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BIRKHOFF'S VARIETY THEOREM WITH AND WITHOUT FREE ALGEBRAS

To the memory of Jan Reiterman

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ABSTRACT. For large signatures Σ we prove that Birkhoff's Variety Theorem holds (i.e., equationally presentable collections of Σ-algebras are precisely those closed under limits, subalgebras, and quotient algebras) iff the universe of small sets is not measurable. Under that limitation Birkhoff's Variety Theorem holds in fact for F-algebras of an arbitrary endofunctor F of the category **Class** of classes and functions.

For endofunctors F of **Set**, the category of small sets, Jan Reiterman proved that if F is a varietor (i.e., if free F-algebras exist) then Birkhoff's Variety Theorem holds for F-algebras. We prove the converse, whenever F preserves preimages: if F is not a varietor, Birkhoff's Variety Theorem does not hold. However, we also present a non-varietor satisfying Birkhoff's Variety Theorem. Our most surprising example is two varietors whose coproduct does not satisfy Birkhoff's Variety Theorem.

1. Introduction

Garrett Birkhoff characterized in 1935 varieties of finitary algebras; in his proof he used free algebras substantially. In the present paper we investigate the limitations of generalizations of Birkhoff's result beyond "finitary" and, to a restricted extent, beyond "over **Set**". Recall the original statement of Birkhoff's Variety Theorem: a full subcategory of $\mathbf{Alg}\,\Sigma$, the category of algebras on a finitary signature Σ, is presentable by equations iff it is closed under

(a) regular quotients,

(b) subobjects, and

(c) products

in $\mathbf{Alg}\,\Sigma$. Well, this is not the original formulation: instead of full subcategory Birkhoff used classes of Σ-algebras, and he spoke about homomorphic images in (a), and subalgebras in (b). However, for algebras over **Set** it is well-known and easy to prove that

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regular epimorphisms are precisely the surjective homomorphisms, and subobjects are precisely the subalgebras. Notice also that in (c) above closure under limits may be felt as more "natural"—but in the presence of (b) this is equivalent to closure under products, of course. This fact is (dramatically) no longer true if we move from sets to classes, as we mention below.

There are good reasons for attempting to generalize Birkhoff's Variety Theorem in Categorical Algebra: for some "juicy" examples of categories that we consider algebraic, e.g., the category $\mathbf{Comp}\,T_2$ of compact Hausdorff spaces, the corresponding signature is not finitary, and, in fact, it is not a small signature. There are two ways of dealing with this: we either substitute signatures by endofunctors of **Set** (e.g., for $\mathbf{Comp}\,T_2$ the canonical endofunctor to consider is the ultrafilter functor β, see Example 2.5 (iii) and 3.4) or we admit that a signature may be a class of operation symbols—and then we move from **Set** to **Class**, the category of classes. In the present paper we do both:

(1) For F-algebras where F is an endofunctor of **Set** we study the question of when Birkhoff's Variety Theorem holds. Jan Reiterman observed that whenever F is a *varietor*, i.e., free F-algebras exist, then Birkhoff's Variety Theorem holds. We prove the converse implication for all functors preserving nonempty preimages (which is a weak requirement fulfilled e.g. by all functors weakly preserving pullbacks): whenever such a functor is not a varietor than Birkhoff's Variety Theorem does not hold for F-algebras. On the other hand, we present an example of a set functor which is not a varietor, although it satisfies Birkhoff's Variety Theorem. A full characterization of all set functors for which Birkhoff's Variety Theorem holds is an open problem. We also present varietors F and G such that Birkhoff's Variety Theorem fails for $F+G$, where algebras of $F+G$ are just "bialgebras" carrying a structure of an F-algebra as well as that of a G-coalgebra; we consider this to be very surprising.

(2) For endofunctors of **Class**, Birkhoff's Variety Theorem (with closure under limits in (c) above) is proved to hold whenever the universe **Set** of small sets is not "too large": what we need is the set-theoretical assumption that **Set** is not measurable. In fact, this assumption is also necessary: if **Set** is measurable then even Σ-algebras do not satisfy Birkhoff's Variety Theorem for large signatures Σ.

Standard categorical concepts and facts used in the text without explanation can be found in Mac Lane [15] or Adámek-Herrlich-Strecker [5].

Acknowledgement. In his seminal dissertation entitled "Categorical Algebraic Constructions" [17] Jan Reiterman developed a theory of algebraized chains, and later he summarized the main ideas in his paper [18]. That paper was a basic inspiration for us: the reader will see that the part concerning endofunctors of **Set** uses ideas of [18] extensively. In particular, algebraized chains play a crucial role in the proof of the Main Theorem in Section 3.

2. Preliminary Facts About Set Functors

2.1. Throughout this section F denotes an endofunctor of **Set**, the category of small sets. We also use the notation

$$\mathbf{Set}^*$$

for the full subcategory of **Set** on all nonempty sets.

2.2. Fact. (See [20] or [9].) For every set functor F there exists a set functor F^* preserving

(i) monomorphisms and

(ii) finite intersections (i.e., pullbacks of monomorphisms)

such that F^* coincides on $\mathbf{Set}^*$ with F.

2.3. Notation. We denote by C_{01} the functor with $\emptyset \longmapsto \emptyset$ and $X \longmapsto 1$ for all nonempty sets X.

A natural transformation $\varrho\colon C_{01} \longrightarrow F$ is called a *distinguished point* of F. In 2.2 we can define F^* by taking as $F^*\emptyset$ the set of all distinguished points (and, for $f\colon \emptyset \longrightarrow X$ defining F^*f by $\varrho \longmapsto \varrho_X$).

2.4. Remark. Recall from the survey paper [19] that in coalgebra one often works with set functors which *preserve weak pullbacks* (i.e. turn pullback squares to squares having the same factorization property except that factorizations need not be unique). We will use a weaker assumption: F *preserve nonempty preimages*, i.e., it preserve pullbacks along a monomorphism in $\mathbf{Set}^*$. The implication

$$\text{weakly preserves pullbacks} \implies \text{preserves nonempty preimages}$$

is easily verified: given a preimage

$$\begin{tikzcd} X_0 \arrow[r, tail, "m"] \arrow[d, "f_0"'] & X \arrow[d, "f"] \\ Y_0 \arrow[r, tail, "n"'] & Y \end{tikzcd}$$

with $X_0 \neq \emptyset$, then m is a split monomorphism, therefore, Fm is a monomorphism. Thus, the F-image of the square, being a weak pullback, is a pullback.

2.5. Examples. (i) Preservation of weak pullbacks is a relatively weak assumption: the collection of all set functors preserving weak pullbacks is closed under

product (in particular power-to-n for any cardinal n)

coproduct, and

composition.

Since that collection contains the identity functor Id and the constant functors, it contains all the *polynomial functors* H_Σ where Σ is a signature: recall that H_Σ is given on objects X by

$$H_\Sigma X = \coprod_{\sigma\in\Sigma} X^n \qquad n = \text{arity of } \sigma$$

and is thus a coproduct of limit-preserving functors.

(ii) Also the *power-set functor* $\mathscr{P}$ weakly preserves pullbacks. Recall that $\mathscr{P}$ is defined on objects X by $\mathscr{P}X = \exp X$ and on morphisms $f\colon X \longrightarrow Y$ by $\mathscr{P}f\colon M \longmapsto f[M]$. The composite functor $\mathscr{P}(\Sigma \times -)$ whose coalgebras are the labelled transition system with action set Σ, see [19], consequently also preserves pullbacks.

(iii) Another example: the *ultrafilter functor* β defined on objects X by

$$\beta X = \text{the set of all ultrafilters on } X$$

and on morphisms $f\colon X \longrightarrow Y$ by

$$\beta f\colon \mathscr{U} \longmapsto \{V \subseteq Y; f^{-1}(V) \in \mathscr{U}\}$$

weakly preserves pullbacks. In fact, let

$$\begin{tikzcd} X \arrow[r, "\bar g"] \arrow[d, "\bar f"'] & Y \arrow[d, "f"] \\ Z \arrow[r, "g"'] & T \end{tikzcd}$$

be a pullback. We are to show that given ultrafilters $\mathscr{U} \in \beta Y$ and $\mathscr{V} \in \beta Z$ with $\beta f(\mathscr{U}) = \beta g(\mathscr{V})$, there exists an ultrafilter $\mathscr{W} \in \beta X$ with $\beta\bar f(\mathscr{W}) = \mathscr{V}$ and $\beta\bar g(\mathscr{W}) = \mathscr{U}$. Observe that the collection of all sets

$$D_{A,B} = \bar f^{-1}(A) \cap \bar g^{-1}(B)$$

with $A \in \mathscr{V}$ and $B \in \mathscr{U}$ fulfilling $f[B] = g[A]$ is closed under finite intersections, and does not contain $\emptyset$. (In fact, choose $C \in \beta f(\mathscr{U})$ with $f^{-1}(C) \subseteq B$ and $g^{-1}(C) \subseteq A$, then $D_{A,B}$ contains the preimage of C.) Therefore, there exists an ultrafilter $\mathscr{W}$ containing all $D_{A,B}$'s. It is easy to verify that $\mathscr{W}$ is mapped to $\mathscr{V}$ by $\beta\bar f$ and to $\mathscr{U}$ by $\beta\bar g$.

