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THIN ELEMENTS AND COMMUTATIVE SHELLS IN CUBICAL ω-CATEGORIES

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ABSTRACT. The relationships between thin elements, commutative shells and connections in cubical ω-categories are explored by a method which does not involve the use of pasting theory or nerves of ω-categories (both of which were previously needed for this purpose; see [2], Section 9). It is shown that composites of commutative shells are commutative and that thin structures are equivalent to appropriate sets of connections; this work extends to all dimensions the results proved in dimensions 2 and 3 in [7, 6].

Introduction

Thin structures in simplicial sets were introduced by Dakin in [8] and were applied to cubical sets in [3, 4, 5]. In the cubical case a thin structure is equivalent to an ω-groupoid structure.

In this paper we use the term *thin structure* in a weaker sense which is appropriate for cubical ω-categories and does not imply the existence of inverses. This concept was introduced in the 2-dimensional case in [13], as arising from a pair of connections in dimension 2, and the equivalence of these notions in this dimension was shown in [7]. We extend this result to all dimensions (Theorem 3.1).

The definition of thin structure depends on the notion of *commutative cube* or *commutative shell.* This was studied in the 3-dimensional case in [13] for certain 'special' double categories, and in [6] as part of the proof of a van Kampen theorem for homotopy double groupoids. A key result there (see also [13, Proposition 3.11]) was that any composition of commutative 3-shells is commutative; we prove this result in all dimensions. We also prove in the general case that thin elements are precisely those which can be expressed as composites using only identity elements and connections (Theorem 2.8).

These results, which were proved for ω-groupoids in [4], can be deduced for ω-categories from results in [2]. However, the methods used in [2] depend on the use of pasting theory and nerves of ω-categories which tend to obscure the intuitive nature of thinness and commutativity. The approach used below is simpler and more direct. The basic simplification is the use of a "partial folding operation" Ψ in place of the full folding operation Φ_n used in [2]. The operation Φ_n is needed to prove the equivalence between cubical ω-categories with connections and globular ω-categories, but the simpler Ψ suffices

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for a detailed study of thinness and commutativity. This direct approach should facilitate applications to homotopy theory (cf.[7, 6, 12, 13]) and to concurrency theory in computer science (cf. [10, 11]).

1. Composing the faces of a cube

Let C be a cubical ω-category as defined in [1] and [2]; for the moment we do not assume the existence of connections. If $x \in \mathsf{C}_n$ is an n-cube in C one may ask which of its $(n-1)$-faces have common $(n-2)$-faces and can be composed in C_{n-1}. The answer is that the following pairs of faces (and in general only these pairs) can be composed:

$$(\partial_i^- x, \partial_{i+1}^+ x), \qquad (\partial_i^+ x, \partial_{i+1}^- x), \qquad i = 1, 2, \ldots, n-1.$$

Thus the faces of x (by which we mean its $(n-1)$-faces) divide naturally into two sequences

$$(\partial_1^- x, \partial_2^+ x, \partial_3^- x, \ldots, \partial_n^\pm x) \text{ and } (\partial_1^+ x, \partial_2^- x, \partial_3^+ x, \ldots, \partial_n^\mp x)$$

in which neighbouring pairs can be composed. We call these respectively the *negative* and the *positive* faces of x. This agrees with the terminology of [2].

We will frequently use 2-dimensional arrays of elements of C_n, and these will be shown in tabular form such as

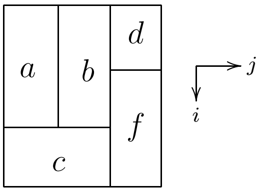
$$\begin{array}{|c|c|c|}\hline a & b & c \\ \hline d & f & g \\ \hline \end{array} \qquad \begin{tikzcd} \cdot \arrow[r, "j"] \arrow[d, "i"'] & {} \\ {} & \end{tikzcd}$$

When using this notation we will always assume that pairs of adjacent elements in the table have a common face in the relevant direction, in this case i or j. Thus in the array above we assume that $\partial_j^+ a = \partial_j^- b$, $\partial_i^+ a = \partial_i^- d$, etc. The array can then be composed *by rows*: $(a \circ_j b \circ_j c) \circ_i (d \circ_j f \circ_j g)$, or *by columns*: $(a \circ_i d) \circ_j (b \circ_i f) \circ_j (c \circ_i g)$. It is a consequence of the interchange law that these two elements of C_n are equal, and we call their common value the *composite* of the array. To emphasise our implicit assumptions, we will use the term *composable array.*

More general composites in the form of rectangular partitions of a rectangle will also be used. The simplest is of the form

$$\begin{array}{|c|c|}\hline a & b \\ \hline \multicolumn{2}{|c|}{c} \\ \hline \end{array} \qquad \begin{tikzcd} \cdot \arrow[r, "j"] \arrow[d, "i"'] & {} \\ {} & \end{tikzcd}$$

Here the implicit assumptions are that $\partial_j^+ a = \partial_j^- b$ and $\partial_i^+(a \circ_j b) = \partial_i^- c$, and so the composite $(a \circ_j b) \circ_i c$ can be formed. For a more general (finite) partition of a rectangle by rectangles, labelled by members of C_n, we will assume that two elements, or composites of elements, with a common edge in the partition, have a common face at this edge, so allowing the composition in the corresponding direction. For example, in the partition



we assume, in addition to the relations above, that

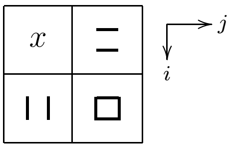
$$\partial_i^+ d = \partial_i^- f \qquad \text{and} \qquad \partial_j^+[(a \circ_j b) \circ_i c] = \partial_j^-(d \circ_i f)$$

and so the composite $[(a \circ_j b) \circ_i c] \circ_j (d \circ_i f)$ can be formed. We shall call such diagrams *composable partitions* provided that some sequence of compositions exist which combines all the elements to form a composite of the partition. We will specify the sequence if there is any ambiguity.

