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MONADS OF EFFECTIVE DESCENT TYPE AND COMONADICITY

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ABSTRACT. We show, for an arbitrary adjunction $F \dashv U : \mathcal{B} \to \mathcal{A}$ with $\mathcal{B}$ Cauchy complete, that the functor F is comonadic if and only if the monad $\mathbf{T}$ on $\mathcal{A}$ induced by the adjunction is of effective descent type, meaning that the free $\mathbf{T}$-algebra functor $F^{\mathbf{T}} : \mathcal{A} \to \mathcal{A}^{\mathbf{T}}$ is comonadic. This result is applied to several situations: In Section 4 to give a sufficient condition for an exponential functor on a cartesian closed category to be monadic, in Sections 5 and 6 to settle the question of the comonadicity of those functors whose domain is **Set**, or $\mathbf{Set}_{\star}$, or the category of modules over a semisimple ring, in Section 7 to study the effectiveness of (co)monads on module categories. Our final application is a descent theorem for noncommutative rings from which we deduce an important result of A. Joyal and M. Tierney and of J.-P. Olivier, asserting that the effective descent morphisms in the opposite of the category of commutative unital rings are precisely the pure monomorphisms.

1. Introduction

Let A and B be two (not necessarily commutative) rings related by a ring homomorphism $i : B \to A$. The problem of Grothendieck's descent theory for modules with respect to $i : B \to A$ is concerned with the characterization of those (right) A-modules Y for which there is $X \in \mathrm{Mod}_B$ and an isomorphism $Y \simeq X \otimes_B A$ of right A-modules. Because of a fundamental connection between descent and monads discovered by Beck (unpublished) and Bénabou and Roubaud [6], this problem is equivalent to the problem of the comonadicity of the extension-of-scalars functor $-\otimes_B A : \mathrm{Mod}_B \to \mathrm{Mod}_A$. There are several results obtained along this line. For example, a fundamental result, due to A. Grothendieck [22] in the case of commutative rings and extended by Cipolla [14] and Nuss [40] to the noncommutative setting, states that the extension-of-scalars functor $-\otimes_B A$ is comonadic provided A is a faithfully flat left B-module. Descent theory for modules w.r.t. morphism of (not necessarily commutative) rings can be generalized by considering the situation in which the rings A and B are related by a (B, A)-bimodule M. This case is considered by L. El Kaoutit and J. Gómez Torrecillas in [16] and S. Caenepeel, E. De Groot and J. Vercruysse in [13]. They have in particular showed that when ${}_B M$ is faithfully flat, the induction functor $-\otimes_B M : \mathrm{Mod}_B \to \mathrm{Mod}_A$ is comonadic. We observe that in this case, the left B-module $S = \mathrm{End}_A(M)$ of the A-endomorphisms of M is also faithfully flat,

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and hence the extension-of-scalars functor $-\otimes_B S : \mathrm{Mod}_B \to \mathrm{Mod}_S$ corresponding to the canonical ring extension

$$B \to S,\, b \to (m \to bm)$$

is comonadic. This rises the question of whether the converse is also true. Thus, it seems natural and, indeed, desirable to pursue the relations between the (pre)comonadicity of the functor $-\otimes_B M : \mathrm{Mod}_B \to \mathrm{Mod}_A$ and that of $-\otimes_B S : \mathrm{Mod}_B \to \mathrm{Mod}_S$. Since $-\otimes_B S$ can be considered as the monad on the category Mod_B arising from the adjunction $-\otimes_B M \dashv \mathrm{Mod}(M,-) : \mathrm{Mod}_A \to \mathrm{Mod}_B$ and Mod_S as the Eilenberg-Moore category of algebras over this monad, this motivates to consider an arbitrary adjunction $F \dashv U : \mathcal{B} \to \mathcal{A}$ and to investigate what is the connection between the (pre)comonadicity of the functor F and that of the free $\mathbf{T}$-algebra functor $F^{\mathbf{T}} : \mathcal{A} \to \mathcal{A}^{\mathbf{T}}$, for $\mathbf{T}$ being the monad on $\mathcal{A}$ arising from this adjunction. The first observation in this direction is that F is precomonadic iff so is $F^{\mathbf{T}}$, and when $\mathcal{B}$ is Cauchy complete, our main result, given in Theorem 3.20, asserts that F is comonadic iff $F^{\mathbf{T}}$ is, from which we deduce in Theorem 7.5 that the question of the comonadicity of the induction functor $-\otimes_B M$ reduces to the comonadicity of the extension-of-scalars functor $-\otimes_B S$ corresponding to the ring extension $B \to S$.

The outline of this paper is as follows. After recalling in Section 2 those notions and aspects of the theory of (co)monads that will be needed, we introduce the notion of monad of (effective) descent type and give a necessary and sufficient condition for a monad on a category having an injective cogenerator to be of descent type.

In Section 3 we present our main result and obtain a necessary and sufficient condition for an arbitrary monad on a Cauchy complete category to be of effective descent type.

In Section 4 we illustrate how to use the results of the previous section to obtain a sufficient condition on an adjoint (co)monad in order its functor-part be (co)monadic.

In Section 5 we analyze when (co)monads on (co)exact categories in which every object is projective (injective) are of effective descent type.

In Section 6 the question of the comonadicity of those functors whose domain is the category of sets is fully answered.

In Section 7 we find some conditions on a (B, A)-bimodule M under which the corresponding induction functor $-\otimes_B M : \mathrm{Mod}_B \to \mathrm{Mod}_A$ is comonadic.

Finally, in Section 8 we make some applications to the descent problem for modules with respect to morphisms of (not necessarily commutative) rings.

2. Preliminaries

In this section we collect some basic facts on (co)monads and their (co)algebras as well as fix notation and terminology. We shall follow usual conventions as, for example, in [1] and [33].

We write $\eta, \epsilon \colon F \dashv U \colon \mathcal{B} \to \mathcal{A}$ to denote that $F : \mathcal{A} \to \mathcal{B}$ and $U : \mathcal{B} \to \mathcal{A}$ are functors where F is left adjoint to U with unit $\eta : 1 \to UF$ and counit $\epsilon : FU \to 1$.

A monad $\mathbf{T} = (T, \eta, \mu)$ on a given category $\mathcal{A}$ is an endofunctor $T : \mathcal{A} \to \mathcal{A}$ equipped with natural transformations $\eta : 1 \to T$ and $\mu : T^2 \to T$ satisfying

$$\mu \cdot T\mu = \mu \cdot \mu T \quad \text{and} \quad \mu \cdot \eta T = \mu \cdot T\eta = 1.$$

Comonads are dual to monads. Namely, $\mathbf{G} = (G, \epsilon, \delta)$ is a comonad on a given category $\mathcal{B}$ if $G : \mathcal{B} \to \mathcal{B}$ is an endofunctor, $\epsilon : G \to 1$ and $\delta : G \to G^2$ are natural transformations satisfying axioms formally dual to those of a monad.

Let $\mathbf{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{A}$ with the Eilenberg-Moore category of $\mathbf{T}$-algebras $\mathcal{A}^{\mathbf{T}}$ and the corresponding forgetful-free adjunction $\eta^{\mathbf{T}}, \epsilon^{\mathbf{T}} \colon F^{\mathbf{T}} \dashv U^{\mathbf{T}} \colon \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$, where $F^{\mathbf{T}} : \mathcal{A} \to \mathcal{A}^{\mathbf{T}}$ is the free $\mathbf{T}$-algebra functor, and $U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$ is the underlying object functor. Recall that $U^{\mathbf{T}}(a, \xi) = a$, where $(a, \xi) = (a, \xi : T(a) \to a)$ is a $\mathbf{T}$-algebra, while $F^{\mathbf{T}}(a) = (T(a), \mu_a)$. We have $U^{\mathbf{T}} F^{\mathbf{T}} = T = UF$, and $\eta^{\mathbf{T}} = \eta$; while $F^{\mathbf{T}} U^{\mathbf{T}}(a, \xi) = (T(a), \mu_a)$ and $\epsilon^{\mathbf{T}}(a, \xi) : (T(a), \mu_a) \to (a, \xi)$ is the $\mathbf{T}$-algebra morphism $\xi : T(a) \to a$. If $\mathbf{T} = (T, \eta, \mu)$ is the monad generated on $\mathcal{A}$ by an adjoint pair $\eta, \epsilon \colon F \dashv U \colon \mathcal{B} \to \mathcal{A}$ (so that, $T = UF$ and $\mu = U\epsilon F$), then there is the comparison functor $K_{\mathbf{T}} : \mathcal{B} \to \mathcal{A}^{\mathbf{T}}$ which assigns to each object $b \in \mathcal{B}$ the $\mathbf{T}$-algebra $(U(b), U(\epsilon_b))$, and to each morphism $f : b \to b'$ the morphism $U(f) : U(b) \to U(b')$, and for which $U^{\mathbf{T}} K_{\mathbf{T}} \simeq U$ and $K_{\mathbf{T}} F \simeq F^{\mathbf{T}}$. This situation is illustrated by the following diagram

$$\begin{tikzcd}
 & \mathcal{B} \arrow[dl, shift left, "U"] \arrow[dr, "K_{\mathbf{T}}"] & \\
\mathcal{A} \arrow[ur, shift left, "F"] \arrow[rr, shift right, "F_{\mathbf{T}}"'] & & \mathcal{A}^{\mathbf{T}}. \arrow[ll, shift right, "U_{\mathbf{T}}"']
\end{tikzcd}$$

The functor U is called *monadic* (resp. *premonadic*) if the comparison functor $K_{\mathbf{T}}$ is an equivalence of categories (resp. full and faithful).

Dually, any comonad $\mathbf{G} = (G, \epsilon, \delta)$ on a category $\mathcal{B}$ gives rise to an adjoint pair $F_{\mathbf{G}} \dashv U_{\mathbf{G}} \colon \mathcal{B}_{\mathbf{G}} \to \mathcal{B}$, where $\mathcal{B}_{\mathbf{G}}$ is the category of $\mathbf{G}$-coalgebras, $U_{\mathbf{G}}(a) = (G(a), \delta_a)$ and $F_{\mathbf{G}}(a, h) = a$. If $F \dashv U \colon \mathcal{B} \to \mathcal{A}$ is an adjoint pair and $\mathbf{G} = (FU, \epsilon, F\eta U)$ is the comonad on $\mathcal{B}$ associated to (U, F), then one has the comparison functor

$$K_{\mathbf{G}} : \mathcal{A} \to \mathcal{B}_{\mathbf{G}}, \ a \to (F(a), F(\delta_a))$$

for which $F_{\mathbf{G}} \cdot K_{\mathbf{G}} \simeq F$ and $K_{\mathbf{G}} \cdot U \simeq U_{\mathbf{G}}$. One says that the functor F is *precomonadic* if $K_{\mathbf{G}}$ is full and faithful, and it is *comonadic* if $K_{\mathbf{G}}$ is an equivalence of categories.

Beck's monadicity theorem gives a necessary and sufficient condition for a right adjoint functor to be (pre)monadic. Before stating this result, we need the following definitions (see [33]). A coequalizer $b \overset{f}{\underset{g}{\rightrightarrows}} b' \xrightarrow{h} b''$ is said to be *split* if there are morphisms $k : b' \to b$ and $l : b'' \to b'$ with

$$hl = 1, \ fk = 1 \text{ and } gk = lh.$$

Given a functor $U : \mathcal{B} \to \mathcal{A}$, a pair of morphisms $(f, g : b \rightrightarrows b')$ in $\mathcal{B}$ is U-*split* if the pair $(U(f), U(g))$ is part of a split coequalizer in $\mathcal{A}$, and U preserves coequalizers of U-split pairs if for any U-split pair $(f, g : b \rightrightarrows b')$ in $\mathcal{B}$, and any coequalizer $h : b' \to b''$ of f and g, $U(h)$ is a coequalizer (necessarily split) of $U(f)$ and $U(g)$.