2.6. FACT. Preservation of nonempty preimages is even weaker than preservation of weak pullbacks: The collection of all set functors preserving nonempty preimages is closed under

product,

coproduct,

composition, and

subfunctor.

And the collection of all functors preserving weak pullbacks is its a proper subcollection. The proofs of the four closure properties are trivial and we omit them.

Recall from [1] the subfunctor $(-)^3_2$ of the polynomial functor $(-)^3$ assigning to every set X the subset $X^3_2 \subseteq X \times X \times X$ of all triples whose components are not pairwise distinct. Then $(-)^3_2$ does not preserve weak pullbacks, but it does preserve nonempty preimages because $(-)^3$ does.

2.7. Examples. (1) The polynomial functors, the power-set functor $\mathscr{P}$ and the functor $\mathscr{P}(\Sigma \times -)$ preserve weak pullbacks, thus, preserve nonempty preimages.

(2) Our insistence on "nonempty" is inspired by the fact that the implication

$$\text{preserves weak pullbacks} \implies \text{preserves preimages}$$

is false in general. A counterexample is $C_{2,1}$, taking $\emptyset$ to a two-element set and all nonempty sets to 1. This functor preserves weak pullbacks (obviously), but it does not preserve the preimage

$$\begin{tikzcd} \emptyset \arrow[r] \arrow[d] & 1 \arrow[d, "\mathrm{inl}"] \\ 1 \arrow[r, "\mathrm{inr}"'] & 1+1 \end{tikzcd}$$

(where inl or inr denote the left and the right coproduct injections).

(3) A functor not preserving nonempty preimages: let D be the quotient of $(-)^2$ modulo the diagonal. That is, on objects X we have $DX = X \times X/_\sim$ where $\sim$ is the smallest equivalence relation with the diagonal $\triangle \subseteq X \times X$ as one class. For every function $f\colon X \longrightarrow 1$ where X has at least two elements the functor D does not preserve the preimage

$$\begin{tikzcd} X \arrow[r, "\mathrm{inl}_X"] \arrow[d, "f"'] & X+X \arrow[d, "f+f"] \\ 1 \arrow[r, "\mathrm{inl}_1"'] & 1+1 \end{tikzcd}$$

2.8. Remark. (i) Let us call a set functor *preconstant* if its restriction to the category $\mathbf{Set}^*$ of all nonempty sets is naturally isomorphic to a constant functor. It follows from results of V. Koubek [13], see also 3.4.7 in [9], that every set functor F is either preconstant, or there exists a cardinal γ such that for every set X we have:

$$\operatorname{card} X \geq \gamma \qquad \text{implies} \qquad \operatorname{card} FX \geq \operatorname{card} X.$$

(In [9] preconstant functors are called constant.)

(ii) Every set functor F is a coproduct

$$F = \coprod_{s \in S} F_s$$

of indecomposable functors, called the *components* of F. (In fact, put $S = F1$ and for $s \in F1$ let $F_s X \subseteq FX$ be the preimage of s under Ft_X for the unique $t_X \colon X \longrightarrow 1$. See [20] or [9] for more details.)

2.9. Lemma. *Every component of a set functor preserving nonempty preimages is faithful or preconstant.*

Proof. It is easy to see that a set functor preserves nonempty preimages iff every component does. Therefore, we can restrict ourselves to indecomposable set functor F (i.e., with $\operatorname{card} F1 = 1$).

Suppose that F is a non-faithful indecomposable functor preserving nonempty preimages. Every non-faithful set functor has a distinguished point $\varphi \colon C_{01} \longrightarrow F$, see [20] or [9]. Consider the preimage as in 2.7 (3). (We denote by inl_X and inr_X the left and right coproduct injections $X \longrightarrow X + X$.) Since Ff is a constant function, it follows that $F(f+f)$ maps both of the subobjects $F\,\mathrm{inl}_X$ and $F\,\mathrm{inr}_X$ onto the element $\varphi_{1+1} \in F(1+1)$, and so does $F\,\mathrm{inl}_1$. Thus, the preimage of $F\,\mathrm{inl}_1$ under $F(f+f)$ contains $F\,\mathrm{inr}_X$. However, this preimage is $F\,\mathrm{inl}_X$, and since F is indecomposable, the intersection of $F\,\mathrm{inl}_X$ and $F\,\mathrm{inr}_X$ has one element. This proves that FX has one element. Thus, F is preconstant. ∎

2.10. Remark. Faithful components G of any set functor preserve finite intersections, see [9]. In particular, they preserve disjointness of subobjects: if $m_1 \colon M_1 \longrightarrow X$ and $m_2 \colon M_2 \longrightarrow X$ are disjoint subobjects of X, then Gm_1 and Gm_2 are disjoint subobjects of GX.

3. Birkhoff's Variety Theorem in **Set**

3.1. We work here with algebras in the category **Set** (of small sets) whose type is determined by an endofunctor F of that category. That is, an *algebra* is a set A equipped by a mapping $\alpha \colon FA \longrightarrow A$. Given algebras (A, α) and (B, β), a *homomorphism* is a function $f \colon A \longrightarrow B$ such that the square

$$\begin{tikzcd} FA \arrow[r, "\alpha"] \arrow[d, "Ff"'] & A \arrow[d, "f"] \\ FB \arrow[r, "\beta"'] & B \end{tikzcd}$$

commutes. These algebras were first studied by J. Lambek [14]. We denote the category of these algebras by $\mathbf{Alg}\,F$.

3.2. Example. The classical Σ-algebras (for a *signature* Σ, i.e., a set Σ of operation symbols with prescribed arities which are arbitrary cardinals) are captured by the *polynomial functor*, see 2.5.

3.3. Example. The category $\mathbf{Alg}\,\mathscr{P}$ where $\mathscr{P}$ is the power-set functor, see 2.5 (iii), contains e.g. the variety of complete join semilattices as a full subcategory: to every such semilattice A assign the function $\alpha\colon \mathscr{P}A \longrightarrow A$ of formation of joins, then $\mathscr{P}$-homomorphisms are precisely the functions preserving joins.

3.4. Example. As in 2.5 (iii), let β denote the *ultrafilter functor*. Then the category $\mathbf{Alg}\,\beta$ contains e.g. the category of compact Hausdorff spaces as a full subcategory: to every such space A assign the function $\alpha\colon \beta A \longrightarrow A$ taking every ultrafilter to its unique limit point in A. Then β-homomorphisms are precisely the continuous functions (because, among compact Hausdorff spaces, these are precisely the functions preserving limit points of ultrafilters).

3.5. Example. Let F be a preconstant functor. Then $\mathbf{Alg}\,F$ is the category of Σ-algebras for a signature Σ consisting of nullary operation symbols, except that the empty algebra is possibly added.

In fact, a preconstant functor F with $F\emptyset = \emptyset$ has the property that $\operatorname{card} FX = n$, for $X \neq \emptyset$, independent of X. Then $\mathbf{Alg}\,F$ is the category of algebras with n nullary operation symbols plus the empty algebra.

If a preconstant functor F fulfils $F\emptyset \neq \emptyset$ then $\mathbf{Alg}\,F$ contains only nonempty algebras; they are, again, just algebras on n nullary operations for the above n.

3.6. Definition. (See [9].) *An endofunctor F is called a* ***varietor*** *if free F-algebras exist, i.e., if the forgetful functor of $\mathbf{Alg}\,F$ has a left adjoint.*

3.7. Theorem. (See [21] and [9].) *A set functor F is a varietor iff*

(a) *F has arbitrarily large fixed points (i.e., arbitrarily large sets X with $FX \cong X$), or*

(b) *F is preconstant.*

3.8. Example. H_Σ is a varietor, but neither $\mathscr{P}$ nor β are.

3.9. Remark. (i) Let F be a varietor. By an *equation* is understood a pair (u, v) of elements in a free F-algebra $\phi(X)$ on set X (of variables). Notation: $u = v$.

An F-algebra A *satisfies* the equation $u = v$ if for every map $f\colon X \longrightarrow A$ ("interpretation" of the variables in A) the corresponding homomorphism $f^\#\colon \phi(X) \longrightarrow A$ fulfills $f^\#(u) = f^\#(v)$.

(ii) What do we mean by equations in case F is not a varietor? For a general endofunctor F a transfinite chain "approximating" the free algebra $\phi(X)$ has been defined in [2] (see also [9]). We denote this chain by

$$W(X)\colon \mathbf{Ord} \longrightarrow \mathbf{Set}$$

and call it the *free-algebra chain*. Its objects $W_i(X)$ and connecting morphisms $w_{i,j}\colon W_i(X) \longrightarrow W_j(X)$ (for $i \le j$ in **Ord**) are determined uniquely up-to natural isomorphism by the following rules:

a) $W_0(X) = X$ and $W_1(X) = X + FX$; $w_{0,1}\colon X \longrightarrow X + FX$ is the coproduct injection,

b) $W_{j+1}(X) = X + FW_j(X)$ and $w_{i+1,j+1} = X + Fw_{i,j}$ for all ordinals $i \le j$,

and

c) $W_j(X) = \mathrm{colim}_{i<j}\, W_i(X)$ with the colimit cocone $(w_{i,j})_{i<j}$ for all limit ordinals j.

