Theory and Applications of Categories, Vol. 15, No. 3, 2005, pp. 66–94.

MODEL STRUCTURES FOR HOMOTOPY OF INTERNAL CATEGORIES

T. EVERAERT, R.W. KIEBOOM AND T. VAN DER LINDEN

ABSTRACT. The aim of this paper is to describe Quillen model category structures on the category $\mathsf{Cat}\mathcal{C}$ of internal categories and functors in a given finitely complete category $\mathcal{C}$. Several non-equivalent notions of internal equivalence exist; to capture these notions, the model structures are defined relative to a given Grothendieck topology on $\mathcal{C}$.

Under mild conditions on $\mathcal{C}$, the regular epimorphism topology determines a model structure where we is the class of weak equivalences of internal categories (in the sense of Bunge and Paré). For a Grothendieck topos $\mathcal{C}$ we get a structure that, though different from Joyal and Tierney's, has an equivalent homotopy category. In case $\mathcal{C}$ is semi-abelian, these weak equivalences turn out to be homology isomorphisms, and the model structure on $\mathsf{Cat}\mathcal{C}$ induces a notion of homotopy of internal crossed modules. In case $\mathcal{C}$ is the category Gp of groups and homomorphisms, it reduces to the case of crossed modules of groups.

The trivial topology on a category $\mathcal{C}$ determines a model structure on $\mathsf{Cat}\mathcal{C}$ where we is the class of strong equivalences (homotopy equivalences), fib the class of internal functors with the homotopy lifting property, and cof the class of functors with the homotopy extension property. As a special case, the “folk” Quillen model category structure on the category $\mathsf{Cat} = \mathsf{Cat}\mathsf{Set}$ of small categories is recovered.

Contents

The first author's research is financed by a Ph.D. grant of the Institute of Promotion of Innovation through Science and Technology in Flanders (IWT-Vlaanderen).

Received by the editors 2004-11-05 and, in revised form, 2005-06-09.

Published on 2005-06-23 in the volume of articles from CT2004.

2000 Mathematics Subject Classification: Primary 18G55 18G50 18D35; Secondary 20J05 18G25 18G30.

Key words and phrases: internal category, Quillen model category, homotopy, homology.

1. Introduction

It is very well-known that the following choices of morphisms define a Quillen model category [38] structure—known as the "folk" structure—on the category Cat of small categories and functors between them: we is the class of equivalences of categories, cof the class of functors, injective on objects and fib the class of functors $p : E \longrightarrow B$ such that for any object e of E and any isomorphism $\beta : b \longrightarrow p(e)$ in B there exists an isomorphism ϵ with codomain e such that $p(\epsilon) = \beta$; this notion was introduced for groupoids by R. Brown in [14]. We are unaware of who first proved this fact; certainly, it is a special case of Joyal and Tierney's structure [30], but it was probably known before. A very explicit proof may be found in an unpublished paper by Rezk [39].

Other approaches to model category structures on Cat exist: Golasiński uses the homotopy theory of cubical sets to define a model structure on the category of pro-objects in Cat [21]; Thomason uses an adjunction to simplicial sets to acquire a model structure on Cat itself [41]. Both are very different from the folk structure. Related work includes folk-style model category structures on categories of 2-categories and bicategories (Lack [34], [33]) and a Thomason-style model category structure for 2-categories (Worytkiewicz, Hess, Parent and Tonks [42]).

If $\mathcal{E}$ is a Grothendieck topos there are two model structures on the category $\mathsf{Cat}\mathcal{E}$ of internal categories in $\mathcal{E}$. On can define the cofibrations and weak equivalences "as in Cat", and then define the fibrations via a right lifting property. This gives Joyal and Tierney's model structure [30]. Alternatively one can define the fibrations and weak equivalences "as in Cat" and than define the cofibrations via a left lifting property. This gives the model structure in this paper. The two structures coincide when every object is projective, as in the case $\mathcal{E} = \mathsf{Set}$.

More generally, if $\mathcal{C}$ is a full subcategory of $\mathcal{E}$, one gets a full embedding of $\mathsf{Cat}\mathcal{C}$ into $\mathsf{Cat}\mathcal{E}$, and one can then define the weak equivalences and fibrations in $\mathsf{Cat}\mathcal{C}$ "as in $\mathsf{Cat}\mathcal{E}$", and the cofibrations via a left lifting property. In particular one can do this when $\mathcal{E} = \mathsf{Sh}(\mathcal{C}, \mathcal{T})$, for a subcanonical Grothendieck topology $\mathcal{T}$ on an arbitrary category $\mathcal{C}$. Starting with such a $\mathcal{C}$, one may also view this as follows: the notions of fibration and weak equivalence in the folk structure may be internalized, provided that one specifies what is meant by essential surjectivity and the existence claim in the definition of fibration. Both of them require some notion of surjection; this will be provided by a topology $\mathcal{T}$ on $\mathcal{C}$.

There are three main obstructions on a site $(\mathcal{C}, \mathcal{T})$ for such a model category structure to exist. First of all, by definition, a model category has finite colimits. We give some sufficient conditions on $\mathcal{C}$ for $\mathsf{Cat}\mathcal{C}$ to be finitely cocomplete: either $\mathcal{C}$ is a topos with natural numbers object; or it is a locally finitely presentable category; or it is a finitely cocomplete regular Mal'tsev category. Next, in a model category, the class we of weak equivalences has the *two-out-of-three* property. This means that if two out of three morphisms f, g, $g \circ f$ belong to we then the third also belongs to we. A sufficient condition for this to be the case is that $\mathcal{T}$ is subcanonical. Finally, we want $\mathcal{T}$ to induce a weak factorization system in the following way. Let $Y_{\mathcal{T}} : \mathcal{C} \longrightarrow \mathsf{Sh}(\mathcal{C}, \mathcal{T})$ denote the composite of the Yoneda embedding with the sheafification functor. A morphism $p : E \longrightarrow B$ in

$\mathcal{C}$ will be called a $\mathcal{T}$*-epimorphism* if $Y_{\mathcal{T}}(p)$ is an epimorphism in $\mathsf{Sh}(\mathcal{C},\mathcal{T})$. The class of $\mathcal{T}$-epimorphisms is denoted by $\mathcal{E}_{\mathcal{T}}$. If $({}^{\square}\mathcal{E}_{\mathcal{T}},\mathcal{E}_{\mathcal{T}})$ forms a weak factorization system, we call it *the weak factorization system induced by* $\mathcal{T}$. This is the case when $\mathcal{C}$ has enough $\mathcal{E}_{\mathcal{T}}$-projective objects.

Joyal and Tierney's model structure [30] is defined as follows. Let $(\mathcal{C},\mathcal{T})$ be a site and $\mathsf{Sh}(\mathcal{C},\mathcal{T})$ its category of sheaves. Then a weak equivalence in $\mathsf{CatSh}(\mathcal{C},\mathcal{T})$ is a *weak equivalence* of internal categories in the sense of Bunge and Paré [15]; a cofibration is a functor, monic on objects; and a fibration has the right lifting property with respect to trivial cofibrations. Using the functor $Y_{\mathcal{T}}$ we could try to transport Joyal and Tierney's model structure from $\mathsf{CatSh}(\mathcal{C},\mathcal{T})$ to $\mathcal{C}$ as follows. For a subcanonical topology $\mathcal{T}$, the Yoneda embedding, considered as a functor $\mathcal{C} \longrightarrow \mathsf{Sh}(\mathcal{C},\mathcal{T})$, is equal to $Y_{\mathcal{T}}$. It follows that $Y_{\mathcal{T}}$ is full and faithful and preserves and reflects limits. Hence it induces a 2-functor $\mathsf{Cat}Y_{\mathcal{T}} : \mathsf{Cat}\mathcal{C} \longrightarrow \mathsf{CatSh}(\mathcal{C},\mathcal{T})$. Say that an internal functor $\boldsymbol{f} : \mathbf{A} \longrightarrow \mathbf{B}$ is an equivalence or cofibration, resp., if and only if so is the induced functor $\mathsf{Cat}Y_{\mathcal{T}}(\boldsymbol{f})$ in $\mathsf{Sh}(\mathcal{C},\mathcal{T})$, and define fibrations using the right lifting property.

We shall, however, consider a different structure on $\mathsf{Cat}\mathcal{C}$, mainly because of its application in the semi-abelian context. The weak equivalences, called $\mathcal{T}$*-equivalences*, are the ones described above. (As a consequence, in the case of a Grothendieck topos, we get a structure that is different from Joyal and Tierney's, but has an equivalent homotopy category.) Where Joyal and Tierney internalize the notion of cofibration, we do so for the fibrations: $\boldsymbol{p} : \mathbf{E} \longrightarrow \mathbf{B}$ is called a $\mathcal{T}$*-fibration* if and only if in the diagram

$$\begin{tikzcd}
\mathrm{iso}(E) \arrow[rd, dotted, "(\boldsymbol{r_p})_0"] \arrow[rrd, bend left, "\mathrm{iso}(p)_1"] \arrow[rdd, bend right, "\delta_1"'] & & \\
& (P_p)_0 \arrow[r, "\overline{p_0}"] \arrow[d, "\overline{\delta_1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{iso}(B) \arrow[d, "\delta_1"] \\
& E_0 \arrow[r, "p_0"'] & B_0
\end{tikzcd}$$

where $\mathrm{iso}(E)$ denotes the object of invertible arrows in the category $\mathbf{E}$, the induced universal arrow $(r_p)_0$ is in $\mathcal{E}_{\mathcal{T}}$. $\mathcal{T}$*-cofibrations* are defined using the left lifting property.

The paper is organized as follows. In Section 3, we study a cocylinder on $\mathsf{Cat}\mathcal{C}$ that characterizes homotopy of internal categories, i.e. such that two internal functors are homotopic if and only if they are naturally isomorphic. This cocylinder is used in Section 4 where we study the notion of internal equivalence, relative to the Grothendieck topology $\mathcal{T}$ on $\mathcal{C}$ defined above. For the trivial topology (the smallest one), a $\mathcal{T}$-equivalence is a *strong equivalence*, i.e. a homotopy equivalence with respect to the cocylinder. We recall that the strong equivalences are exactly the adjoint equivalences in the 2-category $\mathsf{Cat}\mathcal{C}$. If $\mathcal{T}$ is the regular epimorphism topology (generated by covering families consisting of a single pullback-stable regular epimorphism), $\mathcal{T}$-equivalences are the so-called *weak equivalences* [15]. There is no topology $\mathcal{T}$ on Set for which the $\mathcal{T}$-equivalences are the equivalences of Thomason's model structure on Cat: any adjoint is an equivalence in the latter sense, whereas a $\mathcal{T}$-equivalence is always fully faithful.

In Section 5 we study $\mathcal{T}$-fibrations. We prove—this is Theorem 5.5—that the $\mathcal{T}$-equivalences form the class $\mathrm{we}(\mathcal{T})$ and the $\mathcal{T}$-fibrations the class $\mathrm{fib}(\mathcal{T})$ of a model category structure on $\mathsf{Cat}\mathcal{C}$, as soon as the three obstructions mentioned above are taken into account.

Two special cases are subject to a more detailed study: in Section 6, the model structure induced by the regular epimorphism topology; in Section 7, the one induced by the trivial topology. In the first case we give special attention to the situation where $\mathcal{C}$ is a semi-abelian category, because then weak equivalences turn out to be homology isomorphisms, and the fibrations, Kan fibrations. Moreover, the category of internal categories in a semi-abelian category $\mathcal{C}$ is equivalent to Janelidze's category of internal crossed modules in $\mathcal{C}$ [25]. Reformulating the model structure in terms of internal crossed modules (as is done in Theorem 6.7) simplifies its description. If $\mathcal{C}$ is the category of groups and homomorphisms, we obtain the model structures on the category CatGp of categorical groups and the category XMod of crossed modules of groups, as described by Garzón and Miranda in [20].

The second case models the situation in Cat, equipped with the folk model structure, in the sense that here, weak equivalences are homotopy equivalences, fibrations have the homotopy lifting property (Proposition 7.3) and cofibrations the homotopy extension property (Proposition 7.6) with respect to the cocylinder defined in Section 3.

We used Borceux [6] and Mac Lane [35] for general category theoretic results. Lots of information concerning internal categories (and, of course, topos theory) may be found in Johnstone [27]. Other works on topos theory we used are Mac Lane and Moerdijk [36], Johnstone's Elephant [29] and SGA4 [1]. The standard work on "all things semi-abelian" is Borceux and Bourn's book [7].

Acknowledgements. Twice the scope of this paper has been considerably widened: first by Stephen Lack, who pointed out the difference between notions of internal equivalence, and incited us to consider weak equivalences; next by George Janelidze, who explained us how to use Grothendieck topologies instead of projective classes. Many thanks also to the referee, Dominique Bourn, Marino Gran and Tor Lowen for lots of very useful comments and suggestions.

2. Preliminaries

2.1. Internal categories and groupoids. If $\mathcal{C}$ is a finitely complete category then $\mathsf{RG}\mathcal{C}$ (resp. $\mathsf{Cat}\mathcal{C}$, $\mathsf{Grpd}\mathcal{C}$) denotes the category of internal reflexive graphs (resp. categories, groupoids) in $\mathcal{C}$. Let

$$\mathsf{Grpd}\mathcal{C} \xrightarrow{J} \mathsf{Cat}\mathcal{C} \xrightarrow{I} \mathsf{RG}\mathcal{C}$$

denote the forgetful functors. It is well-known that J embeds $\mathsf{Grpd}\mathcal{C}$ into $\mathsf{Cat}\mathcal{C}$ as a coreflective subcategory. Carboni, Pedicchio and Pirovano prove in [17] that, if $\mathcal{C}$ is Mal'tsev, then I is full, and J is an isomorphism. Moreover, an internal reflexive graph carries at most one structure of internal groupoid; hence $\mathsf{Grpd}\mathcal{C}$ may be viewed as a subcategory of $\mathsf{RG}\mathcal{C}$. As soon as $\mathcal{C}$ is, moreover, finitely cocomplete and regular, this subcategory is reflective (see Borceux and Bourn [7, Theorem 2.8.13]). In her article [37], M. C. Pedicchio shows that, if $\mathcal{C}$ is an exact Mal'tsev category with coequalizers, then the category $\mathsf{Grpd}\mathcal{C}$ is {regular epi}-reflective in $\mathsf{RG}\mathcal{C}$. This implies that $\mathsf{Grpd}\mathcal{C}$ is closed in $\mathsf{RG}\mathcal{C}$ under subobjects. In [22], Gran adds to this result that $\mathsf{Cat}\mathcal{C}$ is closed in $\mathsf{RG}\mathcal{C}$ under quotients. It follows that $\mathsf{Cat}\mathcal{C}$ is Birkhoff [26] in $\mathsf{RG}\mathcal{C}$. This, in turn, implies that if $\mathcal{C}$ is semi-abelian, so is $\mathsf{Cat}\mathcal{C}$ [18, Remark 5.4]. Gran and Rosický [23] extend these results to the context of modular varieties. For any variety $\mathcal{V}$, the category $\mathsf{RG}\mathcal{V}$ is equivalent to a variety. They show that, if, moreover, $\mathcal{V}$ is modular, $\mathcal{V}$ is Mal'tsev if and only if $\mathsf{Grpd}\mathcal{V}$ is a subvariety of $\mathsf{RG}\mathcal{V}$ [23, Proposition 2.3].