Now suppose that C is a *cubical ω-category with connections*, that is, it has for each n and for $i = 1, 2, \ldots, n$, extra structure maps $\Gamma_i^+, \Gamma_i^- : \mathsf{C}_n \to \mathsf{C}_{n+1}$ (called connections) satisfying the identities set out, for example, in [1] and in Section 2 of [2]. We shall make free use of the defining identities in [2] without further comment. (In fact, the existence of connections in C implies that all composites of a given composable partition are equal. This is because, using connections, any composable partition can be refined to a composable *array*, and any composite of the partition must be equal to the unique composite of this array: see [7] and [9]. We do not need to use this general theorem.)

The 'degenerate' elements $\Gamma_k^+ x, \Gamma_k^- x$ and $\varepsilon_k x$ will sometimes be represented in composable arrays by the symbols $\ulcorner$, $\lrcorner$ and ($\|$ or $=$) respectively. The symbol $\square$ will be used to denote an element that is an identity for both the horizontal and vertical compositions. These symbols will be used only where the elements of C_n they represent are uniquely determined by the composability of the array. (The lines in these symbols are designed to indicate that the corresponding faces are identities in the direction of these edges.)

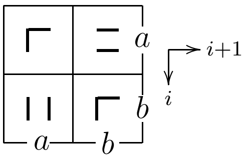
For example, the array



denotes

$$\begin{array}{|c|c|} \hline x & \varepsilon_j \partial_j^+ x \\ \hline \varepsilon_i \partial_i^+ x & \varepsilon_i \partial_i^+ \varepsilon_j \partial_j^+ x \\ \hline \end{array}$$

whose composite is x itself; and the array

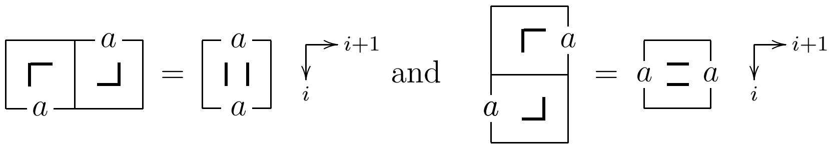


denotes

$$\begin{array}{|c|c|} \hline \Gamma_i^+ a & \varepsilon_{i+1} a \\ \hline \varepsilon_i a & \Gamma_i^+ b \\ \hline \end{array}$$

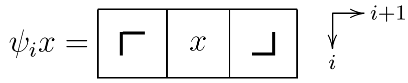
whose composite, according to the transport law [2, (2.6)], is $\Gamma_i^+(a \circ_i b)$. (Here the labels on the edges denote the corresponding faces of elements of the array.)

The following identities (see [2, (2.7)]) will be important in what follows:



that is, $\Gamma_i^+ a \circ_{i+1} \Gamma_i^- a = \varepsilon_i a$ and $\Gamma_i^+ a \circ_i \Gamma_i^- a = \varepsilon_{i+1} a$.

We now define the *elementary folding operations* $\psi_i : \mathsf{C}_n \to \mathsf{C}_n$, $(i = 1, 2, \ldots, n-1)$ by



These operations ψ_i were introduced in [2] as a categorical version of the folding operations Φ_j defined for ω-groupoids in [4]. The chief identities satisfied by the ψ_i are set out in Proposition 3.3(i) of [2]; we shall not need the "braid relations" proved in Theorem 5.2 of that paper. As the picture suggests, the effect of ψ_i is to "fold" the faces $\partial_{i+1}^- x$ and $\partial_{i+1}^+ x$ to the i^{th} direction so that they abut the faces $\partial_i^+ x$ and $\partial_i^- x$ respectively, and to compose the two pairs of faces. The composition of all the negative (and positive) faces of x, together with certain faces of the connections, can therefore be achieved by the folding operation $\Psi : \mathsf{C}_n \to \mathsf{C}_n$ defined by $\Psi = \psi_1 \psi_2 \ldots \psi_{n-1}$.

We note that Ψ is not the full folding operation Φ_n of [2] (which maps C_n into its globular part (see Proposition 3.5 of [2]), but Ψ is sufficient for the study of thin elements and commutative shells. We will return to this point later. For now, the two most important faces of Ψx, namely

$$Px = \partial_1^+ \Psi x \quad \text{and} \quad Nx = \partial_1^- \Psi x,$$

are to be viewed as convenient embodiments of the positive and negative boundaries of x.

1.1. Lemma.

(i) $\psi_1 \psi_2 \ldots \psi_{r-1} \varepsilon_r = \varepsilon_1 : \mathsf{C}_{n-1} \to \mathsf{C}_n$ *for* $1 \leqslant r \leqslant n-1$.

(ii) *If* $y \in \mathsf{C}_{n-1}$ *then* $\Psi \varepsilon_j y \in \varepsilon_1 \mathsf{C}_{n-1}$ *for* $1 \leqslant j \leqslant n-1$.

Proof. (i) This follows immediately from the identities $\psi_i \varepsilon_{i+1} = \varepsilon_i$.
(ii) similarly, using $\psi_i \varepsilon_j = \varepsilon_j \psi_{i-1}$ for $j < i$ and $\psi_j \varepsilon_j = \varepsilon_j$, we have

$$\begin{aligned}
\Psi \varepsilon_j &= \psi_1 \psi_2 \ldots \psi_{n-1} \varepsilon_j \\
&= \psi_1 \psi_2 \ldots \psi_j \varepsilon_j \psi_j \psi_{j+1} \ldots \psi_{n-2} \\
&= \psi_1 \psi_2 \ldots \psi_{j-1} \varepsilon_j \psi_j \psi_{j+1} \ldots \psi_{n-2} \\
&= \varepsilon_j \psi_j \psi_{j+1} \ldots \psi_{n-2} \text{ by (i).}
\end{aligned}$$
□

If $x \in \mathsf{C}_n$, the *shell* $\boldsymbol{\partial} x$ of x is the family consisting of all its faces $\partial_i^\alpha x$ $(i = 1, 2, \ldots, n; \alpha = +, -)$.

