*Theory and Applications of Categories,* Vol. 15, No. 7, 2006, pp. 178–193.

# A GALOIS THEORY WITH STABLE UNITS FOR SIMPLICIAL SETS

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Abstract. We recall and reformulate certain known constructions, in order to make a convenient setting for obtaining generalized monotone-light factorizations in the sense of A. Carboni, G. Janelidze, G. M. Kelly and R. Paré. This setting is used to study the existence of monotone-light factorizations both in categories of simplicial objects and in categories of internal categories. It is shown that there is a non-trivial monotone-light factorization for simplicial sets, such that the monotone-light factorization for reflexive graphs via reflexive relations is a special case of it, obtained by truncation. More generally, we will show that there exists a monotone-light factorization associated with every full subcategory $Mono(F_n)$, $n \geq 0$, consisting of all simplicial sets whose unit morphisms are monic for the localization $F_n : \mathbf{Set}^{\Delta^{op}} \to \mathbf{Set}^{\Delta^{op}_n}$, which truncates each simplicial set after the object of n-simplices. The monotone-light factorization for categories via preorders is as well derived from the proposed setting. We also show that, for regular Mal'cev categories, the reflection of internal groupoids into internal equivalence relations necessarily produces monotone-light factorizations. It turns out that all these reflections do have stable units, in the sense of C. Cassidy, M. Hébert and G. M. Kelly, giving rise to Galois theories.

## 1. Introduction

Essentially every reflection $\mathcal{C} \to \mathcal{X}$, from a category $\mathcal{C}$ into its full subcategory $\mathcal{X}$, gives rise to a factorization system $(\mathrm{E}, \mathrm{M})$. Then, by respectively stabilizing and localizing the classes E and M of morphisms in $\mathcal{C}$, in the sense of [1], one may obtain another one $(\mathrm{E}', \mathrm{M}^*)$, to be called a monotone-light factorization system. The main result of [1] gives a necessary and sufficient condition for what seems to be a quite rare occurrence. Recall also from [1], that $\mathrm{M}^*$ is exactly the class of covering morphisms in the sense of Galois theory of G. Janelidze.

In my PhD thesis [10] (see also [11]), I studied in particular the reflection $\mathbf{Cat} \to \mathbf{Preord}$ from categories into preordered sets. It proved to be another example giving a Galois theory for the category **Cat** of all categories and a non-trivial monotone-light factorization (i.e., $(\mathrm{E}', \mathrm{M}^*) \neq (\mathrm{E}, \mathrm{M})$). A unit morphism $\eta_A : A \to I(A)$ of this reflection is the coequalizer of the kernel pair of another unit morphism $\varphi_A : A \to F(A)$, associated to

The author acknowledges partial financial assistance by Unidade de Investigação e Desenvolvimento Matemática e Aplicações da Universidade de Aveiro/FCT

Received by the editors 2004-11-05 and, in revised form, 2006-06-22.

Published on 2006-07-07 in the volume of articles from CT2004.

2000 Mathematics Subject Classification: 18A32, 18A40, 18G30, 12F10, 55U10, 08B05, 18B25.

Key words and phrases: simplicial object, simplicial set, internal category, internal preorder, regular category, Mal'cev category, descent theory, Galois theory, reflection with stable units, monotone-light factorization, Kan extension, elementary topos, geometric morphism.

the reflection of categories into indiscrete categories, as displayed in the pullback diagram 2.1.

In the present paper we study this *coequalizer of the kernel pair process* in a more general fashion, by beginning not with a reflection but simply with a pointed endofunctor $(F, \varphi)$, i.e., a natural transformation $\varphi : 1_{\mathcal{C}} \to F$, from the identity functor of a category $\mathcal{C}$ to an endofunctor $F : \mathcal{C} \to \mathcal{C}$. We impose then additional conditions, under which the regular epimorphisms $\eta_A : A \to I(A)$ define a reflection $(I, \eta)$. After that we give sufficient conditions for $(I, \eta)$ to have stable units, in the sense of [2] and [1].

In another instance of this process, the pointed endofunctor $(F, \varphi)$, associated with the reflection **RGrphs** $\to$ **LEqRel** of reflexive graphs into connected equivalence relations, gives rise to the reflection of reflexive graphs into reflexive relations. This reflection **RGrphs** $\to$ **RRel** has stable units, as follows from Corollary 4.4 (see Section 4).

It is known that if a reflection $(I, \eta)$ has stable units it is necessarily admissible in the sense of categorical Galois theory [5], also called semi-left exact in [2]. There is therefore a Galois theory for reflexive graphs via reflexive relations.

Furthermore, if a reflection $\mathcal{C} \to \mathcal{X}$ from a finitely-complete category $\mathcal{C}$ has stable units then there exists an associated monotone-light factorization in $\mathcal{C}$, provided that for each object $B$ there is an effective descent morphism $p : E \to B$ in $\mathcal{C}$ such that its domain $E$ belongs to the full subcategory $\mathcal{X}$ (see Corollary 6.2 in [9], which follows from the main result of [1]). This is really the case for both **Cat** $\to$ **Preord** and **RGrphs** $\to$ **RRel**.

The two reflections just considered are of course respectively a special case of $\mathbf{Cat}(\mathcal{S}) \to \mathbf{Preord}(\mathcal{S})$, categories in $\mathcal{S}$ into preorders in $\mathcal{S}$, and $\mathbf{RGrphs}(\mathcal{S}) \to \mathbf{RRel}(\mathcal{S})$, reflexive graphs in $\mathcal{S}$ into reflexive relations in $\mathcal{S}$, when $\mathcal{S} = \mathbf{Set}$. Sufficient conditions on $\mathcal{S}$ for successful internalizations of the former reflections were given in [9]. G. Janelidze suggested me to extend the results obtained in that paper [9] to simplicial objects, looking at them as higher dimensional graphs, with possible future applications in homotopy theory. And so I did, applying the coequalizer of the kernel pair process to the reflection of the category of simplicial sets **Smp** into its full subcategory of the nerves of indiscrete categories; and obtaining as well a new reflection with stable units and a non-trivial monotone-light factorization, into the category **OSmC** of ordered simplicial complexes. Note that there is a non-trivial monotone-light factorization system associated to the reflection $\mathbf{Smp}_n \to \mathbf{OSmC}_n$, for each integer $n \geq 1$. This reflection is just a special case of **Smp** $\to$ **OSmC** after truncating the simplicial sets right after the object of n-simplices. The previous reflection **RGrphs** $\to$ **RRel** is therefore simply the special case $\mathbf{Smp}_1 \to \mathbf{OSmC}_1$ having $n = 1$.