Let $\mathcal{C}$ be finitely complete. Sending an internal category

$$\mathbf{A} = \left(\begin{tikzcd} A_1 \times_{A_0} A_1 \arrow[r, "m"] & A_1 \arrow[r, shift left=2, "d_1"] \arrow[r, shift right=2, "d_0"'] & A_0 \arrow[l, "i" description] \end{tikzcd} \right)$$

to its object of objects A_0 and an internal functor $\boldsymbol{f} = (f_0, f_1) : \mathbf{A} \longrightarrow \mathbf{B}$ to its object morphism f_0 defines a functor $(\cdot)_0 : \mathsf{Cat}\mathcal{C} \longrightarrow \mathcal{C}$. Here $A_1 \times_{A_0} A_1$ denotes a pullback of d_1 along d_0; by convention, $d_1 \circ \mathrm{pr}_1 = d_0 \circ \mathrm{pr}_2$. It is easily seen that $(\cdot)_0$ has both a left and a right adjoint, resp. denoted L and $R : \mathcal{C} \longrightarrow \mathsf{Cat}\mathcal{C}$. Given an object X of $\mathcal{C}$ and an internal category $\mathbf{A}$, the natural bijection $\psi : \mathcal{C}(X, A_0) \longrightarrow \mathsf{Cat}\mathcal{C}(L(X), \mathbf{A})$ maps a morphism $f_0 : X \longrightarrow A_0$ to the internal functor

$$\boldsymbol{f} = \psi(f_0) = (f_0, i \circ f_0) : \mathbf{X} = L(X) \longrightarrow \mathbf{A},$$

where $\mathbf{X}$ is the *discrete* internal groupoid $d_0 = d_1 = i = m = 1_X : X \longrightarrow X$.

The right adjoint R maps an object X to the *indiscrete* groupoid $R(X)$ on X, i.e. $R(X)_0 = X$, $R(X)_1 = X \times X$, d_0 is the first and d_1 the second projection, i is the diagonal and $m : R(X)_1 \times_{R(X)_0} R(X)_1 \longrightarrow R(X)_1$ is the projection on the first and third factor.

Sending an internal category $\mathbf{A}$ to its object of arrows A_1 defines a functor $(\cdot)_1 : \mathsf{Cat}\mathcal{C} \longrightarrow \mathcal{C}$. Since limits in $\mathsf{Cat}\mathcal{C}$ are constructed by first taking the limit in $\mathsf{RG}\mathcal{C}$, then equipping the resulting reflexive graph with the unique category structure such that the universal cone in $\mathsf{RG}\mathcal{C}$ becomes a universal cone in $\mathsf{Cat}\mathcal{C}$, the functor $I : \mathsf{Cat}\mathcal{C} \longrightarrow \mathsf{RG}\mathcal{C}$ creates limits. Hence $(\cdot)_1$ is limit-preserving.

2.2. When is $\mathsf{Cat}\mathcal{C}$ (co)complete? One of the requirements for a category to be a model category is that it is finitely complete and cocomplete. Certainly the completeness poses no problems since it is a pretty obvious fact that $\mathsf{Cat}\mathcal{C}$ has all limits $\mathcal{C}$ has (see e.g. Johnstone [27, Lemma 2.16]); hence $\mathsf{Cat}\mathcal{C}$ is always finitely complete.

The case of cocompleteness is entirely different, because in general cocompleteness of $\mathcal{C}$ need not imply the existence of colimits in $\mathsf{Cat}\mathcal{C}$. (Conversely, $\mathcal{C}$ has all colimits $\mathsf{Cat}\mathcal{C}$ has, because $(\cdot)_0 : \mathsf{Cat}\mathcal{C} \longrightarrow \mathcal{C}$ has a right adjoint.) As far as we know, no characterization exists of those categories $\mathcal{C}$ which have a finitely cocomplete $\mathsf{Cat}\mathcal{C}$; we can only give sufficient conditions for this to be the case.

We get a first class of examples by assuming that $\mathcal{C}$ is a topos with a natural numbers object (or, in particular, a Grothendieck topos, like Joyal and Tierney do in [30]). As explained to us by George Janelidze, for a topos $\mathcal{C}$, the existence of a NNO is equivalent to $\mathsf{Cat}\mathcal{C}$ being finitely cocomplete. Certainly, if $\mathsf{Cat}\mathcal{C}$ has *countable* coproducts, then so has $\mathcal{C}$, hence it has a NNO: take a countable coproduct of 1. But the situation is much worse, because $\mathsf{Cat}\mathcal{C}$ does not even have arbitrary *coequalizers* if $\mathcal{C}$ lacks a NNO. Considering the ordinals 1 and $2 = 1 + 1$ (equipped with the appropriate order) as internal categories $\mathbf{1}$ and $\mathbf{2}$, the coproduct inclusions induce two functors $\mathbf{1} \longrightarrow \mathbf{2}$. If their coequalizer in $\mathsf{Cat}\mathcal{C}$ exists, it is the free internal monoid on 1, considered as a one-object category (its object of objects is equal to 1). But by Remark D5.3.4 in [29], this implies that $\mathcal{C}$ has a NNO! Conversely, in a topos with NNO, mimicking the construction in Set, the functor I may be seen to have a left adjoint; using this left adjoint, we may construct arbitrary finite colimits in $\mathsf{Cat}\mathcal{C}$.

Locally finitely presentable categories form a second class of examples. Indeed, every l.f.p. category is cocomplete, and if a category $\mathcal{C}$ if l.f.p., then so is $\mathsf{Cat}\mathcal{C}$—being a category of models of a sketch with finite diagrams [3, Proposition 1.53]. (Note that in particular, we again find the example of Grothendieck topoi.)

A third class is given by supposing that $\mathcal{C}$ is finitely cocomplete and regular Mal'tsev. Then $\mathsf{Cat}\mathcal{C} = \mathsf{Grpd}\mathcal{C}$ is a reflective [7, Theorem 2.8.13] subcategory of the functor category $\mathsf{RG}\mathcal{C}$, and hence has all finite colimits. This class, in a way, dualizes the first one, because the dual of any topos is a finitely cocomplete (exact) Mal'tsev category [16], [7, Example A.5.17], [10].

2.3. Weak factorization systems and model categories. In this paper we use the definition of model category as presented by Adámek, Herrlich, Rosický and Tholen [2]. For us, next to its elegance, the advantage over Quillen's original definition [38] is its explicit use of weak factorization systems. We briefly recall some important definitions.

2.4. Definition. *Let $l : A \longrightarrow B$ and $r : C \longrightarrow D$ be two morphisms of a category $\mathcal{C}$. l is said to have the* left lifting property *with respect to r and r is said to have the* right lifting property *with respect to l if every commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "l"'] & C \arrow[d, "r"] \\ B \arrow[r] \arrow[ur, dotted, "h"] & D \end{tikzcd}$$

has a lifting $h : B \longrightarrow C$. This situation is denoted $l \square r$ (and has nothing to do with double equivalence relations).

If $\mathcal{H}$ is a class of morphisms then $\mathcal{H}^{\square}$ is the class of all morphisms r with $h \square r$ for all $h \in \mathcal{H}$; dually, ${}^{\square}\mathcal{H}$ is the class of all morphisms l with $l \square h$ for all $h \in \mathcal{H}$.

2.5. Definition. *A* weak factorization system *in $\mathcal{C}$ is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms such that*

1. *every morphism f has a factorization $f = r \circ l$ with $r \in \mathcal{R}$ and $l \in \mathcal{L}$;*
2. *$\mathcal{L}^{\Box} = \mathcal{R}$ and $\mathcal{L} = {}^{\Box}\mathcal{R}$.*

In the presence of condition 1., 2. is equivalent to the conjunction of $\mathcal{L} \Box \mathcal{R}$ and the closedness in the category of arrows $\mathcal{C}^{\rightarrow}$ of $\mathcal{L}$ and $\mathcal{R}$ under the formation of retracts.

2.6. Definition. [Remark 3.6 in [2]] *Let $\mathcal{C}$ be a finitely complete and cocomplete category. A* model structure *on $\mathcal{C}$ is determined by three classes of morphisms,* fib *(*fibrations*),* cof *(*cofibrations*) and* we *(*weak equivalences*), such that*

1. we *has the* 2-out-of-3 property*, i.e. if two out of three morphisms f, g, $g \circ f$ belong to* we *then the third morphism also belongs to* we*, and* we *is closed under retracts in $\mathcal{C}^{\rightarrow}$;*
2. $(\mathrm{cof}, \mathrm{fib} \cap \mathrm{we})$ *and* $(\mathrm{cof} \cap \mathrm{we}, \mathrm{fib})$ *are weak factorization systems.*

A category equipped with a model structure is called a model category*. A morphism in* fib∩we *(resp.* cof∩we*) is called a* trivial fibration *(resp.* trivial cofibration*). Let* 0 *denote an initial and* 1 *a terminal object of $\mathcal{C}$. A* cofibrant *object A is such that the unique arrow $0 \longrightarrow A$ is a cofibration; A is called* fibrant *if $A \longrightarrow 1$ is in* fib*.*

2.7. Grothendieck topologies. We shall consider model category structures on $\mathsf{Cat}\mathcal{C}$ which are defined relative to some *Grothendieck topology* $\mathcal{T}$ on $\mathcal{C}$. Recall that such is a function that assigns to each object C of $\mathcal{C}$ a collection $\mathcal{T}(C)$ of sieves on C (a sieve S on C being a class of morphisms with codomain C such that $f \in S$ implies that $f \circ g \in S$, whenever this composite exists), satisfying

1. the maximal sieve on C is in $\mathcal{T}(C)$;
2. (stability axiom) if $S \in \mathcal{T}(C)$ then its pullback $h^*(S)$ along any arrow $h : D \longrightarrow C$ is in $\mathcal{T}(D)$;
3. (transitivity axiom) if $S \in \mathcal{T}(C)$ and R is a sieve on $\mathcal{C}$ such that $h^*(R) \in \mathcal{T}(D)$ for all $h : D \longrightarrow C$ in S, then $R \in \mathcal{T}(C)$.

A sieve in some $\mathcal{T}(C)$ is called *covering*. We would like to consider sheaves over arbitrary sites $(\mathcal{C}, \mathcal{T})$, not just small ones (i.e. where $\mathcal{C}$ is a small category). For this to work flawlessly, a standard solution is to use the theory of universes, as introduced in [1]. The idea is to extend the Zermelo-Fraenkel axioms of set theory with the axiom (U) “every set is an element of a universe”, where a universe $\mathcal{U}$ is a set satisfying

1. if $x \in \mathcal{U}$ and $y \in x$ then $y \in \mathcal{U}$;
2. if $x, y \in \mathcal{U}$ then $\{x, y\} \in \mathcal{U}$;

3. if $x \in \mathcal{U}$ then the powerset $\mathcal{P}(x)$ of x is in $\mathcal{U}$;

4. if $I \in \mathcal{U}$ and $(x_i)_{i\in I}$ is a family of elements of $\mathcal{U}$ then $\bigcup_{i\in I} x_i \in \mathcal{U}$.

A set is called $\mathcal{U}$-*small* if it has the same cardinality as an element of $\mathcal{U}$. (We sometimes, informally, use the word *class* for a set that is not $\mathcal{U}$-small.) We shall always consider universes containing the set $\mathbb{N}$ of natural numbers, and work in ZFCU (with the ZF axioms + the axiom of choice + the universe axiom). A category consists of a set of objects and a set of arrows with the usual structure; $\mathcal{U}\mathsf{Set}$ ($\mathcal{U}\mathsf{Cat}$) denotes the category whose objects are elements of $\mathcal{U}$ (categories with sets of objects and arrows in $\mathcal{U}$) and whose arrows are functions (functors) between them. Now given a site $(\mathcal{C}, \mathcal{T})$, the category $\mathcal{C}$ is in $\mathcal{U}\mathsf{Cat}$ for some universe $\mathcal{U}$; hence it makes sense to consider the category of presheaves $\mathcal{U}\mathsf{Pr}\mathcal{C} = \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{U}\mathsf{Set})$ and the associated category $\mathcal{U}\mathsf{Sh}(\mathcal{C}, \mathcal{T})$ of sheaves. In what follows, we shall not mention the universe $\mathcal{U}$ we are working with and just write Set, Cat, $\mathsf{Pr}\mathcal{C}$, $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$, etc.

2.8. Examples. On a finitely complete category $\mathcal{C}$, the *regular epimorphism topology* is generated by the following basis: a covering family on an object A consists of a single pullback-stable regular epimorphism $A' \longrightarrow A$. It is easily seen that this topology is subcanonical, i.e. that every representable functor is a sheaf. Hence the Yoneda embedding $Y : \mathcal{C} \longrightarrow \mathsf{Pr}\mathcal{C}$ may be considered as a functor $\mathcal{C} \longrightarrow \mathsf{Sh}(\mathcal{C}, \mathcal{T})$.

The *trivial topology* is the smallest one: the only covering sieve on an object A is the sieve of all morphisms with codomain A. Every presheaf if a sheaf for the trivial topology.

The largest topology is called *cotrivial*: every sieve is covering. The only sheaf for this topology is the terminal presheaf.

We shall consider the weak factorization system on a category $\mathcal{C}$, generated by a Grothendieck topology in the following way.

2.9. Definition. *Let $\mathcal{T}$ be a topology on a category $\mathcal{C}$ and let $Y_{\mathcal{T}} : \mathcal{C} \longrightarrow \mathsf{Sh}(\mathcal{C}, \mathcal{T})$ denote the composite of the Yoneda embedding $Y : \mathcal{C} \longrightarrow \mathsf{Pr}\mathcal{C}$ with the sheafification functor $\mathsf{Pr}\mathcal{C} \longrightarrow \mathsf{Sh}(\mathcal{C}, \mathcal{T})$. A morphism $p : E \longrightarrow B$ will be called a* $\mathcal{T}$-epimorphism *if $Y_{\mathcal{T}}(p)$ is an epimorphism in $\mathsf{Sh}(\mathcal{C}, \mathcal{T})$. The class of $\mathcal{T}$-epimorphisms is denoted by $\mathcal{E}_{\mathcal{T}}$. If $({}^{\square}\mathcal{E}_{\mathcal{T}}, \mathcal{E}_{\mathcal{T}})$ forms a weak factorization system, we call it* the weak factorization system induced by $\mathcal{T}$.

2.10. Remark. Note that if $\mathcal{T}$ is subcanonical, then $Y_{\mathcal{T}}$ is equal to the Yoneda embedding; hence it is a full and faithful functor.

2.11. Remark. The only condition a subcanonical $\mathcal{T}$ needs to fulfil, for it to induce a model structure on $\mathsf{Cat}\mathcal{C}$, is that $({}^{\square}\mathcal{E}_{\mathcal{T}}, \mathcal{E}_{\mathcal{T}})$ is a weak factorization system. When $\mathcal{C}$ has binary coproducts, this is equivalent to $\mathcal{C}$ having enough $\mathcal{E}_{\mathcal{T}}$-projectives [2].