(iii) Given an F-algebra $FA \xrightarrow{\alpha} A$ and an interpretation $f\colon X \longrightarrow A$ of variables, we denote by $f_i^\#\colon W_i(X) \longrightarrow A$ the unique cocone with

$$f_0^\# = f \qquad \text{and} \qquad f_{i+1}^\# = [f, \alpha \cdot F f_i^\#]\colon X + FW_i(X) \longrightarrow A.$$

(iv) By an *equation* $u = v$ is meant a pair $u, v \in W_i(X)$ for a set X of variables and an ordinal i.

The equation is called *trivial* if u and v is the same element of $W_i(X)$. And it is called *reduced* if it is non-trivial, and u or v lies in the complement of $\bigcup_{j<i} w_{j,i}[W_j(X)]$.

3.10. Definition. *We say that an algebra A **satisfies** an equation $u = v$ ($u, v \in W_i(X)$) provided that for every map $f\colon X \longrightarrow A$ we have*

$$f_i^\#(u) = f_i^\#(v).$$

*By a **variety** of F-algebras is meant a full subcategory $\mathscr{V}$ of* $\mathbf{Alg}\,F$ *which is equationally presentable (i.e., for which a class of equations exists such that $\mathscr{V}$ consists of precisely all F-algebras satisfying each of the given equations).*

3.11. Remark. Every non-trivial equation $u = v$, $u, v \in W_i(X)$ determines a reduced equation $u' = v'$ logically equivalent to it; an algebra satisfies $u = v$ iff it satisfies $u' = v'$. In fact, find the smallest j such that $w_{j,i}(u') = u$ and $w_{j,i}(v') = v$ for some $u', v' \in W_j(X)$. Then $u' = v'$ is reduced.

3.12. Remark. As proved in [21], for every varietor a free algebra on X always has the form $W_i(X)$ for some ordinal i. Thus, the above concept of equation is, for varietors, just the usual one.

3.13. Example. Complete lattices. These can be canonically represented as a full subcategory of $\mathbf{Alg}(\mathscr{P} + \mathscr{P})$: to every complete lattice A assign the function $[\mathrm{join}, \mathrm{meet}]\colon \mathscr{P}A + \mathscr{P}A \longrightarrow A$. Then homomorphisms in $\mathbf{Alg}(\mathscr{P} + \mathscr{P})$ are precisely the functions preserving joins and meets.

This full subcategory is a variety of $\mathbf{Alg}(\mathscr{P} + \mathscr{P})$. For example, the axiom $\mathrm{join}\{x\} = x$ is expressed by forming, for $X = \{x\}$, the elements $u = x$ and $v = (\{x\}, 1)$ in $W_1(X) =$

$X+\mathscr{P}X\times\{1,2\}$ (where $\mathscr{P}X\times\{1,2\}$ represents $\mathscr{P}X+\mathscr{P}X$) and considering an equation $u=v$. Analogously for all other axioms expressing the fact that an abstract algebra $\alpha\colon \mathscr{P}A+\mathscr{P}A \longrightarrow A$ has the form $\alpha=[\mathrm{join},\mathrm{meet}]$ for a complete lattice on A. In this case we will need a proper class of equations.

3.14. Example. Monadic algebras. For every monad (T,η,μ) the category of Eilenberg-Moore algebras is a variety of $\mathbf{Alg}\,T$. Conversely, whenever a variety of F-algebras has free algebras, then it is monadic. See [8] (where the terminology was slightly different: the concept of variety was called equational subcategory there).

3.15. Remark. (i) The above Definition 3.10 was introduced in [3].

(ii) Jan Reiterman uses (in [18]) algebraized chains, rather than the free-algebra construction, to define varieties (which he calls "equational varieties"). For the present case where the base category is **Set** he remarks (see his 3.3) that his concept is obviously equivalent to that of 3.10 above.

(iii) Whenever a free F-algebra $X^{\#}$ exists, then the free-algebra construction *stops*, i.e., $w_{i,i+1}\colon W_i \longrightarrow X+FW_i$ is an isomorphism for some i, and $X^{\#}=W_i$ (where the universal arrow $X\longrightarrow X^{\#}$ and the algebra structure $FX^{\#}\longrightarrow X^{\#}$ form the components of $w_{i,i+1}^{-1}$).

Conversely, whenever the free-algebra construction stops after i steps, then it yields a free algebra $X^{\#}=W_i$. See [21] or also [9].

3.16. Lemma. *Every variety of F-algebras is closed in* $\mathbf{Alg}\,F$ *under regular quotients, subobjects and products.*

The proof is trivial.

3.17. Definition. *We say that a set functor F* ***satisfies Birkhoff's Variety Theorem*** *if varieties of F-algebras are precisely the full subcategories of* $\mathbf{Alg}\,F$ *closed under regular quotients, subobjects, and products.*

3.18. Examples. (See [18].)

(i) Every varietor in **Set** satisfies Birkhoff's Variety Theorem.

(ii) The power-set functor does not satisfy Birkhoff's Variety Theorem. In fact, a counter-example is the class of all $\mathscr{P}$-algebras A for which there exists a cardinal α such that A satisfies $u=v$ whenever u,v lie in $W_i(X)$ but not in $W_\alpha(X)$ for some $i>\alpha$. We use the same idea in the proof of the Main Theorem.

3.19. Example. For every small set $\mathscr{A}$ of F-algebras the closure $\overline{\mathscr{A}}$ of $\mathscr{A}$ under products, subalgebras and regular quotient algebras is a variety.

To prove this, let $\mathscr{A}'$ be a closure of $\mathscr{A}$ under limits and subalgebras. Then the canonical forgetful functor $U\colon \mathscr{A}'\longrightarrow \mathbf{Set}$ is a right adjoint. In fact, the category of F-algebras has small limits and large intersections created by its forgetful functor. Therefore, $\mathscr{A}'$ has limits and large intersections, and U preserves them. Since $\mathscr{A}$ is a small cogenerator of $\mathscr{A}'$, the Special Adjoint Functor Theorem guarantees a left adjoint $\Phi_{\mathscr{A}}$ of U. Now $\overline{\mathscr{A}}$ is the closure of $\mathscr{A}'$ under regular quotient algebras, which implies that the algebras $\Phi_{\mathscr{A}} X$ are free on X in $\overline{\mathscr{A}}$ (recall that epimorphisms split in **Set**). By Theorem 4.4 in [18] we conclude that $\overline{\mathscr{A}}$ is a variety.

3.20. Example. All compact Hausdorff spaces, see 3.4, have the property that the *trivial ultrafilter* $\dot{x}$ (of all sets containing x) converges to x. Thus, they satisfy the equation

$$x = \dot{x} \qquad \text{for } x, \dot{x} \in W_1(X) = X + \beta X.$$

In fact, compact Hausdorff spaces form a variety of β-algebras. This follows from the fact, established by E. Manes [16], that they are Eilenberg-Moore algebras of the well-known monad whose underlying functor is β, see 3.14.

3.21. Lemma. *For every set functor and every set $X \neq \emptyset$ of variables the connecting morphisms $w_{i,j} \colon W_i(X) \longrightarrow W_j(X)$ are always monomorphisms.*

Proof. This follows easily by transfinite induction: in the isolated step

$$w_{i+1,j+1} = \mathrm{id}_X + F w_{i,j}$$

use the fact that $F w_{i,j}$ is a monomorphism since $w_{i,j}$ is a split monomorphism (in fact, $X \neq \emptyset$ implies $W_i(X) \neq \emptyset$), and coproducts of monomorphisms in **Set** are monomorphisms. In the limit steps use the fact that

(a) a colimit cone $w_{i,j} \colon W_i(X) \longrightarrow W_j(X)$ (for $i < j$) of a chain of monomorphisms in **Set** is formed by monomorphisms, and

(b) the connecting morphism $w_{j,j+1} \colon W_j(X) \longrightarrow W_{j+1}(X)$ of a cone

$$W_{i+1}(X) = X + FW_i(X) \xhookrightarrow{\mathrm{id}_X + Fw_{i,j}} X + FW_j(X) = W_{j+1}(X)$$

of monomorphisms is a monomorphism. ■

3.22. Observation. (i) The free-algebra construction $W_i(X)$ of 3.9 yields a chain of endofunctors W_i on **Set**. The definition of W_i on morphisms $y \colon Y \longrightarrow X$ is an obvious transfinite induction: $W_0(y) = y$ and $W_{i+1}(y) = y + FW_i(y)$.

(ii) Given an algebra A and two functions

$$Y \xrightarrow{y} X \xrightarrow{f} A,$$

then for every ordinal i we have

$$f_i^{\#} \cdot W_i(y) = (fy)_i^{\#} \colon W_i(Y) \longrightarrow A.$$

This is trivial to prove by transfinite induction.

In the isolated step we use the triangle

$$\begin{tikzcd}
Y + FW_i(Y) \arrow[rr, "y+FW_i(y)"] \arrow[dr, "{[fy,\alpha\cdot F(fy)_i^{\#}]}"'] & & X + FW_i(X) \arrow[dl, "{[f,\alpha\cdot Ff_i^{\#}]}"] \\
& A &
\end{tikzcd}$$

which commutes by the induction hypothesis. Limit steps just use the uniqueness of factorizations through colimit cones.