1.2. Proposition. *Let* C *be a cubical* ω*-category with connections and* $x \in \mathsf{C}_n$. *Then*

(i) *all faces* $\partial_i^\alpha \Psi x$ *with* $i \geqslant 2$ *are of the form* $\varepsilon_1 z_i^\alpha$, *where* $z_i^\alpha \in \mathsf{C}_{n-2}$;

(ii) $\boldsymbol{\partial} Nx = \boldsymbol{\partial} Px$;

(iii) $\boldsymbol{\partial}\Psi x$ *is uniquely determined by* Nx *and* Px.

Proof.

(i) We have $\partial_i^\alpha \psi_j = \psi_j \partial_i^\alpha$ for $i > j+1$, so for $i \geqslant 2$,

$$\partial_i^\alpha \Psi x = \partial_i^\alpha \psi_1 \dots \psi_{n-1} x = \psi_1 \psi_2 \dots \psi_{i-2} \partial_i^\alpha \psi_{i-1}, \dots, \psi_{n-1} x,$$

(if $i = 2$, the $\psi_1 \dots, \psi_{i-2}$ are missing). But $\partial_i^\alpha \psi_{i-1} = \varepsilon_{i-1} \partial_{i-1}^\alpha \partial_i^\alpha$, so

$$\begin{aligned}\partial_i^\alpha \Psi x &= \psi_1 \psi_2 \dots \psi_{i-2} \varepsilon_{i-1} z_i^\alpha \text{ where } z_i^\alpha \in \mathsf{C}_{n-2}\\ &= \varepsilon_1 z_i^\alpha \text{ by Lemma1.1 (i) .}\end{aligned}$$

(ii) This follows from (i) because

$$\partial_i^\alpha N x = \partial_i^\alpha \partial_1^- \Psi x = \partial_1^- \partial_{i+1}^\alpha \Psi x = \partial_1^- \varepsilon_1 z_{i+1}^\alpha = z_{i+1}^\alpha$$

and similarly $\partial_i^\alpha P_x = z_{i+1}^\alpha$.

(iii) The faces of Ψx are Nx, Px and the elements $\varepsilon_1 z$ where z is a face of Nx (or Px). ∎

In the abstract, an n-*shell* in C is a family $\mathsf{s} = \{s_i^\alpha; s_i^\alpha \in \mathsf{C}_{n-1}; i = 1, 2, \dots, n; \alpha = +, -\}$ where the s_i^α satisfy the incidence relation

$$\partial_j^\beta s_i^\alpha = \partial_{i-1}^\alpha s_j^\beta \text{ for } 1 \leqslant j < i \leqslant n \text{ and } \alpha, \beta \in \{+, -\}.$$

We denote by $\square\, \mathsf{C}_{n-1}$ the set of such shells. The usual cubical incidence relations imply that $\boldsymbol{\partial} x \in \square\, \mathsf{C}_{n-1}$ for all $x \in \mathsf{C}_n$.

If $\{\mathsf{C}_0, \mathsf{C}_1, \dots, \mathsf{C}_{n-1}\}$ is a cubical $(n-1)$-category with connections, then $\square \mathsf{C}_{n-1}$ has naturally defined operations $\circ_i$ $(1 \leqslant i \leqslant n)$, and connections $\boldsymbol{\Gamma}_j^\alpha : \mathsf{C}_{n-1} \to \square \mathsf{C}_{n-1}$ $(1 \leqslant j \leqslant n-1)$ which, together with the obvious structure maps $\partial_i^\alpha : \square\, \mathsf{C}_{n-1} \to \mathsf{C}_{n-1}$ and $\boldsymbol{\varepsilon}_j : \mathsf{C}_{n-1} \to \square\, \mathsf{C}_{n-1}$, make $\{\mathsf{C}_0, \mathsf{C}_1, \dots, \mathsf{C}_{n-1}, \square\, \mathsf{C}_{n-1}\}$ a cubical n-category with connections (cf. [4], section 5). Thus we can define folding maps $\psi_i, \Psi : \square \mathsf{C}_{n-1} \to \square \mathsf{C}_{n-1}$ which obviously satisfy:

1.3. Lemma. *In a cubical* n*-category* $(\mathsf{C}_1, \mathsf{C}_2, \dots, \mathsf{C}_n)$ *with connections, the map*

$$x \mapsto \boldsymbol{\partial} x : \mathsf{C}_n \to \square \mathsf{C}_{n-1},$$

together with identity maps in lower dimensions, gives a morphism of cubical n*-categories with connections from* $(\mathsf{C}_1, \mathsf{C}_2, \dots, \mathsf{C}_n)$ *to* $(\mathsf{C}_1, \mathsf{C}_2, \dots, \mathsf{C}_{n-1}, \square \mathsf{C}_{n-1})$. *In particular*

$$\boldsymbol{\Gamma}_i^\alpha x = \boldsymbol{\partial} \Gamma_i^\alpha x,\ \psi_j \boldsymbol{\partial} x = \boldsymbol{\partial} \psi_j x,\ \Psi \boldsymbol{\partial} x = \boldsymbol{\partial} \Psi x,$$

$$N \boldsymbol{\partial} x = \boldsymbol{\partial} N x,\ P \boldsymbol{\partial} x = \boldsymbol{\partial} P x \text{ and } \boldsymbol{\partial} \varepsilon_j x = \boldsymbol{\varepsilon}_j x.$$

∎

1.4. THEOREM. *Let C be a cubical ω-category (or a cubical m-category) with connections. Let $a \in \mathsf{C}_n$ and $\mathsf{s} \in \square\mathsf{C}_{n-1}$. A necessary and sufficient condition for the existence of $x \in \mathsf{C}_n$ such that*

$$\boldsymbol{\partial} x = \mathsf{s} \text{ and } \Psi x = a$$

is that

$$\Psi \mathsf{s} = \boldsymbol{\partial} a.$$

If x exists, it is unique.