More generally, we could begin with the localizations $F_n : \mathbf{Smp} \to \mathbf{Smp}_n$, $n \geq 0$ ($n = 0$ corresponding to the reflection of simplicial sets into the nerves of indiscrete categories analyzed in the last paragraph), apply the coequalizer of the kernel pair process and obtain reflections with stable units and monotone-light factorizations for **Smp**. In fact, these localizations $F_n$ are examples of geometric morphisms $F^* \dashv F_* : \mathcal{E} \to \mathcal{F}$ between elementary topoi which are embeddings having $Mono(F^*)$ dense in $\mathcal{F}$ (i.e., every object of $\mathcal{F}$ is a colimit of objects of $Mono(F^*)$). This turns out to be enough to conclude that there is a reflection $I : \mathcal{F} \to Mono(F^*)$ with stable units and a monotone-light factorization (see Proposition 7.3).

If $\mathcal{S}$ is a Mal'cev variety of universal algebras then every internal category in $\mathcal{S}$ is a groupoid. Hence, the above-mentioned process produces in such a case a reflection with stable units $\mathbf{Grpd}(\mathcal{S}) \to \mathbf{EqRel}(\mathcal{S})$, from internal groupoids into equivalence relations. As M. Gran pointed out to me, for each internal groupoid $G$ in $\mathcal{S}$, there is an internal functor $(\sigma, d_1) : Eq(d_0) \to G$ from an equivalence relation (see Example 5.2). This guarantees the existence of monotone-light factorizations for internal groupoids via equivalence relations, exactly as in the other reflections studied. The category of groupoids in groups is known to be equivalent to the category of crossed modules, $\mathbf{Grpd}(\mathbf{Grp}) \simeq \mathbf{CrossMod}$. So, in particular, there is a monotone-light factorization for crossed modules via normal subgroup inclusions.

## 2. The coequalizer of a pointed endofunctor's kernel pair is well-pointed

Throughout all this paper, $(F, \varphi)$ will denote a pointed endofunctor on a finitely-complete category $\mathcal{C}$, such that for every object $A$ in $\mathcal{C}$ the kernel pair of $\varphi_A : A \to F(A)$ has a coequalizer. That is:

- $\varphi : 1_{\mathcal{C}} \to F$ is a natural transformation from the identity functor of a finitely-complete category $\mathcal{C}$ to the endofunctor $F : \mathcal{C} \to \mathcal{C}$;
- for every object $A$ in $\mathcal{C}$, the kernel pair of the morphism $\varphi_A$ does have a coequalizer $\eta_A = coeq(ker(\varphi_A))$, and the morphisms involved will be displayed as

$$\begin{tikzcd} A \times_{F(A)} A \arrow[rr, "\pi_{2,A}"] \arrow[dd, "\pi_{1,A}"'] & & A \arrow[dl, "\eta_A"] \arrow[dd, "\varphi_A"] \\ & I(A) \arrow[dr, "\mu_A"] & \\ A \arrow[ur, "\eta_A"] \arrow[rr, "\varphi_A"] & & F(A) \end{tikzcd} \quad . \qquad (2.1)$$

Our next lemma follows immediately from the fact that, in the category $\mathcal{C}^{\mathcal{C}}$ of endofunctors on $\mathcal{C}$, both the kernel pair $(\pi_1, \pi_2)$ of $\varphi$ and its coequalizer $\eta = coeq(ker(\varphi)) : 1_{\mathcal{C}} \to I$ are computed pointwise.

2.1. Lemma. *Diagram* 2.1 *defines a pointed endofunctor* $(I, \eta)$ *on* $\mathcal{C}$.

2.2. Proposition. *The pair* $(I, \eta)$ *obtained as above is a well-pointed endofunctor in the sense of [7], i.e.,* $I\eta = \eta I$.

Proof. By naturality of $\eta$, $I(\eta_A)\eta_A = \eta_{I(A)}\eta_A$. Hence, being $\eta_A$ an epimorphism, one obtains $I(\eta_A) = \eta_{I(A)}$ for each $A$ in $\mathcal{C}$. ∎

2.3. Example. Consider the category $\mathbf{Smp}(\mathcal{S}) = \mathcal{S}^{\Delta^{op}}$ of simplicial objects in $\mathcal{S}$, where $\Delta$ is the category of positive ordinal numbers $[n]$ $(n \geq 0)$.

If $\mathcal{S}$ is a finitely-complete category with coequalizers of kernel pairs, so is $\mathbf{Smp}(\mathcal{S})$, since it is well known that in any functor category the limits and colimits can be calculated pointwise. Hence, by the results above, one can state that, for any pointed endofunctor $(F,\varphi)$ on $\mathbf{Smp}(\mathcal{S})$, the pair $(I,\eta)$ obtained by the coequalizer of the kernel pair process is well-pointed, i.e., $I\eta = \eta I$.

2.4. Example. We are now beginning the analysis of an example already studied in [9]. In what follows, we will just adapt the results known for this case to the convenient setting introduced in the present paper. More details are given in [9].

For any finitely-complete category $\mathcal{S}$ there is the category $\mathbf{Cat}(\mathcal{S})$ of categories in $\mathcal{S}$. That is, the category whose objects are the diagrams in $\mathcal{S}$ of the form

$$\begin{tikzcd} C = C_1 \times_{C_0} C_1 \arrow[r, shift left=3, "\pi_2"] \arrow[r, "\gamma"] \arrow[r, shift right=3, "\pi_1"'] & C_1 \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "d_1"'] & C_0 \arrow[l, "i"'] \end{tikzcd} \tag{2.4}$$

satisfying the conditions

$$d_0 i = 1_{C_0} = d_1 i,\ d_0\pi_1 = d_1\pi_2,\ d_0\gamma = d_0\pi_2,\ \textit{and}\ d_1\gamma = d_1\pi_1,$$

where the square represented by the second equation is a pullback and the *composition operation* $\gamma$ satisfies the associative and unit laws.

Consider now, for a finitely-complete category $\mathcal{S}$ with coequalizers of kernel pairs, the pointed endofunctor $(F,\varphi)$ on $\mathbf{Cat}(\mathcal{S})$ for which:

$$\begin{tikzcd} F(C) = C_0 \times C_0 \times C_0 \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & C_0 \times C_0 \arrow[r, shift left=3] \arrow[r, shift right=3] & C_0 \arrow[l] \end{tikzcd} \tag{2.5}$$

where $C_0$ is the object of objects of $C$, and the morphisms are the obvious ones between the powers of $C_0$; for every $C$ in $\mathbf{Cat}(\mathcal{S})$, $\varphi_C = (d_C, 1_{C_0}) : C \to F(C)$, where $d_C = \langle d_0, d_1\rangle$ is the morphism determined by the commutative diagram

$$\begin{tikzcd} & C_1 \arrow[dl, "d_0"'] \arrow[d, "d_C"] \arrow[dr, "d_1"] & \\ C_0 & C_0 \times C_0 \arrow[l] \arrow[r] & C_0 \end{tikzcd} \tag{2.6}$$

By the results above, in order to conclude that there is a well-pointed endofunctor $(I,\eta)$ obtained by the coequalizer of the kernel pair process, one needs only to show that the coequalizer $\eta_C$ of the kernel pair of every internal functor $\varphi_C$ does exist in $\mathbf{Cat}(\mathcal{S})$.