We are now ready to state Beck's monadicity theorem.

2.1. Theorem. (Beck, [5]) *Let $\eta, \epsilon \colon F \dashv U \colon \mathcal{B} \to \mathcal{A}$ be an adjunction, and let $\boldsymbol{T} = (UF, \eta, U\epsilon F)$ be the corresponding monad on $\mathcal{A}$. Then:*

1. *the comparison functor $K_{\mathbf{T}} : \mathcal{B} \to \mathcal{A}^{\mathbf{T}}$ has a left adjoint $L_{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{B}$ if and only if for each $(a, h) \in \mathcal{A}^{\mathbf{T}}$, the pair of morphisms $(F(h), \epsilon_{F(a)})$ has a coequalizer in $\mathcal{B}$.*
2. *When the left adjoint $L_{\mathbf{T}}$ of $K_{\mathbf{T}}$ exists (as it surely does when $\mathcal{B}$ has coequalizers of reflexive pairs of morphisms), then:*

   (i) the unit $1 \to K_{\mathbf{T}}L_{\mathbf{T}}$ of the adjunction $L_{\mathbf{T}} \dashv K_{\mathbf{T}}$ is an isomorphism (or, equivalently, the functor U is premonadic) if and only if each

   $$FUFU(b) \underset{\epsilon_{FU(b)}}{\overset{FU(\epsilon_b)}{\rightrightarrows}} FU(b) \xrightarrow{\epsilon_b} b$$

   is a coequalizer diagram, if and only if each $\epsilon_b : FU(b) \to b$ is a regular epimorphism;

   (ii) the counit $L_{\mathbf{T}}K_{\mathbf{T}} \to 1$ of the adjunction $L_{\mathbf{T}} \dashv K_{\mathbf{T}}$ is an isomorphism if and only if for all $(a, h) \in \mathcal{A}^{\mathbf{T}}$, the functor U preserves the coequalizer of $F(h)$ and $\epsilon_{F(a)}$;

   (iii) if the counit $L_{\mathbf{T}}K_{\mathbf{T}} \to 1$ is an isomorphism, then the unit $1 \to K_{\mathbf{T}}L_{\mathbf{T}}$ is an isomorphism if and only if the functor U is conservative (that is, reflects invertibility of morphisms).

Thus, U is monadic if and only if it is conservative and for each $(a, h) \in \mathcal{A}^{\mathbf{T}}$, the pair of morphisms $(F(h), \epsilon_{F(a)})$ has a coequalizer and this coequalizer is preserved by U.

It is well-known (see, for example, [9]) that if $\eta, \epsilon : F \dashv U : \mathcal{B} \to \mathcal{A}$ is an adjunction with $\mathcal{B}$ admitting equalizers, then U is conservative iff each $\epsilon_b : FU(b) \to b$ is an extremal epimorphism; that is, an epimorphism that factorizes through no proper subobject of its codomain. Quite obviously, every regular epimorphism is extremal, hence U is conservative, provided that it is premonadic. Moreover, it is easy to see that if every extremal epimorphism in $\mathcal{B}$ is regular, then U is premonadic iff it is conservative. Hence:

2.2. PROPOSITION. *Let $F \dashv U : \mathcal{B} \to \mathcal{A}$ be an adjunction with $\mathcal{B}$ admitting equalizers. If U is premonadic, then it is conservative. The converse holds when every extremal epimorphism in $\mathcal{B}$ is regular.*

Consider now a monad $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathcal{A}$ and the corresponding adjoint pair $\eta^{\mathbf{T}}, \epsilon^{\mathbf{T}} : F^{\mathbf{T}} \dashv U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$. This adjunction gives rise to a comonad $\mathbf{G} = (G = F^{\mathbf{T}}U^{\mathbf{T}}, \epsilon = \epsilon^{\mathbf{T}}, \delta = F^{\mathbf{T}}\eta^{\mathbf{T}}U^{\mathbf{T}})$ on $\mathcal{A}^{\mathbf{T}}$. Write $(\mathcal{A}^{\mathbf{T}})_{\mathbf{G}}$ for the category of $\mathbf{G}$-coalgebras, and write $K : \mathcal{A} \longrightarrow (\mathcal{A}^{\mathbf{T}})_{\mathbf{G}}$ for the comparison functor. We record this information in the following diagram

$$\begin{tikzcd}
 & \mathcal{A} \arrow[dl, shift left, "F^{\mathbf{T}}"] \arrow[dr, "K"] & \\
\mathcal{A}^{\mathbf{T}} \arrow[ur, shift left, "U^{\mathbf{T}}"] \arrow[rr, shift right, "(U^{\mathbf{T}})_{\mathbf{G}}"'] & & (\mathcal{A}^{\mathbf{T}})_{\mathbf{G}} \arrow[ll, shift right, "(F^{\mathbf{T}})_{\mathbf{G}}"']
\end{tikzcd}$$

where

- $F^{\mathbf{T}}(a) = (T(a), \eta_a), \quad F^{\mathbf{T}}(f) = f$
- $U^{\mathbf{T}}(a, h) = a, \quad U^{\mathbf{T}}(f) = f$
- $(U^{\mathbf{T}})_{\mathbf{G}}(a, h) = ((T(a), \mu_a), T(\eta_a)), \quad U_{\mathbf{G}}(f) = T(f)$
- $(F^{\mathbf{T}})_{\mathbf{G}}((a, h), \theta) = (a, h), \quad (F^{\mathbf{T}})_{\mathbf{G}}(f) = f$
- $(F^{\mathbf{T}})_{\mathbf{G}}$ is a left adjoint of $(U^{\mathbf{T}})_{\mathbf{G}}$
- $(F^{\mathbf{T}})_{\mathbf{G}} \cdot K \simeq F^{\mathbf{T}}$ and $K \cdot U^{\mathbf{T}} \simeq (U^{\mathbf{T}})_{\mathbf{G}}$.

The category $(\mathcal{A}^{\mathbf{T}})_{\mathbf{G}}$ is called the *category of descent data with respect to the monad* $\mathbf{T}$ and denoted by $\mathfrak{Des}_{\mathcal{A}}(\mathbf{T})$.

We say that the monad $\mathbf{T} = (T, \eta, \mu)$ is *of descent type* when K is full and faithful (or, equivalently, when the functor $F^{\mathbf{T}} : \mathcal{A} \to \mathcal{A}^{\mathbf{T}}$ is precomonadic), and it is *of effective descent type* if K is an equivalence of categories (or, equivalently, if $F^{\mathbf{T}}$ is comonadic).

In order to state conditions on $\mathbf{T}$ so that it be of (effective) descent type, we apply the dual of Beck's theorem to our present situation and conclude that:

2.3. THEOREM. *let $\boldsymbol{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{A}$. Then:*

(i) $\mathbf{T}$ *is of descent type iff each component η_a of the unit $\eta : 1 \to T$ is a regular monomorphism.*

(ii) *$\boldsymbol{T} = (T, \eta, \mu)$ is of effective descent type iff the functor T is conservative and for each $((a, h), \theta) \in \mathfrak{Des}_{\mathcal{A}}(\mathbf{T})$, the pair of morphisms $(\eta_a, \theta : a \rightrightarrows T(a))$ has an equalizer in $\mathcal{A}$ and this equalizer is preserved by T.*

Moreover, applying Proposition 2.2 to the adjunction $F^{\mathbf{T}} \dashv U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$, we get:

2.4. Proposition. *Let $\boldsymbol{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{A}$ which admits coequalizers. If $\mathbf{T}$ is of descent type, then the functor T is conservative. Moreover, if extremal monomorphisms in $\mathcal{A}$ coincide with regular ones, the converse is also true.*

The next result determines exactly what is needed to guarantee that a left adjoint functor whose domain has an injective cogenerator be precomonadic.

2.5. Proposition. *Let $F : \mathcal{A} \to \mathcal{B}$ be a functor with right adjoint $U : \mathcal{B} \to \mathcal{A}$ and unit $\eta : 1 \to UF$, and suppose that $\mathcal{A}$ has an injective cogenerator Q. Then F is precomonadic if and only if the morphism $\eta_Q : Q \to UF(Q)$ is a (split) monomorphism.*

Proof. In view of (the dual of) Theorem 2.1(2)(i), F is precomonadic if and only if each $\eta_a : a \to UF(a)$ is a regular monomorphism; and since Q is injective in $\mathcal{A}$, one direction is clear. For the converse, suppose that $\eta_Q : Q \to UF(Q)$ is a split monomorphism. Then it follows from the Yoneda lemma that the natural transformation

$$\mathcal{A}(\eta,\, Q) : \mathcal{A}(UF(-), Q) \to \mathcal{A}(-, Q)$$

is a split epimorphism. Hence, for any $a \in \mathcal{A}$, the morphism

$$\mathcal{A}(\eta,\, Q) : \mathcal{A}(UF(a), Q) \to \mathcal{A}(a, Q)$$

is a split epimorphism and since Q is an injective cogenerator for $\mathcal{A}$, the morphism $\eta_a : a \to UF(a)$ is a regular monomorphism. ∎

As an immediate consequence we observe that

2.6. Corollary. *Let $\boldsymbol{T} = (T, \eta, \mu)$ be a monad on a category $\mathcal{A}$ and suppose that $\mathcal{A}$ has an injective cogenerator Q. Then $\boldsymbol{T}$ is of descent type if and only if the morphism $\eta_Q : Q \to T(Q)$ is a (split) monomorphism.*

3. Separable functors and comonadicity

We begin by recalling the notion of a separable functor of the second kind introduced in [12].

Given a diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "I"'] & \mathcal{B} \\ \mathcal{X} \end{tikzcd}$$

of categories and functors, consider the following diagram of functors and natural transformations

$$\begin{tikzcd} \mathcal{A}(-,-) \arrow[r, "F_{-,-}"] \arrow[d, "I_{-,-}"'] & \mathcal{B}(F(-), F(-)) \\ \mathcal{X}(I(-), I(-)). \end{tikzcd}$$

The functor F is called *I-separable* (or a *separable functor of the second kind*) if there exists a natural transformation

$$\tau : \mathcal{B}(F(-), F(-)) \to \mathcal{X}(I(-), I(-))$$

with $\tau \circ F_{-,-} = I_{-,-}$.

Note that F is $1_{\mathcal{A}}$-separable if and only if it is separable in the sense of [39].

The following result gives a useful way to check that a left adjoint functor is separable, using the unit of the adjunction.

3.1. Theorem. (Rafael, [43]) *Let $\eta, \epsilon : F \dashv U : \mathcal{B} \to \mathcal{A}$ be an adjunction and $I : \mathcal{A} \to \mathcal{X}$ a functor. Then F is I-separable if and only if the natural transformation $I\eta : I \to IUF$ is a split monomorphism.*

Given a monad $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathcal{A}$, an adjunction $\sigma, \epsilon : F \dashv U : \mathcal{B} \to \mathcal{A}$ is said to be a $\mathbf{T}$-adjunction on $\mathcal{A}$ if $T = UF$, $\sigma = \eta$ and $\mu = U\epsilon F$. In other words, $F \dashv U : \mathcal{B} \to \mathcal{A}$ is a $\mathbf{T}$-adjunction on $\mathcal{A}$ if the monad generated by this adjunction is the given monad $\mathbf{T}$. Clearly the adjunction $F^{\mathbf{T}} \dashv U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$ is a $\mathbf{T}$-adjunction on $\mathcal{A}$. Given an arbitrary functor $I : \mathcal{A} \to \mathcal{X}$, we say that the monad $\mathbf{T}$ is I-separable if its functor-part T is I-separable.