PROOF. The necessity of $\Psi\mathsf{s} = \boldsymbol{\partial} a$ follows from $\boldsymbol{\partial}\Psi x = \Psi\boldsymbol{\partial} x$ (Lemma 1.2). The existence and uniqueness will be deduced from:

1.5. LEMMA. *Let $a \in \mathsf{C}_n$ and $\mathsf{s} \in \square\mathsf{C}_{n-1}$ satisfy $\boldsymbol{\partial} a = \psi_j \mathsf{s}$ for some $j \in \{1, 2, \ldots, n-1\}$. Then there is a unique x in C_n such that $\boldsymbol{\partial} x = \mathsf{s}$ and $\psi_j x = a$.*

PROOF. First suppose that x exists, and consider the array

$$A : \quad \begin{array}{|c|c|c|} \hline \square & \| & \ulcorner \\ \hline \ulcorner & x & \lrcorner \\ \hline \lrcorner & \| & \square \\ \hline \end{array} \qquad \begin{array}{l} \rightarrow j+1 \\ \downarrow j \end{array}$$

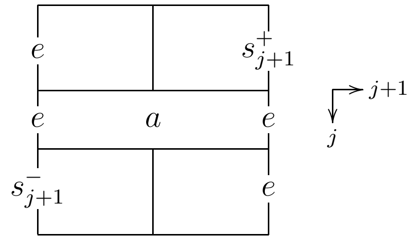
where the elements surrounding x are determined by the faces of x so that all rows and columns are composable. The composite of the middle row is $\psi_j x = a$. The elements of the first and third rows are determined by the faces of x, i.e. by s. Hence the composite of A is determined by a and s. But if we compose A by columns and use the law $\boldsymbol{\Gamma}_j^+ t \circ_j \boldsymbol{\Gamma}_j^- t = \boldsymbol{\varepsilon}_{j+1} t$, we see that the composite of A is x itself. Hence x is unique.

To prove existence, we note that the array A gives a formula for x in terms of a and s. So, given a and s we define x to be the composite of the composable partition

$$x = \quad \begin{array}{|c|c|} \hline \varepsilon_j s_j^- & \Gamma_j^+ s_{j+1}^+ \\ \hline \multicolumn{2}{|c|}{a} \\ \hline \Gamma_j^- s_{j+1}^- & \varepsilon_j s_j^+ \\ \hline \end{array} \qquad \begin{array}{l} \rightarrow j+1 \\ \downarrow j \end{array}$$

Here the first and third rows are the same as those in the array A, except that the 2-fold identities $\square$ have been omitted, being redundant. Because we are assuming that $\boldsymbol{\partial} a = \psi_j \mathsf{s}$, the faces $\partial_j^- a$ and $\partial_j^+ a$ are the same as the upper and lower faces of the composite $\boxed{\ulcorner \mid x \mid \lrcorner}$ in A. Hence the partition is composable and we can compute x from a and s, by composing rows first. It remains to verify that $\psi_j x = a$ and $\boldsymbol{\partial} x = \mathsf{s}$.

The faces $\partial^{\pm}_{j+1}x$ of x are $\circ_j$-composites as indicated in the diagram

$x =$ 

where the faces labelled e are identities for $\circ_j$. (Note that $\partial^{\pm}_{j+1}a$ are identities because $\boldsymbol{\partial} a = \psi_j \mathsf{s}$). Hence

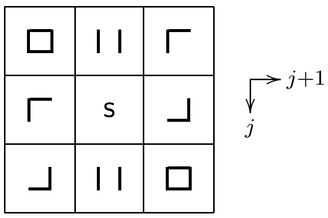
$$\psi_j x = \begin{array}{|c|c|c|c|} \hline \square & \varepsilon_j s^-_j & \Gamma^+_j s^+_{j+1} & \Gamma^-_j s^+_{j+1} \\ \hline \square & \multicolumn{2}{c|}{a} & \square \\ \hline \Gamma^+_j s^-_{j+1} & \Gamma^-_j s^-_{j+1} & \varepsilon_j s^+_j & \square \\ \hline \end{array} \qquad \begin{array}{l} \to j+1 \\ \downarrow j \end{array}$$

Here, the first and last columns are the relevant connections expanded by the transport law and then simplified. The diagram can be viewed as a 3×3 array in which two elements happen to be horizontal composites; therefore we may compute $\psi_j x$ by composing the rows first instead of the columns. Using the law $\Gamma^+_j t \circ_{j+1} \Gamma^-_j t = \varepsilon_j t$, we find that $\psi_j x = a$, as required.