3.23. Main Theorem. *For a set functor which preserves nonempty preimages the following conditions are equivalent:*

(i) *Birkhoff's Variety Theorem holds for F-algebras, and*

(ii) *F is a varietor.*

Proof. Let F preserve nonempty preimages. If it is not a varietor, then we find a class $\mathscr{V}$ of F-algebras which is not a variety although it is closed under regular quotients, subalgebras, and products.

Given a cardinal α, we choose a set X of cardinality α and denote, for every $i > \alpha$, by $\sim_i$ the equivalence relation on $W_i(X)$ given by

$$t \sim_i s \qquad \text{iff} \qquad t = s \text{ or } t, s \notin w_{\alpha,i}\big[W_\alpha(X)\big].$$

Denote by $\mathscr{V}_\alpha$ the variety of F-algebras presented by all equations

$$u = v \qquad \text{for} \qquad u, v \in W_i(X) \text{ with } i > \alpha \text{ and } u \sim_i v.$$

Observe that $\alpha \le \beta$ implies $\mathscr{V}_\alpha \subseteq \mathscr{V}_\beta$. Consequently, since each $\mathscr{V}_\alpha$ is closed under regular quotients, subalgebras and products, so is the union

$$\mathscr{V} = \bigcup_{\alpha \in \mathbf{Card}} \mathscr{V}_\alpha.$$

We now prove that $\mathscr{V}$ is not a variety by verifying that

(a) for every nontrivial equation $u = v$ there exists an algebra in $\mathscr{V}$ which does not satisfy $u = v$

and

(b) $\mathscr{V}$ is a proper subcategory of $\mathbf{Alg}\, F$.

Proof of (a).

For every cardinal $\alpha > 0$ choose a set X of cardinality α and denote by $(W(X), p)$ the free algebra construction considered as an algebraized chain in the sense of [18], i.e., p is the collection $p_i\colon FW_i(X) \longrightarrow W_{i+1}(X) = X + FW_i(X)$ of the right coproduct injections. Define a quotient functor

$$e\colon W(X) \longrightarrow \overline{W}(X)$$

of this functor $W(X)\colon \mathbf{Ord} \longrightarrow \mathbf{Set}$ by the following rule: given an ordinal i then $e_i\colon W_i(X) \longrightarrow W_i(X)/_{\sim_i}$ is the canonical quotient map of the above equivalence $\sim_i$ on $W_i(X)$. It is obvious that this defines a functor $\overline{W}(X)\colon \mathbf{Ord} \longrightarrow \mathbf{Set}$ and a natural transformation $e\colon W(X) \longrightarrow \overline{W}(X)$. We will prove that this is a quotient object of the algebraized chain $(W(X), p)$, i.e., that for every $i \in \mathbf{Ord}$ there exists $\bar{p}_i$ such that the square

$$\begin{tikzcd}
FW_i(X) \arrow[r, "p_i"] \arrow[d, "Fe_i"'] & X + FW_i(X) = W_{i+1}(X) \arrow[d, "e_{i+1}"] \\
F\overline{W}_i(X) \arrow[r, "\bar{p}_i"'] & \overline{W}_{i+1}(X)
\end{tikzcd}$$

commutes. Here it is where our assumption that F preserves nonempty preimages is applied. For $i \leq \alpha$ there is nothing to prove; also all $i > \alpha$ with $W_i(X) = w_{\alpha,i}[W_\alpha(X)]$ are obvious: in this case, e_i is a bijection hence Fe_i is also a bijection. In the remaining case, i.e. $i > \alpha$ and $W_i(X) \supsetneq w_{\alpha,i}[W_\alpha(X)]$, we have a canonical isomorphism $r\colon W_\alpha(X) + 1 \longrightarrow \overline{W}_i(X)$ such that the square

$$\begin{tikzcd}
W_\alpha(X) \arrow[r, "w_{\alpha,i}"] \arrow[d, "\mathrm{inl}"'] & W_i(X) \arrow[d, "e_i"] \\
W_\alpha(X) + 1 \arrow[r, "r"'] & \overline{W_i(X)}
\end{tikzcd}$$

commutes (where inl denotes the left coproduct injection). Here we can observe that $W_\alpha(X) \neq \emptyset$ (because $\alpha > 0$ and X has cardinality α) and the above square is a preimage. Consequently, the F-image of that square is a preimage. This clearly implies that whenever Fe_i merges two distinct elements of $FW_i(X)$, then they both lie outside $Fw_{\alpha,i}[FW_\alpha(X)]$, i.e., as elements of $W_{i+1}(X)$ they are equivalent (under $\sim_{i+1}$). Consequently, $e_{i+1}{\cdot}p_i$ merges such elements too. This proves the existence of $\bar{p}_i$ above.

Thus, we define an algebraized chain $(\overline{W}(X), \bar{p})$ which is a quotient of the free-algebra chain, and is stationary (i.e., the connecting morphisms of $\overline{W}(X)$ are, from some ordinal on, isomorphisms). From 3.9 in [18] we conclude that the chain $(\overline{W}(X), \bar{p})$ has a reflection A^α in the full category $\mathbf{Alg}\,F$. This F-algebra A^α lies in the variety $\mathscr{V}_\alpha$ and does not satisfy any non-trivial equation $u = v$ with $u, v \in W_i(Y)$ for any set Y with $\operatorname{card} Y \leq \alpha$

and any ordinal $i \leq \alpha$. See the detailed argument of Example 4.6 in [18] (formulated for the power-set functor, but valid generally). Since α was any cardinal, this proves the claim (a).

Proof of (b).

We construct a set D and an F-algebra on it, denoted also by D, such that for every ordinal α there exists an ordinal $i > \alpha$ and elements

$$t_1, t_2 \in W_{i+1}(D) - w_{\alpha,i+1}\big[W_\alpha(D)\big] \tag{1}$$

such that

$$D \text{ does not satisfy } t_1 = t_2, \tag{2}$$

This proves $D \notin \mathscr{V}_\alpha$. Consequently, D lies in $\mathbf{Alg}\, F - \mathscr{V}$.

From Lemma 2.9 we know that $F = G + H$ where

G has all components faithful

and

H is preconstant.

It follows that

$$G \text{ preserves disjointness of subobjects,} \tag{3}$$

see Remark 2.10. Next, there exists an infinite cardinal γ such that for every set X we have

$$\operatorname{card} X \geq \gamma \qquad \text{implies} \qquad \operatorname{card} GX > \operatorname{card} X. \tag{4}$$

In fact, since H is preconstant and F is not a varietor, it follows from Theorem 3.7 that G is not a varietor; i.e., it does not have arbitrarily large fixed points. On the other hand, since G is faithful, for every set $X \neq \emptyset$ we have $\operatorname{card} GX \geq \operatorname{card} X$, see [9]—thus, (4) follows. Since H is preconstant, we can choose the above γ so that

$$\operatorname{card} X \geq \gamma \qquad \text{implies} \qquad \operatorname{card} FX = \operatorname{card} GX. \tag{5}$$

Moreover, from (4) we conclude that

$$\operatorname{card} X \geq \gamma \qquad \text{implies} \qquad \operatorname{card} W_i(X) < \operatorname{card} W_{i+1}(X) \text{ for all } i \in \mathbf{Ord}. \tag{6}$$

In fact, we have $W_{i+1}(X) = X + FW_i(X)$ and we can apply (4) to $W_i(X)$ in place of X (since by 3.21 $\operatorname{card} W_i(X) \geq \operatorname{card} X \geq \gamma$). Finally observe that 3.21 and (4), (5) yield that

$$\operatorname{card} X \geq \gamma \qquad \text{implies} \qquad \operatorname{card} W_{i+1}(X) = \operatorname{card} FW_i(X). \tag{7}$$

Choose a set Z of cardinality γ. We define an F-algebra on the coproduct

$$D = Z + Z \qquad \text{with injections} \qquad u_l, u_r \colon Z \longrightarrow D$$

for which we prove that the identity function $f = \mathrm{id}_D$ has for every ordinal i the following property:

$$\text{the preimages of } u_l \text{ and } u_r \text{ under } f^{\#}_{i+1}\colon W_{i+1}(D) \longrightarrow D \text{ have cardinalities at least } \operatorname{card} W_{i+1}(D). \tag{8}$$

We then derive $D \notin \mathscr{V}_\alpha$ by choosing t_1 in the preimage of u_l under $f^{\#}_{i+1}$ and t_2 in the preimage of u_r. It follows that $f^{\#}_{i+1}$ does not merge t_1 and t_2, consequently, (2) follows. And from (6) we derive that $\operatorname{card} W_{i+1}(D) > \operatorname{card} W_\alpha(D)$, therefore we can choose t_1, t_2 so that (1) holds.