It follows from Theorem 3.1 that:

3.2. Proposition. *Let $\eta, \epsilon : F \dashv U : \mathcal{B} \to \mathcal{A}$ be an adjunction, $\boldsymbol{T}$ the corresponding monad on $\mathcal{A}$ and $I : \mathcal{A} \to \mathcal{X}$ an arbitrary functor. The following assertions are equivalent:*

(i) *The functor F is I-separable.*

(ii) *The free $\mathbf{T}$-algebra functor $F^{\boldsymbol{T}} : \mathcal{A} \to \mathcal{A}^{\boldsymbol{T}}$ is I-separable.*

(iii) *For any $\boldsymbol{T}$-adjunction $F' \dashv U' : \mathcal{B}' \to \mathcal{A}$, the functor F' is I-separable.*

(iv) *The monad $\boldsymbol{T}$ is I-separable.*

Since any functor V is at once seen to be V-separable, a corollary follows immediately:

3.3. Corollary. *Let $F \dashv U : \mathcal{B} \to \mathcal{A}$ be an adjunction and $\boldsymbol{T}$ the corresponding monad on $\mathcal{A}$. Then, for any $\mathbf{T}$-adjunction $F' \dashv U' : \mathcal{B}' \to \mathcal{A}$, the functor F' is F-separable and the monad $\mathbf{T}$ is F-separable. In particular, F is $F^{\mathbf{T}}$-separable and $\mathbf{T}$ is $F^{\mathbf{T}}$-separable.*

We shall need the following elementary lemmas on separable functors. The first one is evident.

3.4. Lemma. *If $F : \mathcal{A} \to \mathcal{B}$ is the composite of functors $I : \mathcal{A} \to \mathcal{X}$ and $K : \mathcal{X} \to \mathcal{B}$, then the functor I is F-separable.*

3.5. Lemma. *Let $F : \mathcal{A} \to \mathcal{B}$ and $I : \mathcal{A} \to \mathcal{X}$ be functors such that F is I-separable. If I is conservative, then so is F.*

Proof. Let $F(f)$ is an isomorphism. Then $I(f)$ is also an isomorphism by Proposition 2.4 in [12], and since I is conservative by hypothesis, f is an isomorphism as well. So F is conservative. ∎

The two preceding lemmas yield:

3.6. PROPOSITION. *In the situation of Lemma* 3.4*, suppose further that* F *is* I*-separable. Then* F *is conservative if and only if* I *is.*

Recall [33] that a pair of morphisms $(f, g : a' \rightrightarrows a'')$ in a category $\mathcal{A}$ is called *contractible* if there exists a morphism $t : a'' \to a'$ with $tf = 1$ and $ftg = gtg$; and that if $F : \mathcal{A} \to \mathcal{B}$ is a functor, $(f, g : a' \rightrightarrows a'')$ is called F-contractible if the pair of morphisms $(F(f), F(g) : F(a') \rightrightarrows F(a''))$ in $\mathcal{B}$ is contractible.

3.7. PROPOSITION. *Let* $F{:}\mathcal{A}{\to}\mathcal{B}$ *and* $I{:}\mathcal{A}{\to}\mathcal{X}$ *be functors and assume that* F *is* I*-separable. Then any* F*-contractible pair is also* I*-contractible. Moreover, when* $\mathcal{X}$ *is Cauchy complete, in the sense that every idempotent endomorphism* e *in* $\mathcal{X}$ *has a factorization* $e = ij$ *where* $ji = 1$*, any* F*-split pair is also* I*-split.*

PROOF. Since F is I-separable, there exists a natural transformation $\tau : \mathcal{B}(F, F) \to \mathcal{X}(I, I)$ such that $\tau \circ F_{-,-} \simeq I_{-,-}$. Now, if $(f, g : a' \rightrightarrows a'')$ is a F-contractible pair, there exists a morphism $q : F(a'') \to F(a')$ such that $q \cdot F(f) = 1$ and $F(f) \cdot q \cdot F(g) = F(g) \cdot q \cdot F(g)$. Put $t = \tau_{a'',a'}(q)$. Then, using the naturality of τ, we compute

$$t \cdot I(f) = \tau_{a'',a'}(q) \cdot I(f) = \tau_{a',a'}(q \cdot F(f)) = \tau_{a',a'}(1_{F(a')}) = I_{a',a'}(1_{a'}) = 1_{I(a')},$$

and

$$\begin{aligned} I(f) \cdot t \cdot I(g) &= I(f) \cdot \tau_{a'',a'}(q) \cdot I(g) = \tau_{a',a''} \cdot (F(f) \cdot q \cdot F(g) \\ &= \tau_{a',a''}(F(g) \cdot q \cdot F(g)) = I(g) \cdot \tau_{a'',a'}(q) \cdot I(g) = I(g) \cdot t \cdot I(g). \end{aligned}$$

Hence that pair $(I(f), I(g))$ is contractible, so that (f, g) is I-contractible.

For the second assertion of the proposition, we recall (for instance from [1]) that an equalizer $x \xrightarrow{h} y \overset{f}{\underset{g}{\rightrightarrows}} z$ in an arbitrary category $\mathcal{X}$ is split if and only if the pair (f, g) is contractible; and if one assumes that $\mathcal{X}$ is Cauchy complete, then any contractible pair has an equalizer (see, for example, [4]), and this equalizer is split. Now, if $(f, g : a' \rightrightarrows a'')$ is F-split, then the pair (f, g) is F-contractible and it follows from the previous part of the proof that it is also I-contractible. By hypothesis, $\mathcal{X}$ is Cauchy complete, so that $I(f)$ and $I(g)$ have an equalizer and this equalizer splits. Consequently, the pair (f, g) is I-split. ∎

Combining this with Lemma 3.4 gives:

3.8. PROPOSITION. *Let* $F : \mathcal{A} \to \mathcal{B}$ *be a composite* KI*, where* $I : \mathcal{A} \to \mathcal{X}$ *and* $K : \mathcal{X} \to \mathcal{B}$*. If* F *is* I*-separable, then a pair of morphisms in* $\mathcal{A}$ *is* F*-contractible if and only if it is* I*-contractible. When* $\mathcal{X}$ *is Cauchy complete, then a pair of morphisms in* $\mathcal{A}$ *is* F*-split if and only if it is* I*-split.*

PROOF. Just observe that any I-split pair is also $F \simeq KI$-split. ∎

A corollary follows immediately:

3.9. Corollary. *Let $F : \mathcal{A} \to \mathcal{B}$ be a composite KI, where $I : \mathcal{A} \to \mathcal{X}$ and $K : \mathcal{X} \to \mathcal{B}$. Suppose that F is I-separable and that the category $\mathcal{X}$ is Cauchy complete. Then $\mathcal{A}$ has equalizers of F-split pairs if and only if it has equalizers of I-split pairs.*

Call a conservative functor $I : \mathcal{A} \to \mathcal{X}$ *quasi-comonadic* if $\mathcal{A}$ has and I preserves equalizers of I-split pairs. Clearly a quasi-comonadic functor is comonadic iff it has a right adjoint.

3.10. Proposition. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left adjoint functor and let $I : \mathcal{A} \to \mathcal{X}$ be a quasi-comonadic functor such that F is I-separable. Suppose that $\mathcal{X}$ is Cauchy complete and that F preserves equalizers of I-split pairs (available by our assumption on I). Then F is comonadic.*

Proof. The functor I, being quasi-comonadic, is conservative and it follows from Lemma 3.5 that F is also conservative. Moreover, F has a right adjoint by hypothesis. Then we conclude from (the dual of) Beck's Theorem that F is comonadic if and only if $\mathcal{A}$ has and F preserves equalizers of F-split pairs. But since $\mathcal{A}$ has and F preserves equalizers of I-split pairs by our assumption on I, it suffices to show that every F-split pair is also I-split. Since $\mathcal{X}$ is assumed to be Cauchy complete, this follows from Proposition 3.8. ∎

3.11. Proposition. *Let $\boldsymbol{T} = (T, \eta, \mu)$ be a monad on a Cauchy complete category $\mathcal{A}$, and let $F^{\mathbf{T}} \dashv U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$ be the corresponding adjunction. Then:*

(i) *The functor T is conservative iff $F^{\mathbf{T}}$ is.*

(ii) *$\mathcal{A}$ has and T preserves equalizers of T-split pairs iff $\mathcal{A}$ has and $F^{\mathbf{T}}$ preserves equalizers of $F^{\mathbf{T}}$-split pairs.*

(iii) *The monad $\mathbf{T}$ is of effective descent type if and only if the functor T is quasi-comonadic.*

Proof. (i). Since $T = U^{\mathbf{T}} F^{\mathbf{T}}$ and since $F^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$ is conservative, T is conservative iff so is $F^{\mathbf{T}}$.

(ii). As $\mathcal{A}$ is Cauchy complete by hypothesis, so too is $\mathcal{A}^{\mathbf{T}}$, and since T is $F^{\mathbf{T}}$-separable by Corollary 3.3, it follows from Corollary 3.9 applied to the diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "T"] \arrow[d, "F^{\mathbf{T}}"'] & \mathcal{A} \\
\mathcal{A}^{\mathbf{T}} \arrow[ur, "U^{\mathbf{T}}"'] &
\end{tikzcd}$$

that $\mathcal{A}$ has equalizers of $F^{\mathbf{T}}$-split pairs iff it has equalizers of T-split pairs. The result now follows from the fact that $U^{\mathbf{T}}$ preserves and reflects all limit that exist in $\mathcal{A}^{\mathbf{T}}$.

(iii) is an immediate consequence of (i) and (ii). ∎

3.12. Corollary. *Let* $\mathbf{T}$ *be a monad on a Cauchy complete category* $\mathcal{A}$*. If the functor* T *has a right adjoint, then the following are equivalent:*

(i) $\mathbf{T}$ *is of effective descent type.*

(ii) T *is conservative.*

In order to proceed, we shall need the following

3.13. Lemma. *Let* $H, H' : \mathcal{X} \to \mathcal{Y}$ *be functors and assume that there are natural transformations* $p : H' \to H$ *and* $q : H \to H'$ *such that* $q \cdot p = 1$*. Then, for any diagram* $x \xrightarrow{h} y \underset{g}{\overset{f}{\rightrightarrows}} z$ *in* $\mathcal{X}$ *such that* $fh = gh$*, with* $H(x) \xrightarrow{H(h)} H(y) \underset{H(g)}{\overset{H(f)}{\rightrightarrows}} H(z)$ *also* $H'(x) \xrightarrow{H'(h)} H'(y) \underset{H'(g)}{\overset{H'(f)}{\rightrightarrows}} H'(z)$ *is a (split) equalizer.*

Proof. Let us first assume that, in the following commutative diagram

$$\begin{tikzcd}
H'(x) \arrow[r, "H'(h)"] \arrow[d, "p_x"'] & H'(y) \arrow[r, shift left, "H'(f)"] \arrow[r, shift right, "H'(g)"'] \arrow[d, "p_y"'] & H'(z) \arrow[d, "p_z"'] \\
H(x) \arrow[r, "H(h)"] & H(y) \arrow[r, shift left, "H(f)"] \arrow[r, shift right, "H(g)"'] & H(z),
\end{tikzcd}$$

the bottom row is an equalizer diagram. We have to show that the top row is also an equalizer diagram. An easy diagram chasing shows that the desired result will follow if the left square is shown to be a pullback, and that this is indeed the case follows from Lemma 10 of [8] applied to the commutative diagram

$$\begin{tikzcd}
H'(x) \arrow[r, "H'(h)"] \arrow[d, shift right, "p_x"'] & H'(y) \arrow[d, shift right, "p_y"'] \\
H(x) \arrow[r, "H(h)"'] \arrow[u, shift right, "q_x"'] & H(y). \arrow[u, shift right, "q_y"']
\end{tikzcd}$$

The second assertion of the lemma is immediate from the fact that split equalizers are absolute, in the sense that they are preserved by any functor. ∎

Note that, although the lemma is stated for (split) equalizers, it would remain true if that were replaced by any (absolute) limit.