Sufficient conditions for the existence of such coequalizers $\eta_C$ were given in [9, Proposition 3.3]. In particular, if the coequalizer $e_C : C_1 \to I(C)_1$ of the kernel pair $(p_C, q_C)$ of $d_C : C_1 \to C_0 \times C_0$ is a pullback stable regular epimorphism in $\mathcal{S}$, then the following diagram displays the coequalizer $\eta_C = (e_C, 1_{C_0})$ of the kernel pair of $\varphi_C = (d_C, 1_{C_0})$ in $\mathbf{Cat}(\mathcal{S})$:

$$
\begin{tikzcd}[column sep=large, row sep=huge]
(C_1 \times_{C_0\times C_0} C_1) \times_{C_0} (C_1 \times_{C_0\times C_0} C_1) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d, shift right, "p_C \times p_C"'] \arrow[d, shift left, "q_C \times q_C"] & C_1 \times_{C_0\times C_0} C_1 \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[d, shift right, "p_C"'] \arrow[d, shift left, "q_C"] & C_0 \arrow[l] \arrow[d, "1_{C_0}"] \\
C_1 \times_{C_0} C_1 \arrow[r, shift left=3, "\pi_2"] \arrow[r, "\gamma"] \arrow[r, shift right=3, "\pi_1"'] \arrow[d, "e_C \times e_C"] & C_1 \arrow[r, shift left=3, "d_0"] \arrow[r, shift right=3, "d_1"'] \arrow[d, "e_C"] & C_0 \arrow[l, "i"'] \arrow[d, "1_{C_0}"] \\
I(C)_1 \times_{C_0} I(C)_1 \arrow[r, shift left=3, "\pi_2^{I(C)}"] \arrow[r, "\gamma^{I(C)}"] \arrow[r, shift right=3, "\pi_1^{I(C)}"'] & I(C)_1 \arrow[r, shift left=3, "d_0^{I(C)}"] \arrow[r, shift right=3, "d_1^{I(C)}"'] & C_0 \arrow[l, "e_C i"'] 
\end{tikzcd}
\quad . \tag{2.7}
$$

## 3. Idempotency of $(I, \eta)$

It is known that a pointed endofunctor $(I, \eta)$ on $\mathcal{C}$ is idempotent (i.e., $I\eta = \eta I$ and $\eta I$ is an isomorphism) if and only if the full replete subcategory $Fix(I, \eta)$ of $(I, \eta)$-fixed objects in $\mathcal{C}$ is reflective in $\mathcal{C}$ with reflection $\eta$ (an object $A$ is $(I, \eta)$-fixed if $\eta_A$ is an isomorphism; see [6]).

Proposition 3.1 below states that, if $(I, \eta)$ is the well-pointed endofunctor obtained from the pointed endofunctor $(F, \varphi)$ by the coequalizer of the kernel pair process, then $Fix(I, \eta)$ is equal to $Mono(F, \varphi)$, the full subcategory of $\mathcal{C}$ formed by the objects $A$ with $\varphi_A$ monic. It follows trivially that $\eta I$ is an isomorphism if and only if $\varphi I$ is a monomorphism in $\mathcal{C}^{\mathcal{C}}$. This new characterization of the idempotency of the well-pointed endofunctor $(I, \eta)$ will be given in Corollary 3.2.

3.1. Proposition. *Consider the well-pointed endofunctor $(I, \eta)$ on $\mathcal{C}$ obtained from a pointed endofunctor $(F, \varphi)$, through the coequalizer of the kernel pair process displayed in diagram* 2.1.

*Then, the two full subcategories $Fix(I, \eta)$ and $Mono(F, \varphi)$ of $\mathcal{C}$ are identical, $Fix(I, \eta) = Mono(F, \varphi)$.*

Proof. Consider the pullback diagram 2.1.

If $\varphi_A$ is a monomorphism then $\eta_A$ must also be a monomorphism. This implies that $\eta_A$ is an isomorphism, since it is in addition a regular epimorphism.

Conversely, if $\eta_A$ is an isomorphism then $\pi_{1,A} = \pi_{2,A}$ is an isomorphism, since $\eta_A$ is the coequalizer of the kernel pair $(\pi_{1,A}, \pi_{2,A})$. Therefore, $\varphi_A$ is a monomorphism provided $\eta_A$ is an isomorphism. ∎

3.2. Corollary. *The endofunctor $(I, \eta)$ on $\mathcal{C}$, obtained by the process displayed in diagram* 2.1*, is idempotent, in the sense of [7] and [6], if and only if $\varphi_{I(A)}$ is a monomorphism for each object $A$ in $\mathcal{C}$.*

Corollaries 3.3 and 3.4 below will be useful in the examples.

3.3. Corollary. *If $\mu$ is a monomorphism then $(I, \eta)$ is idempotent provided either $F\varphi$ or $F\eta$ is a monomorphism.*

Proof. Let us suppose first that both $\mu$ and $F\eta$ are monomorphisms. By naturality of $\varphi$ one has $F(\eta_A)\varphi_A = \varphi_{I(A)}\eta_A$ for each object $A$ in $\mathcal{C}$. Therefore $F(\eta_A)\mu_A = \varphi_{I(A)}$, since $\varphi_A = \mu_A\eta_A$ and $\eta_A$ is a regular epimorphism. Hence, $\varphi_{I(A)}$ is a monomorphism because it is the composite of two monomorphisms, for each object $A$ in $\mathcal{C}$. It follows then from Corollary 3.2 that $(I, \eta)$ is idempotent.

To complete this proof, one needs only to note that if the composite $F\varphi = F\mu \cdot F\eta$ is a monomorphism then $F\eta$ is necessarily also a monomorphism. ∎

3.4. Corollary. *If $\mathcal{C}$ is a regular category or, more generally, admits a (regular epi, mono)-factorization, and $(F, \varphi)$ is idempotent then $(I, \eta)$ is idempotent.*

Proof. Since $\mathcal{C}$ is regular, $\mu$ is a monomorphism because $\varphi_A = \mu_A\eta_A$ is a regular epi-mono factorization. $F\varphi$ is an isomorphism since $(F, \varphi)$ is idempotent. Hence, $(I, \eta)$ is idempotent by Corollary 3.3. ∎

3.5. Example. For any finitely-complete category $\mathcal{S}$ with coequalizers of kernel pairs, consider the pointed endofunctor $(F, \varphi)$ on $\mathbf{Smp}(\mathcal{S}) = \mathcal{S}^{\Delta^{op}}$ for which

$$\begin{tikzcd} F(A) = \cdots\ A_0 \times A_0 \times A_0 \arrow[r, shift left=3] \arrow[r, shift left=1, leftarrow] \arrow[r, shift right=1] \arrow[r, shift right=3, leftarrow] \arrow[r, shift right=5] & A_0 \times A_0 \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & A_0 \end{tikzcd} \quad . \qquad (3.1)$$