Finally $\boldsymbol{\partial} x$ is the composite (by rows) of

$$\begin{array}{|c|c|} \hline \boldsymbol{\partial}\varepsilon_j s^-_j & \boldsymbol{\partial}\Gamma^+_j s^+_{j+1} \\ \hline \multicolumn{2}{|c|}{\boldsymbol{\partial} a} \\ \hline \boldsymbol{\partial}\Gamma^-_j s^-_{j+1} & \boldsymbol{\partial}\varepsilon_j s^+_j \\ \hline \end{array} \qquad \begin{array}{l} \to j+1 \\ \downarrow j \end{array}$$

Since $\boldsymbol{\partial} a = \psi_j \mathsf{s}$, this is the composite in $\square\, \mathsf{C}_{n-1}$ of the array



which, as before, is just s. ∎

We now use induction on $r \leqslant n-1$ to show that if $\boldsymbol{\partial} a = \psi_1\psi_2\dots\psi_r \mathsf{s}$, then there is a unique $x \in \mathsf{C}_n$ such that $\psi_1\psi_2\dots\psi_r x = a$ and $\boldsymbol{\partial} x = \mathsf{s}$. The case $r = 1$ is covered by Lemma 1.5. Suppose that the result is true when $r = t-1 < n-1$ and that $\boldsymbol{\partial} a = \psi_1\psi_2\dots\psi_r \mathsf{s}$. Then, by induction hypothesis, there is a unique $y \in \mathsf{C}_n$ such that $\psi_1\psi_2\dots\psi_{r-1} y = a$ and $\boldsymbol{\partial} y = \psi_t \mathsf{s}$. But, again by Lemma 1.5, there is then a unique $x \in \mathsf{C}_n$ with $\psi_t x = y$ and $\boldsymbol{\partial} x = \mathsf{s}$, completing the induction. The case $r = n-1$, completes the proof of the theorem. ∎

2. Thin elements and commutative shells

We say that an element $x \in \mathsf{C}_n$ is *thin* if $\Psi x \in \varepsilon_1 \mathsf{C}_{n-1}$. Then $\Psi x = \varepsilon_1 N x = \varepsilon_1 P x$.

We say that a shell $\mathsf{s} \in \square\, \mathsf{C}_{n-1}$ is *commutative* if $N\mathsf{s} = P\mathsf{s}$.

2.1. PROPOSITION. *Let C be a cubical ω-category (or cubical m-category) with connections.*

(i) *The shell of a thin element of C_n is a commutative n-shell.*

(ii) *A commutative n-shell is the same thing as a thin element of $\square\, \mathsf{C}_{n-1}$.*

(iii) *Any commutative n-shell s has a unique thin filler (i.e. a thin element $x \in \mathsf{C}_n$ with $\boldsymbol{\partial} x = \mathsf{s}$).*

PROOF. (i) Let $x \in \mathsf{C}_n$ be thin. Then $\Psi x = \varepsilon_1 z$ for some $z \in \mathsf{C}_{n-1}$, so $Nx = Px = z$. Therefore $N\boldsymbol{\partial} x = \boldsymbol{\partial} N x = \boldsymbol{\partial} P x = P \boldsymbol{\partial} x$, by Lemma 1.3.
(ii) If $\mathsf{s} \in \square\, \mathsf{C}_{n-1}$ is commutative, then $N\mathsf{s} = P\mathsf{s} = u$, say. The other faces of $\Psi\mathsf{s}$ are all of the form $\varepsilon_1 v$, where v is a face of u. These faces determine the shell $\Psi\mathsf{s}$ and identify it as $\boldsymbol{\varepsilon}_1 u$, so s is thin. The converse is obvious.
(iii) Let s be a commutative n-shell. Let $u = N\mathsf{s} = P\mathsf{s}$ and put $a = \varepsilon_1 u \in \mathsf{C}_n$. Then $\boldsymbol{\partial} a = \boldsymbol{\varepsilon}_1 u = \Psi \mathsf{s}$, by (ii) and Lemma 1.3. By Theorem 1.4, there is a unique $x \in \mathsf{C}_n$ with $\boldsymbol{\partial} x = \mathsf{s}$ and $\Psi x = a = \varepsilon_1 u$. ∎

2.2. PROPOSITION. *Let C be a cubical ω-category (or a cubical m-category) with connections.*

(i) *Elements in C of the form $\varepsilon_i c$ or $\Gamma_i^\alpha c$ are thin.*

(ii) *If $a, b \in \mathsf{C}_n$ are thin and $c = a \circ_i b$ then c is thin.*

PROOF. (i) The thinness of $\varepsilon_i c$ has been proved in Lemma 1.1. To prove thinness of $\Gamma_i^\alpha c$, we first establish some formulae involving the ψ_i and Γ_i^α.

2.3. Lemma.

(i) $\psi_i \Gamma_i^\alpha = \varepsilon_i$.

(ii) $\psi_j \Gamma_i^\alpha = \Gamma_i^\alpha \psi_{j-1}$, *if* $j > i+1$.

(iii) $\psi_i \psi_{i+1} \Gamma_i^+ c = \varepsilon_i(\Gamma_i^+ \partial_{i+1}^- c \ \circ_{i+1} \ c)$.
$\psi_i \psi_{i+1} \Gamma_i^- c = \varepsilon_i(c \ \circ_{i+1} \ \Gamma_i^- \partial_{i+1}^+ c)$.

Proof. (i)
$$\begin{aligned}\psi_i \Gamma_i^+ c &= \Gamma_i^+ \partial_{i+1}^- \Gamma_i^+ c \ \circ_{i+1} \ \Gamma_i^+ c \ \circ_{i+1} \ \Gamma_i^- \partial_{i+1}^+ \Gamma_i^+ c. \\ &= \Gamma_i^+ \varepsilon_i \partial_\alpha^- c \ \circ_{i+1} \ \Gamma_i^+ c \ \circ_{i+1} \ \Gamma_i^- c \\ &= \varepsilon_{i+1} \varepsilon_i \partial_\alpha^- c \ \circ_{i+1} \ \Gamma_i^+ c \ \circ_{i+1} \ \Gamma_i^- c \\ &= \Gamma_i^+ c \ \circ_{i+1} \ \Gamma_i^- c \\ &= \varepsilon_i c.\end{aligned}$$