With the aid of this lemma, we can now prove:

3.14. PROPOSITION. *In a diagram of categories and functors*

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "F"'] \arrow[d, "I"'] & \mathcal{B} \arrow[l, shift right, "U"'] \\ \mathcal{X} \arrow[ur, "K"'] & \end{tikzcd}$$

suppose that

- *U is a right adjoint of F,*
- *$F \simeq KI$,*
- *F is I-separable.*

Then:

(i) *A pair $(f, g : a' \rightrightarrows a'')$ of morphisms in $\mathcal{A}$ is F-contractible iff it is I-contractible.*

(ii) *For any diagram*

$$a \xrightarrow{h} a' \underset{g}{\overset{f}{\rightrightarrows}} a''$$

in $\mathcal{A}$,

$$I(a) \xrightarrow{I(h)} I(a') \underset{I(g)}{\overset{I(f)}{\rightrightarrows}} I(a'') \tag{3.1}$$

is a split equalizer if and only if so is

$$UF(a) \xrightarrow{UF(h)} UF(a') \underset{UF(g)}{\overset{UF(f)}{\rightrightarrows}} UF(a''). \tag{3.2}$$

(iii) *If $\mathcal{A}$ has and F preserves equalizers of F-split pairs, then $\mathcal{A}$ has and I preserves equalizers of I-split pairs. The converse is also true when $\mathcal{X}$ is Cauchy complete.*

PROOF. (i) is a direct consequence of Proposition 3.8.

(ii). Suppose that (3.1) is a split equalizer. Applying the composite UK to this split equalizer and using that KI is (isomorphic to) F, we see that (3.2) is a split equalizer diagram.

Conversely, suppose that (3.2), and hence also

$$IUF(a) \xrightarrow{IUF(h)} IUF(a') \underset{IUF(g)}{\overset{IUF(f)}{\rightrightarrows}} IUF(a''), \tag{3.3}$$

is a split equalizer diagram. Since F is I-separable and since F admits as a right adjoint the functor U with unit say $\eta : 1 \to UF$, the natural transformation $I\eta : I \to IUF$

is a split monomorphism (see Theorem 3.1). It now follows from Lemma 3.13 that the diagram

$$I(a) \xrightarrow{I(h)} I(a') \underset{I(g)}{\overset{I(f)}{\rightrightarrows}} I(a'') \tag{3.4}$$

is a split equalizer.

(iii) follows by applying part (ii), together with Corollary 3.9.

■

3.15. Proposition. *Let*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "I"'] & \mathcal{B} \\ \mathcal{X} \arrow[ur, "K"'] & \end{tikzcd}$$

be a diagram of categories and functors commuting to within a natural isomorphism and suppose that

- *F is I-separable,*
- *the functors F and I have right adjoints.*

Then:

(i) *If F is comonadic, then so is I.*

(ii) *If X is Cauchy complete, then F is comonadic if and only if I is.*

Proof. In view of part (iii) of Proposition 3.14, it suffices to show that under these conditions, F is conservative if and only if I is; but this is indeed so, as Proposition 3.6 shows. ■

Taking $I = 1$ and $K = F$ in this theorem, we obtain the following result which relates comonadic functors to separable ones.

3.16. Proposition. (cf. Proposition 7.7 in [42] and Theorem 2.3 in [25]) *Let $\eta, \epsilon : F \dashv U : \mathcal{B} \to \mathcal{A}$ be an adjunction with $\mathcal{A}$ Cauchy complete. Then F is comonadic provided that it is separable; in other words (by Theorem 3.1), if the natural transformation $\eta : 1 \to FU$ is a split monomorphism, then F is comonadic.*

As a special case we have:

3.17. Corollary. *Let* $\boldsymbol{T} = (T, \eta, \mu)$ *be a monad on a Cauchy complete category* $\mathcal{A}$*. If the natural transformation* $\eta : 1 \to T$ *is a split monomorphism, then* $\mathbf{T}$ *is a monad of effective descent type.*

Let $\mathbf{T}$ be a monad on a category $\mathcal{A}$ and let $\eta, \epsilon \colon F \dashv U : \mathcal{B} \to \mathcal{A}$ be a $\mathbf{T}$-adjunction. Applying Proposition 3.15 to the commutative diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift right, "F^{\mathbf{T}}"'] \arrow[d, "F"'] & \mathcal{A}^{\mathbf{T}} \arrow[l, shift right, "U^{\mathbf{T}}"'] \\ \mathcal{B} \arrow[ur, "K_{\mathbf{T}}"'] & \end{tikzcd}$$

we get:

3.18. Proposition. *Let* $\mathbf{T}$ *be a monad on a category* $\mathcal{A}$ *and let* $F \dashv U : \mathcal{B} \to \mathcal{A}$ *be a* $\mathbf{T}$*-adjunction. If* $\mathbf{T}$ *is of effective descent type, then the functor* F *is comonadic. The converse is true when* $\mathcal{B}$ *is Cauchy complete.*

We now come to the main result of this section:

3.19. Theorem. *Let* $\mathbf{T}$ *be a monad on a category* $\mathcal{A}$*. Then:*

(i) $\mathbf{T}$ *is of descent type if and only if any* $\mathbf{T}$*-adjunction* $F \dashv U : \mathcal{B} \to \mathcal{A}$ *has* F *precomonadic.*

(ii) *If* $\mathbf{T}$ *is of effective descent type, then any* $\mathbf{T}$*-adjunction* $F \dashv U : \mathcal{B} \to \mathcal{A}$ *has* F *comonadic.*

(iii) *If there exists a* $\mathbf{T}$*-adjunction* $F \dashv U : \mathcal{B} \to \mathcal{A}$ *with* $\mathcal{B}$ *Cauchy complete and with* F *comonadic, then* $\mathbf{T}$ *is of effective descent type.*

(iv) *If* $\mathcal{A}$ *is Cauchy complete, then* $\mathbf{T}$ *is of effective descent type if and only if there exists a* $\mathbf{T}$*-adjunction* $F \dashv U : \mathcal{B} \to \mathcal{A}$ *with* $\mathcal{B}$ *Cauchy complete and with* F *comonadic.*

Proof. (i) follows by comparing the dual of Theorem 2.1 (2(i)) and Theorem 2.3 (i).

(ii) (resp. (iii)) follows by applying Proposition 3.15 (i) (resp. Proposition 3.15 (ii)) to the diagram

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F^{\mathbf{T}}"] \arrow[d, "F"'] & \mathcal{A}^{\mathbf{T}} \\ \mathcal{B}, \arrow[ur, "K_{\mathbf{T}}"'] & \end{tikzcd}$$

together with the fact that $F^{\mathbf{T}}$ is F-separable (see Corollary 3.3).

(iv). If $\mathbf{T}$ is of effective descent type, then $\mathcal{A}^{\mathbf{T}}$ is Cauchy complete (since $\mathcal{A}$ is so) and the $\mathbf{T}$-adjunction $F^{\mathbf{T}} \dashv U^{\mathbf{T}} : \mathcal{A}^{\mathbf{T}} \to \mathcal{A}$ has $F^{\mathbf{T}}$ comonadic. The converse is just part (iii). ∎

An important consequence of this theorem is

3.20. Theorem. *Let $F \dashv U : \mathcal{B} \to \mathcal{A}$ be an adjunction with $\mathcal{B}$ Cauchy complete and let $\boldsymbol{T} = (T, \eta, \mu)$ be the corresponding monad on $\mathcal{A}$. Then the functor F is (pre)comonadic if and only if the monad $\mathbf{T}$ is of (effective) descent type.*

3.21. Proposition. *Let $\mathcal{A}$ and $\mathcal{B}$ be categories with equalizers, $F \dashv U : \mathcal{B} \to \mathcal{A}$ an adjunction and $\mathbf{T}$ (resp. $\boldsymbol{G}$) the corresponding monad (resp. comonad) on $\mathcal{A}$ (resp. on $\mathcal{B}$). Then the following are equivalent:*

(i) *The functor F is conservative and preserves equalizers (equivalently, preserves and reflects equalizers).*

(ii) *The functor F is comonadic and the functor $G = FU$ preserves equalizers.*

(iii) *The functor G preserves equalizers and the functor $T = UF$ preserves and reflects equalizers.*

Proof. (i) $\Rightarrow$ (ii). If the functor F preserves and reflects equalizers, then a straightforward application of the dual of Beck's theorem shows that F is comonadic. Moreover, since the functor U (being right adjoint) preserves all limits, the composite $G = FU$ preserves and reflects equalizers provided that so does F.

(ii) $\Rightarrow$ (i). The comonadicity of F implies that it is conservative and that the comparison functor $K_{\mathbf{G}} : \mathcal{A} \to \mathcal{B}_{\mathbf{G}}$ is an equivalence of categories. Then, since $F_{\mathbf{G}} K_{\mathbf{G}} \simeq F$, F preserves equalizers iff $F_{\mathbf{G}}$ does. But if G preserves equalizers, then so too does $F_{\mathbf{G}}$, since it is well known (see, for instance, [7]) that if the category $\mathcal{B}$ has some type of limits preserved by FU, then the category $\mathcal{B}_{\mathbf{G}}$ has the same type of limits and these are preserved by the functor $F_{\mathbf{G}}$.

(i) $\Rightarrow$ (iii). If F preserves and reflects equalizers, then it is comonadic and the functor G preserves equalizers by the proof of the implication (i) $\Rightarrow$ (ii); and since any category admitting equalizers is Cauchy complete, it follows from Proposition 3.18 that the monad $\mathbf{T}$ is of effective descent type. Then, by Proposition 3.11(iii), the functor T is quasi-comonadic and hence conservative. Moreover, the functor T as a composite of the equalizer-preserving F and the right adjoint (and hence equalizer-preserving) U, does preserve equalizers. Thus the functor T preserves and reflects equalizers.

(iii) $\Rightarrow$ (ii). We have only to show that F is comonadic. Since the functor T preserves and reflects equalizers, it is of effective descent type by Theorem 3.11(iii), and it follows from Proposition 3.18 that F is comonadic. ∎

We conclude this section with the following necessary and sufficient condition for a monad on a Cauchy complete category to be of effective descent type.

3.22. Theorem. *A monad* $\boldsymbol{T} = (T, \eta, \mu)$ *on a Cauchy complete category* $\mathcal{A}$ *is of effective descent type if and only if there exists a commutative diagram (up to natural isomorphism)*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "T"] \arrow[d, "I"'] & \mathcal{A} \arrow[d, "I'"] \\
\mathcal{B} \arrow[r, "T'"'] & \mathcal{C}
\end{tikzcd}$$

of categories and functors such that

(i) the functor I *is quasi-comonadic,*

(ii) the category $\mathcal{B}$ *is Cauchy complete,*

(iii) the functor I' *is conservative,*

(iv) the monad $\mathbf{T}$ *is* I*-separable.*

Proof. To prove the "only if" part, just take $I = T$, and $T' = I' = 1_B$ and observe that

- the functor T is quasi-comonadic by Proposition 3.11(iii);
- like any functor, T is T-separable.

Turning to the converse, assume that the conditions of the theorem are satisfied. Then, since $T = U^{\mathbf{T}}F^{\mathbf{T}}$, one has the following commutative diagram (up to natural isomorphism)

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F^{\mathbf{T}}"] \arrow[dd, "I"'] & \mathcal{A}^{\mathbf{T}} \arrow[d, "U^{\mathbf{T}}"] \\
 & A \arrow[d, "I'"] \\
\mathcal{B} \arrow[r, "T'"'] & \mathcal{C}
\end{tikzcd}$$

in which

- the functor $I'U^{\mathbf{T}}$, being the composite two conservative functors, is conservative;
- since $\mathbf{T}$ is I-separable, $F^{\mathbf{T}}$ is I-separable by Proposition 3.2.