Due to (3) there exists an algebra structure $\delta\colon FD \longrightarrow D$ for which the following diagram

$$\begin{tikzcd}
GZ \arrow[r, "Gu_l"] \arrow[dd, "v_l"'] & GD \arrow[d, "\mathrm{inl}"] & GZ \arrow[l, "Gu_r"'] \arrow[dd, "v_r"] \\
 & FD \arrow[d, "\delta"] & \\
Z \arrow[r, "u_l"'] & D & Z \arrow[l, "u_r"]
\end{tikzcd}$$

commutes, where v_l, v_r are arbitrarily chosen and inl denotes the left coproduct injection $GD \longrightarrow FD = GD + HD$. This algebra fulfils (8) for all ordinals i: The proof for u_l will be performed by transfinite induction on i. The proof for u_r follows by symmetry.

First step: this follows from our choice of Z and of $f = \mathrm{id}$ above.

Isolated step: we have, by induction hypothesis, subobjects

$$m_l\colon M_l \longrightarrow W_i(D) \qquad \text{and} \qquad m_r\colon M_r \longrightarrow W_i(D)$$

such that $f^{\#}_i \cdot m_l$ factorizes through u_l and $f^{\#}_i \cdot m_r$ through u_r:

$$\begin{tikzcd}
M_l \arrow[r, "m_l"] \arrow[d, "p_l"'] & W_i(D) \arrow[d, "f^{\#}_i"] & M_r \arrow[l, "m_r"'] \arrow[d, "p_r"] \\
Z \arrow[r, "u_l"'] & D & Z \arrow[l, "u_r"]
\end{tikzcd}$$

and $\operatorname{card} M_l = \operatorname{card} M_r = \operatorname{card} W_i(D)$. From this we prove that the preimage of u_l under $f^{\#}_{i+1}$ has cardinality $\operatorname{card} W_{i+1}(D)$, analogously for u_r. In fact, consider the following subobject of $W_{i+1}(D)$

$$GM_l \xrightarrow{Gm_l} GW_i(D) \xrightarrow{\mathrm{inl}} FW_i(D) \xrightarrow{\mathrm{inr}} W_{i+1}(D)$$

(with inl or inr being the left or the right coproduct injections) whose cardinality is, due to (5) and (7),

$$\operatorname{card} GM_l = \operatorname{card} FM_l = \operatorname{card} FW_i(D) = \operatorname{card} W_{i+1}(D).$$

This subobject when composed with $f^{\#}_{i+1} = [f, \delta{\cdot}Ff^{\#}]$ factorizes through u_l:

$$
\begin{tikzcd}
GM_l \arrow[r, "Gm_l"] \arrow[d, "Gp_l"'] & GW_i(D) \arrow[r, "\mathrm{inl}"] \arrow[d, "Gf^{\#}_i"'] & FW_i(D) \arrow[r, "\mathrm{inr}"] \arrow[d, "Ff^{\#}_i"] & W_{i+1}(D) \arrow[dd, "f^{\#}_{i+1}"] \\
GZ \arrow[r, "Gu_l"'] \arrow[d, "v_l"'] & GD \arrow[r, "\mathrm{inl}"'] & FD \arrow[dr, "\delta"'] & \\
Z \arrow[rrr, "u_l"'] & & & D
\end{tikzcd}
$$

Limit step: for every limit ordinal i the preimage of u_l under $f^{\#}_i = \operatorname{colim}_{j<i} f^{\#}_j$ is a colimit of the subchain of $W_j(D)$ $(j < i)$ formed by the preimages of u_l under $f^{\#}_j$ $(j < i)$ — in fact, pullbacks commute with chain colimits in **Set**. The cardinality of the colimit of all preimages of u_l is, by induction hypothesis, the supremum of $\operatorname{card} W_j(D)$ for $j < i$, and this is precisely $\operatorname{card} W_i(D)$. ∎

3.24. Corollary. *For a functor $F\colon$ **Set** $\longrightarrow$ **Set** preserving nonempty preimages the following conditions are equivalent*:

(i) *F is a varietor*,

(ii) *Birkhoff's Variety Theorem holds for F-algebras*,

(iii) *the category of F-algebras is cocomplete*,

(iv) *F has arbitrarily large fixed points.*

In fact, (ii) $\longrightarrow$ (i) by Theorem 3.23, (i) $\longrightarrow$ (ii) due to Theorem 4.4 in [18], the equivalence (i) $\longleftrightarrow$ (iii) was proved in [6] and (i) $\longleftrightarrow$ (iv) in [21].

3.25. Example. We present two endofunctors F_1 and F_2 of **Set** such that

(i) F_1 and F_2 are varietors

therefore, Birkhoff's Variety Theorem holds for F_1 and F_2, but

(ii) $F_1 + F_2$ is not a varietor and does not satisfy Birkhoff's Variety Theorem.

This example, based on ideas of Václav Koubek (private communication) was already presented in [9], Example 4.4.4, but there we only mentioned the fact that $F_1 + F_2$ is not a varietor.

We begin by choosing a class $C_1 \subseteq$ **Card** of cardinal numbers such that both C_1 and $C_2 =$ **Card** $- C_1$ have the property that, for $k = 1$ or 2,

$$\text{there exist arbitrarily large cardinal } \beta \text{ with } (\beta, 2^\beta] \subseteq C_k \qquad (*)$$

(where $(\beta, 2^\beta]$ denotes the cardinal interval of all $\alpha \in \mathbf{Card}$ with $\beta < \alpha \leq 2^\beta$). The class C_1 can be defined as $C_1 = \bigcup_{i \in \mathbf{Ord}} D_i$ where $D_0 = (\aleph_0, 2^{\aleph_0}]$, $D_{i+1} = D_i \cup (2^\delta, 2^{2^\delta}]$ for $\delta = \sup D_i$, and $D_i = \bigcup_{j<i} D_j$ for limit ordinals i. We then define a functor F_k ($k = 1$ or 2) on objects X by

$$F_k X = \{M \subseteq X; M = \emptyset \text{ or } \operatorname{card} M \in C_k\}$$

and on morphisms $f\colon X \longrightarrow Y$ by

$$F_k f\colon M \longmapsto \begin{cases} f[M] & \text{if } f \text{ restricted to } M \text{ is monic} \\ \emptyset & \text{else} \end{cases}$$

This is well defined: F_k preserves composition and identity maps obviously. Observe that F_1 is a varietor by 3.7: whenever C_2 contains $(\beta, 2^\beta]$ with β infinite, then a set X of cardinality 2^β is a fixed point of F_1: for every cardinal $\alpha \leq \beta$ we have $\operatorname{card} X^\alpha = (2^\beta)^\alpha = 2^\beta = \operatorname{card} X$, thus, $F_1 X \cong X$. By symmetry, F_2 is a varietor. But $F_1 + F_2$ is not a varietor since it has no fixed point: for every set X with $\operatorname{card} X \in C_k$ we have $\operatorname{card} F_k X \geq \operatorname{card} X^X > \operatorname{card} X$.

3.26. Proposition. *The functor* $F_1 + F_2$ *in* Example 3.25 *does not satisfy Birkhoff's Variety Theorem.*

Proof. This is a modification of the proof of Theorem 3.23 in which we have to overcome the difficulty that $F = F_1 + F_2$ does not preserve nonempty preimages.

Modification of the proof of (a). The only argument in the proof of 3.23 (a) using preservation of preimages concerned the existence of $\bar{p}_i\colon F\overline{W}_i(X) \longrightarrow \overline{W}_{i+1}(X)$. This is clear in our case: given distinct elements $a, b \in FW_i(X)$ with $Fe_i(a) = Fe_i(b)$, we verify directly that $e_{i+1} \cdot p_i$ also merges a, b. Suppose, e.g., that $a, b \in F_1 W_i(X)$ (these elements clearly lie in the same component of F).

(i) Let $F_1 e_i(a) = \emptyset$. Then $F_1 e_i(b) = \emptyset$ which means that a, b are either empty or contain distinct elements equivalent under $\sim_i$. In the latter case the image under p_i contains elements equivalent under $\sim_{i+1}$, thus, $e_{i+1} \cdot p_i$ merges a and b.

(ii) Let $F_1 e_i(a) \neq \emptyset$. Then a contains no pair of distinct elements equivalent under $\sim_i$, and the same holds for b. Therefore a and b have the same intersection with $w_{\alpha,i}[W_\alpha(X)]$ and at most one element is outside $w_{\alpha,i}[W_\alpha(X)]$. Moreover, from $F_1 e_i(a) = F_1 e_i(b)$ it follows that if one of the sets a, b is a subset of $w_{\alpha,i}[W_\alpha(X)]$ then so is the other one. This implies that $e_{i+1} \cdot p_i$ merges a and b.

Modification of the proof of (b). Choose a cardinal γ such that for every set X and every $k = 1, 2$ we have

$$\operatorname{card} X \geq \gamma \qquad \text{implies} \qquad \operatorname{card} F_k X \geq \operatorname{card} X.$$

Define an algebra on $D = Z + Z$ (with $\operatorname{card} Z = \gamma$ and with injections u_l and u_r) so that the diagram

$$\begin{tikzcd}
F_1D \arrow[r, "\mathrm{inl}"] \arrow[d, "v_l"'] & F_1D + F_2D \arrow[d, "\delta"] & F_2D \arrow[l, "\mathrm{inr}"'] \arrow[d, "v_r"] \\
Z \arrow[r, "u_l"'] & D & Z \arrow[l, "u_r"]
\end{tikzcd}$$

commutes. We will prove that $f = \mathrm{id}_D$ has the analogous property to (8) in 3.23:

the preimages of u_l and u_r under $f^{\#}_{i+2}\colon W_{i+2}(D) \longrightarrow D$ have cardinalities at least $\operatorname{card} W_{i+1}(D)$. (9)

This implies $D \notin \mathscr{V}$ as in the proof in 3.23.