One way of avoiding universes is by avoiding sheaves: indeed, $\mathcal{T}$-epimorphisms have a well-known characterization in terms of the topology alone.

2.12. PROPOSITION. [Corollary III.7.5 and III.7.6 in [36]] *Let $\mathcal{T}$ be a topology on a category $\mathcal{C}$. Then a morphism $p : E \longrightarrow B$ in $\mathcal{C}$ is $\mathcal{T}$-epic if and only if for every $g : X \longrightarrow B$ there exists a covering family $(f_i : U_i \longrightarrow X)_{i\in I}$ and a family of morphisms $(u_i : U_i \longrightarrow E)_{i\in I}$ such that for every $i \in I$, $p\circ u_i = g\circ f_i$.* ∎

2.13. EXAMPLES. If $\mathcal{T}$ is the trivial topology, it is easily seen that the $\mathcal{T}$-epimorphisms are exactly the split epimorphisms.

When $\mathcal{T}$ is the cotrivial topology, every morphism is $\mathcal{T}$-epic.

In case $\mathcal{T}$ is the regular epimorphism topology, a $\mathcal{T}$-epimorphism is nothing but a pullback-stable regular epimorphism: certainly, every pullback-stable regular epimorphism is $\mathcal{T}$-epic; conversely, one shows that if $p\circ u = f$ is a pullback-stable regular epimorphism then so is p.

3. A cocylinder on $\mathsf{Cat}\mathcal{C}$

One way of defining homotopy in a category $\mathcal{C}$ is relative to a *cocylinder* on $\mathcal{C}$. Recall (e.g. from Kamps [31] or Kamps and Porter [32]) that this is a structure

$$((\cdot)^I : \mathcal{C} \longrightarrow \mathcal{C}, \quad \epsilon_0, \epsilon_1 : (\cdot)^I \Longrightarrow 1_{\mathcal{C}}, \quad s : 1_{\mathcal{C}} \Longrightarrow (\cdot)^I)$$

such that $\epsilon_0 \bullet s = \epsilon_1 \bullet s = 1_{1_{\mathcal{C}}}$. Given a cocylinder $((\cdot)^I, \epsilon_0, \epsilon_1, s)$ on $\mathcal{C}$, two morphisms $f, g : X \longrightarrow Y$ are called *homotopic* (or, more precisely, *right homotopic*, to distinguish with the notion of *left* homotopy defined using a cylinder) if there exists a morphism $H : X \longrightarrow Y^I$ such that $\epsilon_0(Y)\circ H = f$ and $\epsilon_1(Y)\circ H = g$. The morphism H is called a *homotopy from f to g* and the situation is denoted $H : f \simeq g$.

Let $\mathcal{C}$ be a finitely complete category. In this section, we describe a cocylinder on $\mathsf{Cat}\mathcal{C}$ such that two internal functors are homotopic if and only if they are naturally isomorphic. We follow the situation in Cat very closely. Let $\mathcal{I}$ denote the *interval groupoid*, i.e. the category with two objects $\{0, 1\}$ and the following four arrows.

$$\begin{tikzcd} 0 \arrow[loop left, "1_0"] \arrow[r, bend left, "\tau"] & 1 \arrow[loop right, "1_1"] \arrow[l, bend left, "\tau^{-1}"] \end{tikzcd}$$

Then putting $\mathcal{C}^{\mathcal{I}} = \mathsf{Fun}(\mathcal{I}, \mathcal{C})$, the category of functors from $\mathcal{I}$ to $\mathcal{C}$, defines a cocylinder on Cat. It is easily seen that an object of $\mathcal{C}^{\mathcal{I}}$, being a functor $\mathcal{I} \longrightarrow \mathcal{C}$, is determined by the choice of an isomorphism in $\mathcal{C}$; a morphism of $\mathcal{C}^{\mathcal{I}}$, being a natural transformation $\mu : F \Longrightarrow G : \mathcal{I} \longrightarrow \mathcal{C}$ between two such functors, is determined by a commutative square

$$\begin{tikzcd} F(0) \arrow[r, "\mu_0"] \arrow[d, "F(\tau)"', "\cong"] & G(0) \arrow[d, "\cong"', "G(\tau)"] \\ F(1) \arrow[r, "\mu_1"'] & G(1) \end{tikzcd}$$

in $\mathcal{C}$ with invertible downward-pointing arrows.

It is well-known that the category $\mathsf{Grpd}\mathcal{C}$ is coreflective in $\mathsf{Cat}\mathcal{C}$; let iso : $\mathsf{Cat}\mathcal{C}\longrightarrow\mathsf{Grpd}\mathcal{C}$ denote the right adjoint of the inclusion $J : \mathsf{Grpd}\mathcal{C}\longrightarrow\mathsf{Cat}\mathcal{C}$. Given a category $\mathbf{A}$ in $\mathcal{C}$, the functor iso may be used to describe the object $\mathrm{iso}(A)$ of "isomorphisms in $\mathbf{A}$" (cf. Bunge and Paré [15]) as the object of arrows of $\mathrm{iso}(\mathbf{A})$, the couniversal groupoid associated with $\mathbf{A}$. The counit $\epsilon_{\mathbf{A}} : \mathrm{iso}(\mathbf{A})\longrightarrow\mathbf{A}$ at $\mathbf{A}$ is a monomorphism, and will be denoted

$$\begin{tikzcd}
\mathrm{iso}(A) \arrow[r, tail, "j"] \arrow[d, shift right=2, "\delta_0"'] \arrow[d, shift left=2, "\delta_1"] & A_1 \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] \\
A_0 \arrow[u] \arrow[r, equal] & A_0. \arrow[u, "i"]
\end{tikzcd}$$

The object A_1^I of "commutative squares with invertible downward-pointing arrows in $\mathbf{A}$" is given by the the pullback

$$\begin{tikzcd}
A_1^I \arrow[r, "\mathrm{pr}_2"] \arrow[d, "\mathrm{pr}_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & A_1 \times_{A_0} \mathrm{iso}(A) \arrow[d, "m\circ(1_{A_1}\times_{1_{A_0}} j)"] \\
\mathrm{iso}(A) \times_{A_0} A_1 \arrow[r, "m\circ(j\times_{1_{A_0}} 1_{A_1})"'] & A_1.
\end{tikzcd}$$

The unique morphism induced by a cone on this diagram, represented by $(f, g, h, k) : X \longrightarrow \mathrm{iso}(A) \times A_1 \times A_1 \times \mathrm{iso}(A)$, will be denoted by

$$\begin{tikzcd}
\cdot \arrow[r, "h"] \arrow[d, "f"'] & \cdot \arrow[d, "k"] \\
\cdot \arrow[r, "g"'] & \cdot
\end{tikzcd} : X \longrightarrow A_1^I.$$

Put $A_0^I = \mathrm{iso}(A)$. Horizontal composition

$$\mathrm{comp} = \begin{tikzcd}
\cdot \arrow[r, "m\circ(\mathrm{pr}_1\circ\mathrm{pr}_2\times_{\delta_0}\mathrm{pr}_1\circ\mathrm{pr}_2)"] \arrow[d, "\mathrm{pr}_1\circ\mathrm{pr}_1\circ\mathrm{pr}_1"'] & \cdot \arrow[d, "\mathrm{pr}_2\circ\mathrm{pr}_2\circ\mathrm{pr}_2"] \\
\cdot \arrow[r, "m\circ(\mathrm{pr}_2\circ\mathrm{pr}_1\times_{\delta_1}\mathrm{pr}_2\circ\mathrm{pr}_1)"'] & \cdot
\end{tikzcd}, \qquad \mathrm{id} = \begin{tikzcd}
\cdot \arrow[r, "i\circ\delta_0"] \arrow[d, "1_{\mathrm{iso}(A)_1}"'] & \cdot \arrow[d, "1_{\mathrm{iso}(A)_1}"] \\
\cdot \arrow[r, "i\circ\delta_1"'] & \cdot
\end{tikzcd},$$

$\mathrm{dom} = \mathrm{pr}_1 \circ \mathrm{pr}_1$ and $\mathrm{cod} = \mathrm{pr}_2 \circ \mathrm{pr}_2$ now define an internal category

$$\mathbf{A^I} = \Big(A_1^I \times_{A_0^I} A_1^I \xrightarrow{\mathrm{comp}} A_1^I \overset{\xrightarrow{\mathrm{cod}}}{\underset{\xrightarrow[\mathrm{dom}]{}}{\xleftarrow{\mathrm{id}}}} A_0^I \Big).$$

Thus we get a functor $(\cdot)^{\mathbf{I}} : \mathsf{Cat}\mathcal{C} \longrightarrow \mathsf{Cat}\mathcal{C}$. Putting

$$\epsilon_0(\mathbf{A}) = (\delta_0, \mathrm{pr}_1 \circ \mathrm{pr}_2) : \mathbf{A^I} \longrightarrow \mathbf{A}, \qquad \epsilon_1(\mathbf{A}) = (\delta_1, \mathrm{pr}_2 \circ \mathrm{pr}_1) : \mathbf{A^I} \longrightarrow \mathbf{A}$$

and $s(\mathbf{A}) = (\iota, s(\mathbf{A})_1)$ with

$$s(\mathbf{A})_1 = \begin{tikzcd} \cdot \arrow[r, "1_{A_1}"] \arrow[d, "\iota\circ d_0"'] & \cdot \arrow[d, "\iota\circ d_1"] \\ \cdot \arrow[r, "1_{A_1}"'] & \cdot \end{tikzcd} : A_1 \longrightarrow A_1^I$$

gives rise to natural transformations $\epsilon_0, \epsilon_1 : (\cdot)^{\mathbf{I}} \Longrightarrow 1_{\mathsf{Cat}\mathcal{C}}$ and $s : 1_{\mathsf{Cat}\mathcal{C}} \Longrightarrow (\cdot)^{\mathbf{I}}$ such that $\epsilon_0 \bullet s = \epsilon_1 \bullet s = 1_{1_{\mathsf{Cat}\mathcal{C}}}$.

Recall that, for internal functors $\boldsymbol{f}, \boldsymbol{g} : \mathbf{A} \longrightarrow \mathbf{B}$, an *internal natural transformation* $\mu : \boldsymbol{f} \Longrightarrow \boldsymbol{g}$ is a morphism $\mu : A_0 \longrightarrow B_1$ such that $d_0 \circ \mu = f_0$, $d_1 \circ \mu = g_0$ and $m \circ (f_1, \mu \circ d_1) = m \circ (\mu \circ d_0, g_1)$. Categories, functors and natural transformations in a given category $\mathcal{C}$ form a 2-category $\mathsf{Cat}\mathcal{C}$. For two internal natural transformations $\mu : \boldsymbol{f} \Longrightarrow \boldsymbol{g}$ and $\nu : \boldsymbol{g} \Longrightarrow \boldsymbol{h}$, $\nu \bullet \mu = m \circ (\nu, \mu)$ is their (vertical) composition; for $\mu : \boldsymbol{f} \Longrightarrow \boldsymbol{g} : \mathbf{A} \longrightarrow \mathbf{B}$ and $\mu' : \boldsymbol{f}' \Longrightarrow \boldsymbol{g}' : \mathbf{B} \longrightarrow \mathbf{C}$, the (horizontal) composition is

$$\mu' \circ \mu = m \circ (\mu' \circ f_0, g_1' \circ \mu) = m \circ (f_1' \circ \mu, \mu' \circ g_0) : \boldsymbol{f}' \circ \boldsymbol{f} \Longrightarrow \boldsymbol{g}' \circ \boldsymbol{g} : \mathbf{A} \longrightarrow \mathbf{C}.$$

An internal natural transformation $\mu : \boldsymbol{f} \Longrightarrow \boldsymbol{g} : \mathbf{A} \longrightarrow \mathbf{B}$ is an *internal natural isomorphism* if and only if an internal natural transformation $\mu^{-1} : \boldsymbol{g} \Longrightarrow \boldsymbol{f}$ exists such that $\mu \bullet \mu^{-1} = 1_{\boldsymbol{g}} = i \circ g_0$ and $\mu^{-1} \bullet \mu = 1_{\boldsymbol{f}} = i \circ f_0$. Hence an internal natural isomorphism is nothing but an isomorphism in a hom-category $\mathsf{Cat}\mathcal{C}(\mathbf{A}, \mathbf{B})$. Moreover, this is the case, exactly when μ factors over $j : \mathrm{iso}(A) \longrightarrow A_1$. Note that, if $\mathbf{B}$ is a groupoid, and $\mathrm{tw} : B_1 \longrightarrow B_1$ denotes its "twisting isomorphism", then $\mu^{-1} = \mathrm{tw} \circ \mu$.

3.1. Example. For every internal category $\mathbf{A}$ of $\mathcal{C}$, the morphism

$$\begin{tikzcd} \cdot \arrow[r, "i\circ\delta_0"] \arrow[d, "\iota\circ\delta_0"'] & \cdot \arrow[d, "1_{\mathrm{iso}(A)}"] \\ \cdot \arrow[r, "j"'] & \cdot \end{tikzcd} : \mathrm{iso}(A) = A_0^I \longrightarrow A_1^I$$

is a natural isomorphism $s(\mathbf{A}) \circ \epsilon_0(\mathbf{A}) \Longrightarrow 1_{\mathbf{A}^{\mathbf{I}}} : \mathbf{A}^{\mathbf{I}} \longrightarrow \mathbf{A}^{\mathbf{I}}$.

As expected:

3.2. Proposition. [cf. Exercise 2.3 in Johnstone [27]] *If $\mu : \boldsymbol{f} \Longrightarrow \boldsymbol{g} : \mathbf{A} \longrightarrow \mathbf{B}$ is an internal natural isomorphism, then $\boldsymbol{H} = (\mu, H_1) : \mathbf{A} \longrightarrow \mathbf{B}^{\mathbf{I}}$ with*

$$H_1 = \begin{tikzcd} \cdot \arrow[r, "f_1"] \arrow[d, "\mu\circ d_0"'] & \cdot \arrow[d, "\mu\circ d_1"] \\ \cdot \arrow[r, "g_1"'] & \cdot \end{tikzcd} : A_1 \longrightarrow B_1^I$$

is a homotopy $H : f \simeq g$. If $\boldsymbol{H} : \mathbf{A} \longrightarrow \mathbf{B}^{\mathbf{I}}$ is a homotopy $\boldsymbol{f} \simeq \boldsymbol{g} : \mathbf{A} \longrightarrow \mathbf{B}$ then $j \circ H_0 : A_0 \longrightarrow B_1$ is an internal natural isomorphism $\boldsymbol{f} \Longrightarrow \boldsymbol{g}$. Hence the homotopy relation $\simeq$ is an equivalence relation on every $\mathsf{Cat}\mathcal{C}(\mathbf{A}, \mathbf{B})$. ∎

3.3. Proposition. *For any internal category* $\mathbf{A}$ *of* $\mathcal{C}$, *putting* $d_0 = \epsilon_0(\mathbf{A}), d_1 = \epsilon_1(\mathbf{A}) : \mathbf{A}^{\mathbf{I}} \longrightarrow \mathbf{A}$ *and* $i = s(\mathbf{A}) : \mathbf{A} \longrightarrow \mathbf{A}^{\mathbf{I}}$ *defines a reflexive graph in* $\mathsf{Cat}\mathcal{C}$ *which carries a structure of internal groupoid; hence it is a double category in* $\mathcal{C}$. ∎

The following well-known construction will be very useful.