The 0-component of $\varphi(A) : A \to F(A)$ is the identity morphism $1_{A_0} : A_0 \to A_0$, as shown in the diagram

$$\begin{tikzcd} \cdots\ A_2 \arrow[d, "d_A \times d_A"] \arrow[r, shift left=3] \arrow[r, shift left=1, leftarrow] \arrow[r, shift right=1] \arrow[r, shift right=3, leftarrow] \arrow[r, shift right=5] & A_1 \arrow[d, "d_A"] \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & A_0 \arrow[d, "1_{A_0}"] \\ \cdots\ A_0 \times A_0 \times A_0 \arrow[r, shift left=3] \arrow[r, shift left=1, leftarrow] \arrow[r, shift right=1] \arrow[r, shift right=3, leftarrow] \arrow[r, shift right=5] & A_0 \times A_0 \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & A_0 \end{tikzcd} \quad . \qquad (3.2)$$

Then, one knows that the pair $(I, \eta)$ is a well-pointed endofunctor, by the reasons given in Section 2.

Remark now that:

- $F(A)$ and $F(F(A))$ are the same simplicial objects, since their objects of points are the same, $(F(A))_0 = A_0 = (F(F(A)))_0$, for every simplicial object $A$;

- $F(\varphi_A)$ is the identity morphism of $F(A)$ for every simplicial object $A$, since $(F(\varphi_A))_0 = (\varphi_A)_0 = 1_{A_0}$ (any morphism of simplicial objects with codomain $F(A)$ is completely determined by its 0-component!).

So, according to Corollary 3.3, one has only to show that every morphism $\mu_A$ in $\mathbf{Smp}(\mathcal{S})$ is a monomorphism in order to prove that $(I, \eta)$ is idempotent. But that is the case when $\mathcal{S}$ admits a (regular epi, mono)-factorization, for then the factorization $\varphi_A = \mu_A \eta_A$ is calculated pointwise using the regular epi-mono factorizations in $\mathcal{S}$.

One could as well arrive to the same conclusion, for a category $\mathcal{S}$ which is regular or, more generally, admits a (regular epi, mono)-factorization, by using only Corollary 3.4: a functor category like $\mathbf{Smp}(\mathcal{S})$ admits regular epi-mono factorizations (respectively, $\mathbf{Smp}(\mathcal{S})$ is regular) whenever $\mathcal{S}$ admits regular epi-mono factorizations (respectively, $\mathcal{S}$ is regular); it is also easy to show that $(F, \varphi)$ is idempotent in this example.

3.6. Example. Let us consider again, for a finitely-complete category $\mathcal{S}$ with coequalizers of kernel pairs, the pointed endofunctor $(F, \varphi)$ on $\mathbf{Cat}(\mathcal{S})$ of Example 2.4. It is also required for each internal category $C$ in $\mathcal{S}$ that the coequalizer $\eta_C$ of the kernel pair of $\varphi_C$ does exist in $\mathbf{Cat}(\mathcal{S})$, which is known to be the case when in particular $\mathcal{S}$ is a regular category (see [9, Proposition 3.3]).

Note that for this example $F(\varphi_C)$ is the identity morphism of $F(C)$. Indeed, one has that $F(C) = F(F(C))$ since they have the same object of objects $(F(C))_0 = C_0 = (F(F(C)))_0$, and $(F(\varphi_C))_0 = (\varphi_C)_0 = 1_{C_0}$ the identity morphism of $C_0$ in $\mathcal{S}$.

Hence, by Corollary 3.3, if $\mu_C = (m_C, 1_{C_0}) : I(C) \to F(C)$ is a monomorphism for each internal category $C$ in $\mathbf{Cat}(\mathcal{S})$, then $(I, \eta)$ is idempotent. Equivalently, if $m_C$ is a monomorphism for each $C$ in $\mathcal{C}$, then $Fix(I, \eta)$ is a full reflexive subcategory of $\mathbf{Cat}(\mathcal{S})$. Remark that this is obviously the case when $\mathcal{S}$ is regular, since then $d_C = m_C e_C$ is a regular epi-mono factorization in $\mathcal{S}$ (the morphisms $d_C : C_1 \to F(C)_1$, $e_C : C_1 \to I(C)_1$ and $m_C : I(C)_1 \to F(C)_1$ in $\mathcal{S}$, are respectively the first components of the internal functors $\varphi_C = (d_C, 1_{C_0})$, $\eta_C = (e_C, 1_{C_0})$ and $\mu_C = (m_C, 1_{C_0})$; see diagram 2.7).

## 4. Stabilization of the idempotent $(I, \eta)$

Our next proposition gives sufficient conditions for the coequalizer of the kernel pair process to produce a reflection $(I, \eta)$ such that, for every pullback $g^*(\eta_A)$ of a unit morphism $\eta_A$ along any morphism $g$ in $\mathcal{C}$, $I(g^*(\eta_A))$ is an isomorphism.

4.1. Proposition. *Consider the well-pointed endofunctor $(I, \eta)$, obtained from the pointed endofunctor $(F, \varphi)$ by the coequalizer of the kernel pair process displayed in diagram* 2.1.

*Then, $(I, \eta)$ is idempotent with stable units, in the sense of [2] and [1], provided the following four conditions hold:*

*(i)* $\mu$ *is a monomorphism;*

*(ii)* $F\eta$ *is an isomorphism;*

*(iii) the functor* $F$ *preserves the pullback diagrams of the form*

$$\begin{tikzcd} C \times_{I(A)} A \arrow[r] \arrow[d] & A \arrow[d, "\eta_A"] \\ C \arrow[r, "g"] & I(A) \end{tikzcd} \quad ;$$

*(iv) all morphisms* $\eta_A$ *are pullback stable regular epimorphisms.*

Proof. Consider the commutative diagram

$$\begin{tikzcd} C \times_{I(A)} A \arrow[r, "\eta_{C\times_{I(A)}A}"] \arrow[d, "g^*(\eta_A)"] & I(C \times_{I(A)} A) \arrow[r, "\varphi_{I(C\times_{I(A)}A)}"] \arrow[d, "I(g^*(\eta_A))"] & FI(C \times_{I(A)} A) \arrow[d, "FI(g^*(\eta_A))"] \\ C \arrow[r, "\eta_C"] & I(C) \arrow[r, "\varphi_{I(C)}"] & FI(C) \end{tikzcd} \quad ,$$

where $g^*(\eta_A)$ is the pullback of the unit morphism $\eta_A : A \to I(A)$ along $g : C \to I(A)$.

