrm{op}} \to \mathcal{V}$ preserves the limit $\{\phi, T^{\mathrm{op}}\}$ in $\mathcal{N}^{\mathrm{op}}$. To ask this for each Φ-colimit $\phi * T$ in $\mathcal{N}$ is just to ask $\tilde{G}b$ to lie in Φ-$\mathbf{Conts}(\mathcal{N}^{\mathrm{op}}, \mathcal{V})$. ∎

When we take $G : \mathcal{N} \to \mathcal{B}$ in 8.4 to be $K : \mathcal{N} \to \Phi$-$\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$, it is trivial that the inclusion $\tilde{K} : \Phi$-$\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}] \to [\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ takes its values in Φ-$\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$; so the lemma gives:

8.5. Corollary. *For any Φ-cocomplete $\mathcal{N}$, the inclusion $K : \mathcal{N} \to \Phi$-$\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ preserves Φ-colimits.*

Another useful lemma is the following:

8.6. Lemma. *Let $\mathcal{C}$ be a full subcategory of $\mathcal{A}$, and write $\mathcal{B}$ for the full subcategory of $\mathcal{A}$ given by those objects of $\mathcal{A}$ which admit a reflexion into $\mathcal{C}$. Let $\mathcal{C}$ and $\mathcal{A}$ admit Φ-colimits. Then $\mathcal{B}$ is closed in $\mathcal{A}$ under Φ-colimits.*

PROOF. Let the colimit $\phi * T$ exist in $\mathcal{A}$, where $\phi : \mathcal{K}^{\mathrm{op}} \to \mathcal{V}$ lies in Φ and where $T : \mathcal{K} \to \mathcal{A}$ takes its values in $\mathcal{B}$. Write $R : \mathcal{B} \to \mathcal{C}$ for the functor sending each $b \in \mathcal{B}$ to its reflexion Rb in $\mathcal{C}$. Then for $c \in \mathcal{C}$ we have

$$\begin{aligned}\mathcal{A}(\phi * T, c) &\cong [\mathcal{K}^{\mathrm{op}}, \mathcal{V}](\phi, \mathcal{A}(T-, c)), \text{ by the definition of } \phi * T;\\ &\cong [\mathcal{K}^{\mathrm{op}}, \mathcal{V}](\phi, \mathcal{C}(RT-, c))), \text{ since } R \text{ is the reflexion;}\\ &\cong \mathcal{C}(\phi * RT, c), \text{ by the definition of } \phi * RT;\end{aligned}$$

thus $\phi * T$ admits the reflexion $\phi * RT$ in $\mathcal{C}$. ∎

8.7. REMARK. Any full subcategory $\mathcal{C}$ of $[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ is of course cocomplete like $[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ if it is reflective. It is an old and classical observation that the converse is also true whenever $\mathcal{C}$ contains the representables (so that, once again, the inclusion $\mathcal{C} \to [\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ is isomorphic to $\tilde{K}$, where $K : \mathcal{N} \to \mathcal{C}$ is the inclusion). For to say that *the object* ϕ *of* $[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ *admits a reflexion* d *in* $\mathcal{C}$ is to say that we have, naturally in c, an isomorphism

$$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}](\phi, \tilde{K}c) \cong \mathcal{C}(d, c),$$

and this is to say that d is the colimit $\phi * K$ in $\mathcal{C}$.

We shall adopt the following notation: for a full subcategory $\mathcal{B}$ of $\mathcal{A}$ and a class Φ of weights: we write $\Phi\{\mathcal{B}\}$ for the closure of $\mathcal{B}$ in $\mathcal{A}$ under Φ-colimits. Of course $\mathcal{A}$ must be understood if this notation is to suffice: otherwise we should use $\Phi\{\mathcal{B} \mid \mathcal{A}\}$.

8.8. PROPOSITION. *Still supposing* $\mathcal{N}$ *to be small and* Φ*-cocomplete, write* $\Phi^+\{\Phi(\mathcal{N})\}$ *for the closure of* $\Phi(\mathcal{N})$ *in* $[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ *under* Φ^+*-colimits. Then each object of* $[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ *that lies in* $\Phi^+\{\Phi(\mathcal{N})\}$ *has a reflexion in* Φ-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$*. In fact this reflexion lies in* $\Phi^+(\mathcal{N})$*, and the reflexion of an object of* $\Phi(\mathcal{N})$ *lies in* $\mathcal{N}$.