The proof for Γ_i^- is similar.
(ii) This is proved similarly by using standard laws from pp. 80-81 of [2].
(iii) When we try to compute $\psi_{i+1}\Gamma_i^+$, we are hindered by the lack of a simple law involving $\Gamma_{i+1}^- \Gamma_i^+ c$. However, when we compute $\psi_i \psi_{i+1} \Gamma_i^+ c$, this difficulty disappears. We note that if $i+1 < j$ then $\psi_i(a \ \circ_j \ b) = \psi_i a \ \circ_j \ \psi_i b$. So

$$\begin{aligned}\psi_i \psi_{i+1} \Gamma_i^+ c &= \psi_i(\Gamma_{i+1}^+ \partial_{i+2}^- \Gamma_i^+ c \ \circ_{i+2} \ \Gamma_i^+ c \ \circ_{i+2} \ \Gamma_{i+1}^- \partial_{i+2}^+ \Gamma_i^+ c) \\ &= \psi_i \Gamma_{i+1}^+ \partial_{i+2}^- \Gamma_i^+ c \ \circ_{i+2} \ \psi_i \Gamma_i^+ c \ \circ_{i+2} \ \psi_i \Gamma_{i+1}^- \partial_{i+2}^+ \Gamma_i^+ c \\ &= \psi_i \Gamma_{i+1}^+ \Gamma_i^+ \partial_{i+1}^- c \ \circ_{i+2} \ \varepsilon_i c \ \circ_{i+2} \ \psi_i \Gamma_{i+1}^- \Gamma_i^+ \partial_{i+1}^+ c.\end{aligned}$$

We calculate the first and last terms separately:

$$\psi_i \Gamma_{i+1}^+ \Gamma_i^+ = \psi_i \Gamma_i^+ \Gamma_i^+ = \varepsilon_i \Gamma_i^+ \text{ by (i)}$$

and

$$\begin{aligned}\psi_i \Gamma_{i+1}^- \Gamma_i^+ y &= \Gamma_i^+ \partial_{i+1}^- \Gamma_{i+1}^- \Gamma_i^+ y \ \circ_{i+1} \ \Gamma_{i+1}^- \Gamma_i^+ y \ \circ_{i+1} \ \Gamma_i^- \partial_{i+1}^+ \Gamma_{i+1}^- \Gamma_i^+ y \\ &= (\Gamma_i^+ \Gamma_i^+ y \ \circ_{i+1} \ \Gamma_{i+1}^- \Gamma_i^+ y) \ \circ_{i+1} \ \Gamma_i^- \varepsilon_{i+1} \partial_{i+1}^+ \Gamma_i^+ y \\ &= (\Gamma_{i+1}^+ \Gamma_i^+ y \ \circ_{i+1} \ \Gamma_{i+1}^- \Gamma_i^+ y) \ \circ_{i+1} \ \Gamma_i^- \varepsilon_{i+1} y \\ &= \varepsilon_{i+2} \Gamma_i^+ y \ \circ_{i+1} \ \varepsilon_{i+2} \Gamma_i^- y \\ &= \varepsilon_{i+2}(\Gamma_i^+ y \ \circ_{i+1} \ \Gamma_i^- y) \\ &= \varepsilon_{i+2} \varepsilon_i y.\end{aligned}$$

Hence

$$\begin{aligned}\psi_i \psi_{i+1} \Gamma_i^+ c &= (\varepsilon_i \Gamma_i^+ \partial_{i+1}^- c \ \circ_{i+2} \ \varepsilon_i c) \ \circ_{i+2} \ \varepsilon_{i+2} \varepsilon_i \partial_{i+1}^+ c \\ &= \varepsilon_i(\Gamma_i^+ \partial_{i+1}^- c \ \circ_{i+1} \ c).\end{aligned}$$

The proof is similar for Γ_i^-. This completes the proof of Lemma 2.3. □

Returning to Proposition 2.2(i), the proof that $\Gamma_i^\alpha c$ (for $c \in \mathsf{C}_{n-1}$) is thin is now straightforward:

$$\begin{aligned}
\Psi\Gamma_i^\alpha c &= \psi_1\psi_2\dots\psi_{n-1}\Gamma_i^\alpha c\\
&= \psi_1\psi_2\dots\psi_i\psi_{i+1}\Gamma_i^\alpha c\psi_{i+1}\dots\psi_{n-2}c, \text{ by 2.3(ii)}\\
&= \psi_1\psi_2\dots\psi_{i-1}\varepsilon_i z \text{ for some } z \in \mathsf{C}_{n-1}, \text{ by 2.3(iii)}\\
&= \varepsilon_1 z \quad \text{by 1.1(i), as required.}
\end{aligned}$$

(ii) To prove that composites of thin elements are thin we introduce a subsidiary definition: if $x \in \mathsf{C}_n$ and $1 \leqslant j \leqslant n-1$, we say that x is *j-thin* if $\psi_1\psi_2\dots\psi_j x \in \varepsilon_1\mathsf{C}_{n-1}$. Thus, for elements of C_n, $(n-1)$-*thin* means *thin*. We take “$x \in \mathsf{C}_n$ is 0-thin” to mean $x \in \varepsilon_1\mathsf{C}_{n-1}$.

2.4. Lemma. *If $j \geqslant 1$, then x is j-thin if and only if $\psi_j x$ is $(j-1)$-thin.* ■

2.5. Lemma. *If $x = y \circ_i z$ in C_n and $y, z \in \varepsilon_j\,\mathsf{C}_{n-1}$ then $x \in \varepsilon_j\,\mathsf{C}_{n-1}$.*

(Note that in the case $i = j$, the hypotheses imply $x = y = z$.) ■

2.6. Lemma. *If $x \in \varepsilon_k\,\mathsf{C}_{n-1}$, then x is $(k-1)$-thin.*

Proof. $\psi_1\psi_2\dots\psi_{k-1}x = \psi_1\psi_2\dots\psi_{k-1}\varepsilon_k y = \varepsilon_1 y$ by (1.1)(i). ■

We now use induction on j to prove:

$$\text{If } a, b \in \mathsf{C}_n \text{ are } j\text{-thin and } c = a \circ_i b \text{ for some } 1 \leqslant i \leqslant n \text{ then } c \text{ is } j\text{-thin.} \tag{*}$$

The case $j = 0$ is contained in Lemma 2.5.