As, by Corollary 3.3, conditions (i) and (ii) in the statement imply that $(I, \eta)$ is idempotent, it only remains to verify that $I(g^*(\eta_A))$ is an isomorphism.

Conditions (ii) and (iii) in the statement imply that both $F(\eta_C)$ and $F(g^*(\eta_A))$ are isomorphisms in $\mathcal{C}$. $FI(g^*(\eta_A))$ is therefore also an isomorphism, since $FI(g^*(\eta_A))F(\eta_{C\times_{I(A)}A}) = F(\eta_C)F(g^*(\eta_A))$. Hence, we conclude that $I(g^*(\eta_A))$ is a monomorphism, since $\varphi_{I(C)}I(g^*(\eta_A)) = FI(g^*(\eta_A))\varphi_{I(C\times_{I(A)}A)}$ and $\varphi_{I(C\times_{I(A)}A)}$ is a monomorphism by Corollary 3.2.

Furthermore, by condition (iv) in the statement, $g^*(\eta_A)$, $\eta_{C\times_{I(A)}A}$ and $\eta_C$ are all pullback stable regular epimorphisms, which are known to be closed under composition and to have the strong right cancellation property. $I(g^*(\eta_A))$ is therefore a regular epimorphism since $I(g^*(\eta_A))\eta_{C\times_{I(A)}A} = \eta_C g^*(\eta_A)$.

Hence, being $I(g^*(\eta_A))$ a regular epimorphism which is simultaneously a monomorphism, it is necessarily an isomorphism. ∎

Corollary 4.3 below, which follows trivially from our next lemma and Proposition 4.1, restates the latter in terms of factorization systems.

We will now suppose that both pointed endofunctors $(F, \varphi)$ and $(I, \eta)$, respectively the "input" and "output" of the coequalizer of the kernel pair process, are idempotent. It is well known that in this case the full reflective subcategories $Fix(F, \varphi)$ and $Fix(I, \eta)$ give rise respectively to prefactorization systems $(\mathrm{E}_F, \mathrm{M}_F)$ and $(\mathrm{E}_I, \mathrm{M}_I)$, where $\mathrm{E}_F$, respectively $\mathrm{E}_I$, is the class of morphisms $f$ in $\mathcal{C}$ such that $Ff$, respectively $If$, is an isomorphism (see [1]). We have already showed in Proposition 3.1 that $Fix(I, \eta) = Mono(F, \varphi)$, which implies $Fix(F, \varphi) \subseteq Fix(I, \eta)$. Hence, by the properties of prefactorization systems associated to reflective subcategories, $\mathrm{M}_F \subseteq \mathrm{M}_I$ and $\mathrm{E}_I \subseteq \mathrm{E}_F$ (see [1, §3]).

4.2. Lemma. *If the pointed endofunctors $(F, \varphi)$ and $(I, \eta)$ are idempotent then $\mu$ is a monomorphism and $F\eta$ is an isomorphism.*

Proof. It is known that every unit morphism $\eta_A$ belongs to $\mathrm{E}_I$ (see [1]). Therefore, $\eta_A$ also belongs to $\mathrm{E}_F \supseteq \mathrm{E}_I$, i.e., $F(\eta_A)$ is an isomorphism.

Then, as $F\varphi$ is an isomorphism by hypothesis, it is easy to conclude that $F\mu$ must also be an isomorphism, since $\varphi = \mu \cdot \eta$. This implies that $\mu$ is a monomorphism, since $F(\mu_A)\varphi_{I(A)} = \varphi_{F(A)}\mu_A$ by naturality of $\varphi$, for every object $A$ in $\mathcal{C}$, and $\varphi I$ is a monomorphism by hypothesis (see Corollary 3.2). ∎

4.3. Corollary. *Suppose that the pointed endofunctors $(F, \varphi)$ and $(I, \eta)$ of Section 2 are both idempotent, i.e., they are in fact reflections with the respective associated prefactorization systems $(\mathrm{E}_F, \mathrm{M}_F)$ and $(\mathrm{E}_I, \mathrm{M}_I)$ (see [1]).*

*Then, the reflection $(I, \eta)$ does have stable units, in the sense of [2], provided the following two conditions hold:*

*(i) the functor $F$ preserves the pullback diagrams of the form*

$$\begin{tikzcd} C \times_{I(A)} A \arrow[r] \arrow[d] & A \arrow[d, "\eta_A"] \\ C \arrow[r, "g"] & I(A) \end{tikzcd} ,$$

*where $\eta_A$ is any unit morphism of the reflection $(I, \eta)$;*

*(ii) all unit morphisms $\eta_A$ are pullback stable regular epimorphisms.*

*Condition (i) is equivalent to the following condition (i') provided $(F, \varphi)$ and $(I, \eta)$ are both idempotent:*

*(i') for every object $A$ in $\mathcal{C}$, the unit morphism $\eta_A$ of $I$ belongs to $\mathrm{E}'_F$, the largest class of morphisms contained in $\mathrm{E}_F$ which is stable under pullbacks.*

Our next corollary is a direct consequence of Corollaries 3.3 and 4.3.

4.4. Corollary. *If $\mathcal{C}$ is a regular category, $(F, \varphi)$ is idempotent and $F$ is a left exact functor, then $(I, \eta)$ is idempotent with stable units.*

Proof. If $\mathcal{C}$ is a regular category and $(F, \varphi)$ is idempotent, then Corollary 3.4 states that $(I, \eta)$ is also idempotent.

The left exactness of $F$ is equivalent to the condition that every pullback of a morphism in $\mathrm{E}_F$ is in $\mathrm{E}_F$ (see [2, Th. 4.7]), i.e., $\mathrm{E}_F = \mathrm{E}'_F$, which implies that every unit morphism $\eta_A$ is in $\mathrm{E}'_F$, since it is known to be in $\mathrm{E}_I \subseteq \mathrm{E}_F$. We have just checked that condition (i') in Corollary 4.3 holds.

Finally, the fact that $\mathcal{C}$ is a regular category validates condition (ii) in Corollary 4.3. ∎

4.5. Example. In Examples 3.5 and 3.6, of pointed endofunctors $(F, \varphi)$ on $\mathbf{Smp}(\mathcal{S})$ and $\mathbf{Cat}(\mathcal{S})$ respectively, for a finitely-complete category $\mathcal{S}$ with coequalizers of kernel pairs, it is easy to check that $(F, \varphi)$ is idempotent and $F$ is a left exact functor.

Hence, if $\mathcal{S}$ is a regular category then all the conditions in Proposition 4.1 hold for those examples. We can conclude that the two general examples do give rise to $(I, \eta)$ idempotent with stable units, provided $\mathcal{S}$ is a regular category.