We may therefore apply Theorem 6 in [36] to conclude that the functor $F^{\mathbf{T}}$ is comonadic, or equivalently, that the monad $\mathbf{T}$ is of effective descent type. ∎

In the rest of the paper, we shall give some applications of the results of this section by studying the (co)monadicity of certain special classes of functors.

4. Adjoint (co)monads

We follow [15] in calling a comonad $\mathbf{G} = (G, \epsilon, \delta)$ on a category $\mathcal{A}$ left adjoint to a monad $\mathbf{T} = (T, \eta, \mu)$ on $\mathcal{A}$ if there exists an adjunction

$$\alpha : \mathcal{A}(G(a), a') \to \mathcal{A}(a, T(a'))$$

such that the following two diagrams are commutative for all $a, a' \in \mathcal{A}$:

$$\begin{tikzcd} & \mathcal{A}(a,a') \arrow[dl, "\mathcal{A}(\epsilon_a, a')"'] \arrow[dr, "\mathcal{A}(a, \eta_{a'})"] & \\ \mathcal{A}(G(a),a') \arrow[rr, "\alpha_{a,a'}"] & & \mathcal{A}(a, T(a')) \end{tikzcd} \tag{4.1}$$

$$\begin{tikzcd} \mathcal{A}(G^2(a),a') \arrow[r, "\alpha^2_{a,a'}"] \arrow[d, "\mathcal{A}(\delta_a, a')"'] & \mathcal{A}(a, T^2(a')) \arrow[d, "\mathcal{A}(a, \mu_{a'})"] \\ \mathcal{A}(G(a),a') \arrow[r, "\alpha_{a,a'}"] & \mathcal{A}(a, T(a')) \end{tikzcd} \tag{4.2}$$

Specializing $a = a'$ in (4.1), we see that the diagram

$$\begin{tikzcd} & \mathcal{A}(a,a) \arrow[dl, "\mathcal{A}(\epsilon_a, a)"'] \arrow[dr, "\mathcal{A}(a, \eta_a)"] & \\ \mathcal{A}(G(a),a) \arrow[rr, "\alpha_{a,a}"] & & \mathcal{A}(a, T(a)) \end{tikzcd}$$

commutes. Chasing the identity morphism $1_a : a \to a$ around the last diagram gives the equality

$$\mathcal{A}(a, \eta_a)(1_a) = \alpha_{a,a}(\mathcal{A}(\epsilon_a, a)(1_a))$$

which, using $\mathcal{A}(a, \eta_a)(1_a) = \eta_a$ and $\mathcal{A}(\epsilon_a, a)(1_a) = \epsilon_a$, may be written as

$$\epsilon_a = \alpha^{-1}_{a,a}(\eta_a),$$

or equivalently, since $\alpha^{-1}_{a,a}(\eta_a) = \epsilon'_a \cdot G(\eta_a)$ (where $\epsilon' : GT \to 1$ is the counit of the adjunction $G \dashv T$), as

$$\epsilon_a = \epsilon'_a \cdot G(\eta_a).$$

Hence

$$\epsilon = \epsilon' \cdot G(\eta).$$

Similarly,

$$\eta = T(\epsilon) \cdot \eta',$$

where $\eta' : 1 \to TG$ is the unit of the adjunction $G \dashv T$, and we have:

4.1. Proposition. *Let $\mathcal{A}$ be a category and let a comonad $\boldsymbol{G} = (G, \epsilon, \delta)$ on $\mathcal{A}$ be left adjoint to a monad $\boldsymbol{T} = (T, \eta, \mu)$. Write $\eta' : 1 \to TG$ and $\epsilon' : GT \to 1$ for the unit and counit of the adjunction $G \dashv T$. Then:*

- *if ϵ is a split epimorphism, then so too is ϵ';*
- *if η is a split monomorphism, then so too is η'.*

Combining Corollary 3.17 and its dual with Proposition 4.1 yields:

4.2. Theorem. *Let $\boldsymbol{T} = (T, \eta, \mu)$ be a monad on a Cauchy complete category $\mathcal{A}$ and let $\boldsymbol{G} = (G, \epsilon, \delta)$ be a comonad on $\mathcal{A}$ which is left adjoint to $\boldsymbol{T}$. Then:*

- *If $\epsilon : G \to 1$ is a split epimorphism, then the functor $T : \mathcal{A} \to \mathcal{A}$ is monadic.*
- *If $\eta : 1 \to T$ is a split monomorphism, then the functor $G : \mathcal{A} \to \mathcal{A}$ is comonadic.*

Recall [33] that a category $\mathcal{A}$ with all finite products is called *cartesian closed* when each functor

$$a \times - : \mathcal{A} \to \mathcal{A}$$

has a right adjoint

$$(-)^a : \mathcal{A} \to \mathcal{A}.$$

It is well known that, for any object $a \in \mathcal{A}$, one can equip $(-)^a$ and $a \times -$ with monad and comonad structures, respectively, so that the comonad $a \times -$ becomes a left adjoint to the monad $(-)^a$.

Recall that an object $a \in \mathcal{A}$ has a global element, if there exists a morphism from the terminal object t of $\mathcal{A}$ into a.

4.3. Proposition. *Let $\mathcal{A}$ be a cartesian closed category and suppose that $\mathcal{A}$ is Cauchy complete. Then, for any object $a \in \mathcal{A}$ with a global element, the functor*

$$(-)^a : \mathcal{A} \to \mathcal{A}$$

is monadic.

Proof. Immediate from Theorem 4.2 using the fact that the counit of the comonad $a \times -$, which is induced by the projection $a \times x \to x$, is a split epimorphism if and only if the unique morphism $a \to t$ is a split epimorphism, i.e. iff a has a global element. ■

As a special case of the above theorem, we obtain the following result of F. Métayer on the monadicity of the functor-part of a state monad (see, [37]):

4.4. Proposition. *Let $\mathcal{A}$ be a cartesian closed category with a proper factorization system. Then, for any object $a \in \mathcal{A}$ with a global element, the functor*

$$(-)^a : \mathcal{A} \to \mathcal{A}$$

is monadic.

(Recall [21] that a factorization system $(\mathbb{E}, \mathbb{M})$ is *proper* when each member of $\mathbb{E}$ is an epimorphisms and each member of $\mathbb{M}$ is a monomorphism.)

Proof. It suffices by the previous proposition to show that a category with a proper factorization system is Cauchy complete. Suppose, therefore, that $\mathcal{A}$ is a category with a proper factorization system $(\mathbb{E}, \mathbb{M})$, and consider an idempotent morphism e in $\mathcal{A}$. Let $e = sr$ be its $(\mathbb{E}, \mathbb{M})$-factorization. Then $sr = e = e^2 = (sr)(sr) = (srs)r$, and since r is an epimorphisms, we have $s = srs$, and since s is a monomorphism, we get the equality $rs = 1$. It means that in $\mathcal{A}$ every idempotent splits, so that $\mathcal{A}$ is Cauchy complete. ∎

5. (Co)monads on (co)exact categories in which every object is projective (injective)

We use the notion of an exact category in the sense of Barr [2], with the convention that "exact" always includes the existence of finite limits.

We start with the following variation of Duskin's theorem (see, for example, [1]).

5.1. Theorem. *Let $\mathcal{A}$ be a category admitting kernel-pairs of split epimorphisms and let $\mathcal{B}$ be an exact category. Then the following two assertions are equivalent for any right adjoint functor $U : \mathcal{B} \to \mathcal{A}$:*

(i) *U is monadic.*

(ii) *U is conservative and U preserves those regular epimorphisms whose kernel-pairs are U-split.*

Since split epimorphisms and split monomorphisms are preserved by any functor, Theorem 5.1 and its dual give:

5.2. Proposition. *A (co)monad on a (co)exact category in which every object is projective (injective) is of effective (co)descent type if and only if its functor-part is conservative.*

Recall that a category that admits equalizers or coequalizers is Cauchy complete; in particular, any exact or coexact category is Cauchy complete and the previous theorem and Proposition 3.18 give:

5.3. Theorem. *A right adjoint functor from an exact category in which every object is projective to any category admitting kernel-pairs of split epimorphisms is monadic if and only if it is conservative. Dually, a left adjoint functor from a coexact category in which every object is injective into any category admitting cokernel-pairs of split monomorphisms is comonadic if and only if it is conservative.*

Since the categories of pointed sets, $\mathbf{Set}_*$, and modules over an arbitrary semisimple ring are both exact and coexact and since in these categories every object is both projective and injective, we recover Theorem 6 in [3] as a special case of Theorem 5.3.

6. (Co)monads on **Set**

In this section, we will discuss when a (co)monad on **Set** is of effective (co)descent type. We start by considering comonads on **Set**. Recall that a comonad $\mathbf{G} = (G, \epsilon, \delta)$ on **Set** is called *non-degenerate* if there exists a set X for which $G(X) \neq \emptyset$. It is easy to see that $\mathbf{G}$ is non-degenerate iff $G(1) \neq \emptyset$. Since the identity functor $\mathrm{Id} : \mathbf{Set} \to \mathbf{Set}$ is representable by the set 1, it follows from the Yoneda lemma that the natural transformations $\mathrm{Id} \to G$ are in bijection with the elements of the set $G(1)$; hence there exists at least one natural transformation $\epsilon' : \mathrm{Id} \to G$, provided that $\mathbf{G}$ is non-degenerate. It is clear then that the composite $\mathrm{Id} \xrightarrow{\epsilon'} G \xrightarrow{\epsilon} \mathrm{Id}$ is the identity natural transformation. So we have proved that for any non-degenerate comonad $\mathbf{G} = (G, \epsilon, \delta)$ on **Set**, the natural transformation $\epsilon : G \to 1$ is a split epimorphism and since **Set** is well-known to be Cauchy complete, it follows from (the dual of) Corollary 3.17 that

6.1. Theorem. *Every non-degenerate comonad on **Set** is of effective codescent type.*

In view of Theorem 3.19, we have the following

6.2. Corollary. *Let $F \dashv U : \mathbf{Set} \to \mathcal{A}$ be an adjunction with $\mathcal{A}$ Cauchy complete. Then U is monadic if and only if $FU(1) \neq \emptyset$.*

Our next task is to derive a necessary and sufficient condition under which a monad on **Set** is of effective descent type. In contrast to the case of comonads, the situation with monads on **Set** is more complicated, as we shall see below.

Let $\mathcal{A}$ be a locally small category and let a be an object of $\mathcal{A}$ admitting all small copowers. Given a small set X and an element $x \in X$, write $X \bullet a$ for the X-indexed copower of a, and write $i_x : a \to X \bullet a$ for the canonical x-th injection into the coproduct. Then one has an adjunction

$$\eta, \epsilon : - \bullet a \dashv \mathcal{A}(a, -) : \mathcal{A} \to \mathbf{Set}$$

whose unit η has the components

$$\eta_X : X \to \mathcal{A}(a, X \bullet a), \quad \eta_X(x) = i_x,$$

while the components $\epsilon_{a'} : \mathcal{A}(a, a') \bullet a \to a$ of the counit ϵ are given by $\epsilon_{a'} \cdot i_f = f$ for all $f : a \to a'$.

Conversely, it is well known that a functor $U : \mathcal{A} \to \mathbf{Set}$ has a left adjoint $F : \mathbf{Set} \to \mathcal{A}$ if and only if it is representable, say $U = \mathcal{A}(a, -)$, for some object $a \in \mathcal{A}$ for which the category $\mathcal{A}$ has all set-indexed copowers, in which case F may be given by $F(X) = X \bullet a$, for each $X \in \mathbf{Set}$.