PROOF. Since $\mathcal{N}$ is Φ-cocomplete by hypothesis, it admits for each $\phi \in \Phi(\mathcal{N}) = \Phi[\mathcal{N}]$ the colimit $\phi * 1_{\mathcal{N}}$ of $1_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}$. So since $K : \mathcal{N} \to \Phi$-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ preserves Φ-colimits by Corollary 8.5, the object $K(\phi * 1)$ of Φ-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ is the colimit $\phi * K$; and accordingly it is, by Remark 8.7, the reflexion in Φ-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ of $\phi \in [\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$. Thus every $\phi \in \Phi(\mathcal{N})$ has a reflexion in Φ-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$, which in fact lies in $\mathcal{N}$ (embedded by $K : \mathcal{N} \to \Phi$-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$). Since Φ-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ admits Φ^+-colimits by Proposition 8.1, it follows from Lemma 8.6 that the objects of $[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ admitting a reflexion in Φ-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ are closed under Φ^+-colimits; accordingly they include all the objects of $\Phi^+\{\Phi(\mathcal{N})\}$. Since the reflexion preserves Φ^+-colimits, which are (by Proposition 8.1) formed in Φ-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ as they are in $[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$, the reflexions all lie in the Φ^+-closure $\Phi^+(\mathcal{N})$ of $\mathcal{N}$ in $[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$. ∎

8.9. THEOREM. *For a small* Φ*-cocomplete* $\mathcal{N}$*, the inclusion* $\Phi^+(\mathcal{N}) \subset \Phi$-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ *of Proposition 8.1 is an equality if and only if* Φ-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}] \subset \Phi^+\{\Phi(\mathcal{N})\}$*. In particular* $\Phi^+(\mathcal{N}) = \Phi$-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ *whenever* $\Phi^+\{\Phi(\mathcal{N})\}$ *is all of* $[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$.

PROOF. Since $\mathcal{N} \subset \Phi(\mathcal{N})$ we have $\Phi^+(\mathcal{N}) \subset \Phi^+\{\Phi(\mathcal{N})\}$, so that certainly Φ-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ $\subset \Phi^+\{\Phi(\mathcal{N})\}$ if $\Phi^+(\mathcal{N}) = \Phi$-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$. Now if Φ-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}] \subset \Phi^+\{\Phi(\mathcal{N})\}$, then by Proposition 8.8 each object of Φ-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ has for its reflexion in Φ-**Conts**$[\mathcal{N}^{\mathrm{op}}, \mathcal{V}]$ – namely itself – an object of $\Phi^+(\mathcal{N})$. ∎

We may express the above by saying that, in these circumstances, the Φ-flat weights coincide with the Φ-continuous ones.

It is convenient to introduce the following definition.

8.10. Definition. *A class Φ of weights is said to be* locally small *if each $\Phi(\mathcal{K})$ with $\mathcal{K}$ small is also small.*

Since $\Phi^*(\mathcal{K}) = \Phi(\mathcal{K})$ for any $\mathcal{K}$, a class Φ is locally small if and only if its saturation Φ^* is so. Moreover, when Φ is saturated, since we have $\Phi(\mathcal{K}) = \Phi[\mathcal{K}]$, to say that Φ is locally small is to say that each $\Phi[\mathcal{K}]$ is small. For a general class Φ, it was observed in Section 3.5 of [Kel82] that Φ is locally small when the class Φ is in fact a small set. For example, when $\mathcal{V} = \mathbf{Set}$ and Φ consists of the three weights giving initial objects, binary coproducts and coequalizers, Φ^* is locally small; here $\Phi(\mathcal{K})$ is the free finitely-cocomplete category on $\mathcal{K}$, and Φ^* is the saturation of the weights for finite colimits. Similarly when $\mathcal{V}$ is locally finitely presentable as a closed category, as in [Kel82-2]; what are there called "the finite indexing types" form a small set Φ, so that Φ^* is locally small; here $\Phi(\mathcal{K})$ is again the free finitely-cocomplete category on $\mathcal{K}$, and Φ^* is the saturation of the weights for finite colimits; compare Examples 3.13 and 5.7. We may note that the class Φ of Example 8.2 is locally small.

For a *locally small* saturated class Φ, the last statement of 8.9 has a converse. First note that for any $K : \mathcal{A} \to \mathcal{C}$ with $\mathcal{A}$ small, the left Kan extension along K of $Y : \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is $\tilde{K}$, by 2.9. In particular $\mathrm{Lan}_Y Y$ is the identity. However $Y = WZ$, where $Z : \mathcal{A} \to \Phi(\mathcal{A})$ and $W : \Phi(\mathcal{A}) \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ are the inclusions. Thus $\mathrm{Lan}_Y Y \cong \mathrm{Lan}_W(\mathrm{Lan}_Z Y) \cong \mathrm{Lan}_W W$, since $\mathrm{Lan}_Z Y \cong \tilde{Z} \cong W$. (That is, in the language of [Kel82], the functor W is *dense*).

8.11. Theorem. *When the saturated class Φ is locally small, the following are equivalent:*
(i) Φ-$\mathbf{Conts}[\mathcal{N}^{\mathrm{op}}, \mathcal{V}] = \Phi^+(\mathcal{N})$ for any small Φ-cocomplete $\mathcal{N}$;
(ii) For any small $\mathcal{A}$, every presheaf $\mathcal{A}^{\mathrm{op}} \to \mathcal{V}$ is a Φ^+-colimit of a diagram in $\Phi(\mathcal{A})$;
(iii) For any small $\mathcal{A}$, $\Phi^+\{\Phi(\mathcal{A})\} = [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$.