## 5. Monotone-light factorization for the idempotent $(I, \eta)$

Proposition 5.1 below is a weaker version of [9, Corollary 6.2]. The latter is a consequence of the main result of [1], which gives necessary and sufficient conditions for the existence of a monotone-light factorization system $(\mathrm{E}', \mathrm{M}^*)$.

5.1. Proposition. *[9] Suppose that the following two conditions hold:*

- *the pointed endofunctor $(I, \eta)$ on $\mathcal{C}$, obtained as in Section 2, is idempotent and does have stable units;*
- *for each object $B$ in $\mathcal{C}$, there is an effective descent morphism $p : E \to B$ in $\mathcal{C}$ such that its domain $E$ is an object of $Fix(I, \eta)$(=$Mono(F, \varphi)$).*

*Then,* $(\mathrm{E}, \mathrm{M})$ *and* $(\mathrm{E}', \mathrm{M}^*)$ *are factorization systems. The pair* $(\mathrm{E}, \mathrm{M})$ *stands for the prefactorization system associated to the reflection* $(I, \eta)$*. The latter pair* $(\mathrm{E}', \mathrm{M}^*)$ *is obtained from the former by simultaneous stabilization of* $\mathrm{E}$ *and localization of* $\mathrm{M}$*, in the sense of [1], and it is called a monotone-light factorization system.*

5.2. Example. For $\mathcal{S} = \mathbf{Set}$ the category of sets, the idempotent $(I, \eta)$ with stable units on $\mathbf{Cat}(\mathbf{Set})$ is simply the reflection $\mathbf{Cat} \to \mathbf{Preord}$ of small categories into preordered sets, which was studied in [11]. For each object $B$ in $\mathbf{Cat}$, there is the effective descent morphism $\varepsilon_B : \mathbf{Cat}([3], B) \cdot [3] \to B$ in $\mathbf{Cat}$, the obvious projection from the coproduct of sufficiently many copies of the ordinal number [3]. Note that the morphism $\varepsilon_B$ is the counit of the adjunction $(-) \cdot [3] \dashv \mathbf{Cat}([3], -) : \mathbf{Cat} \to \mathbf{Set}$.

Hence, the conditions of Proposition 5.1 hold and one can conclude that there is a non-trivial monotone-light factorization on $\mathbf{Cat}$ which arises from the process described in this paper. The idempotent $(F, \varphi)$ is in this case the reflection of the category $\mathbf{Cat}$ of all categories into the category of indiscrete categories.

As M. Gran pointed out to me, Proposition 5.1 applies to the reflection $\mathbf{Grpd}(\mathcal{S}) \to \mathbf{EqRel}(\mathcal{S})$ between internal groupoids and equivalence relations, for any regular category $\mathcal{S}$. This is exactly the reflection $\mathbf{Cat}(\mathcal{S}) \to \mathbf{Preord}(\mathcal{S})$ in the case $\mathcal{S}$ is a (regular) Mal'cev category (i.e., a category with finite limits in which the internal equivalence relations coincide with the reflexive relations, $\mathbf{EqRel}(\mathcal{S}) = \mathbf{RRel}(\mathcal{S})$), simply because $\mathbf{Cat}(\mathcal{S}) = \mathbf{Grpd}(\mathcal{S})$ and $\mathbf{Preord}(\mathcal{S}) = \mathbf{EqRel}(\mathcal{S})$ in that case. In fact, for each internal groupoid $G$ in $\mathcal{S}$, there is an internal functor $(\sigma, d_1) : Eq(d_0) \to G$, with $\sigma = \gamma(1_{G_1} \times s)$, as shown in the following diagram

$$
\begin{tikzcd}
G_1 \times_{G_0} G_1 \times_{G_0} G_1 \arrow[r, "p_1 \times p_2"] \arrow[d, "\sigma \times \sigma"] & G_1 \times_{G_0} G_1 \arrow[r, shift left=2, "p_2"] \arrow[r, shift right=2, "p_1"'] \arrow[d, "\sigma"] & G_1 \arrow[l, "< 1, 1 >"'] \arrow[d, "d_1"] \\
G_1 \times_{G_0} G_1 \arrow[r, "\gamma"] & G_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] \arrow[loop below, "s"] & G_0 \arrow[l, "i"']
\end{tikzcd}
.
$$

The internal functor $(\sigma, d_1)$ is an effective descent morphism in the category $\mathbf{Grpd}(\mathcal{S})$ of internal groupoids in $\mathcal{S}$. This is so because $\sigma < 1_{G_1}, id_0 >= 1_{G_1}$ and $d_1 i = 1_{G_0}$.

Hence, being $\mathcal{S}$ a regular Mal'cev category, we conclude from Proposition 5.1 that there is a monotone-light factorization associated to the reflection $(I, \eta)$. This is of course the case when $\mathcal{S}$ is a Mal'cev variety of universal algebras. In particular, if $\mathcal{S} = \mathbf{Grp}$ the category of groups, the coequalizer of the kernel pair process produces a reflection with stable units and monotone-light factorization for crossed modules. Since $\mathbf{Cat}(\mathbf{Grp}) = \mathbf{Grpd}(\mathbf{Grp})$ is equivalent to the category **CrossMod** of crossed modules.

## 6. The monotone-light factorization for simplicial sets via ordered simplicial complexes

The category of ordered simplicial complexes in $\mathcal{S}$ will be denoted by $\mathbf{OSmC}(\mathcal{S})$; obviously we can simply define it as $Mono(F, \varphi)$, where $F$ and $\varphi$ are as in Example 3.5. The following proposition gives an equivalent reformulation; although the readers familiar with simplicial objects will find it straightforward, we will recall the proof.

6.1. Proposition. *For a regular category $\mathcal{S}$, a simplicial object $A$ in $\mathbf{Smp}(\mathcal{S})$ is in its full reflective subcategory $\mathbf{OSmC}(\mathcal{S})$ if and only if its face maps $d_i^A : A_{j+1} \to A_j$, $0 \le i \le j+1$, are jointly monic, for each $j \ge 0$.*

Proof. Consider the commutative diagram

$$
\begin{tikzcd}
A_{j+1} \arrow[r, "d_i^A"] \arrow[d, "\varphi_{j+1}"] & A_j \arrow[d, "\varphi_j"] \\
A_0^{j+2} \arrow[r, "d_i^{F(A)}"] & A_0^{j+1}
\end{tikzcd}
, \tag{6.1}
$$

associated to the pointed endofunctor $(F, \varphi)$ of Example 3.5. We know from Proposition 3.1 that the simplicial object $A$ belongs to $\mathbf{OSmC}(\mathcal{S}) = Fix(I, \eta)$ if and only if $\varphi_j : A_j \to A_0^{j+1}$ is a monomorphism for every $j \ge 0$, i.e., it belongs to $Mono(F, \varphi)$.