3.4. Definition. [Mapping path space construction] *Let* $\boldsymbol{f} : \mathbf{A} \longrightarrow \mathbf{B}$ *be an internal functor. Pulling back the split epimorphism* $\epsilon_1(\mathbf{B})$ *along* $\boldsymbol{f}$ *yields the following diagram, where both the upward and downward pointing squares commute, and* $\overline{\epsilon_1(\mathbf{B})}\circ\overline{s(\mathbf{B})} = 1_{\mathbf{A}}$.

$$\begin{tikzcd}
\mathbf{P}_{\boldsymbol{f}} \arrow[r, "\bar{\boldsymbol{f}}"] \arrow[d, shift left, "\overline{\epsilon_1(\mathbf{B})}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{B}^{\mathbf{I}} \arrow[d, shift left, "\epsilon_1(\mathbf{B})"] \\
\mathbf{A} \arrow[r, "\boldsymbol{f}"'] \arrow[u, shift left, "\overline{s(\mathbf{B})}"] & \mathbf{B} \arrow[u, shift left, "s(\mathbf{B})"]
\end{tikzcd} \tag{I}$$

The object $\mathbf{P}_{\boldsymbol{f}}$ *is called a* mapping path space *of* $\boldsymbol{f}$. *We denote the universal arrow induced by the commutative square* $\epsilon_1(\mathbf{B})\circ\boldsymbol{f}^{\mathbf{I}} = \boldsymbol{f}\circ\epsilon_1(\mathbf{A})$ *by* $\boldsymbol{r_f} : \mathbf{A}^{\mathbf{I}} \longrightarrow \mathbf{P}_{\boldsymbol{f}}$.

4. $\mathcal{T}$-equivalences

Let $\mathcal{C}$ be a finitely complete category. Recall (e.g. from Bunge and Paré [15]) that an internal functor $\boldsymbol{f} : \mathbf{A} \longrightarrow \mathbf{B}$ in $\mathcal{C}$ is called *full* (resp. *faithful*, *fully faithful*) when, for any internal category $\mathbf{X}$ of $\mathcal{C}$, the functor

$$\mathsf{Cat}\mathcal{C}(\mathbf{X}, \boldsymbol{f}) : \mathsf{Cat}\mathcal{C}(\mathbf{X}, \mathbf{A}) \longrightarrow \mathsf{Cat}\mathcal{C}(\mathbf{X}, \mathbf{B})$$

is full (resp. faithful, fully faithful). There is the following well-known characterization of full and faithful functors.

4.1. Proposition. *Let* $\boldsymbol{f} : \mathbf{A} \longrightarrow \mathbf{B}$ *be a functor in a finitely complete category* $\mathcal{C}$.

1. *If* $\boldsymbol{f}$ *is full, then the square*

$$\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "{(d_0,d_1)}"'] & B_1 \arrow[d, "{(d_0,d_1)}"] \\
A_0 \times A_0 \arrow[r, "f_0 \times f_0"'] & B_0 \times B_0
\end{tikzcd} \tag{II}$$

   is a weak pullback in $\mathcal{C}$.

2. $\boldsymbol{f}$ *is faithful if and only if the morphisms* $d_0, d_1 : A_1 \longrightarrow A_0$ *together with* $f_1 : A_1 \longrightarrow B_1$ *form a monosource.*

3. $\boldsymbol{f}$ *is fully faithful if and only if* II *is a pullback.* ∎

4.2. Remark. Since fully faithful functors reflect isomorphisms, the Yoneda Lemma (e.g. in the form of Metatheorem 0.1.3 in [7]) implies that the functor iso : $\mathsf{Cat}\mathcal{C} \longrightarrow \mathsf{Grpd}\mathcal{C}$ preserves fully faithful internal functors. Quite obviously, they are also stable under pulling back.

The following lifting property of fully faithful functors will prove very useful.

4.3. Proposition. [cf. the proof of Lemma 2.1 in Joyal and Tierney [30]] *Consider a commutative square*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{j}"'] & \mathbf{E} \arrow[d, "\boldsymbol{p}"] \\ \mathbf{X} \arrow[r, "\boldsymbol{g}"'] \arrow[ur, dotted, "\boldsymbol{h}"] & \mathbf{B} \end{tikzcd} \tag{III}$$

in $\mathsf{Cat}\mathcal{C}$ *with* $\boldsymbol{p}$ *fully faithful. This square has a lifting* $\boldsymbol{h} : \mathbf{X} \longrightarrow \mathbf{E}$ *if and only if there exists a morphism* $h_0 : X_0 \longrightarrow E_0$ *such that* $p_0 \circ h_0 = g_0$ *and* $h_0 \circ j_0 = f_0$. ■

For us, the notion of essential surjectivity has several relevant internalizations, resulting in different notions of internal equivalence. Our weak equivalences in $\mathsf{Cat}\mathcal{C}$ will be defined relative to some class of morphisms $\mathcal{E}$ in $\mathcal{C}$, which in practice will be the class of $\mathcal{T}$-epimorphisms for a topology $\mathcal{T}$ on $\mathcal{C}$.

4.4. Definition. *Let* $\mathcal{E}$ *be a class of morphisms and* $\boldsymbol{f} : \mathbf{A} \longrightarrow \mathbf{B}$ *an internal functor in* $\mathcal{C}$. *If the morphism* $\delta_0 \circ \overline{f_0}$ *in the diagram*

$$\begin{tikzcd} (P_f)_0 \arrow[r, "\overline{f_0}"] \arrow[d, "\overline{\delta_1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{iso}(B) \arrow[r, "\delta_0"] \arrow[d, "\delta_1"] & B_0 \\ A_0 \arrow[r, "f_0"'] & B_0 & \end{tikzcd}$$

is in $\mathcal{E}$, *then* $\boldsymbol{f}$ *is called* essentially $\mathcal{E}$-surjective. *An* $\mathcal{E}$-equivalence *is an internal functor which is full, faithful and essentially* $\mathcal{E}$*-surjective. If* $\mathcal{E} = \mathcal{E}_{\mathcal{T}}$ *is the class of* $\mathcal{T}$*-epimorphisms for a Grothendieck topology* $\mathcal{T}$ *on* $\mathcal{C}$, *the respective notions become* essentially $\mathcal{T}$-surjective *and* $\mathcal{T}$-equivalence. *The class of* $\mathcal{T}$*-equivalences for a topology* $\mathcal{T}$ *is denoted by* $\mathrm{we}(\mathcal{T})$.

4.5. Example. In case $\mathcal{T}$ is the cotrivial topology, any functor is essentially $\mathcal{T}$-surjective, and hence the $\mathcal{T}$-equivalences are exactly the fully faithful functors.

4.6. Example. If $\mathcal{T}$ is the trivial topology then an internal functor $\boldsymbol{f} : \mathbf{A} \longrightarrow \mathbf{B}$ is essentially $\mathcal{T}$-surjective if and only if the functor $\mathsf{Cat}\mathcal{C}(\mathbf{X}, \boldsymbol{f})$ is essentially surjective for all $\mathbf{X}$. If $\boldsymbol{f}$ is moreover fully faithful, it is called a *strong equivalence*. This name is justified by the obvious fact that a strong equivalence is a $\mathcal{T}$-equivalence for every topology $\mathcal{T}$. If $\boldsymbol{f}$ is a strong equivalence, a functor $\boldsymbol{g} : \mathbf{B} \longrightarrow \mathbf{A}$ exists and natural isomorphisms $\epsilon : \boldsymbol{f} \circ \boldsymbol{g} \Longrightarrow 1_{\mathbf{B}}$ and $\eta : 1_{\mathbf{A}} \Longrightarrow \boldsymbol{g} \circ \boldsymbol{f}$; hence $\boldsymbol{f}$ is a homotopy equivalence with respect to the cocylinder from Section 3. There is even more:

Recall that an *internal adjunction* is a quadruple

$$(\boldsymbol{f} : \mathbf{A} \longrightarrow \mathbf{B}, \quad \boldsymbol{g} : \mathbf{B} \longrightarrow \mathbf{A}, \quad \epsilon : \boldsymbol{f} \circ \boldsymbol{g} \Longrightarrow 1_{\mathbf{B}}, \quad \eta : 1_{\mathbf{A}} \Longrightarrow \boldsymbol{g} \circ \boldsymbol{f})$$

such that the *triangular identities* $(\epsilon \circ 1_{\boldsymbol{f}}) \bullet (1_{\boldsymbol{f}} \circ \eta) = 1_{\boldsymbol{f}}$ and $(1_{\boldsymbol{g}} \circ \epsilon) \bullet (\eta \circ 1_{\boldsymbol{g}}) = 1_{\boldsymbol{g}}$ hold. Then $\boldsymbol{f}$ is *left adjoint* to $\boldsymbol{g}$, $\boldsymbol{g}$ *right adjoint* to $\boldsymbol{f}$, ϵ the *counit* and η the *unit* of the adjunction. Using J. W. Gray's terminology [24], we shall call *lali* a left adjoint left inverse functor, and, dually, *rari* a right adjoint right inverse functor. In case $\boldsymbol{f}$ is left adjoint left inverse to $\boldsymbol{g}$, we denote the situation $\boldsymbol{f} = \mathrm{lali}\, \boldsymbol{g}$ or $\boldsymbol{g} = \mathrm{rari}\, \boldsymbol{f}$.

4.7. Remark. Since then $\boldsymbol{f} \circ \boldsymbol{g} = 1_{\mathbf{B}}$ and $\epsilon = 1_{1_{\mathbf{B}}} : 1_{\mathbf{B}} \Longrightarrow 1_{\mathbf{B}}$, the triangular identities reduce to $1_{\boldsymbol{f}} = (1_{1_{\mathbf{B}}} \circ 1_{\boldsymbol{f}}) \bullet (1_{\boldsymbol{f}} \circ \eta) = 1_{\boldsymbol{f}} \circ \eta$, which means that

$$f_1 \circ i = m \circ (f_1 \circ i, f_1 \circ \eta) = f_1 \circ m \circ (i, \eta) = f_1 \circ m \circ (i \circ d_0, 1_{\mathbf{A}}) \circ \eta = f_1 \circ \eta,$$

and $1_{\boldsymbol{g}} = (1_{\boldsymbol{g}} \circ 1_{1_{\mathbf{A}}}) \bullet (\eta \circ 1_{\boldsymbol{g}}) = \eta \circ 1_{\boldsymbol{g}}$, meaning that $i \circ g_0 = \eta \circ g_0$.

An *adjoint equivalence* is a (left and right) adjoint functor with unit and counit natural isomorphisms. It is well known that every equivalence of categories is an adjoint equivalence; see e.g. Borceux [6] or Mac Lane [35]. It is somewhat less known that this is still the case for strong equivalences of internal categories. In fact, in any 2-category, an equivalence between two objects is always an adjoint equivalence; see Blackwell, Kelly and Power [5]. More precisely, the following holds.

4.8. Proposition. [Blackwell, Kelly and Power, [5]] *Let $\mathfrak{C}$ be a 2-category and $f : C \longrightarrow D$ a 1-cell of $\mathfrak{C}$. Then f is an adjoint equivalence if and only if for every object X of $\mathfrak{C}$, the functor $\mathfrak{C}(X, f) : \mathfrak{C}(X, C) \longrightarrow \mathfrak{C}(X, D)$ is an equivalence of categories.* ■

Hence, in the 2-category $\mathsf{Cat}\mathcal{C}$ of internal categories in a given finitely complete category $\mathcal{C}$, every strong equivalence is adjoint; and in the 2-category $\mathsf{Grpd}\mathcal{C}$ of internal groupoids in $\mathcal{C}$, the notions “adjunction”, “strong equivalence” and “adjoint equivalence” coincide.

4.9. Remark. If $\boldsymbol{f} : \mathbf{A} \longrightarrow \mathbf{B}$ is a split epimorphic fully faithful functor, it is always a strong equivalence. Denote $\boldsymbol{g} = \mathrm{rari}\, \boldsymbol{f} : \mathbf{B} \longrightarrow \mathbf{A}$ its right adjoint right inverse. Then the unit η of the adjunction induces a homotopy $\boldsymbol{H} : \mathbf{A} \longrightarrow \mathbf{A}^{\mathbf{I}}$ from $1_{\mathbf{A}}$ to $\boldsymbol{g} \circ \boldsymbol{f}$. It is easily checked that the triangular identities now amount to $\boldsymbol{f}^{\mathbf{I}} \circ s(\mathbf{A}) = \boldsymbol{f}^{\mathbf{I}} \circ \boldsymbol{H}$ and $s(\mathbf{A}) \circ \boldsymbol{g} = \boldsymbol{H} \circ \boldsymbol{g}$.

4.10. Example. Example 3.1 implies that for any internal category $\mathbf{A}$, $s(\mathbf{A})$ is a right adjoint right inverse of $\epsilon_0(\mathbf{A})$ and $\epsilon_1(\mathbf{A})$. *A fortiori*, the three internal functors are strong equivalences.

4.11. Example. If $\mathcal{T}$ is the regular epimorphism topology then an internal functor $\boldsymbol{f}$ is in $\mathrm{we}(\mathcal{T})$ if and only if it is a *weak equivalence* in the sense of Bunge and Paré [15]. In case $\mathcal{C}$ is semi-abelian, weak equivalences may be characterized using homology (Proposition 6.5).

In order, for a class of morphisms in a category, to be the class of weak equivalences in a model structure, it needs to satisfy the two-out-of-three property (Definition 2.6). The following proposition gives a sufficient condition for this to be the case.

4.12. PROPOSITION. *If $\mathcal{T}$ is a subcanonical topology on a category $\mathcal{C}$ then the class of $\mathcal{T}$-equivalences has the two-out-of-three property.*

PROOF. For a subcanonical topology $\mathcal{T}$, the Yoneda embedding, considered as a functor $\mathcal{C} \longrightarrow \mathsf{Sh}(\mathcal{C},\mathcal{T})$, is equal to $Y_{\mathcal{T}}$. It follows that $Y_{\mathcal{T}}$ is full and faithful and preserves and reflects limits. Hence it induces a 2-functor $\mathsf{Cat}Y_{\mathcal{T}} : \mathsf{Cat}\mathcal{C} \longrightarrow \mathsf{CatSh}(\mathcal{C},\mathcal{T})$. Moreover, this 2-functor is such that an internal functor $\boldsymbol{f} : \mathbf{A} \longrightarrow \mathbf{B}$ in $\mathcal{C}$ is a $\mathcal{T}$-equivalence if and only if the functor $\mathsf{Cat}Y_{\mathcal{T}}(\boldsymbol{f})$ in $\mathsf{CatSh}(\mathcal{C},\mathcal{T})$ is a weak equivalence. According to Joyal and Tierney [30], weak equivalences in a Grothendieck topos have the two-out-of-three property; the result follows. ■

Not every topology induces a class of equivalences that satisfies the two-out-of-three property, as shows the following example.