Proof. (*iii*) implies (*i*) by Theorem 8.9 and (*ii*) implies (*iii*) trivially; so it remains to prove (*i*) implies (*ii*). For any presheaf $F : \mathcal{A}^{\mathrm{op}} \to \mathcal{V}$, we have $F \cong \mathrm{Lan}_W W(F) \cong [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](W-, F) * W$ using 2.9. However the presheaf $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}](W-, F) : \Phi(\mathcal{A})^{\mathrm{op}} \to \mathcal{V}$ is Φ-continuous as W preserves Φ-colimits and the representable $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}](-, F)$ is continuous. But this presheaf is a weight since $\Phi(\mathcal{A})$ like $\mathcal{A}$ is small; so by (*i*), it belongs to Φ^+. ∎

8.12. Remark. A special case of Theorem 8.11 forms part of [ABLR02] Theorem 2.4.

Notice that $\Phi^+\{\Phi(\mathcal{A})\}$ in the theorem above is different from $\Phi^+(\Phi(\mathcal{A}))$, which is the closure of $\Phi(\mathcal{A})$ under Φ^+-colimits in $[\Phi(\mathcal{A})^{\mathrm{op}}, \mathcal{V}]$. Nevertheless since $\Phi \subset \Phi^{+-}$, $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]_{\Phi^+}$ is closed in $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ under Φ-colimits by Proposition 5.6 and since $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]_{\Phi^+}$ also contains the representables, it contains $\Phi(\mathcal{A})$. So Proposition 4.3 gives:

8.13. Observation. Let Φ be a locally small class satisfying the equivalent conditions of Theorem 8.11. Then for each small $\mathcal{A}$, the category $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is equivalent to $\Phi^+(\Phi(\mathcal{A}))$.

Note that this result may also be deduced from Theorem 3.6 which, because $\mathcal{V}$ is complete, gives Φ-$\mathbf{Conts}[\Phi(\mathcal{A})^{\mathrm{op}}, \mathcal{V}] \simeq [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ for any small $\mathcal{A}$; for we have Φ-$\mathbf{Conts}[\Phi(\mathcal{A})^{\mathrm{op}}, \mathcal{V}] = \Phi^+(\Phi(\mathcal{A}))$ from 8.9.

8.14. Example. Let $\mathcal{V}$ be locally finitely presentable as a closed category, in the sense of [Kel82-2], and let Φ be the saturation of the weights for finite colimits. In this context, the weights in Φ^+ are said to be *flat*. For a small $\mathcal{A}$, the full subcategory $\Phi(\mathcal{A})$ of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ consisting of the finite colimits of the representables is also, as mentioned previously in 3.13, the subcategory of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ given by the finitely presentable objects; moreover, by [Kel82-2] Theorem 7.2 again, every object of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ is a filtered colimit of a diagram in $\Phi(\mathcal{A})$. However (conical) filtered colimits in $\mathcal{V}$ commute, by [Kel82-2] Proposition 4.9, with finite limits; so the weight for a conical filtered colimit is flat – that is, belongs to Φ^+. Thus $\Phi^+\{\Phi(\mathcal{A})\}$ is all of $[\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$ for any small $\mathcal{A}$, so that Theorem 8.11 applies for this Φ.

8.15. Example. Everything in Example 8.14 continues to hold when we take for Φ not the weights for finite colimits but those for α-colimits, where α is a regular cardinal; see [Kel82-2], Section 7.4.

8.16. Example. Let Φ be the saturation of the class Ψ of Example 5.8, so that a Φ-cocomplete category is one with an initial object. Here $\Phi(\mathcal{A})$ consists of the representables along with the initial object 0 of $[\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$. Now any presheaf $F : \mathcal{A}^{\mathrm{op}} \to \mathbf{Set}$ is the conical colimit of the canonical $Y/F \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, where Y/F is the comma category of $Y : \mathcal{A} \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ and $F : 1 \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$; so it is also the conical colimit of the canonical $W/F \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$, where $W : \Phi(\mathcal{A}) \to [\mathcal{A}^{\mathrm{op}}, \mathbf{Set}]$ is the inclusion; for W/F differs from Y/F only by the addition of an initial object, namely the unique map $0 \to F$. Since W/F is accordingly connected, F lies in $\Phi^+\{\Phi(\mathcal{A})\}$ by Example 5.8. So again Theorem 8.11 applies.

8.17. Remark. We get a trivial case where Theorem 8.11 applies if we take Φ to be $\mathcal{Q}$. Since $\mathcal{Q} = \mathcal{P}^- = 0^{+\,-}$, we have $\mathcal{Q}^+ = 0^{+\,-\,+} = 0^+ = \mathcal{P}$. By [Joh89], the class $\mathcal{Q}$ is locally small if $\mathcal{V}_0$ is locally presentable.

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