We are going to suppose first that $\varphi_j : A_j \to A_0^{j+1}$ is a monomorphism for every $j \geq 0$. Let $d_i^A f = d_i^A g$, for every $0 \leq i \leq j+1$. Then, $\varphi_j d_i^A f = \varphi_j d_i^A g$, which implies by the commutativity of the diagram just above that $d_i^{F(A)} \varphi_{j+1} f = d_i^{F(A)} \varphi_{j+1} g$, having $0 \leq i \leq j+1$. This last equality and the fact that the face maps (of any fixed dimension) of $F(A)$ are jointly monic imply that $\varphi_{j+1} f = \varphi_{j+1} g$. Hence, we have just proved that every object in $\mathbf{Smp}(\mathcal{S})$ of the form $I(A)$ does have jointly monic face maps.

We have to prove the converse now. Suppose that the face maps $d_i^A : A_{j+1} \to A_j$, $0 \leq i \leq j+1$, are jointly monic, for each $j \geq 0$. Under such an assumption, one has to show that if $\varphi_j$ is a monomorphism (induction hypothesis) then so is $\varphi_{j+1}$, since $\varphi_0$ is the identity morphism of $A_0$. In this way, if $\varphi_{j+1} f = \varphi_{j+1} g$ then $d_i^{F(A)} \varphi_{j+1} f = d_i^{F(A)} \varphi_{j+1} g$, having $0 \leq i \leq j+1$, and $\varphi_j d_i^A f = \varphi_j d_i^A g$ by the commutativity of diagram 6.1. Hence, $d_i^A f = d_i^A g$, having $0 \leq i \leq j+1$, since $\varphi_j$ is a monomorphism by the induction hypothesis. It follows that $\varphi$ is a monomorphism provided the morphisms $d_i^A$ are jointly monic, $0 \leq i \leq j+1$. ∎

According to Proposition 5.1, there is a monotone-light factorization associated with the reflection $\mathbf{Smp} \to \mathbf{OSmC}$ of simplicial sets into ordered simplicial complexes, provided there is an effective descent morphism $\varepsilon_B : E \to B$ such that $E$ is in $\mathbf{OSmC}$, for each simplicial set $B$ in $\mathbf{Smp}$.

Indeed, we may choose $\varepsilon_B$ to be the $B$-component of the counit of the adjunction $(-) \cdot \omega \dashv \mathbf{Smp}(\omega, -) : \mathbf{Smp} \to \mathbf{Set}$, where $\omega$ is the first infinite ordinal considered as a simplicial set via the usual nerve functor. That is, $\varepsilon_B : \mathbf{Smp}(\omega, B) \cdot \omega \to B$ is the canonical morphism from the coproduct of "sufficiently many" copies of $\omega$ to $B$.

In fact, as $\mathbf{Smp} = \mathbf{Set}^{\Delta^{op}}$ is a presheaf category, a morphism of simplicial sets is an effective descent morphism if it is an epimorphism. Every counit morphism $\varepsilon_B : \mathbf{Smp}(\omega, B) \cdot \omega \to B$, of the adjunction $\mathbf{Smp} \to \mathbf{Set}$, is an effective descent morphism if and only if the right adjoint $\mathbf{Smp}(\omega, -)$ is a faithful functor. As this latter statement is easy to check, we have guaranteed the existence of a monotone-light factorization for simplicial sets via ordered simplicial complexes.

Note also that, as will see below, such a monotone-light factorization $(\mathrm{E}', \mathrm{M}^*)$ is a non-trivial one, i.e., it does not coincide with the reflective factorization $(\mathrm{E}, \mathrm{M})$.

For each $n \geq 0$, the reflection $\mathbf{Smp}_n(\mathcal{S}) \to \mathbf{OSmC}_n(\mathcal{S})$, of the presheaf category of n-truncated simplicial objects to n-truncated ordered simplicial complexes in $\mathcal{S}$, is induced by the reflection $\mathbf{Smp}(\mathcal{S}) \to \mathbf{OSmC}(\mathcal{S})$. For the case $\mathcal{S} = \mathbf{Set}$, there is a monotone-light factorization associated to each reflection $\mathbf{Smp}_n \to \mathbf{OSmC}_n$, $n \geq 0$, which is a straightforward restriction of the one for $\mathbf{Smp}$. In these cases one can replace $\omega$ by the ordinal number $[n]$, when displaying the suitable effective descent morphisms for each object $B$ in $\mathbf{Smp}_n$.

In particular, having $n = 1$, the reflection $\mathbf{Smp}_1 \to \mathbf{OSmC}_1$ is just the reflection $\mathbf{RGrphs} \to \mathbf{RRel}$ of reflexive graphs into reflexive relations, which has a non-trivial monotone-light factorization

$$(\mathrm{E}', \mathrm{M}^*) = (\textit{Bijections on Vertices and Surjections on Arrows}, \text{“}\textit{Faithful}\text{”}),$$

similar to the one for **Cat** $\to$ **Preord**. This gives the desired conclusion of non-triviality for all monotone-light factorizations above except the case $n = 0$, where the reflection itself is trivial, i.e., it is the identity functor of the category of sets.

## 7. Geometric morphisms and monotone-light factorizations

In last Section 6, we have started with the reflection of the presheaf category $\mathbf{Smp} = \mathbf{Set}^{\Delta^{op}}$ into its full subcategory in which the objects are the nerves of indiscrete categories, $F_0 : \mathbf{Smp} \to \mathbf{Smp}_0 = \mathbf{Set}^{\Delta_0^{op}} (\simeq \mathbf{Set})$. This can be generalized to the reflections $F_n$ : $\mathbf{Smp} \to \mathbf{Smp}_n = \mathbf{Set}^{\Delta_n^{op}}$, $n \geq 0$, which truncate each simplicial set right after the object of n-simplices:

$$A \longmapsto A_n \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \vdots\\ \longrightarrow\end{array} \cdots\ A_1 \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} A_0 \tag{7.1}$$

(Note that for the obvious localizations

$$F_n^k : \mathbf{Smp}_k \to \mathbf{Smp}_n, k \geq n \geq 0,$$

we could copy all the results obtained below for the adjunctions 7.1).

Given a functor $K : \mathcal{B} \to \mathcal{A}$, let $\mathcal{S}$ be a regular and complete category. Then, the induced functor

$$\mathcal{S}^K : \mathcal{S}^{\mathcal{A}} \to \mathcal{S}^{\mathcal{B}}$$

is continuous, i.e., preserves all limits, and has a right adjoint, since every object $T$ : $\mathcal{B} \to \mathcal{S}$ of the presheaves category $\mathcal{S}^{\mathcal{B}}$ does have a right Kan extension along $K : \mathcal{B} \to \mathcal{A}$. Furthermore, it is known that if $K$ is fully faithful then the right adjoint of $\mathcal{S}^K$ is also so. Hence, one can apply Corollary 4.4 factorizing the localization $\mathcal{S}^K : \mathcal{S}^{\mathcal{A}} \to \mathcal{S}^{\mathcal{B}}$ through a reflection with stable units. Remark that we could just ask for finitely completeness of $\mathcal{S}$, provided $\mathcal{B}$ were a finite category and $\mathcal{A}$ had finite hom-sets, as is the case for the full inclusion $\Delta_n^{op} \subset \Delta^{op}, n \geq 0$ (see Theorem 1 in [8, X.3]: right Kan extension as a point-wise limit).