4.13. EXAMPLE. Let $\boldsymbol{g} : \mathbf{B} \longrightarrow \mathbf{C}$ a functor between small categories which preserves terminal objects. Let $\boldsymbol{f} : \mathbf{1} \longrightarrow \mathbf{B}$ be a functor from a terminal category to $\mathbf{B}$ determined by the choice of a terminal object in $\mathbf{B}$. Then $\boldsymbol{g} \circ \boldsymbol{f}$ and $\boldsymbol{f}$ are fully faithful functors, whereas $\boldsymbol{g}$ need not be fully faithful. Hence the class of $\mathcal{T}$-equivalences induced by the cotrivial topology on Set does not satisfy the two-out-of-three property.

5. The $\mathcal{T}$-model structure on $\mathsf{Cat}\mathcal{C}$

In this section we suppose that $\mathcal{C}$ is a finitely complete category such that $\mathsf{Cat}\mathcal{C}$ is finitely complete and cocomplete.

5.1. DEFINITION. *Let $\mathcal{E}$ be a class of morphisms in $\mathcal{C}$ and $\boldsymbol{p} : \mathbf{E} \longrightarrow \mathbf{B}$ an internal functor. $\boldsymbol{p}$ is called an* $\mathcal{E}$-fibration *if and only if in the left hand side diagram*

$$
\begin{tikzcd}
\mathrm{iso}(E) \arrow[dr, dotted, "(r_p)_0"] \arrow[drr, bend left, "\mathrm{iso}(p)_1"] \arrow[ddr, bend right, "\delta_1"'] & & \\
& (P_p)_0 \arrow[r, "\overline{p_0}"] \arrow[d, "\overline{\delta_1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{iso}(B) \arrow[d, "\delta_1"] \\
& E_0 \arrow[r, "p_0"'] & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
X \arrow[ddrrr, bend left, "\beta"] \arrow[dddrr, bend right, "e"'] & & & \\
& U_i \arrow[ul, dotted, "f_i"'] \arrow[dr, dotted, "\varepsilon_i"] & & \\
& & \mathrm{iso}(E) \arrow[r, "\mathrm{iso}(p)_1"] \arrow[d, "\delta_1"'] \arrow[dr, phantom, "(\mathtt{i})"] & \mathrm{iso}(B) \arrow[d, "\delta_1"] \\
& & E_0 \arrow[r, "p_0"'] & B_0
\end{tikzcd}
\qquad (\mathtt{IV})
$$

the induced universal arrow $(r_p)_0$ *is in* $\mathcal{E}$*. If* $\mathcal{E} = \mathcal{E}_{\mathcal{T}}$ *comes from a topology* $\mathcal{T}$ *on* $\mathcal{C}$ *we say that* $\boldsymbol{p}$ *is a* $\mathcal{T}$*-fibration. The functor* $\boldsymbol{p}$ *is said to be* star surjective, relative to $\mathcal{T}$ *if, given an object* X *in* $\mathcal{C}$ *and arrows* e *and* β *such as in the right hand side diagram above, there exists a covering family* $(f_i : U_i \longrightarrow X)_{i \in I}$ *and a family of morphisms* $(\epsilon_i : U_i \longrightarrow \mathrm{iso}(E))_{i \in I}$ *keeping it commutative for all* $i \in I$.

By Proposition 2.12, an internal functor $\boldsymbol{p}$ is a $\mathcal{T}$-fibration if and only if it is star surjective, relative to $\mathcal{T}$.

5.2. Example. If $\mathcal{T}$ is the trivial topology then an internal functor $\boldsymbol{p}$ is a $\mathcal{T}$-fibration if and only if the square $\mathtt{i}$ is a weak pullback. Such a $\boldsymbol{p}$ is called a *strong fibration.* In case $\mathcal{C}$ is Set, the strong fibrations are the *star-surjective* functors [14]. It is easily seen that the unique arrow $\mathbf{A} \longrightarrow \mathbf{1}$ from an arbitrary internal category $\mathbf{A}$ to a terminal object $\mathbf{1}$ of $\mathsf{Cat}\mathcal{C}$ is always a strong fibration; hence every object of $\mathsf{Cat}\mathcal{C}$ is strongly fibrant.

5.3. Example. Obviously, if $\mathcal{T}$ is the cotrivial topology, any functor is a $\mathcal{T}$-fibration.

5.4. Example. An internal functor $\boldsymbol{p} : \mathbf{E} \longrightarrow \mathbf{B}$ is called a *discrete fibration* if the square

$$\begin{array}{ccc} E_1 & \xrightarrow{p_1} & B_1 \\ {\scriptstyle d_1}\big\downarrow & & \big\downarrow{\scriptstyle d_1} \\ E_0 & \xrightarrow[p_0]{} & B_0 \end{array}$$

is a pullback. Every discrete fibration is a strong fibration. Note that this is obvious in case $\mathbf{E}$ is a groupoid; in general, one proves it by considering morphisms $e : X \longrightarrow E_0$ and $\beta : X \longrightarrow \mathrm{iso}(B)$ such that $p_0 \circ e = \delta_1 \circ \beta = d_1 \circ j \circ \beta$. Then a unique morphism $\epsilon : X \longrightarrow E_1$ exists such that $p_1 \circ \epsilon = j \circ \beta$ and $d_1 \circ \epsilon = e$. This ϵ factors over $\mathrm{iso}(E)$: indeed, since $e' = d_0 \circ \epsilon$ is such that $p_0 \circ e' = d_0 \circ p_1 \circ \epsilon = d_1 \circ j \circ \mathrm{tw} \circ \beta$, there exists a unique arrow $\epsilon' : X \longrightarrow E_1$ such that $d_1 \circ \epsilon' = e'$ and $p_1 \circ \epsilon' = j \circ \mathrm{tw} \circ \beta$. Using the fact that the square above is a pullback, it is easily shown that ϵ' is the inverse of ϵ.

Given a topology $\mathcal{T}$ on $\mathcal{C}$, we shall consider the following structure on $\mathsf{Cat}\mathcal{C}$: $\mathrm{we}(\mathcal{T})$ is the class of $\mathcal{T}$-weak equivalences; $\mathrm{fib}(\mathcal{T})$ is the class of $\mathcal{T}$-fibrations; $\mathrm{cof}(\mathcal{T})$ is the class ${}^{\square}(\mathrm{fib}(\mathcal{T}) \cap \mathrm{we}(\mathcal{T}))$ of $\mathcal{T}$*-cofibrations*, internal functors having the left lifting property with respect to all *trivial* $\mathcal{T}$*-fibrations.*

The aim of this section is to prove the following

5.5. Theorem. *If* $\mathrm{we}(\mathcal{T})$ *has the two-out-of-three property and* $\mathcal{C}$ *has enough* $\mathcal{E}_{\mathcal{T}}$*-projectives then* $(\mathsf{Cat}\mathcal{C}, \mathrm{fib}(\mathcal{T}), \mathrm{cof}(\mathcal{T}), \mathrm{we}(\mathcal{T}))$ *is a model category.*

5.6. Proposition. *A functor* $\boldsymbol{p} : \mathbf{E} \longrightarrow \mathbf{B}$ *is a trivial* $\mathcal{T}$*-fibration if and only if it is fully faithful, and such that* p_0 *is a* $\mathcal{T}$*-epimorphism.*

Proof. If $\boldsymbol{p}$ is a trivial $\mathcal{T}$-fibration then it is a fully faithful functor; now suppose that $b : X \longrightarrow B_0$ is an arbitrary morphism. Since $\boldsymbol{p}$ is essentially $\mathcal{T}$-surjective, a covering family $(f_i : U_i \longrightarrow X)_{i\in I}$ exists and families of morphisms $(\beta_i : U_i \longrightarrow B_1)_{i\in I}$ and $(e_i : U_i \longrightarrow E_0)_{i\in I}$ such that $p_0\circ e_i = \delta_1\circ\beta_i$ and $\delta_0\circ\beta_i = b\circ f_i$. Since $\boldsymbol{p}$ is a $\mathcal{T}$-fibration, for every $i \in I$, e_i and β_i induce a covering family $(g_{ij} : V_{ij} \longrightarrow U_i)_{j\in I_i}$ and a family of morphisms $(\epsilon_{ij} : V_{ij} \longrightarrow \mathrm{iso}(E))_{j\in I_i}$. Put $b'_{ij} = \delta_0\circ\epsilon_{ij} : V_{ij} \longrightarrow E_0$, then $p_0\circ b'_{ij} = b\circ f_i\circ g_{ij}$; because by the transitivity axiom, $(f_i\circ g_{ij} : V_{ij} \longrightarrow X)_{j\in I_i, i\in I}$ forms a covering family, this shows that p_0 is a $\mathcal{T}$-epimorphism.

Conversely, we have to prove that $\boldsymbol{p}$ is an essentially $\mathcal{T}$-surjective $\mathcal{T}$-fibration. Given $b : X \longrightarrow B_0$, the fact that $p_0 \in \mathcal{E}_{\mathcal{T}}$ induces a covering family $(f_i : U_i \longrightarrow X)_{i\in I}$ and a family of morphisms $(e_i : U_i \longrightarrow E_0)_{i\in I}$ such that $p_0\circ e_i = b\circ f_i$. Using the equality $\delta_0\circ(\iota\circ b) = \delta_1\circ(\iota\circ b) = b$, this shows that $\boldsymbol{p}$ is essentially $\mathcal{T}$-surjective. To prove $\boldsymbol{p}$ a $\mathcal{T}$-fibration, consider the right hand side commutative diagram of solid arrows `IV` above. Because p_0 is a $\mathcal{T}$-epimorphism, there is a covering family $(f_i : U_i \longrightarrow X)_{i\in I}$ and a family $(e'_i : U_i \longrightarrow E_0)_{i\in I}$ such that $p_0\circ e'_i = \delta_0\circ\beta\circ f_i$. This gives rise to a diagram

$$
\begin{tikzcd}
U_i \arrow[dr, dotted, "\epsilon_i"] \arrow[drr, bend left, "\beta\circ f_i"] \arrow[ddr, bend right, "(e\circ f_i, e'_i)"'] & & \\
& \mathrm{iso}(E) \arrow[r, "\mathrm{iso}(p)_1"] \arrow[d, "(\delta_0,\delta_1)"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{iso}(B) \arrow[d, "(\delta_0,\delta_1)"] \\
& E_0 \times E_0 \arrow[r, "p_0\times p_0"'] & B_0 \times B_0
\end{tikzcd}
$$

for every $i \in I$. $\boldsymbol{p}$ being fully faithful, according to Remark 4.2, its square is a pullback, and induces the needed family of dotted arrows $(\epsilon_i : U_i \longrightarrow \mathrm{iso}(E))_{i\in I}$. ■

5.7. Example. For any internal category $\mathbf{A}$, $\epsilon_0(\mathbf{A})$ and $\epsilon_1(\mathbf{A})$, being strong equivalences which are split epimorphic on objects, are strong fibrations.

5.8. Corollary. *A functor $\boldsymbol{p} : \mathbf{E} \longrightarrow \mathbf{B}$ is a $\mathcal{T}$-fibration if and only if the universal arrow $\boldsymbol{r_p} : \mathbf{E^I} \longrightarrow \mathbf{P_p}$ is a trivial $\mathcal{T}$-fibration.*

Proof. This is an immediate consequence of Proposition 5.6, Example 4.10 and the fact that strong equivalences have the two-out-of-three property (Proposition 4.12). ■

5.9. Corollary. *An internal functor $\boldsymbol{j} : \mathbf{A} \longrightarrow \mathbf{X}$ is a $\mathcal{T}$-cofibration if and only if $j_0 \in {}^{\Box}\mathcal{E}_{\mathcal{T}}$.*

Proof. This follows from Proposition 4.3 and Proposition 5.6. ■

5.10. Proposition. *Every internal functor of $\mathcal{C}$ may be factored as a strong equivalence (right inverse to a strong trivial fibration) followed by a strong fibration.*

Proof. This is an application of K. S. Brown's Factorization Lemma [13]. To use it, we must show that $(\mathsf{Cat}\mathcal{C}, \mathcal{F}, \mathcal{W})$, where $\mathcal{F}$ is the class of strong fibrations and $\mathcal{W}$ is the class of strong equivalences, forms a category of fibrant objects. Condition (A) is just Proposition 4.12 and (B) follows from the fact that split epimorphisms are stable under pulling back. Proving (C) that strong fibrations are stable under pulling back is easy, as is the stability of $\mathcal{F} \cap \mathcal{W}$, the class of split epimorphic fully faithful functors. The path space needed for (D) is just the cocylinder from Section 3. Finally, according to Example 5.2, every internal category is strongly fibrant, which shows condition (E). ∎

5.11. Proposition. *Any trivial $\mathcal{T}$-cofibration is a split monic adjoint equivalence.*

Proof. Using Proposition 5.10, factor the trivial $\mathcal{T}$-cofibration $\boldsymbol{j} : \mathbf{A} \longrightarrow \mathbf{X}$ as a strong equivalence $\boldsymbol{f} : \mathbf{A} \longrightarrow \mathbf{B}$ (in fact, a right adjoint right inverse) followed by a strong fibration $\boldsymbol{p} : \mathbf{B} \longrightarrow \mathbf{X}$. By the two-out-of-three property of weak equivalences, $\boldsymbol{p}$ is a trivial $\mathcal{T}$-fibration; hence the commutative square

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\boldsymbol{f}"] \arrow[d, "\boldsymbol{j}"'] & \mathbf{B} \arrow[d, "\boldsymbol{p}"] \\ \mathbf{X} \arrow[r, equal] \arrow[ur, dotted, "\boldsymbol{s}"] & \mathbf{X} \end{tikzcd}$$

has a lifting $\boldsymbol{s} : \mathbf{X} \longrightarrow \mathbf{B}$. It follows that $\boldsymbol{j}$ is a retract of $\boldsymbol{f}$. The class of right adjoint right inverse functors being closed under retracts, we may conclude that $\boldsymbol{j}$ is a split monic adjoint equivalence. ∎

5.12. Proposition. [Covering Homotopy Extension Property] *Consider the commutative diagram of solid arrows*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\boldsymbol{H}"] \arrow[d, "\boldsymbol{j}"'] & \mathbf{E}^{\mathbf{I}} \arrow[r, "\boldsymbol{p}^{\mathbf{I}}"] \arrow[d, "\epsilon_1(\mathbf{E})"] & \mathbf{B}^{\mathbf{I}} \arrow[d, "\epsilon_1(\mathbf{B})"] \\ \mathbf{X} \arrow[r, "\boldsymbol{f}"'] \arrow[ur, dotted, "\boldsymbol{L}"] \arrow[urr, "\boldsymbol{K}"] & \mathbf{E} \arrow[r, "\boldsymbol{p}"'] & \mathbf{B}. \end{tikzcd}$$

If $\boldsymbol{j} \in \mathrm{cof}(\mathcal{T})$ and $\boldsymbol{p} \in \mathrm{fib}(\mathcal{T})$, then a morphism $\boldsymbol{L} : \mathbf{X} \longrightarrow \mathbf{E}^{\mathbf{I}}$ exists keeping the diagram commutative.