6.3. Proposition. *Let $\mathcal{A}$ be a locally small category and let a be an object of $\mathcal{A}$ admitting all small copowers. Write $\boldsymbol{T}$ for the monad on $\boldsymbol{Set}$ arising from the adjunction $F = -\bullet a \dashv U = \mathcal{A}(a, -) : \mathcal{A} \to \boldsymbol{Set}$. Then the following statements are equivalent:*

(i) *The functor $- \bullet a : \boldsymbol{Set} \to \mathcal{A}$ is faithful.*

(ii) *The functor $- \bullet a : \boldsymbol{Set} \to \mathcal{A}$ is conservative.*

(iii) *The functor $- \bullet a : \boldsymbol{Set} \to \mathcal{A}$ is precomonadic.*

(iv) *The functor-part T of the monad $\mathbf{T}$ is faithful.*

(v) *The functor-part T of the monad $\mathbf{T}$ is conservative.*

(vi) *The monad $\mathbf{T}$ is of descent type.*

(vii) *The morphism $\eta_2 : 2 \to T(2)$, where η is the unit of the adjunction $- \bullet a \dashv \mathcal{A}(a, -)$ and 2 denotes the two-element set $\{0, 1\}$, is injective.*

(viii) *The coproduct injections $i_0, i_1 : a \to a + a$ are distinct morphisms.*

(ix) *The codiagonal morphism $\nabla : a \to a + a$ is not an isomorphism.*

(x) *a is not a partial initial object. (Recall [2] that an object is called partial initial if it equalizes every parallel pair out of it.)*

(xi) *There exists at least two distinct morphisms with domain a.*

(xii) *There exists an object of $\boldsymbol{Set}^{\boldsymbol{T}}$ whose underlying set has more than one element.*

Proof. It is well known (see, for example [28], p. 44) that a left adjoint functor whose domain admits coequalizers is faithful (resp. conservative, resp. precomonadic) if and only if the unit of the adjunction is componentwise a monomorphism (resp. an extremal monomorphism, resp. a regular monomorphism). Since every regular monomorphism is extremal and since in $\mathbf{Set}$, every monomorphism is regular, it follows that the functor $- \bullet a : \mathbf{Set} \to \mathcal{A}$ is faithful iff it is conservative, iff it is precomonadic. Thus, (i), (ii) and (iii) are equivalent.

Since $T = U^{\mathbf{T}} F^{\mathbf{T}}$, where, recall, $F^{\mathbf{T}} : \mathbf{Set} \to \mathbf{Set}^{\mathbf{T}}$ is the free $\mathbf{T}$-algebra functor and $U^{\mathbf{T}} : \mathbf{Set}^{\mathbf{T}} \to \mathbf{Set}$ is the underlying object functor, and since the functor $U^{\mathbf{T}}$ is conservative and faithful, T is faithful (resp. conservative) if and only if the functor $F^{\mathbf{T}}$ is. It follows that (i) is equivalent to (iv) and (ii) is equivalent to (v).

(iii) and (vi) are equivalent by definition.

Since 2 is an injective cogenerator for **Set**, we get from Corollary 2.6 that (vi) is equivalent to (vii).

The equivalence of (viii), (ix), (x) and (xi) is proved in Barr [2].

(vi) and (xii) are equivalent by Lemma 3 of [20].

Finally, since the functor-part T of $\mathbf{T}$ is the functor $\mathcal{A}(a, - \bullet a)$, to say that the map η_2 is injective is to say that the map $2 \to \mathcal{A}(a, 2 \bullet a) = \mathcal{A}(a, a+a)$, which sends $k \in \{0,1\}$ the k-th injection $a \to a + a$, is injective, whence the equivalence of (vii) and (viii). ∎

6.4. Remark. It follows at once from the definition of a partial initial object that, for any such object, the unique morphism $0_a : 0 \to a$, where 0 is the initial object of the category $\mathcal{A}$ (or, equivalently, $0 = \emptyset \bullet a$), is an epimorphism; thus this morphism is an isomorphism iff it is a regular monomorphism.

Recall that in a category, a pair of morphisms $f, g : X \rightrightarrows Y$ is called coreflexive, if there exists a morphism $s : Y \to X$ such that $sf = sg = 1$.

6.5. Lemma. *In **Set**, any coreflexive equalizer diagram*

$$X \xrightarrow{h} Y \underset{\underset{s}{\longleftarrow}}{\overset{f}{\underset{g}{\rightrightarrows}}} Z \tag{6.1}$$

with $X \neq \emptyset$ can be given the structure of a split equalizer.

Proof. Let $q_1, q_2 : Y \rightrightarrows D$ be the cokenel-pair of h, and $r : D \to Z$ the unique map for which $rq_1 = f$ and $rq_2 = g$. As the pair (f, g) is coreflexive, the map r is easily seen to be injective; hence, since $D \neq \emptyset$ because $X \neq \emptyset$ by hypothesis, there exists a map $r' : Z \to D$ with $rr' = 1$. Similarly, h is a split injection, and hence the diagram

$$X \xrightarrow{h} Y \overset{q_1}{\underset{q_2}{\rightrightarrows}} D$$

is a split equalizer (see, for example, [7]). Now, a simple computation shows that if the maps $D \xrightarrow{j} Y \xrightarrow{i} X$ split this diagram, then the maps $Z \xrightarrow{j'=jr'} Y \xrightarrow{i} X$ split (6.1). ∎

6.6. Proposition. *Let $\eta, \epsilon \colon F \dashv U \colon \mathcal{A} \to$ **Set** be an adjunction with $\mathcal{A}$ Cauchy complete. Then the following statements are equivalent:*

(i) *There exists a non-empty set X for which the pair of the coproduct injections $(i_0, i_1 : X \rightrightarrows X + X)$ is F-split.*

(ii) *The pair of maps $(i_0, i_1 : 1 \rightrightarrows 1 + 1)$ is F-split.*

(iii) *The pair of maps $(i_0, i_1 : 1 \rightrightarrows 1 + 1)$ is UF-split.*

(iv) *The equalizer of the maps $UF(i_0)$ and $UF(i_1)$ is non-empty.*

Proof. The implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (iv) are trivial. Since $\mathcal{A}$ is assumed to be Cauchy complete, taking $I = F$ in Proposition 3.14 (ii) gives that (ii) and (iii) are equivalent. The implication (iv) $\Rightarrow$ (iii) follows from Lemma 6.5. It remains to show that (i) $\Rightarrow$ (ii). If X is non-empty and $x_0 \in X$, then the diagram

$$\begin{tikzcd} 1 \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] \arrow[d, shift right, "x_0"'] & 1+1 \arrow[d, shift right, "x_0+x_0"'] \\ X \arrow[u, shift right, "k_X"'] \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] & X+X, \arrow[u, shift right, "k_X+k_X"'] \end{tikzcd}$$

where $x_0 : 1 \to X$ is the map that picks up the element x_0 in X, and $k_X : X \to 1$ is the unique map, is commutative. It follows from (the dual of) Observation 1.2 of [25] that $1 \underset{i_1}{\overset{i_0}{\rightrightarrows}} 1+1$ is F-split provided that $X \underset{i_1}{\overset{i_0}{\rightrightarrows}} X+X$ is. ■

6.7. Lemma. *Let X be a non-empty set and $\emptyset_X : \emptyset \to X$ the unique map. For an arbitrary functor $F : \mathbf{Set} \to \mathcal{A}$, the morphism $F(\emptyset_X) : F(\emptyset) \to F(X)$ is a regular monomorphism if and only if the morphism $F(\emptyset_1) : F(\emptyset) \to F(1)$ is.*

Proof. Since X is non-empty, there exists an element $x_0 \in X$. Then we can write the map $\emptyset_X : \emptyset \to X$ as the composite of $\emptyset_1 : \emptyset \to 1$ and $x_0 : 1 \to X$. But the map $x_0 : 1 \to X$, and hence also $F(x_0)$, is a split monomorphism and applying Propositions 2.1 and 2.2 of [27], we see that $F(\emptyset_X)$ is a regular monomorphism if and only if $F(\emptyset_1)$ is. ■

Let $F : \mathbf{Set} \to \mathcal{A}$ be an arbitrary functor. Since in **Set**, any injection with non-empty domain splits and since split monomorphisms are preserved by any functor, the only monomorphisms in **Set** F has a chance of not preserving are those with empty domain. But, according to the previous lemma, F preserves such monomorphisms if and only if the morphism $F(\emptyset_1) : F(\emptyset) \to F(1)$ is a regular monomorphism in $\mathcal{A}$. Thus, we have proved that

6.8. Proposition. *A functor $F : \mathbf{Set} \to \mathcal{A}$ preserves all (regular) monomorphisms if and only if the morphism $F(\emptyset_1) : F(\emptyset) \to F(1)$ is a regular monomorphism in $\mathcal{A}$.*

6.9. Proposition. *Let $F \dashv U : \mathcal{A} \to \mathbf{Set}$ be an adjunction with $\mathcal{A}$ admitting equalizers. If U is conservative, the following conditions are equivalent:*

(i) *F preserves all (regular) monomorphisms.*

(ii) *The morphism $F(\emptyset_1) : F(\emptyset) \to F(1)$ is regular monomorphism in $\mathcal{A}$.*

(iii) *F preserves those (regular) monomorphisms whose cokernel-pairs are F-split.*

(iv) *The diagram*

$$UF(\emptyset) \xrightarrow{UF(\emptyset_1)} UF(1) \underset{UF(i_1)}{\overset{UF(i_0)}{\rightrightarrows}} UF(1+1)$$

is an equalizer.

PROOF. (i) and (ii) are equivalent by the previous proposition, (i) implies (iii) trivially.

(iii) $\Rightarrow$ (ii) : Suppose that Condition (iii) holds, and consider the following commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] & F(1) \arrow[r, shift left, "F(i_0)"] \arrow[r, shift right, "F(i_1)"'] & F(1+1) \\ 0 = F(\emptyset) \arrow[u, "j"] \arrow[ur, "F(\emptyset_1)"'] & & \end{tikzcd}$$

where $i : X \to F(1)$ is the equalizer of $F(i_0)$ and $F(i_1)$ and j is the comparison morphism. Applying U to this diagram and using that U preserves limits, we get the following commutative diagram

$$\begin{tikzcd} U(X) \arrow[r, "U(i)"] & UF(1) \arrow[r, shift left, "UF(i_0)"] \arrow[r, shift right, "UF(i_1)"'] & UF(1+1) \\ U(0) = UF(\emptyset) \arrow[u, "U(j)"] \arrow[ur, "UF(\emptyset_1)"'] & & \end{tikzcd}$$

in which the row is an equalizer. We distinguish two cases, according as $U(X)$ is or is not the empty set.

If $U(X) = \emptyset$, then $U(0) = \emptyset$ also and hence $U(j)$ is an isomorphism. But U is conservative by hypothesis, so j is also an isomorphism; it follows that the morphism $F(\emptyset_1)$ is an equalizer of $F(i_0)$ and $F(i_1)$. In particular, $F(i_0)$ is a regular monomorphism in $\mathcal{A}$.

If instead $U(X)$ is non-empty, then, since the pair of maps $(UF(i_0), UF(i_1))$ is coreflexive, it is part of a split equalizer diagram by Lemma 6.5. Thus the pair $(i_0, i_1 : 1 \rightrightarrows 1+1)$ is UF-split, and since any category admitting equalizers is Cauchy complete, it follows from Proposition 6.6 that the pair (i_0, i_1) is F-split. Hence, the morphism $\emptyset_1 : \emptyset \to 1$ is a monomorphism whose cokernel-pair is F-split; and since F is assumed to preserve such monomorphisms, the morphism $F(\emptyset_1) : F(\emptyset) \to F(1)$ is a regular monomorphism in $\mathcal{A}$.