For every ordinal i, recalling $W_{i+1}(D) = D + FW_i(D)$, we have the following subobject of $W_{i+2}(D)$:

$$F_1FW_i(D) \xrightarrow{F_1\,\mathrm{inr}} F_1W_{i+1}(D) \xrightarrow{\mathrm{inl}} FW_{i+1}(D) \xrightarrow{\mathrm{inr}} W_{i+2}(D).$$

This subobject is contained in the preimage of u_l under $f^{\#}_{i+2}$ because its composite with

$$f^{\#}_{i+2} = \left[f, \delta\cdot Ff^{\#}_{i+1}\right]$$

factorizes through u_l:

$$\begin{tikzcd}
F_1FW_i(D) \arrow[r, "F_1\,\mathrm{inr}"] \arrow[d, "F_1Ff^{\#}_i"'] & F_1W_{i+1}(D) \arrow[r, "\mathrm{inl}"] \arrow[d, "F_1f^{\#}_{i+1}"'] & FW_{i+1}(D) \arrow[r, "\mathrm{inr}"] \arrow[d, "Ff^{\#}_{i+1}"] & W_{i+2}(D) \arrow[dd, "f^{\#}_{i+2}"] \\
F_1FD \arrow[r, "F_1\delta"'] & F_1D \arrow[r, "\mathrm{inl}"'] \arrow[d, "v_l"'] & FD \arrow[dr, "\delta"'] & \\
& Z \arrow[rr, "u_l"'] & & D
\end{tikzcd}$$

Consequently, the preimage of u_l has cardinality at least

$$\operatorname{card} F_1FW_i(D) \geq \operatorname{card} FW_i(D) = \operatorname{card} W_{i+1}(D).$$

In fact, the first inequality follows from the above choice of γ and X, since, obviously,

$$\operatorname{card} FW_i(D) \geq \gamma.$$

The proof that preimage of u_r has cardinality at least $\operatorname{card} W_{i+1}(D)$ follows by symmetry. ∎

3.27. Example. A modification $\bar{\beta}$ of the ultrafilter functor, see 2.5 (iii), which

(a) is not a varietor, but

(b) fulfils Birkhoff's Variety Theorem.

Recall that $\dot{x}$ denotes the trivial ultrafilter of all sets containing x. Define $\bar{\beta}$ on objects by

$$\bar{\beta}X = \beta X/{\sim}$$

where $\sim$ is the least equivalence relation merging all trivial ultrafilters. Its equivalence class $\{\dot{x}; x \in X\}$ is denoted by

$$b \in \bar{\beta}X.$$

For a function $f\colon X \longrightarrow Y$ define $\bar{\beta}f$ by

$$\bar{\beta}f(b) = b$$

and given a nontrivial ultrafilter $\mathscr{U} \in \beta X$ then

$$\bar{\beta}f(\mathscr{U}) = \begin{cases} \beta f(\mathscr{U}) & \text{if } f\cdot m \text{ is monic for some member } M \xhookrightarrow{m} X \text{ of } \mathscr{U} \\ b & \text{else} \end{cases}$$

It is clear that since βX has cardinality larger than X for infinite sets X, we have

$$\operatorname{card} \bar{\beta}X > \operatorname{card} X \qquad \text{for } X \text{ infinite.}$$

Thus, $\bar{\beta}$ is not a varietor.

3.28. Proposition. *$\bar{\beta}$ fulfils Birkhoff's Variety Theorem.*

Proof. (1) Let us prove that for every $\bar{\beta}$-algebra (A, α) and the smallest infinite cardinal $c_A > \operatorname{card} A$ we have that A satisfies all reduced (see 3.9 (iv)) equations $t = b$ in $W_i(A)$ with $i \geq c_A$.

We proceed by induction in i, where both the first step ($i = c_A$) and every limit step are trivial by default: given a reduced equation in $W_i(A)$ the ordinal i is not a limit ordinal.

Isolated step: We assume that A satisfies every reduced equation $s = b$ where $s \in W_j(A)$ for some j with $c_A \leq j \leq i$. Then we prove that A satisfies every reduced equation

$$t = b \qquad \text{with } t \in W_{i+1}(A).$$

Assuming the contrary, i.e.,

$$f^{\#}_{i+1}(t) \neq \alpha(b) \qquad \text{for some } f\colon A \longrightarrow A$$

we derive a contradiction. Since $t = b$ is reduced, $t \in A + \bar{\beta}[W_i(A)]$ lies in the right-hand summand, i.e., t is an ultrafilter on $W_i(A)$. Since $f^{\#}_{i+1}(t) \neq \alpha(b)$ there exists

$$m\colon M \hookrightarrow W_i(A) \qquad \text{a member of } t \text{ such that } f^{\#}_i{\cdot}m \text{ is monic.}$$

In particular,

$$\operatorname{card} M \leq \operatorname{card} A < c_A.$$

Moreover, without loss of generality we can assume

$$f^{\#}_i(s) \neq \alpha(b) \qquad \text{for all } s \in M$$

(since $f^{\#}_i{\cdot}m$ is monic of M, and we can substitute M by $M - \{s_0\}$ for any $s_0 \in M$: the ultrafilter t is non-trivial, thus, with M it contains $M - \{s_0\}$). By induction hypothesis none of the equations $s = b$ (with $s \in M$) is reduced in $W_i(A)$. Let $s' = b$ be a logically equivalent reduced equation in $W_{i(s)}(A)$ for some ordinal $i(s) < i$, see Remark in 3.10. From $f^{\#}_i(s) \neq \alpha(b)$ it follows that A does not satisfy $s' = b$ which, by induction hypothesis, implies

$$i(s) < c_A.$$

Moreover, $\operatorname{card} M < c_A$, thus, there exists $j < c_A$ which is an upper bound of all $i(s)$—that is,

$$M \subseteq w_{j,i}\big[W_j(A)\big].$$

Denote by $\bar{t}$ the ultrafilter of all preimages of the members of t under $w_{j,i}$. Since $w_{j,i}$ is a monomorphism, see Lemma 3.21, we get

$$t = \bar{\beta} w_{j,i}(\bar{t}) = w_{j+1,i+1}(\bar{t}).$$

Since $j < c_A$, this contradicts to $t = b$ being reduced.

(2) For every class $\mathscr{A}$ of $\bar{\beta}$-algebras denote by $E(\mathscr{A})$ the class of all reduced equations satisfied by every algebra of $\mathscr{A}$. Assuming that $\mathscr{A}$ is closed under limits, subalgebras and regular quotient algebras, we prove that $\mathscr{A}$ is presented by $E(\mathscr{A})$. Express $\mathscr{A}$ as a union of an increasing chain of small sets

$$\mathscr{A} = \bigcup_{j \in \mathbf{Ord}} \mathscr{A}_j.$$

Each $\mathscr{A}_j$ generates the variety $\overline{\mathscr{A}}_j$ of Example 3.19. Given an algebra B satisfying all equations which hold for all algebras of $\mathscr{A}$:

$$E(B) \supseteq E(\mathscr{A}) = \bigcap_{j \in \mathbf{Ord}} E(\mathscr{A}_j)$$

we prove $B \in \mathscr{A}$. Then $\mathscr{A}$ is a variety.

Denote by Ω the small set of all reduced equations in $W_r(B)$ for all $r < c_B$ (see (1) above). Since $\Omega - E(\mathscr{A}_j)$ is an increasing chain with

$$\Omega - E(B) \subseteq \Omega - E(\mathscr{A}) = \bigcup_{j \in \mathbf{Ord}} \Omega - E(\mathscr{A}_j)$$

and Ω is small, there exists an ordinal k with $\Omega - E(B) \subseteq \Omega - E(\mathscr{A}_k)$. It follows that for every reduced equation $t = b$ which holds in $\mathscr{A}_k$ we have: $t = b$ holds in B. (If $t = b$ lies in Ω, this is obvious from $\Omega - E(B) \subseteq \Omega - E(\mathscr{A}_k)$, if $t = b$ does not lie in Ω, apply (1) above and 3.22.) Consequently, also all non-reduced equations $t = b$ holding in $\mathscr{A}_k$ hold in B, see the Remark in 3.10.

The set $\mathscr{A}_k$ is small, thus we can choose an infinite cardinal

$$d > \operatorname{card} A \qquad \text{for every algebra } A \in \mathscr{A}_k.$$

Given a reduced equation

$$t = s \text{ in } W_i(A) \text{ lying in } E(\mathscr{A}_k) - E(B) \tag{+}$$

(i.e., such that B does not satisfy it but algebras of $\mathscr{A}_k$ do), we conclude that

$$i < d.$$

In fact, for $i \geq d$ assume, without loss of generality, that t is reduced in $W_i(A) - w_{j,i}[W_j(A)]$ for all $j < i$—then $t = b$ holds in $\mathscr{A}_k$, see (1) above. From (+) we conclude that $b = s$ holds in $\mathscr{A}_k$. By our choice of k it follows that $t = b$ and $b = s$ hold in B—in contradiction to $t = s$ not being an element of $E(B)$.