Proof. Since $\boldsymbol{p}$ is a $\mathcal{T}$-fibration, by Corollary 5.8, the associated universal arrow $\boldsymbol{r_p} : \mathbf{E}^{\mathbf{I}} \longrightarrow \mathbf{P}_{\boldsymbol{p}}$ is a trivial $\mathcal{T}$-fibration. Let $\boldsymbol{M} : \mathbf{X} \longrightarrow \mathbf{P}_{\boldsymbol{p}}$ be the unique morphism such that $\overline{\epsilon_1(\mathbf{B})} \circ \boldsymbol{M} = \boldsymbol{f}$ and $\overline{\boldsymbol{p}} \circ \boldsymbol{M} = \boldsymbol{K}$ (cf. Diagram I); then the square

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\boldsymbol{H}"] \arrow[d, "\boldsymbol{j}"'] & \mathbf{E}^{\mathbf{I}} \arrow[d, "\boldsymbol{r_p}"] \\ \mathbf{X} \arrow[r, "\boldsymbol{M}"'] \arrow[ur, dotted, "\boldsymbol{L}"] & \mathbf{P}_{\boldsymbol{p}} \end{tikzcd}$$

commutes, and yields the needed lifting $\boldsymbol{L}$. ∎

5.13. Proposition. [cf. Proposition I.3.11 of Kamps and Porter [32]] *Any $\mathcal{T}$-fibration has the right lifting property with respect to any trivial $\mathcal{T}$-cofibration.*

Proof. Suppose that in square III, $\boldsymbol{j} \in \mathrm{cof}(\mathcal{T}) \cap \mathrm{we}(\mathcal{T})$ and $\boldsymbol{p} \in \mathrm{fib}(\mathcal{T})$. By Proposition 5.11, $\boldsymbol{j}$ is a split monomorphic adjoint equivalence; denote $\boldsymbol{k} = \mathrm{lali}\, \boldsymbol{j} : \mathbf{X} \longrightarrow \mathbf{A}$. According to Remark 4.9, a homotopy $\boldsymbol{H} : \mathbf{X} \longrightarrow \mathbf{X}^{\mathbf{I}}$ from $1_{\mathbf{X}}$ to $\boldsymbol{j} \circ \boldsymbol{k}$ may be found such that $\boldsymbol{H} \circ \boldsymbol{j} = s(\mathbf{X}) \circ \boldsymbol{j} : \mathbf{A} \longrightarrow \mathbf{X}^{\mathbf{I}}$. Because the diagram of solid arrows

$$\begin{tikzcd}
\mathbf{A} \arrow[r, "s(\mathbf{E})\circ \boldsymbol{f}"] \arrow[d, "\boldsymbol{j}"'] & \mathbf{E}^{\mathbf{I}} \arrow[r, "\boldsymbol{p}^{\mathbf{I}}"] \arrow[d, "\epsilon_1(\mathbf{E})"] & \mathbf{B}^{\mathbf{I}} \arrow[d, "\epsilon_1(\mathbf{B})"] \\
\mathbf{X} \arrow[r, "\boldsymbol{f}\circ \boldsymbol{k}"'] \arrow[ur, dotted, "\boldsymbol{L}"] \arrow[urr, "\boldsymbol{g}^{\mathbf{I}}\circ \boldsymbol{H}"] & \mathbf{E} \arrow[r, "\boldsymbol{p}"'] & \mathbf{B}
\end{tikzcd}$$

commutes, the Covering Homotopy Extension Property 5.12 gives rise to a morphism $\boldsymbol{L} : \mathbf{X} \longrightarrow \mathbf{E}^{\mathbf{I}}$; the morphism $\boldsymbol{h} = \epsilon_0(\mathbf{E}) \circ \boldsymbol{L}$ is the desired lifting for Diagram III. ■

5.14. Lemma. [cf. Lemma 2.1 in Johnstone [28]] *Let $\mathbf{B}$ be a category in $\mathcal{C}$ and let $p_0 : E_0 \longrightarrow B_0$ be a morphism in $\mathcal{C}$. Form the pullback*

$$\begin{tikzcd}
E_1 \arrow[r, dotted, "p_1"] \arrow[d, dotted, "{(d_0,d_1)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d, "{(d_0,d_1)}"] \\
E_0 \times E_0 \arrow[r, "p_0 \times p_0"'] & B_0 \times B_0.
\end{tikzcd}$$

Then the left hand side graph $\mathbf{E}$ carries a unique internal category structure such that $\boldsymbol{p} = (p_0, p_1) : \mathbf{E} \longrightarrow \mathbf{B}$ is a functor. Moreover, $\boldsymbol{p}$ is a fully faithful functor, and if p_0 is a $\mathcal{T}$-epimorphism, then $\boldsymbol{p}$ is a trivial $\mathcal{T}$-fibration. ■

5.15. Proposition. *Every internal functor of $\mathcal{C}$ may be factored as a $\mathcal{T}$-cofibration followed by a trivial $\mathcal{T}$-fibration.*

Proof. Let $\boldsymbol{f} : \mathbf{A} \longrightarrow \mathbf{B}$ be an internal functor, and, using Remark 2.11 that $({}^{\square}\mathcal{E}_{\mathcal{T}}, \mathcal{E}_{\mathcal{T}})$ forms a weak factorization system, factor f_0 as an element $j_0 : A_0 \longrightarrow E_0$ of ${}^{\square}\mathcal{E}_{\mathcal{T}}$ followed by a $\mathcal{T}$-epimorphism $p_0 : E_0 \longrightarrow B_0$. Then the construction in Lemma 5.14 yields a trivial $\mathcal{T}$-fibration $\boldsymbol{p} : \mathbf{E} \longrightarrow \mathbf{B}$. Let $j_1 : A_1 \longrightarrow E_1$ be the unique morphism such that $p_1 \circ j_1 = f_1$ and $(d_0, d_1) \circ j_1 = (j_0 \times j_0) \circ (d_0, d_1)$. Since $\boldsymbol{p}$ is faithful, $\boldsymbol{j} = (j_0, j_1) : \mathbf{A} \longrightarrow \mathbf{E}$ is a functor; according to Corollary 5.9, it is a $\mathcal{T}$-cofibration. ■

5.16. Proposition. *Every internal functor of $\mathcal{C}$ may be factored as a trivial $\mathcal{T}$-cofibration followed by a $\mathcal{T}$-fibration.*

Proof. For an internal functor $\boldsymbol{f} : \mathbf{A} \longrightarrow \mathbf{B}$, let $\boldsymbol{f} = \boldsymbol{p} \circ \boldsymbol{j}'$ be the factorization of $\boldsymbol{f}$ from Proposition 5.10. Then $\boldsymbol{p}$ is a $\mathcal{T}$-fibration and $\boldsymbol{j}'$ a $\mathcal{T}$-equivalence. Using Proposition 5.15, this $\boldsymbol{j}'$ may be factored as a (necessarily trivial) $\mathcal{T}$-cofibration $\boldsymbol{j}$ followed by a $\mathcal{T}$-fibration $\boldsymbol{p}'$. Thus we get a trivial $\mathcal{T}$-cofibration $\boldsymbol{j}$ and a $\mathcal{T}$-fibration $\boldsymbol{p} \circ \boldsymbol{p}'$ such that $f = (\boldsymbol{p} \circ \boldsymbol{p}') \circ \boldsymbol{j}$. ■

Proof of Theorem 5.5. We only need to comment on the closedness under retracts of the classes $\mathrm{fib}(\mathcal{T})$, $\mathrm{cof}(\mathcal{T})$ and $\mathrm{we}(\mathcal{T})$. For the $\mathcal{T}$-fibrations, $\mathcal{T}$-cofibrations and essentially $\mathcal{T}$-surjective morphisms, closedness under retracts follows from the closedness of the classes $\mathcal{E}_{\mathcal{T}}$ and ${}^{\square}\mathcal{E}_{\mathcal{T}}$; for fully faithful functors, the property has a straightforward direct proof. ■

6. Case study: the regular epimorphism topology

This section treats the model category structure on $\mathsf{Cat}\mathcal{C}$ induced by choosing $\mathcal{T}$ the regular epimorphism topology on $\mathcal{C}$. Let us recall what we already know about it: the class $\mathrm{we}(\mathcal{T})$ consists of weak equivalences of internal categories; fibrations are $\{\text{regular epi}\}$-fibrations and cofibrations have an object morphism in ${}^{\square}\{\text{regular epi}\}$. Hence all of its objects are fibrant, and an internal category is cofibrant if and only if its object of objects is $\{\text{regular epi}\}$-projective.

We shall be focused mainly on semi-abelian categories and internal crossed modules, but we start by explaining a connection with Grothendieck topoi and Joyal and Tierney's model structure.

6.1. Grothendieck topoi. Let $\mathcal{C}$ be a Grothendieck topos equipped with the regular epimorphism topology $\mathcal{T}$. Since all epimorphisms are regular, it is clear that epimorphisms and $\mathcal{T}$-epimorphisms coincide. It follows that the notion of $\mathcal{T}$-equivalence coincides with the one considered by Joyal and Tierney in [30]. Consequently, the two structures have equivalent homotopy categories. It is, however, clear that in general, Joyal and Tierney's fibrations and cofibrations are different from ours.

6.2. Semi-abelian categories. From now on we suppose that $\mathcal{C}$ is semi-abelian with enough $\{\text{regular epi}\}$-projectives, to give an alternative characterization of weak equivalences, and to describe the induced model category structure on the category of internal crossed modules.

Recall the following notion of homology of simplicial objects in a semi-abelian category (see Everaert and Van der Linden [19]). First of all, a morphism is called *proper* when its image is a kernel, and a chain complex is *proper* whenever all its differentials are. As in the abelian case, the *n-th homology object* of a proper chain complex C with differentials d_n is said to be $H_nC = \mathrm{Cok}[C_{n+1} \longrightarrow K[d_n]]$. The category of proper chain complexes in $\mathcal{C}$ is denoted $\mathsf{PCh}\mathcal{C}$, and $\mathcal{SC} = \mathsf{Fun}(\Delta^{\mathrm{op}}, \mathcal{C})$ is the category of simplicial objects in $\mathcal{C}$. The *Moore* functor $N : \mathcal{SC} \longrightarrow \mathsf{PCh}\mathcal{C}$ maps a simplicial object A in $\mathcal{C}$ with face operators ∂_i and degeneracy operators σ_i to the chain complex $N(A)$ in $\mathcal{C}$ given by

$$N_nA = \bigcap_{i=0}^{n-1} K[\partial_i : A_n \longrightarrow A_{n-1}], \qquad d_n = \partial_n \circ \bigcap_i \mathsf{Ker}\,\partial_i : N_nA \longrightarrow N_{n-1}A,$$

for $n \geq 1$, and $N_nA = 0$, for $n < 0$. Because $N(A)$ is proper [19, Theorem 3.6], it makes sense to consider its homology. Indeed, the *n-th homology object* H_nA is defined to be the n-th homology object $H_nN(A)$ of the associated proper chain complex $N(A)$. This process sends a short exact sequences of simplicial objects to a long exact sequence in $\mathcal{C}$ [19, Corollary 5.7].

Since, via the nerve construction, any internal category may be considered as a simplicial object, we can apply this homology theory to internal categories. More precisely, recall (e.g. from Johnstone [27, Remark 2.13]) that there is an embedding $\mathrm{ner} : \mathsf{Cat}\mathcal{C} \longrightarrow \mathcal{SC}$ of $\mathsf{Cat}\mathcal{C}$ into $\mathcal{SC}$ as a full subcategory. Given a category $\mathbf{A}$ in $\mathcal{C}$, its *nerve* $\mathrm{ner}\,\mathbf{A}$ is the simplicial object defined on objects by n-fold pullback $\mathrm{ner}_n\,\mathbf{A} = A_1 \times_{A_0} \cdots \times_{A_0} A_1$ if $n \geq 2$, $\mathrm{ner}_1\,\mathbf{A} = A_1$ and $\mathrm{ner}_0\,\mathbf{A} = \mathbf{A}_0$; on morphisms by $\partial_0 = d_1, \partial_1 = d_0 : \mathrm{ner}_1\,\mathbf{A} \longrightarrow \mathrm{ner}_0\,\mathbf{A}$, $\sigma_0 = i : \mathrm{ner}_0\,\mathbf{A} \longrightarrow \mathrm{ner}_1\,\mathbf{A}$; $\partial_0 = \mathrm{pr}_2, \partial_1 = m, \partial_2 = \mathrm{pr}_1 : \mathrm{ner}_2\,\mathbf{A} \longrightarrow \mathrm{ner}_1\,\mathbf{A}$ and $\sigma_0 = (i, 1_{A_1}), \sigma_1 = (1_{A_1}, i) : \mathrm{ner}_1\,\mathbf{A} \longrightarrow \mathrm{ner}_2\,\mathbf{A}$; etc. A simplicial object is isomorphic to an object in the image of ner if and only if, as a functor $\Delta^{\mathrm{op}} \longrightarrow \mathcal{C}$, it is left exact.

6.3. Definition. *Suppose that $\mathcal{C}$ is semi-abelian, $\mathbf{A}$ is a category in $\mathcal{C}$ and $n \in \mathbb{Z}$. The object $H_n\mathbf{A} = H_n\,\mathrm{ner}\,\mathbf{A}$ will be called the n-th* homology object *of $\mathbf{A}$ and the functor $H_n = H_n \circ \mathrm{ner} : \mathsf{Cat}\mathcal{C} \longrightarrow \mathcal{C}$ the n-th* homology functor*.*

6.4. Proposition. *Let $\mathbf{A}$ be a category in a semi-abelian category $\mathcal{C}$. If $n \notin \{0,1\}$ then $H_n\mathbf{A} = 0$, $H_1\mathbf{A} = K[(d_0,d_1) : A_1 \longrightarrow A_0 \times A_0]$ and $H_0\mathbf{A} = \mathrm{Coeq}[d_0, d_1 : A_1 \longrightarrow A_0]$. Moreover, $H_1\mathbf{A} = K[(d_0,d_1)]$ is an abelian object of $\mathbf{A}$.*

Proof. Using e.g. the Yoneda Lemma (in the form of Metatheorem 0.2.7 in [7]), it is quite easily shown that $N\mathbf{A} = N\,\mathrm{ner}\,\mathbf{A}$ is the chain complex

$$\cdots \longrightarrow 0 \longrightarrow K[d_1] \xrightarrow{d_0 \circ \mathsf{Ker}\, d_1} A_0 \longrightarrow 0 \longrightarrow \cdots.$$

By Remark 3.3 in [19], $K[d_0 \circ \mathsf{Ker}\, d_1] = K[d_0] \cap K[d_1]$, which is clearly equal to $K[(d_0,d_1) : A_1 \longrightarrow A_0 \times A_0]$. The last equality is an application of [19, Corollary 3.10], and $H_1\mathbf{A}$ being abelian is a consequence of Theorem 5.5 in [19] or Bourn [11, Proposition 3.1]. ∎

6.5. Proposition. *Let $\mathcal{C}$ be a semi-abelian category and $\boldsymbol{f} \colon \mathbf{A} \longrightarrow \mathbf{B}$ a functor in $\mathcal{C}$.*

1. *$\boldsymbol{f}$ is fully faithful if and only if $H_0\boldsymbol{f}$ is mono and $H_1\boldsymbol{f}$ is iso.*
2. *$\boldsymbol{f}$ is essentially* {regular epi}*-surjective if and only if $H_0\boldsymbol{f}$ is a regular epimorphism.*

Hence an internal functor is a weak equivalence exactly when it is a homology isomorphism.