Suppose that (*) is true for $j = 0, 1, \dots, k-1$, where $1 \leqslant k \leqslant n-1$. We will deduce (*) for $j = k$.

Let $c = a \circ_i b$ in C_n and assume that a and b are k-thin. We examine $\psi_k c = \psi_k(a \circ_i b)$ in order to prove that it is $(k-1)$-thin.

Case 1: $k < i-1$ or $k > i$. In this case $\psi_k c = \psi_k a \circ_i \psi_k b$ and $\psi_k a$ and $\psi_k b$ are $(k-1)$-thin. By induction hypothesis, $\psi_k c$ is $(k-1)$-thin and so c is k-thin by 2.4.

Case 2: $k = i - 1$. Then

$$\begin{aligned}
\psi_k c = {} & \psi_{i-1}(a \circ_i b) \\
= {} & \begin{array}{|c|c|c|c|}\hline \ulcorner & a & b & \lrcorner \\ \hline\end{array} \qquad \begin{array}{l} \rightarrow i \\ \downarrow \\ i-1 \end{array} \\
= {} & \begin{array}{|c|c|c|c|c|}\hline \square & \shortparallel & \ulcorner & b & \lrcorner \\ \hline \ulcorner & a & \lrcorner & \shortparallel & \square \\ \hline\end{array} \qquad \begin{array}{l} \rightarrow i \\ \downarrow \\ i-1 \end{array} \\
= {} & \begin{array}{|c|c|}\hline \varepsilon_{i-1}u & \psi_{i-1}b \\ \hline \psi_{i-1}a & \varepsilon_{i-1}v \\ \hline\end{array} \qquad \begin{array}{l} \rightarrow i \\ \downarrow \\ i-1 \end{array} \\
= {} & \begin{array}{|c|c|}\hline \varepsilon_k u & \psi_k b \\ \hline \psi_k a & \varepsilon_k v \\ \hline\end{array} \qquad \begin{array}{l} \rightarrow i \\ \downarrow \\ i-1 \end{array}
\end{aligned}$$

where $u = \partial^-_{i-1}a, v = \partial^+_{i-1}b$. Now $\psi_k a$ and $\psi_k b$ are $(k-1)$-thin, by 2.4, and $\varepsilon_k u, \varepsilon_k v$ are $(k-1)$-thin, by 2.6. So $\psi_k c$, being a composite of these, is $(k-1)$-thin by induction hypothesis. Hence c is k-thin.

Case 3: $k = i$. This is similar using the formula

$$\begin{aligned}
\psi_k c = {} & \psi_i(a \circ_i b) \\
= {} & \begin{array}{|c|c|c|}\hline & a & \\ \cline{2-2} \ulcorner & & \lrcorner \\ & b & \\ \hline\end{array} \qquad \begin{array}{l} \rightarrow i+1 \\ \downarrow \\ i \end{array} \\
= {} & \begin{array}{|c|c|c|c|c|}\hline \ulcorner & = & a & \lrcorner & \shortparallel \\ \hline \shortparallel & \ulcorner & b & = & \lrcorner \\ \hline\end{array} \qquad \begin{array}{l} \rightarrow i+1 \\ \downarrow \\ i \end{array} \\
= {} & \begin{array}{|c|c|}\hline \psi_i a & \varepsilon_i s \\ \hline \varepsilon_i t & \psi_i b \\ \hline\end{array} \qquad \begin{array}{l} \rightarrow i+1 \\ \downarrow \\ i \end{array}
\end{aligned}$$

where $s = \partial^+_{i+1} b, t = \partial^-_{i+1} a$.

Thus, in all cases, c is k-thin, so the induction is complete. The case $j = n - 1$ of (*) completes the proof of Proposition 2.2. ■

2.7. Corollary. *Let C be a cubical ω-category (or m-category) with connections.*

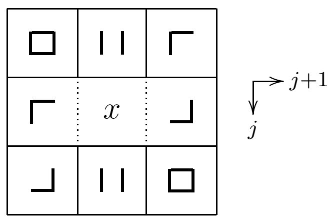
(i) *n-shells of the form $\boldsymbol{\varepsilon}_i c$ or $\boldsymbol{\Gamma}^\alpha_i c$ for $c \in \mathsf{C}_{n-1}$ are commutative.*

(ii) *Composites of commutative shells are commutative.* ■

From Proposition 2.2 we easily deduce

2.8. Theorem. *Let C be a cubical ω-category (or m-category) with connections. An element c in C_n is thin if and only if it is a composite of elements of the form $\varepsilon_i a$ or $\Gamma^\alpha_i a$ $(a \in \mathsf{C}_{n-1})$.*

Proof. Proposition 2.2 shows that any such composite is thin. For the converse, suppose that $c \in \mathsf{C}_n$ is thin. Then $\Psi c = \psi_1 \psi_2 \dots \psi_{n-1} c = \varepsilon_1 z$ for some $z \in \mathsf{C}_{n-1}$. Now we saw in the proof of Lemma 1.5 that any element $x \in \mathsf{C}_n$ can be written as a composite of $\psi_j x$ and elements of type $\varepsilon_i a, \Gamma^\alpha_i a$, namely

$x =$ 

(here the dotted segments indicate that $\psi_j x$ is first partitioned as shown and the 3×3 array is then completed so as to be composable.)