The localizations $F_n : \mathbf{Smp} \to \mathbf{Smp}_n (n \geq 0)$ are in fact examples of *essential geometric morphisms*, between elementary topoi, which are also *embeddings*. We shall call *geometric morphism* to any adjunction between finitely-complete categories such that the left adjoint preserves finite limits; it is called an *embedding* when the right adjoint is fully faithful; it is called *essential* if the left adjoint is also a right adjoint of some other functor.

We will now prove that all these localizations $F_n : \mathbf{Smp} \to \mathbf{Smp}_n$, besides giving rise to reflections $I_n : \mathbf{Smp} \to Mono(F_n)$ with stable units, by applying the coequalizer of the kernel pair process, produce in addition monotone-light factorizations, as for the case $n = 0$ already worked out in last Section 6. According to Proposition 5.1, we still need to exhibit an effective descent morphism $p : E \to B$ (i.e. an epimorphism, since **Smp** is a topos) for each simplicial set $B$, and such that $\varphi_E^n : E \to F_n(E)$ is monic in **Smp**, i.e., $E \in Mono(F_n) = Fix(I_n)$ ($\varphi^n$ is of course the unit of the adjunction associated to $F_n$). We have already shown in Section 6 that there exists an effective descent morphism $p : E \to B$ with $E$ in $\mathbf{Smp}_0$ for every simplicial set $B$; therefore one can state Conclusion 7.2.

Alternatively, we are going to give a more general argument, which also provides the needed effective descent morphisms $p : E \to B$ with $E$ in $\mathbf{Smp}_n$, for each simplicial set $B$. We care to do so since the proof of the following Proposition 7.3 is a simple generalisation of the considerations below.

It is known that every $B \in \mathbf{Smp} = \mathbf{Set}^{\Delta^{op}}$ is a colimit of representable functors. Hence, there is a canonical presentation of $B$,

$$\begin{tikzcd} E' \arrow[r, shift left, "q"] \arrow[r, shift right, "r"'] & E \arrow[r, "p"] & B \end{tikzcd} \quad , \tag{7.2}$$

where $p = ker(q, r)$ is the coequalizer in $\mathbf{Smp}$ of $q$ and $r$, and $E$ is the coproduct of a family of representable functors.

7.1. Lemma. *Consider the reflection $F_n : \mathbf{Smp} \to \mathbf{Smp}_n$ given in diagram 7.1, whose unit morphism is $\varphi^n$.*

*Every unit morphism of any representable functor*

$$\varphi^n_{\Delta(-,[p])} : \Delta(-,[p]) \to F_n(\Delta(-,[p])),\ p \geq 0,$$

*is a monomorphism in $\mathbf{Smp} = \mathbf{Set}^{\Delta^{op}}$.*

Proof. First, note that the face maps

$$d_i : \Delta([j+1],[p]) \to \Delta([j],[p]),\ 0 \leq i \leq j+1,$$

of the simplicial set $\Delta(-,[p])$ are jointly monic for each $j \geq 0$. This is so because $\Delta(-,[p])$ is the nerve of the ordinal number $[p]$ considered as a category.

The fact that the face maps are jointly monic implies that any two simplicial morphisms $f$ and $g$ from any simplicial set $A$ into $\Delta(-,[p])$ are equal if and only if their two 0-components $f_0, g_0 : A_0 \to \Delta([0],[p])$ are equal.

Hence, $\varphi^n_{\Delta(-,[p])}$ is a monomorphism if and only if its 0-component is an injection: $\varphi^n_{\Delta(-,[p])}$ is a monomorphism if and only if each of its components is an injection; the converse is also easy to prove,

$\varphi^n_{\Delta(-,[p])} f = \varphi^n_{\Delta(-,[p])} g \Rightarrow (\varphi^n_{\Delta(-,[p])})_0 f_0 = (\varphi^n_{\Delta(-,[p])})_0 g_0$

$\Rightarrow f_0 = g_0$, because $(\varphi^n_{\Delta(-,[p])})_0$ is an injection by hypothesis

$\Rightarrow f = g$, because two morphisms into a representable functor are completely determined by their 0-components.

Indeed, the 0-component $(\varphi^n_{\Delta(-,[p])})_0$ is always the identity function, since the reflection $\varphi^n_A : A \to F_n(A)$ does not change the first $n$ objects of simplices $A_i$, $0 \leq i \leq n$. ∎

If $F \dashv G : \mathcal{X} \to \mathcal{C}$ is a generic adjunction with unit $\varphi$, then $\coprod_{i\in I} \varphi_{C_i} = \varphi_{\coprod_{i\in I} C_i}$ for any family $\{C_i \in \mathcal{C} \mid i \in I\}$ such that its coproduct does exist. Therefore, $E = \coprod_{i\in I} \Delta(-,[n_i])$ belongs to $Mono(F_n)$ if and only if $\varphi_{\coprod_{i\in I} \Delta(-,[n_i])} = \coprod_{i\in I} \varphi_{\Delta(-,[n_i])}$ is a monomorphism. That is so because Lemma 7.1 tells us that each $\varphi_{\Delta(-,[n_i])}$ is injective componentwise, which implies that $\coprod_{i\in I} \varphi_{\Delta(-,[n_i])}$ is injective componentwise, i.e., a monomorphism.

7.2. Conclusion. *There is a monotone-light factorization for each of the reflections obtained through the coequalizer of the kernel pair process from $(F_n, \varphi^n)$, $n \geq 0$.*

7.3. Proposition. *Let $F : \mathcal{E} \to \mathcal{F}$ be a geometric morphism between regular categories, $F^* \dashv F_* : \mathcal{E} \to \mathcal{F}$, which is an embedding.*

*Then, the reflection $I : \mathcal{F} \to Mono(F^*)$, obtained from the localization $F^* : \mathcal{F} \to \mathcal{E}$ through the coequalizer of the kernel pair process, does have stable units. Moreover, there is a monotone-light factorization associated to the reflection $I : \mathcal{F} \to Mono(F^*)$ provided the following four conditions also hold:*

1. *the category $\mathcal{F}$ is cocomplete;*
2. *the full subcategory $Mono(F^*)$ is dense in $\mathcal{F}$, i.e., every object of $\mathcal{F}$ is a colimit of objects of $Mono(F^*)$.*
3. *in $\mathcal{F}$ the coproduct of monomorphisms is a monomorphism;*
4. *regular epis are effective descent morphisms in $\mathcal{F}$.*

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