Proof. In the diagram

$$\begin{tikzcd}
H_1\mathbf{A} \arrow[r, "\mathsf{Ker}\,(d_0,d_1)"] \arrow[d, "H_1\boldsymbol{f}"'] & A_1 \arrow[r] \arrow[d, "f_1"'] \arrow[rr, bend left, "(d_0,d_1)"] \arrow[dr, phantom, "\mathtt{(i)}"] & R[q] \arrow[r, tail] \arrow[d, dotted] \arrow[dr, phantom, "\mathtt{(ii)}"] & A_0 \times A_0 \arrow[d, "f_0 \times f_0"] \\
H_1\mathbf{B} \arrow[r, "\mathsf{Ker}\,(d_0,d_1)"'] & B_1 \arrow[r] \arrow[rr, bend right, "(d_0,d_1)"'] & R[r] \arrow[r, tail] & B_0 \times B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
R[q] \arrow[r, shift left] \arrow[r, shift right] \arrow[d, dotted] \arrow[dr, phantom, "\mathtt{(iii)}"] & A_0 \arrow[r, "q=\mathsf{Coeq}\,(d_0,d_1)"] \arrow[d, "f_0"] & H_0\mathbf{A} \arrow[d, "H_0\boldsymbol{f}"] \\
R[r] \arrow[r, shift left] \arrow[r, shift right] & B_0 \arrow[r, "r=\mathsf{Coeq}\,(d_0,d_1)"'] & H_0\mathbf{B}
\end{tikzcd}$$

the arrow $H_1\boldsymbol{f}$ is an isomorphism if and only if the square $\mathtt{i}$ is a pullback (Lemmas 4.2.4 and 4.2.5 in [7]); square $\mathtt{ii}$ is a pullback if and only if $\mathtt{iii}$ is a joint pullback, which (by Proposition 1.1 in [9]) is the case exactly when $H_0\boldsymbol{f}$ is mono. This already shows one implication of 1. To prove the other, note that if $\boldsymbol{f}$ is fully faithful, then $H_1\boldsymbol{f}$ is an isomorphism; hence $\mathtt{i}$ is a pullback, $\mathtt{ii}$ is a pullback [26, Proposition 2.5], $\mathtt{iii}$ is a joint pullback and $H_0\boldsymbol{f}$ a monomorphism.

For the proof of 2. consider the following diagrams.

$$\begin{tikzcd}
(P_f)_0 \arrow[r, "\overline{f_0}"] \arrow[d, "\overline{d_1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[r, "d_0"] \arrow[d, "d_1"] & B_0 \\
A_0 \arrow[r, "f_0"'] & B_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
(P_f)_0 \arrow[r, "d_0\circ\overline{f_0}"] \arrow[d, "\overline{d_1}"'] \arrow[dr, phantom, "(\mathtt{iv})"] & B_0 \arrow[d, "r"] \\
A_0 \arrow[r, "H_0f\circ q"'] & H_0B
\end{tikzcd}$$

If H_0f is a regular epimorphism then so is the bottom arrow in diagram $\mathtt{iv}$. Accordingly, we only need to show that the induced arrow $p: (P_f)_0 \longrightarrow P$ to the pullback P of $H_0f\circ q$ along r is a regular epi. Now in the left hand side diagram

$$\begin{tikzcd}
P \arrow[r] \arrow[d, "\overline{r}"'] \arrow[dr, phantom, "(\mathtt{v})"] & R[r] \arrow[r, "p_0"] \arrow[d, "p_1"] \arrow[dr, phantom, "\lrcorner", very near start] & B_0 \arrow[d, "r"] \\
A_0 \arrow[r, "f_0"'] & B_0 \arrow[r, "r"'] & H_0B
\end{tikzcd}
\qquad
\begin{tikzcd}
(P_f) \arrow[r, "p"] \arrow[d, "\overline{f_0}"'] \arrow[dr, phantom, "(\mathtt{vi})"] & P \arrow[r, "\overline{r}"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dr, phantom, "(\mathtt{v})"] & A_0 \arrow[d, "f_0"] \\
B_1 \arrow[r, "o"'] & R[r] \arrow[r, "p_1"'] & B_0
\end{tikzcd}$$

this pullback is the outer rectangle; its left hand side square $\mathtt{v}$ is a pullback. Now so is $\mathtt{vi}$; it follows that and p is a regular epimorphism, so being the arrow o, universal for the equalities $p_0\circ o = d_0$ and $p_1\circ o = d_1$ to hold.

Conversely, note that $r\circ d_0\circ\overline{f_0} = H_0f\circ q\circ\overline{d_1}$; hence if $d_0\circ\overline{f_0}$ is regular epic, so is H_0f. ∎

The notion of *Kan fibration* makes sense in the context of regular categories: see [16] and [19]. We use it to characterize the fibrations in $\mathsf{Cat}\mathcal{C}$.

6.6. Proposition. *A functor $\boldsymbol{p}: \mathbf{E} \longrightarrow \mathbf{B}$ in a semi-abelian category $\mathcal{C}$ is a fibration if and only if* $\mathrm{ner}\,\boldsymbol{p}$ *is a Kan fibration.*

Proof. First note that, because a semi-abelian category is always Mal'tsev, every category in $\mathcal{C}$ is an internal groupoid. We may now use M. Barr's Embedding Theorem for regular categories [4] in the form of Metatheorem A.5.7 in [7]. Indeed, the properties "some internal functor is a fibration" and "some simplicial morphism is a Kan fibration" may be added to the list of properties [7, 0.1.3], and it is well-known that in Set, a functor between two groupoids is a fibration if and only if its nerve is a Kan fibration. ∎

In his paper [25], Janelidze introduces a notion of crossed module in an arbitrary semi-abelian category $\mathcal{C}$. Its definition is based on Bourn and Janelidze's notion of internal semidirect product [12] and Borceux, Janelidze and Kelly's notion of internal object action [8]. Internal crossed modules also generalize the case where $\mathcal{C} = \mathsf{Gp}$ in the sense that an equivalence $\mathsf{XMod}\mathcal{C} \simeq \mathsf{Cat}\mathcal{C}$ still exists. Using this equivalence, we may transport the model structures from Theorem 5.5 to the category $\mathsf{XMod}\mathcal{C}$. In case $\mathcal{C}$ has enough regular projectives and $\mathcal{T}$ is the regular epimorphism topology, this has the advantage that the classes of fibrations, cofibrations and weak equivalences have a very easy description.

We recall from [25] the definition of internal crossed modules. Given $g : B \longrightarrow B'$ and $h : X \longrightarrow X'$, the morphism $g\flat h : B\flat X \longrightarrow B'\flat X'$ is unique in making the diagram

$$\begin{tikzcd}
0 \arrow[r] & B\flat X \arrow[r, tail, "\kappa_{B,X}"] \arrow[d, dotted, "g\flat h"'] & B+X \arrow[r, "{[1_B,0]}"] \arrow[d, "g+h"] & B \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & B'\flat X' \arrow[r, tail, "\kappa_{B',X'}"'] & B'+X' \arrow[r, "{[1_{B'},0]}"'] & B' \arrow[r] & 0
\end{tikzcd}$$

commute. The category $\mathsf{SplitEpi}\mathcal{C}$ of split epimorphisms in $\mathcal{C}$ with a given splitting ($\mathsf{Pt}\mathcal{C}$ in [12] and in Definition 2.1.14 of [7]) is equivalent to the category $\mathsf{Act}\mathcal{C}$ of *actions* in $\mathcal{C}$, of which the objects are triples (B, X, ξ), where $\xi : B\flat X \longrightarrow X$ makes the following diagram commute:

$$\begin{tikzcd}
B\flat(B\flat X) \arrow[r, "\mu^B_X"] \arrow[d, "1_B\flat\xi"'] & B\flat X \arrow[d, "\xi"] & X \arrow[l, "\eta^B_X"'] \arrow[dl, equal] \\
B\flat X \arrow[r, "\xi"'] & X;
\end{tikzcd}$$

here μ^B_X is defined by the exactness of the rows in the diagram

$$\begin{tikzcd}
0 \arrow[r] & B\flat(B\flat X) \arrow[r, tail, "\kappa_{B,B\flat X}"] \arrow[d, dotted, "\mu^B_X"'] & B+(B\flat X) \arrow[r, "{[1_B,0]}"] \arrow[d, "{[\mathrm{in}_1,\kappa_{B,X}]}"] & B \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B\flat X \arrow[r, tail, "\kappa_{B,X}"'] & B+X \arrow[r, "{[1_B,0]}"'] & B \arrow[r] & 0
\end{tikzcd}$$

and η^B_X is unique such that $\kappa_{B,X}\circ\eta^B_X = \mathrm{in}_2 : X \longrightarrow B+X$. A morphism $(B, X, \xi) \longrightarrow (B', X', \xi')$ in $\mathsf{Act}\mathcal{C}$ is a pair (g, h), where $g : B \longrightarrow B'$ and $h : X \longrightarrow X'$ are morphisms in $\mathcal{C}$ with $h\circ\xi = \xi'\circ(g\flat h)$.

An *internal precrossed module* in $\mathcal{C}$ is a 4-tuple (B, X, ξ, f) with (B, X, ξ) in $\mathsf{Act}\mathcal{C}$ and $f : X \longrightarrow B$ a morphism in $\mathcal{C}$ such that the left hand side diagram

$$\begin{tikzcd}
B\flat X \arrow[r, tail, "\kappa_{B,X}"] \arrow[d, "\xi"'] & B+X \arrow[d, "{[1_B,f]}"] \\
X \arrow[r, "f"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
(B+X)\flat X \arrow[r, "{[1_B,f]\flat 1_X}"] \arrow[d, "{[1_{B+X},\mathrm{in}_2]^{\#}}"'] & B\flat X \arrow[d, "\xi"] \\
B\flat X \arrow[r, "\xi"'] & X
\end{tikzcd}$$

commutes. A morphism $(B, X, \xi, f) \longrightarrow (B', X', \xi', f')$ is a morphism $(g, h) : (B, X, \xi) \longrightarrow (B', X', \xi')$ in $\mathsf{Act}\mathcal{C}$ such that $g\circ f = f'\circ h$. There is an equivalence between $\mathsf{RG}\mathcal{C}$ and the category $\mathsf{PreCrossMod}\mathcal{C}$ of precrossed modules in $\mathcal{C}$. An *internal crossed module* in $\mathcal{C}$ is an internal precrossed module (B, X, ξ, f) in $\mathcal{C}$ for which the right hand side diagram above commutes. Here $[1_{B+X}, \mathrm{in}_2]^\#$ is the unique morphism such that $\kappa_{B,X}\circ[1_{B+X}, \mathrm{in}_2]^\# = [1_{B+X}, \mathrm{in}_2]\circ\kappa_{B+X,X}$. If $\mathsf{XMod}\mathcal{C}$ or $\mathsf{CrossMod}\mathcal{C}$ denotes the full subcategory of $\mathsf{PreCrossMod}\mathcal{C}$ determined by the crossed modules, then the last equivalence (co)restricts to an equivalence $\mathsf{Cat}\mathcal{C} \simeq \mathsf{XMod}\mathcal{C}$. This equivalence maps an internal category $\mathbf{A}$ to the internal crossed module

$$(A_0, \quad K[d_1], \quad \xi : A_0 \flat K[d_1] \longrightarrow K[d_1], \quad d_0\circ\mathsf{Ker}\, d_1 : K[d_1] \longrightarrow A_0)$$

where ξ is the pullback of $[i, \mathsf{Ker}\, d_1] : A_0 + K[d_1] \longrightarrow A_1$ along $\mathsf{Ker}\, d_1 : K[d_1] \longrightarrow A_1$. Thus an internal functor $\boldsymbol{f} : \mathbf{A} \longrightarrow \mathbf{B}$ is mapped to the morphism

$$(f_0, N_1\boldsymbol{f}) : (A_0, N_1\mathbf{A}, \xi, d_0\circ\mathsf{Ker}\, d_1) \longrightarrow (B_0, N_1\mathbf{B}, \xi', d_0\circ\mathsf{Ker}\, d_1).$$

If the homology objects of an internal crossed module (B, X, ξ, f) are those of the associated internal category, the only non-trivial ones are $H_0(B, X, \xi, f) = \mathsf{Coker}\, f$ and $H_1(B, X, \xi, f) = \mathsf{Ker}\, f$ (see Proposition 6.4).

6.7. THEOREM. *If $\mathcal{C}$ is a semi-abelian category with enough projectives then a model category structure on $\mathsf{XMod}\mathcal{C}$ is defined by choosing* we *the class of homology isomorphisms,* cof *the class of morphisms (g, h) with g in* $^\square\{\text{regular epi}\}$ *and* fib *the class of morphisms (g, h) with h regular epic.*

PROOF. Only the characterization of fib needs a proof. Given an internal functor $\boldsymbol{p} : \mathbf{E} \longrightarrow \mathbf{B}$, consider the diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & K[d_1] \arrow[r, tail, "\mathsf{Ker}\, d_1"] \arrow[d, "N_1\boldsymbol{p}"'] \arrow[dr, phantom, "\text{(i)}"] & E_1 \arrow[r, two heads, "d_1"] \arrow[d, dotted, "(r_p)_0"] & E_0 \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & K[d_1] \arrow[r, tail] \arrow[d, equal] & (P_p)_0 \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\text{(ii)}"] & E_0 \arrow[r] \arrow[d, "p_0"] & 0 \\
0 \arrow[r] & K[d_1] \arrow[r, tail, "\mathsf{Ker}\, d_1"'] & B_1 \arrow[r, two heads, "d_1"'] & B_0 \arrow[r] & 0
\end{tikzcd}$$

where squares i and ii are a pullbacks (cf. Lemma 4.2.2 and its Corollary 4.2.3 in [7]). By [7, Lemma 4.2.5], $N_1\boldsymbol{p}$ is a regular epimorphism if and only if $(r_p)_0$ is regular epic, i.e. exactly when $\boldsymbol{p}$ is a $\{\text{regular epi}\}$-fibration. ∎

6.8. EXAMPLE. [Crossed modules of groups] In the specific case of $\mathcal{C}$ being the semi-abelian category Gp of groups and group homomorphisms, the structure on $\mathsf{XMod}\mathcal{C}$ coincides with the one considered by Garzón and Miranda in [20]. This category is equivalent to the category $\mathsf{XMod} = \mathsf{XModGp}$ of crossed modules of groups. As such, XMod carries the model structure from Theorem 6.7.