We conclude that the set $E(\mathscr{A}_k) - E(B)$ is small. Since $E(B)$ contains $E(\mathscr{A}) = \bigcap_{i \in \mathbf{Ord}} E(\mathscr{A}_i)$, for every member $t = s$ of $E(\mathscr{A}_k) - E(B)$, there exists an ordinal $l \geq k$ such that $E(\mathscr{A}_l)$ does not contain $t = s$. Now $E(\mathscr{A}_k) - E(B)$ is small, so we can choose l independently of k. Then $E(\mathscr{A}_l)$ is disjoint from $E(\mathscr{A}_k) - E(B)$—that is, $E(B) \supseteq E(\mathscr{A}_l)$. Now $\overline{\mathscr{A}}_l$ is a variety (see 3.19) with $E(\mathscr{A}_l) = E(\overline{\mathscr{A}}_l)$. This proves $B \in \overline{\mathscr{A}}_l \subseteq \mathscr{A}$. ■

3.29. Concluding Remark. For set functors F preserving nonempty preimages the question of Birkhoff's Variety Theorem is completely answered by Theorem 3.22: it is equivalent to F being a varietor. However, by a slight modification of the proof also some interesting functors can be included which do not preserve nonempty preimages. For example the functor $F_1 + F_2$ in 3.26—this functor is a modification of the power-set functor strongly resembling the modification of β performed in 3.27. Yet, the conclusion of 3.27 is opposite to that of 3.26, which indicates that the precise characterization of set functors for which Birkhoff's Variety Theorem holds is a complicated problem.

The functor $\bar{\beta}$ of 3.26 is not faithful. However, by a modification of the proof of 3.28 one can show that the faithful functor $\bar{\beta} + \mathrm{Id}$ also satisfies Birkhoff's Variety Theorem without being a varietor.

4. Birkhoff's Variety Theorem for Classes

4.1. The Category **Class**. We work here in ZFC (Zermelo-Fraenkel Set Theory with the Axiom of Choice) with a choice of a universe of "small" sets. If $\aleph_\infty$ denotes the cardinality of this universe, then the category **Set** of small sets can, as in [7], be identified with the category of all sets of cardinalities smaller than $\aleph_\infty$. And the category **Class** of classes can be identified with that of all sets of cardinalities at most $\aleph_\infty$. More precisely: our foundation is ZFC with a strongly inaccessible cardinal $\aleph_\infty$. Then **Set** is the category of all sets which hereditarily have cardinality smaller than $\aleph_\infty$. This is clearly equivalent to the category of all sets of cardinality smaller than $\aleph_\infty$. Analogously for **Class**. Thus, the above identification does not influence any categorical results.

We denote by **Ord** and **Card** the classes of all small ordinals and small cardinals, respectively. Thus, $\aleph_\infty$ is the first ordinal outside of **Ord**.

4.2. Notation. Given an endofunctor F of **Class**, we denote, analogously to 3.1, by $\mathbf{Alg}\,F$ the category of all F-algebras $FA \xrightarrow{\alpha} A$ and homomorphisms and we call F a *varietor* if the forgetful functor $\mathbf{Alg}\,F \longrightarrow \mathbf{Class}$ has a left adjoint—i.e., every object K generates a free F-algebra $\Phi(K)$. We denote by $\eta_K \colon K \longrightarrow \Phi(K)$ the universal arrow.

4.3. Theorem. (See [7].) *Every endofunctor of* **Class** *is a varietor.*

4.4. Remark. We say that **Set** *is measurable* if the cardinal $\aleph_\infty$ is measurable. In other words, a nontrivial two-valued measure exists on a proper class which is α-additive for all small cardinals α. The negation of this assumption states that every ultrafilter on a class closed under small intersections is trivial (i.e., consists, for some element x, of all classes containing x). As proved by J. Isbell [12], this is equivalent to **Set** being codense in **Class**. In other words: every class is a limit of a large diagram of small sets.

4.5. Definition. *Given an endofunctor F of* **Class**, *by an* ***equation*** *is meant a pair t_1, t_2 of elements of a free algebra $\Phi(K)$. An F-algebra A is said to* ***satisfy*** *the equation provided that for every map $f \colon K \longrightarrow A$ the corresponding homomorphism $f^\# \colon \Phi(K) \longrightarrow A$ fulfils $f^\#(t_1) = f^\#(t_2)$.*

A collection $\mathscr{A}$ of F-algebras is called a ***variety*** *if it can be presented by equations, i.e., if there exists a class $\mathscr{E}$ of equations such that members of the collection $\mathscr{A}$ are precisely the F-algebras which satisfy every equation in $\mathscr{E}$.*

4.6. Remark. A class $\mathscr{E}$ of equations can also be described by an epimorphism in **Class** $e \colon \Phi(K) \twoheadrightarrow E$. An F-algebra A satisfies the equations of $\mathscr{E}$ iff every homomorphism from $\Phi(K)$ to A factorizes through e.

4.7. Example. Let $\Sigma = (\Sigma_n)_{n \in \mathbf{Card}}$ be a (possibly large) *signature*, i.e., a class of "operation" symbols together with an "arity" function assigning to every operation symbol a small cardinal. Then Σ-algebras are precisely the algebras on the polynomial endofunctor

$F_\Sigma\colon$ **Class** $\longrightarrow$ **Class** defined on objects by

$$F_\Sigma X = \coprod_{n\in\mathbf{Card}} \Sigma_n \times X^n.$$

As a concrete example consider $\Sigma = \mathbf{Card}$ where each symbol has arity zero. H. Herrlich observed in [11] that the class $\mathscr{A}$ of all algebras A for which there exists $i \in \mathbf{Card}$ such that all the constants $j \geq i$ form the same element in A is closed under

(i) products

(ii) subalgebras, and

(iii) regular quotient algebras.

And it is clear that $\mathscr{A}$ is not a variety. However, if in (i) "products" are substituted by "limits", it turns out that $\mathscr{A}$ is no longer a counter-example to Birkhoff's Variety Theorem. At least not if **Set** is not (compare Example 4.9):

4.8. Birkhoff's Variety Theorem for Classes. *Assuming that* **Set** *is not measurable, every endofunctor F of the category of classes has the property that varieties are precisely the classes of F-algebras closed under regular quotients, subobjects, and limits.*

4.9. Remark. (i) Limits are not restricted to mean "small limits" but all limits existing in $\mathbf{Alg}\,F$. (In 4.7 (i), products are also not restricted to mean "small products" but all products existing in $\mathbf{Alg}\,F$.)

(ii) Every variety can be presented by a single quotient of a free F-algebra as will be seen in the following proof.

Proof. (1) Let $\mathscr{A}$ be a full subcategory of $\mathbf{Alg}\,F$ closed as above. Let K be a proper class, and denote by $\Phi(K)$ a free F-algebra on K. Let $\sim_i$ $(i \in I)$ be the collection of all congruences on $\Phi(K)$ such that the quotient algebra

$$A_i = \Phi(K)/{\sim_i}$$

lies in $\mathscr{A}$. We prove below that if $\sim$ is the meet of these congruences, then also the algebra

$$A = \Phi(K)/{\sim}$$

lies in $\mathscr{A}$. This implies that $\mathscr{A}$ is a variety presented by the canonical quotient $c\colon \Phi(K) \twoheadrightarrow A$. In fact, let B be an algebra in $\mathscr{A}$, then for every homomorphism $h\colon \Phi(K) \longrightarrow B$ the kernel congruence of h is $\sim_i$ for some i (because $\mathscr{A}$ is closed under subalgebras), thus, $h = m\cdot c_i$ for the canonical quotient $c_i\colon \Phi(K) \longrightarrow \Phi(K)/{\sim_i}$ and for some homomorphism

$m\colon A_i \longrightarrow B$.

$$\begin{tikzcd}
 & \Phi(K) \arrow[r, "h"] \arrow[dl, "c"'] \arrow[dd, "c_i"] & B \\
A \arrow[dr, "\bar{c}_i"'] & & \\
 & A_i = \Phi(K)/{\sim_i} \arrow[uur, "m"'] &
\end{tikzcd}$$

Since $\sim$ refines $\sim_i$, we also have a canonical homomorphism

$$\bar{c}_i\colon A \longrightarrow A_i \quad \text{with} \quad c_i = \bar{c}_i \cdot c \qquad (i \in I).$$

This implies that h factorizes through c, i.e., B satisfies the equations in the kernel of c. Conversely, let B be an F-algebra satisfying them. We prove $B \in \mathscr{A}$. Without loss of generality, $B \neq \emptyset$ (if $B = \emptyset$ then B is a subalgebra of every F-algebra, thus, it lies in $\mathscr{A}$). Let $f\colon K \longrightarrow B$ be an epimorphism (which exists since K is a proper class). Then the corresponding homomorphism $\bar{f}\colon \Phi(K) \longrightarrow B$ factorizes though c:

$$\begin{tikzcd}
\Phi(K) \arrow[r, "\bar{f}"] \arrow[d, "c"'] & B \\
A \arrow[ur, "h"'] &
\end{tikzcd}$$

and h is a homomorphism (because both hc and c are homomorphisms and, since c is a split epimorphism in **Class**, Fc is an epimorphism). Since for the universal arrow η_K we see that $f = h \cdot c \cdot \eta_K$ is an epimorphism in **Class**, so is h. Thus, $A \in \mathscr{A}$ implies $B \in \mathscr{A}$, due to closedness under quotient algebras.