(ii) $\Leftrightarrow$ (iv) : To say that $F(\emptyset_1) : F(\emptyset) \to F(1)$ is a regular monomorphism in $\mathcal{A}$ is (since F, having a right adjoint, preserves cokernel-pairs) to say that the diagram

$$F(\emptyset) \xrightarrow{F(\emptyset_1)} F(1) \underset{F(i_1)}{\overset{F(i_0)}{\rightrightarrows}} F(1+1) \tag{6.2}$$

is an equalizer. But the conservative and right adjoint U preserves and reflects all limits that exists in $\mathcal{A}$. Thus in particular (6.2) is an equalizer diagram if and only if the diagram

$$UF(\emptyset) \xrightarrow{UF(\emptyset_1)} UF(1) \underset{UF(i_1)}{\overset{UF(i_0)}{\rightrightarrows}} UF(1+1)$$

is. ■

Noting that for any monad $\mathbf{T}$ on $\mathbf{Set}$, the category $\mathbf{Set}^{\mathbf{T}}$ is complete and the forgetful functor $U : \mathbf{Set}^{\mathbf{T}} \to \mathbf{Set}$ is conservative, we get the following result:

6.10. Theorem. *A monad $\boldsymbol{T} = (T, \eta, \mu)$ on $\boldsymbol{Set}$ is of effective descent type if and only if the map $\eta_2 : 2 \to T(2)$ is injective and the diagram*

$$T(\emptyset) \xrightarrow{T(\emptyset_1)} T(1) \underset{T(i_1)}{\overset{T(i_0)}{\rightrightarrows}} T(1+1)$$

is an equalizer.

From Theorems 3.19 and 6.10 we get

6.11. Theorem. *Let $\mathcal{A}$ be a locally small category and let a be an object of $\mathcal{A}$ for which $\mathcal{A}$ has all small copowers. Then the functor*

$$F = - \bullet a : \boldsymbol{Set} \to \mathcal{A}$$

is comonadic if a is not a partial initial object and the diagram

$$\mathcal{A}(a,0) \xrightarrow{\mathcal{A}(a,0_a)} \mathcal{A}(a,a) \underset{\mathcal{A}(a,i_1)}{\overset{\mathcal{A}(a,i_0)}{\rightrightarrows}} \mathcal{A}(a,a+a)$$

is an equalizer. When $\mathcal{A}$ is Cauchy complete, the converse is also true.

It is well known that a morphism admitting a cokernel-pair is a regular monomorphism iff it is the equalizer of its cokernel-pair; so, if $a \in \mathcal{A}$ is such that $\mathcal{A}$ admits all small copowers $X \bullet a$, then to say that the unique morphism $0_a : 0 \to a$ is a regular monomorphism is to say that the diagram $0 \xrightarrow{0_a} a \underset{i_1}{\overset{i_0}{\rightrightarrows}} a+a$ is an equalizer and since representable functors preserve limits, it follows from the above theorem that:

6.12. Proposition. *Let $\mathcal{A}$ be a locally small category and let a be an object of $\mathcal{A}$ for which $\mathcal{A}$ admits all small copowers. If a is not a partial initial object and if the morphism $0_a : 0 \to a$ is a regular monomorphism in $\mathcal{A}$, then the functor $F = - \bullet a : \boldsymbol{Set} \to \mathcal{A}$ is comonadic.*

Using Remark 6.4, this proposition can be paraphrased as follows:

6.13. Proposition. *Let $\mathcal{A}$ be a locally small category and let a be an object of $\mathcal{A}$ for which $\mathcal{A}$ admits all small copowers. If a is not (isomorphic to) 0 and if the morphism $0_a : 0 \to a$ is a regular monomorphism in $\mathcal{A}$, then the functor $F = - \bullet a : \boldsymbol{Set} \to \mathcal{A}$ is comonadic.*

6.14. Proposition. *Let $\mathcal{A}$ be a locally small category in which coproducts are disjoint and a an object of $\mathcal{A}$ for which $\mathcal{A}$ admits all small copowers. Under these conditions, the functor*

$$F = - \bullet a : \boldsymbol{Set} \to \mathcal{A}$$

is comonadic if and only if a is not a partial initial object.

Proof. Disjointness says that the diagram $0 \xrightarrow{0_a} a \underset{i_1}{\overset{i_0}{\rightrightarrows}} a+a$ is an equalizer, and so if a is not a partial initial object, then it follows directly from the previous theorem that the functor F is comonadic. The converse is trivial. ∎

Propositions 6.3 and 6.14 together give the following:

6.15. Theorem. *Let $\mathcal{A}$ be a locally small category in which coproducts are disjoint. Then a functor $F = - \bullet a : \boldsymbol{Set} \to \mathcal{A}$ is comonadic if and only if it is precomonadic.*

As a special case of Proposition 6.12, we have the following result of Sobral (see Proposition 2.1 in [44]):

6.16. Proposition. *Let $\mathcal{A}$ be a locally small category and a an object of $\mathcal{A}$ for which $\mathcal{A}$ admits all small copowers. If a is not a partial initial object and if the unique morphism $0_a : 0 \to a$ is a split monomorphism, the functor $F = - \bullet a : \boldsymbol{Set} \to \mathcal{A}$ is comonadic.*

Proof. This is immediate because any split monomorphism is regular. ∎

There is another way of stating the results obtained above.

It is well known (see, [34], Chapter 1) that monads on **Set** and (possibly infinitary) algebraic theories are entirely equivalent concepts. Accordingly we may think of any monad on **Set** as a (possibly infinitary) algebraic theory, and if $\mathcal{T}$ is an algebraic theory corresponding to a monad $\mathbf{T}$ on **Set**, the elements of $T(X)$, $X \in \mathbf{Set}$, can be identified with the set $\mathcal{T}(X,1)$ of X-ary operations of the theory $\mathcal{T}$. Under this identification, $T(\emptyset)$ becomes simply the set of constants of $\mathcal{T}$, while the equalizer $i : P \to T(1)$ of the maps $T(i_0), T(i_1) : T(1) \rightrightarrows T(1+1)$ becomes the set of the pseudo-constants of $\mathcal{T}$. (Recall (for example, from [26]) that a pseudo-constant of a theory is a unary operation u for which $u(x) = u(y)$ is an equation.) One says that an algebraic theory is *degenerate* if the identity unary operation is a pseudo-constant. It is easily seen that this is equivalent to saying that each model of the theory has at most one element.

One can now easily translate Theorems 6.10 and 6.11 and Propositions 6.12-6.16 into the present language. For instance, in the case of Theorem 6.10, one obtains:

6.17. Theorem. (cf. Theorem 11 in [3]) *The monad corresponding to an algebraic theory $\mathcal{T}$ is*

(i) of descent type if and only if $\mathcal{T}$ is non-degenerate;

(ii) of effective descent type if and only if $\mathcal{T}$ is non-degenerate and every pseudo-constant is a constant.

7. (Co)monads on module categories

Let K be an associative commutative ring with unit, fixed throughout the rest of the paper (possible $K = \mathbb{Z}$, the ring of integers). All rings are associative unital K-algebras. A right or left module means a unital module. All bimodules are assumed to be K-symmetric. The K-categories of left and right modules over a ring A are denoted by ${}_A\mathrm{Mod}$ and Mod_A, respectively; while the category of (A, B)-bimodules is ${}_A\mathrm{Mod}_B$. We will use the notation ${}_BM_A$ to indicate that M is a left B, right A-module.

Let A be a ring. Recall that an A-coring is a comonoid in the monoidal category ${}_A\mathrm{Mod}_A$. Thus an A-coring is a triple $\Sigma = (\Sigma, \delta_\Sigma, \epsilon_\Sigma)$, where Σ is an (A, A)-bimodule, and $\delta_\Sigma : \Sigma \to \Sigma \otimes_A \Sigma$ and $\epsilon_\Sigma : \Sigma \to A$ are (A, A)-bimodule morphisms such that

$$(\delta_\Sigma \otimes_A 1_\Sigma)\delta_\Sigma = (1_\Sigma \otimes_A \delta_\Sigma)\delta_\Sigma \ \text{ and } \ (\epsilon_\Sigma \otimes_A 1_\Sigma)\delta_\Sigma = (1_\Sigma \otimes_A \epsilon_\Sigma)\delta_\Sigma = 1_\Sigma.$$

δ_Σ is known as the comultiplication of the A-coring, ϵ_Σ as the counit, and the equations show that δ_Σ and ϵ_Σ obey coassociative and left and right counitary laws.

A right Σ-comodule is a pair (X, σ_X) consisting of a right A-module X and a morphism $\sigma_X : X \to X \otimes_A \Sigma$ (called the Σ-coaction) satisfying

$$(1_X \otimes_A \delta_\Sigma)\sigma_X = (\sigma_X \otimes_A 1_\Sigma)\sigma_X \text{ and } (1_X \otimes_A \epsilon_\Sigma)\sigma_X = 1_X.$$

A morphism of right Σ-comodules (X, σ_X) and $(X', \sigma_{X'})$ is a morphism of right A-modules $f : X \to X'$ such that $(f \otimes_A 1_\Sigma)\sigma_X = \sigma_{X'} f$. The category of right Σ-comodules and their morphisms is denoted by Mod^Σ. Left Σ-comodules and their morphisms are defined analogously, and their category is denoted by ${}^\Sigma\mathrm{Mod}$.

For any right A-module Y, the right A-module morphism

$$Y \otimes_A \delta_\Sigma : Y \otimes_A \Sigma \to Y \otimes_A \Sigma \otimes_A \Sigma$$

makes $Y \otimes_A \Sigma$ a right Σ-comodule. Thus the assignment

$$Y \to (Y \otimes_A \Sigma,\ Y \otimes_A \delta_\Sigma)$$

yields a functor

$$\mathrm{Mod}_A \to \mathrm{Mod}^\Sigma,$$

which is right adjoint to the evident forgetful functor

$$\mathrm{Mod}^\Sigma \to \mathrm{Mod}_A.$$

Similarly, one has that the functor

$${}_A\mathrm{Mod} \to {}^\Sigma\mathrm{Mod}$$

defined, on objects, by

$$Y \to (\Sigma \otimes_A Y,\ \delta_\Sigma \otimes_A Y)$$

is right adjoint to the forgetful functor

$$^{\Sigma}\mathrm{Mod} \to {}_A\mathrm{Mod}.$$

A detailed account of the theory of corings and comodules can be found in [11].

For any A-coring $\Sigma = (\Sigma, \delta_\Sigma, \epsilon_\Sigma)$, we can define on Mod_A a monad

$$\mathbf{T}_\Sigma = (T = \mathrm{Mod}_A(\Sigma, -),\ \eta = \mathrm{Mod}_A(\epsilon_\Sigma, -),\ \mu = \mathrm{Mod}_A(\delta_\Sigma, -))$$

and a comonad

$$\mathbf{G}_\Sigma = (G = - \otimes_A \Sigma,\ \sigma = - \otimes_A \epsilon_\Sigma,\ \rho = - \otimes_A \delta_\Sigma).$$

Note that the category $(\mathrm{Mod}_A)^{\mathbf{G}_\Sigma}$ is just the category of right Σ-comodules over the A-coring Σ and that the free $\mathbf{G}_\Sigma$-coalgebra functor $U_{\mathbf{G}_\Sigma} : \mathrm{Mod}_A \to (\mathrm{Mod}_A)^{\mathbf{G}_\Sigma}$ is just the functor

$$\mathrm{Mod}_A \to \mathrm{Mod}^{\Sigma},\ \ Y \to (Y \otimes_A \Sigma,\ Y \otimes_A \delta_\Sigma).$$

7.1. Proposition. *Let A be a ring and let $\Sigma = (\Sigma, \delta_\Sigma, \epsilon_\Sigma)$ be an A-coring. Then the following are equivalent:*

(i) *The monad $\mathbf{T}_\Sigma$ is of descent type.*

(ii) *The monad $\mathbf{T}_\Sigma$ is of effective descent type.*

(iii) *The comonad $\boldsymbol{G}_\Sigma$ is of codescent type.*

(iv) *The comonad $\boldsymbol{G}_\Sigma$ is of effective codescent type.*

(v) *The morphism $\epsilon_\Sigma : \Sigma \to A$ is a split epimorphism in* Mod_A.