7. Case study: the trivial topology

In this section we suppose that $\mathcal{T}$ is the trivial topology on $\mathcal{C}$ (as a rule, we shall not mention it), and give a more detailed description of the model structure from Theorem 5.5. It turns out to resemble very much Strøm's model category structure on the category Top of topological spaces and continuous maps [40]: its weak equivalences are the homotopy equivalences, its cofibrations are the functors which have the homotopy extension property and its fibrations are the functors which have the homotopy lifting property.

7.1. Proposition. *An internal functor is a trivial fibration if and only if it is a split epimorphic equivalence.*

Proof. This is an immediate consequence of Proposition 5.6. ∎

It follows that every object of this model category is cofibrant. Since we already showed them to be fibrant as well, the notion of homotopy induced by the model structure (see Quillen [38]) is determined entirely by the cocylinder from Section 3. Hence the weak equivalences are homotopy equivalences, also in the model-category-theoretic sense of the word.

7.2. Definition. *Let $\mathcal{C}$ be a finitely complete category. An internal functor $\boldsymbol{p}\colon \mathbf{E} \longrightarrow \mathbf{B}$ is said to have the* homotopy lifting property *if and only if the diagram*

$$\begin{tikzcd} \mathbf{E}^{\mathbf{I}} \arrow[r, "\boldsymbol{p}^{\mathbf{I}}"] \arrow[d, "\epsilon_1(\mathbf{E})"'] & \mathbf{B}^{\mathbf{I}} \arrow[d, "\epsilon_1(\mathbf{B})"] \\ \mathbf{E} \arrow[r, "\boldsymbol{p}"'] & \mathbf{B} \end{tikzcd}$$

is a weak pullback in $\mathsf{Cat}\mathcal{C}$.

7.3. Proposition. [cf. R. Brown, [14]] *An internal functor is a strong fibration if and only if it has the homotopy lifting property.*

Proof. This follows from Proposition 7.1, Corollary 5.8 and the two-out-of-three property of strong equivalences. ∎

We now give a characterization of cofibrations in the following terms.

7.4. Definition. *An internal functor $\boldsymbol{j}\colon \mathbf{A} \longrightarrow \mathbf{X}$ is said to have the* homotopy extension property *if and only if any commutative square*

$$\begin{tikzcd} \mathbf{A} \arrow[r, "\boldsymbol{H}"] \arrow[d, "\boldsymbol{j}"'] & \mathbf{E}^{\mathbf{I}} \arrow[d, "\epsilon_1(\mathbf{E})"] \\ \mathbf{X} \arrow[r, "\boldsymbol{f}"'] \arrow[ur, dotted, "\overline{\boldsymbol{H}}"] & \mathbf{E} \end{tikzcd} \qquad (\mathtt{V})$$

has a lifting $\overline{\boldsymbol{H}}\colon \mathbf{X} \longrightarrow \mathbf{E}^{\mathbf{I}}$.

To show that a functor with the homotopy extension property is a cofibration, we need a way to approximate any trivial fibration $\boldsymbol{p}$ with some $\epsilon_1(\mathbf{E})$. The next construction allows (more or less) to consider p_0 as the object morphism of $\epsilon_1(\boldsymbol{p}\mathbf{E}) : (\boldsymbol{p}\mathbf{E})^{\mathbf{I}} \longrightarrow \boldsymbol{p}\mathbf{E}$.

7.5. Lemma. *If $\boldsymbol{p} : \mathbf{E} \longrightarrow \mathbf{B}$ is an adjoint equivalence with right inverse $\boldsymbol{s} : \mathbf{B} \longrightarrow \mathbf{E}$ then the reflexive graph*

$$\boldsymbol{p}\mathbf{E} = \Big(\begin{tikzcd} E_1 \arrow[r, shift left=2, "p_0\circ d_1"] \arrow[r, shift right=2, "p_0\circ d_0"'] & B_0 \arrow[l, "i\circ s_0" description] \end{tikzcd} \Big)$$

carries an internal category structure.

Proof. The morphism $m\circ(p_1 \times_{1_{B_0}} p_1) : E_1 \times_{B_0} E_1 \longrightarrow B_1$ is such that $d_0\circ m\circ(p_1 \times_{1_{B_0}} p_1) = p_0\circ d_0\circ \mathrm{pr}_1$ and $d_1\circ m\circ(p_1 \times_{1_{B_0}} p_1) = p_0\circ d_1\circ \mathrm{pr}_2$. Since $\boldsymbol{p}$ is an adjoint equivalence, it is a fully faithful functor; by Proposition 4.1, a unique morphism $m : E_1 \times_{B_0} E_1 \longrightarrow E_1$ exists satisfying $m\circ(p_1 \times_{1_{B_0}} p_1) = p_1\circ m$, $d_0\circ m = d_0\circ \mathrm{pr}_1$ and $d_1\circ m = d_1\circ \mathrm{pr}_2$. This is the needed structure of internal category. ■

7.6. Proposition. *An internal functor is a cofibration if and only if it has the homotopy extension property.*

Proof. One implication is obvious, $\epsilon_1(\mathbf{E})$ being a trivial fibration. Now consider a commutative square such as `III` above, where $\boldsymbol{j}$ has the homotopy extension property and $\boldsymbol{p} \in \mathrm{fib} \cap \mathrm{we}$. By the hypothesis on $\boldsymbol{j}$ and Proposition 4.3, j_0 has the left lifting property with respect to all morphisms of the form $\delta_1 : \mathrm{iso}(A) \longrightarrow A_0$ for $\mathbf{A} \in \mathsf{Cat}\mathcal{C}$. In particular, by Lemma 7.5, j_0 has the left lifting property with respect to $p_0\circ d_1\circ j = \epsilon_1(pE)_0 : \mathrm{iso}(pE) \longrightarrow B_0$. If $h_0' : X_0 \longrightarrow \mathrm{iso}(pE)$ denotes a lifting for the commutative square $g_0\circ j_0 = (p_0\circ d_1\circ j)\circ(\iota\circ f_0)$, then $h_0 = d_1\circ j\circ h_0' : X_0 \longrightarrow E_0$ is such that $p_0\circ h_0 = g_0$ and $h_0\circ j_0 = f_0$. This h_0 induces the needed lifting for diagram `III`. ■

Proposition 7.1 may now be dualized as follows:

7.7. Proposition. *An internal functor is a trivial cofibration if and only if it is a split monomorphic equivalence.*

Proof. One impliciation is Proposition 5.11. To prove the other, consider a commutative square such as `V` above, and suppose that $\boldsymbol{j} : \mathbf{A} \longrightarrow \mathbf{X}$ is an equivalence with left adjoint left inverse $\boldsymbol{k} : \mathbf{X} \longrightarrow \mathbf{A}$. Let $\boldsymbol{K} : \mathbf{X} \longrightarrow \mathbf{X}^{\mathbf{I}}$ denote a homotopy from $\boldsymbol{j}\circ\boldsymbol{k}$ to $1_{\mathbf{X}}$. Put $\overline{H}_0 = m\circ(H_0\circ k_0, f_1\circ K_0) : X_0 \longrightarrow \mathrm{iso}(E)$, then $\delta_1\circ\overline{H}_0 = f_0$ and $\overline{H}_0\circ j_0 = H_0$. Proposition 4.3 now yields the needed lifting $\overline{\boldsymbol{H}} : \mathbf{X} \longrightarrow \mathbf{E}^{\mathbf{I}}$. ■

7.8. Theorem. *If $\mathcal{C}$ is a finitely complete category such that $\mathsf{Cat}\mathcal{C}$ is finitely cocomplete (cf. §2.2) then a model category structure is defined on $\mathsf{Cat}\mathcal{C}$ by choosing* we *the class of homotopy equivalences,* cof *the class of functors having the homotopy extension property and* fib *the class of functors having the homotopy lifting property.* ■

References

[1] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, Lecture notes in mathematics, vol. 269, Springer, 1972, Séminaire de Géometrie Algébrique du Bois-Marie 1963-1964 (SGA4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[2] J. Adámek, H. Herrlich, J. Rosický, and W. Tholen, *Weak factorization systems and topological functors*, Appl. Categ. Struct. **10** (2002), 237–249.

[3] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, 1994.

[4] M. Barr, *Exact categories*, Exact categories and categories of sheaves, Lecture notes in mathematics, vol. 236, Springer, 1971, pp. 1–120.

[5] R. Blackwell, G. M. Kelly, and A. J. Power, *Two-dimensional monad theory*, J. Pure Appl. Algebra **59** (1989), 1–110.

[6] F. Borceux, *Handbook of categorical algebra*, Encyclopedia of Mathematics and its Applications, vol. 50, 51 and 52, Cambridge University Press, 1994.

[7] F. Borceux and D. Bourn, *Mal'cev, protomodular, homological and semi-abelian categories*, Mathematics and its Applications, vol. 566, Kluwer Academic Publishers, 2004.

[8] F. Borceux, G. Janelidze, and G. M. Kelly, *Internal object actions*, to appear in Comment. Math. Univ. Carolinae, 2005.

[9] D. Bourn, *The denormalized* 3×3 *lemma*, J. Pure Appl. Algebra **177** (2003), 113–129.

[10] D. Bourn, *Protomodular aspect of the dual of a topos*, Adv. Math. **187** (2004), 240–255.

[11] D. Bourn, *On the direct image of intersections in exact homological categories*, J. Pure Appl. Algebra **196** (2005), 39–52.

[12] D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products*, Theory and Applications of Categories **4** (1998), no. 2, 37–46.

[13] K. S. Brown, *Abstract homotopy theory and generalized sheaf cohomology*, Trans. Amer. Math. Soc. **186** (1973), 419–458.

[14] R. Brown, *Fibrations of groupoids*, J. Algebra **15** (1970), 103–132.

[15] M. Bunge and R. Paré, *Stacks and equivalence of indexed categories*, Cah. Top. Géom. Diff. Catég. **XX** (1979), no. 4, 373–399.

[16] A. Carboni, G. M. Kelly, and M. C. Pedicchio, *Some remarks on Maltsev and Goursat categories*, Appl. Categ. Struct. **1** (1993), 385–421.

[17] A. Carboni, M. C. Pedicchio, and N. Pirovano, *Internal graphs and internal groupoids in Mal'cev categories*, Proceedings of Conf. Category Theory 1991, Montreal, Am. Math. Soc. for the Canad. Math. Soc., Providence, 1992, pp. 97–109.

[18] T. Everaert and T. Van der Linden, *Baer invariants in semi-abelian categories I: General theory*, Theory and Applications of Categories **12** (2004), no. 1, 1–33.

[19] T. Everaert and T. Van der Linden, *Baer invariants in semi-abelian categories II: Homology*, Theory and Applications of Categories **12** (2004), no. 4, 195–224.

[20] A. R. Garzón and J. G. Miranda, *Homotopy theory for (braided) Cat-groups*, Cah. Top. Géom. Diff. Catég. **XXXVIII** (1997), no. 2, 99–139.

[21] M. Golasiński, *Homotopies of small categories*, Fund. Math. **CXIV** (1981), 209–217.

[22] M. Gran, *Internal categories in Mal'cev categories*, J. Pure Appl. Algebra **143** (1999), 221–229.

[23] M. Gran and J. Rosický, *Special reflexive graphs in modular varieties*, Alg. Univ. **52** (2004), 89–102.

[24] J. W. Gray, *Fibred and cofibred categories*, Proceedings of the Conference on Categorical Algebra, La Jolla 1965 (S. Eilenberg, D. K. Harrison, S. MacLane, and H. Röhrl, eds.), Springer, 1966, pp. 21–83.

[25] G. Janelidze, *Internal crossed modules*, Georgian Math. J. **10** (2003), no. 1, 99–114.

[26] G. Janelidze and G. M. Kelly, *Galois theory and a general notion of central extension*, J. Pure Appl. Algebra **97** (1994), 135–161.

[27] P. T. Johnstone, *Topos theory*, L.M.S. Monographs, vol. 10, Academic Press, 1977.

[28] P. T. Johnstone, *The 'closed subgroup theorem' for localic herds and pregroupoids*, J. Pure Appl. Algebra **70** (1991), 97–106.

[29] P. T. Johnstone, *Sketches of an elephant: A topos theory compendium*, Oxford Logic Guides, vol. 43, 44, Oxford University Press, 2002, Volumes 1 and 2.

[30] A. Joyal and M. Tierney, *Strong stacks and classifying spaces*, Category Theory, Proceedings Como 1990, Lecture notes in mathematics, vol. 1488, Springer, 1991, pp. 213–236.

[31] K. H. Kamps, *Kan-Bedingungen und abstrakte Homotopietheorie*, Math. Z. **124** (1972), 215–236.

[32] K. H. Kamps and T. Porter, *Abstract homotopy and simple homotopy theory*, World Scientific Publishing, Singapore, 1997.

[33] S. Lack, *A Quillen model structure for bicategories*, K-theory, to appear.

[34] S. Lack, *A Quillen model structure for 2-categories*, K-theory **26** (2002), 171–205.

[35] S. Mac Lane, *Categories for the working mathematician*, second ed., Graduate texts in mathematics, vol. 5, Springer, 1998.

[36] S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic: A first introduction to topos theory*, Springer, 1992.

[37] M. C. Pedicchio, *A categorical approach to commutator theory*, J. Algebra **177** (1995), 647–657.

[38] D. G. Quillen, *Homotopical algebra*, Lecture notes in mathematics, vol. 43, Springer, 1967.

[39] Ch. Rezk, *A model category for categories*, unpublished, 1996.

[40] A. Strøm, *The homotopy category is a homotopy category*, Arch. Math. **23** (1972), 435–441.

[41] R. W. Thomason, *Cat as a closed model category*, Cah. Top. Géom. Diff. Catég. **XXI** (1980), no. 3, 305–324.

[42] K. Worytkiewicz, K. Hess, P. E. Parent, and A. Tonks, *A model structure à la Thomason on* **2-Cat**, `math.AT/0411154`, 2005.

Vakgroep Wiskunde
Faculteit Wetenschappen
Vrije Universiteit Brussel
Pleinlaan 2
1050 Brussel
Belgium
Email: `teveraer@vub.ac.be`
`rkieboom@vub.ac.be`
`tvdlinde@vub.ac.be`

This article may be accessed via WWW at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/15/3/15-03.{dvi,ps}`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.

Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.

The method of distribution of the journal is via the Internet tools `WWW/ftp`. The journal is archived electronically and in printed paper format.

Subscription information. Individual subscribers receive (by e-mail) abstracts of articles as they are published. Full text of published articles is available in .dvi, Postscript and PDF. Details will be e-mailed to new subscribers. To subscribe, send e-mail to `tac@mta.ca` including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor.

Information for authors. The typesetting language of the journal is TeX, and LaTeX2ε is the preferred flavour. TeX source of articles for publication should be submitted by e-mail directly to an appropriate Editor. They are listed below. Please obtain detailed information on submission format and style files from the journal's WWW server at `http://www.tac.mta.ca/tac/`. You may also write to `tac@mta.ca` to receive details by e-mail.