By iteration, x can be written as a composite of $\varepsilon_1 z$ and elements of type $\varepsilon_i a, \Gamma^\alpha_i a$. ■

2.9. Corollary. *An n-shell is commutative if and only if it can be written as a composite of shells of type $\boldsymbol{\varepsilon}_i a, \boldsymbol{\Gamma}^\alpha_i a$, where $a \in \mathsf{C}_{n-1}$.* ■

Remark 1. It is not clear from the proof of 2.8 whether a thin element can always be written as a composite of an *array* of elements of type $\varepsilon_i a, \Gamma^\alpha_i a$.

Remark 2. The particular folding map Ψ used to define thin elements depends on a number of choices and conventions. Theorem 2.8 shows that the notion of thinness is intrinsic and does not depend on these choices. Thus, for example, one might use $\Psi' = \psi_{n-1} \psi_{n-2} \dots \psi_1$ instead of Ψ but, by symmetry and Theorem 2.8, this would give the same concept. Similarly, the more complicated full folding operation Φ_n used in [2] gives the same concept of thinness (see Section 9 of that paper, especially Proposition 9.2 and Theorem 9.3). It is particularly reassuring that the concept of commutative shell is independent of the choice of foldings.

3. Thin structures and connections

We now consider cubical ω-categories (or m-categories) without the assumption of the extra structure of connections. Of course elements $\varepsilon_i a$ and shells $\boldsymbol{\varepsilon}_i a$ exist for such ω-categories so, in view of Theorem 2.8, it is not surprising that there is a close relationship between existence of thin elements and the existence of connections. An equivalence between them in the 2-dimensional case was proved in [7]. We extend this result to all dimensions.

Let C be a cubical ω-category (or m-category). Suppose that C has connections $\Gamma_i^-, \Gamma_i^+ : \mathsf{C}_{k-1} \to \mathsf{C}_k$, defined for all $k = 1, 2, \ldots, n-1$, satisfying the usual laws, up to that dimension (see [2].) We aim to characterize possible definitions of thin elements in C_n without first introducing more connections there.

As mentioned in Section 1, the n-category $(\mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_{n-1}, \square\, \mathsf{C}_{n-1})$ does have connections in dimension n as well as those in lower dimension, so we can define folding operations $\psi_1, \psi_2, \ldots, \psi_{n-1}, \Psi : \square\, \mathsf{C}_{n-1} \to \square\, \mathsf{C}_{n-1}$. As a result, the idea of a commutative n-shell is available, and we denote by $\boxbslash\, \mathsf{C}_{n-1}$ the set of commutative n-shells in $\square\, \mathsf{C}_{n-1}$. Clearly, by Corollary 2.7, $(\mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_{n-1}, \boxbslash\, \mathsf{C}_{n-1})$ is a sub-(cubical n-category with connections) of $(\mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_{n-1}, \square\, \mathsf{C}_{n-1})$.

Definition A *thin structure* on C_n is a morphism

$$\theta : (\mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_{n-1}, \boxbslash\, \mathsf{C}_{n-1}) \to (\mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_{n-1}, \mathsf{C}_n)$$

of cubical n-categories which is the identity on $\mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_{n-1}$. Such a thin structure defines "thin" elements in C_n, namely elements of the form $\theta(\mathbf{s})$ for a commutative n-shell s. Note that θ is necessarily injective on $\boxbslash\, \mathsf{C}_{n-1}$ (because it preserves faces) and the image of θ must be a sub-n-category of C. Consequently, every commutative n-shell in C has a unique "thin" filler in C_n, and the composites of "thin" elements are "thin". Furthermore, we may now define $\Gamma_i^\alpha : \mathsf{C}_{n-1} \to \mathsf{C}_n$ by $\Gamma_i^\alpha a = \theta \mathbf{\Gamma}_i^\alpha a$. Because θ preserves the lower dimensional Γ_i^α, $\circ_i$, ∂_i^α and ε_i, these newly defined Γ_i^α satisfy the required laws making $(\mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_{n-1}, \mathsf{C}_n)$ a cubical n-category with connections. The thin elements of C_n defined using these Γ_i^α are precisely the same as the "thin" elements defined by θ, because of Proposition 2.1.

Conversely, if we are given connections $\Gamma_i^\alpha : \mathsf{C}_{n-1} \to \mathsf{C}_n$ making $(\mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_n)$ a cubical n-category with connections, there is a unique thin structure θ with $\theta \mathbf{\Gamma}_i^\alpha a = \Gamma_i^\alpha a$ for all $a \in \mathsf{C}_{n-1}, \alpha \in \{+, -\}, i \in \{1, 2, \ldots, n-1\}$. This is because the morphism θ must map $\boldsymbol{\varepsilon}_i a$ to $\varepsilon_i a$ and therefore, when it is defined on the $\mathbf{\Gamma}_i^\alpha a$, it is uniquely determined on all commutative shells, by Corollary 2.9. That such a θ exists is easily deduced from Proposition 2.1, and the thin elements defined by the two methods again coincide. Hence

3.1. Theorem. *Let $\mathsf{C} = (\mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_n)$ be a cubical n-category and suppose that*

$$(\mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_{n-1})$$

has the structure of cubical $(n-1)$*-category with connections. Then there is a natural bijection between thin structures* $\theta : \boxslash\, \mathsf{C}_{n-1} \to \mathsf{C}_n$ *and sets of connections* $\Gamma_i^+, \Gamma_i^- : \mathsf{C}_{n-1} \to \mathsf{C}_n$ *making* $(\mathsf{C}_0, \mathsf{C}_1, \ldots, \mathsf{C}_n)$ *a cubical* n*-category with connections. The thin elements defined by the connections coincide in all cases with the thin elements defined by the corresponding* θ*.*

Remark It would be useful to have a simple description of what is meant by a cubical T-complex, that is a weak thin structure (in all dimensions) on a cubical *complex.* The aim would be to impose axioms on the set of "thin" elements in each dimension which would be equivalent to the existence of a cubical ω-category structure with connections. A simple description does exist in the groupoid case (see [8, 5]), but seems to be more difficult in the category case.

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