It remains to prove $A \in \mathscr{A}$. Let D be the diagram in $\mathscr{A}$ whose objects are A_i for $i \in I$ and whose morphisms are the canonical quotients $c_{ij}\colon A_i \longrightarrow A_j$

$$\begin{tikzcd}
 & A \arrow[dl, "\bar{c}_i"'] \arrow[dr, "\bar{c}_j"] & \\
A_i \arrow[rr, "c_{ij}"'] & & A_j
\end{tikzcd}$$

for all $i, j \in I$ such that $\sim_i$ refines $\sim_j$. We prove that the cone $(\bar{c}_i)_{i \in I}$ is a limit cone of D—then $A \in \mathscr{A}$ since $\mathscr{A}$ is closed under limits. The only nontrivial part is the verification that given a compatible collection

$$x_i \in A_i \qquad (i \in I),$$

there exists a unique $x \in A$ with $x_i = \bar{c}_i(x)$ for all $i \in I$. In fact, consider

$$\mathscr{F} = \{F \subseteq A; x_i \in \bar{c}_i[F] \text{ for all } i \in I\}.$$

This is, obviously, a filter; in fact, an ultrafilter: given a disjoint decomposition $A = V_1 \cup V_2$, then $V_t \in \mathscr{F}$ for $t = 1$ or 2. (Assuming the contrary, we have for $t = 1, 2$ an index $i(t)$ with $x_{i(t)} \notin \bar{c}_{i(t)}[V_t]$. The kernel of $\langle \bar{c}_{i(1)}, \bar{c}_{i(2)} \rangle \colon A \longrightarrow A_{i(1)} \times A_{i(2)}$ has the form $\sim_j$ for some $j \in I$—this follows from $\mathscr{A}$ being closed under finite products and subalgebras. Since $\sim_j$ refines $\sim_{i(1)}$, we have

$$x_{i(1)} = c_{j,i(1)}(x_j),$$

therefore $\bar{c}_{i(1)} = c_{j,i(1)} \cdot \bar{c}_j$ implies

$$x_j \notin \bar{c}_j[V_1].$$

Analogously, $x_j \notin \bar{c}_j[V_2]$—a contradiction, since $\bar{c}_j$ is an epimorphism.) In fact, the same argument as above can be used to prove that for every disjoint decomposition $A = \bigcup_{t \in T} V_t$ where T is a small set we have $V_t \in \mathscr{F}$ for some $t \in T$. In other words, $\mathscr{F}$ is closed under small intersections. Since **Set** is not measurable, $\mathscr{F}$ contains $\{x\}$ for a unique $x \in A$. Then $\bar{c}_i(x) = x_i$.

(2) If $\mathscr{A}$ is a variety, then it is closed under limits. In fact, suppose that $(A, \alpha) \xrightarrow{h_i} (A_i, \alpha_i)$ $(i \in I)$ is a limit cone of a diagram D in $\mathbf{Alg}\, F$ and suppose that each (A_i, α_i) satisfies equations $e \colon \Phi(K) \twoheadrightarrow E$ see 4.6. We prove that (A, α) also satisfies them.

For every homomorphism $\bar{f} \colon \Phi(K) \longrightarrow A$ we have factorizations

$$h_i \cdot \bar{f} = k_i \cdot e \qquad (k_i \colon E \longrightarrow A_i)$$

for all $i \in I$. Since $(k_i \cdot e)_{i \in I}$ is a cone of D and e is an epimorphism in **Class**, $(k_i)_{i \in I}$ is a cone of D in **Class**, thus, we have a unique $k \colon E \longrightarrow A$ with

$$h_i \cdot k = k_i \qquad (i \in I).$$

Here we use the (trivial) fact that the forgetful functor of $\mathbf{Alg}\, F$ preserves limits. From the equalities

$$h_i \cdot \bar{f} = h_i \cdot (k \cdot e) \qquad (i \in I)$$

it follows that $\bar{f} = k \cdot e$. Therefore, A satisfies e.

The verification that $\mathscr{A}$ is closed under subalgebras and regular quotient algebras is straightforward. ∎

4.10. Example. A class of Σ-algebras which is not a variety but is closed under limits, subalgebras, and regular quotient algebras. We assume here that **Set** is measurable, i.e., a proper class K is not a limit of the canonical diagram $K/\mathbf{Set} \longrightarrow \mathbf{Class}$ of all maps $f \colon K \longrightarrow X$ with X small. Consider $\Sigma = K$ as the signature of nullary operation symbols. Let $\mathscr{A}$ be the collection of all algebras $(A, (k^A)_{k \in K})$ such that $A^* = \{k^A; k \in K\}$ is a small set. Then $\mathscr{A}$ is clearly closed under subalgebras and regular quotient algebras. We verify that it is also closed under limits. Let $D \colon \mathscr{D} \longrightarrow \mathscr{A}$ be a diagram with a limit $A \xrightarrow{c_d} A_d$ ($d \in \mathscr{D}^{\mathbf{obj}}$) in $\mathbf{Alg}\,\Sigma$. Then A^* is contained in a limit, L, in **Class**, of the diagram formed by the sets A_d^* and the corresponding restrictions of the connecting morphisms of D. Since each A_d^* is a small set, and since **Set** is measurable, L cannot be isomorphic to K—thus, A^* is a small set. That is, $A \in \mathscr{A}$.

However, $\mathscr{A}$ is not a variety: for every pair $k \neq l$ in K there exists an algebra $A \in \mathscr{A}$ with $k^A \neq l^A$.

4.11. Example. Denote by $\mathscr{P}'$ the extension of the power-set functor to **Class**: $\mathscr{P}'X$ is the class of all small subsets of X. Assuming that **Set** is not measurable, Birkhoff's Variety Theorem holds for $\mathscr{P}'$, but not for $\mathscr{P}$. This is surprising because the equations used for $\mathscr{P}'$ are not more expressive than those used for $\mathscr{P}$. However, given a collection $\mathscr{V}$ of $\mathscr{P}$-algebras closed under products, subalgebras and regular quotients in $\mathbf{Alg}\,\mathscr{P}$, it is in general not closed under limits in $\mathbf{Alg}\,\mathscr{P}'$. Now let $\mathscr{V}'$ be the closure of $\mathscr{V}$ under limits, subalgebras and regular quotients in $\mathbf{Alg}\,\mathscr{P}'$. This is an equational class of $\mathscr{P}'$-algebras, but there is no reason why its intersection with $\mathbf{Alg}\,\mathscr{P}$ should be $\mathscr{V}$. Thus, things are not all that surprising, after all: the collection of all $\mathscr{P}$-algebras lying in $\mathscr{V}'$ is simply the equational hull of $\mathscr{V}$ in $\mathbf{Alg}\,\mathscr{P}$.

5. Conclusions and Related Research

Jan Reiterman proved in [18] that for varietors F on **Set** Birkhoff's Variety Theorem holds. That is, presentability by equations is equivalent to closure under limits, subalgebras and regular quotient algebras. Originally, we wanted to prove the converse, inspired by ideas of [18]. This goal was achieved for endofunctors F preserving nonempty preimages (which is a weak side condition). For such functors we proved that Birkhoff's Variety Theorem holds iff F is a varietor. For general functors the converse is not true, i.e., there exist functors $F\colon \mathbf{Set} \longrightarrow \mathbf{Set}$ which are not varietors and for which Birkhoff's Variety Theorem holds.

> *Open problem.* Characterize endofunctors F of **Set** for which Birkhoff's Variety Theorem holds in $\mathbf{Alg}\,F$.

In ZFC with a choice of small sets, the "standard" set theory plus the axiom of choice, we also proved that all endofunctors of **Class** fulfill Birkhoff's Variety Theorem under the set-theoretic assumption that the cardinality of **Set** is non-measurable. In view of the above result of J. Reiterman, and the fact that all endofunctors of **Class** are varietors, see [7], it seems in fact surprising that any set-theoretical assumptions are needed. But they are: we presented a simple example of a non-equational class of Σ-algebras which is closed under limits, subalgebras and regular quotient algebras, where Σ is just a class of nullary symbols—whenever the set-theoretical assumption above is not fulfilled.

How far can the results above be generalized to categories other than **Set** and **Class**? Nothing much is known for general categories. However, for the dual categories $\mathbf{Set}^{\mathrm{op}}$ and $\mathbf{Class}^{\mathrm{op}}$ the following is proved in [4]:

(a) Every endofunctor of $\mathbf{Class}^{\mathrm{op}}$ is a varietor and Birkhoff's Variety Theorem holds.

(b) There exists an endofunctor of $\mathbf{Set}^{\mathrm{op}}$ for which Birkhoff's Variety Theorem does not hold.

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