Proof. Since the functor $T = \mathrm{Mod}_A(\Sigma, -) : \mathrm{Mod}_A \to \mathrm{Mod}_A$ preserves all limits, the monad $\mathbf{T}_\Sigma$ is of effective descent type iff it is of descent type. But to say that $\mathbf{T}_\Sigma$ is of descent type is, by Theorem 2.3(i), to say that, for each $X \in \mathrm{Mod}_A$, the morphism $\eta_X = \mathrm{Mod}_A(\epsilon_\Sigma, X)$ is a (regular) monomorphism, which in turn is to say that ϵ_Σ is an epimorphism (and then necessarily a split one) in Mod_A. Thus, (i), (ii) and (v) are equivalent.

Next, since the functor $G = - \otimes_A : \mathrm{Mod}_A \to \mathrm{Mod}_A$ preserves all colimits, the comonad $\mathbf{G}_\Sigma$ is of effective codescent type iff it is of codescent type. But since A is a projective generator for the category Mod_A, it follows from the dual of Corollary 2.6 that $\mathbf{G}_\Sigma$ is of codescent type iff the A-component $\sigma_A = A \otimes_A \epsilon_\Sigma \simeq \epsilon_\Sigma$ of the counit $\sigma : G \to 1$ is a split epimorphism in Mod_A. So that, (iii), (iv) and (v) are also equivalent. This completes the proof. ∎

7.2. PROPOSITION. *Let A be a ring and E a Cauchy complete K-category. Then:*

(i) *If $\eta, \epsilon : F \dashv U : Mod_A \to \mathsf{E}$ is an adjunction such that the composite FU preserves all colimits, then the functor U is monadic if and only if the morphism $\epsilon_A : FU(A) \to A$ is a split epimorphism in Mod_A.*

(ii) *If $\eta, \epsilon : F \dashv U : \mathsf{E} \to Mod_A$ is an adjunction such that the composite UF preserves all limits, then the functor F is comonadic if and only if the morphism $\epsilon_A : FU(A) \to A$ is a split epimorphism in Mod_A.*

PROOF. (i). Since E is Cauchy complete, it follows from (the dual of) Theorem 3.19 that U is monadic iff the comonad $\mathbf{G} = (FU, \epsilon, F\eta U)$ arising from the adjunction $F \dashv U$ is of effective codescent type. As FU is assumed to preserve all colimits, there exists an A-coring $\Sigma = (\Sigma, \delta_\Sigma, \epsilon_\Sigma)$, where $\Sigma = FU(A)$, for which one has an isomorphism $\mathbf{G} \simeq \mathbf{G}_\Sigma$ of comonads (see Corollary (3.7)(ii), in [19]). The conclusion now follows from the previous theorem using the fact that $\epsilon_\Sigma = \epsilon_A : FU(A) \to A$.

(ii). The argument here is the same as in (i): Just use Corollary (5.5)(1) instead of Corollary (3.7)(ii) in [19]. ∎

We now turn our attention to monads on module categories.

Let A and B be rings, M a (B, A)-bimodule, $\mathcal{E}_M = \mathrm{Mod}_A(M, M)$ the ring of the right endomorphisms of M and $i_M : B \to \mathcal{E}$, $b \to (m \to bm)$ the corresponding ring extension. (A ring extension means a homomorphism of rings that takes 1 to 1). Consider the induction functor

$$F_M = -\otimes_B M : \mathrm{Mod}_B \to \mathrm{Mod}_A$$

and its right adjoint, the coinduction functor,

$$U_M = \mathrm{Mod}_A(M, -) : \mathrm{Mod}_A \to \mathrm{Mod}_B.$$

The unit $\eta : 1 \to U_M F_M$ and counit $\epsilon : F_M U_M \to 1$ of the adjunction are given by the formulas:

$$\eta_X : X \to \mathrm{Mod}_A(M, X \otimes_B M), \quad \eta_X(x)(m) = x \otimes_B m$$

and

$$\epsilon_Y : \mathrm{Mod}_A(M, Y) \otimes_B M \to Y, \quad \epsilon_Y(f \otimes_B m) = f(m).$$

We write $\mathbf{T}_M$ (resp. $\mathbf{G}_M$) to denote the monad (resp. comonad) on Mod_B (resp. Mod_A) arising from this adjunction.

Dually, if N is an (A, B)-bimodule, then the functor $N\otimes_B- : {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$ is adjoint on the left to the functor ${}_A\mathrm{Mod}(M, -) : {}_A\mathrm{Mod} \to {}_B\mathrm{Mod}$. We write ${}_N\mathbf{T}$ (resp. ${}_N\mathbf{G}$) for the monad (resp. comonad) on ${}_B\mathrm{Mod}$ (resp. ${}_A\mathrm{Mod}$) generated by this adjunction.

Recall (for example from [31]) that a morphism $f : M \to N$ of right B-modules is called *pure* if $f \otimes_B 1_L : M \otimes_B L \to N \otimes_B L$ is injective for every left B-module L. Pure morphisms in the category of left B-modules are defined analogously.

7.3. Proposition. *For any (B, A)-bimodule M, the following are equivalent:*

(i) *the functor* $-\otimes_B M : \mathrm{Mod}_B \to \mathrm{Mod}_A$ *is faithful;*

(ii) *the functor* $-\otimes_B M : \mathrm{Mod}_B \to Mod_A$ *is conservative;*

(iii) *M is totally faithful as a left B-module, i.e. the morphism*

$$\eta_X : X \to \mathrm{Mod}_A(M, X \otimes_B M), \ \ \eta_X(m) = x \otimes_B m,$$

is injective for every $X \in \mathrm{Mod}_B$*;*

(iv) *the functor* $-\otimes_B M : \mathrm{Mod}_B \to \mathrm{Mod}_A$ *is precomonadic;*

(v) *the monad* $\mathbf{T}_M$ *is of descent type;*

(vi) *the functor* $-\otimes_B M : \mathrm{Mod}_B \to \mathrm{Mod}_A$ *reflects the zero object; that is, if* $X \otimes_B M \simeq 0$*, then* $X \simeq 0$.

Each of (i)-(vi) *implies*

(vii) *the ring extension* $i_M : B \to \mathcal{E}_M$ *is a pure morphism of left B-modules;*

and all seven are equivalent if M is finitely generated and projective as a right A-module.

Proof. As in the proof of Proposition 6.3, we see−since the unit of the adjunction $F_M \dashv U_M$ has for its X-component the morphism η_X−that (i)-(iv) are equivalent.

(iv) and (v) are equivalent by definition.

(i) implies (vi), since any faithful functor reflects monomorphisms.

(vi) $\Rightarrow$ (iii). We just observe that, if X' is the kernel of the morphism $\eta_X : X \to \mathrm{Mod}_A(M, X \otimes_B M)$, then $X' \otimes_B M = 0$.

For any $X \in \mathrm{Mod}_B$, consider the composition

$$X \xrightarrow{X \otimes_B i_M} X \otimes_B \mathrm{Mod}_A(M, M) \xrightarrow{t_X} \mathrm{Mod}_A(M, X \otimes_B M),$$

where t_X is given by

$$t_X(x \otimes_B f)(m) = x \otimes_B f(m).$$

Then $t_X \cdot (X \otimes_B i_M) = \eta_X$, whence it follows trivially that $X \otimes_B i_M$ is injective for every $X \in \mathrm{Mod}_B$ (or, equally, i_M is pure in ${}_B\mathrm{Mod}$), provided that every η_X is. Hence each of (i)-(vi) implies that the ring extension $i_M : B \to \mathcal{E}_M$ is pure in ${}_B\mathrm{Mod}$.

Finally, if M is assumed to be finitely generated and projective as a right A-module, then the morphism t_X is an isomorphism and thus η_X is injective if and only if $X \otimes_B i_M$ is.

■

7.4. Remark. When M_A is finitely generated and projective, the equivalence of (i), (ii) and (iii) is proved in [13] (see Proposition 2.3 of [13]).

When M is assumed to be finitely generated and projective as a right A-module, then there is a natural isomorphism of functors $U_M = \mathrm{Mod}_A(M,-) \simeq - \otimes_B M^*$, where $M^* = \mathrm{Mod}_A(M, A)$ is the dual of M_A which is an (A, B)-bimodule in a canonical way. Then $T_M = U_M \circ F_M = \mathrm{Mod}_A(M, - \otimes_B M) \simeq - \otimes_B \mathrm{Mod}_A(M, M) = - \otimes_B \mathcal{E}_M$, while $G_M = F_M U_M = \mathrm{Mod}_A(M,-)\otimes_B M \simeq -\otimes_A M^* \otimes_B M$. Moreover, the unit and the counit of the adjunction $F_M \dashv U_M$ take, in the present case, the forms

$$\eta_X = X \otimes_B i_M : X \simeq X \otimes_B B \to X \otimes_B \mathcal{E}_M$$

and

$$\epsilon_Y = Y \otimes_A \mathrm{ev}_M : Y \otimes_A M^* \otimes_B M \to Y \otimes_A A \simeq Y.$$

It follows that tensoring with the ring $\mathcal{E}_M$ is isomorphic to the monad $\mathbf{T}_M$. It is well known (see, for example, [19]) that $(\mathrm{Mod}_B)^{\mathbf{T}_M}$ is equivalent to the category of right $\mathcal{E}_M$-modules, $\mathrm{Mod}_{\mathcal{E}_M}$, by an equivalence which identifies the comparison functor $K_{\mathbf{T}_M} : \mathrm{Mod}_B \to \mathrm{Mod}_{\mathcal{E}_M}$ with the functor $- \otimes_B \mathcal{E}_M$.

Recall that if $i : B \to A$ is a ring extension, then the natural A-bimodule structure on A gives rise to a (B, B)-bimodule structure on A via i. A then receives the natural (B, A)-bimodule structure and thus induces the functor $-\otimes_B A : \mathrm{Mod}_B \to \mathrm{Mod}_A$. Similarly, viewing A as (A, B)-bimodule, one has the functor $A\otimes_B - : {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$. Theorem 3.20 now gives:

7.5. Theorem. *Let A and B be rings, M a (B, A)-bimodule with M_A finitely generated and projective, $\mathcal{E}_M = \mathrm{Mod}_A(M, M)$ the ring of the right endomorphisms of M and*

$$i_M : B \to \mathcal{E}_M,\ b \longrightarrow [m \to bm]$$

the corresponding ring extension. Then the induction functor $-\otimes_B M : \mathrm{Mod}_B \to \mathrm{Mod}_A$ is (pre)comonadic if and only if the functor $-\otimes_B \mathcal{E}_M : \mathrm{Mod}_B \to \mathrm{Mod}_{\mathcal{E}_M}$ is.

We now consider the functor

$$C = \mathbf{Ab}(-, \mathbb{Q}/\mathbb{Z}) : \mathbf{Ab} \to (\mathbf{Ab})^{\mathrm{op}}$$

(where $\mathbf{Ab}$ is the category of abelian groups and $\mathbb{Q}/\mathbb{Z}$ is the rational circle abelian group) which is well-known to be exact and conservative. Given a ring A and a right A-module M, the abelian group $C(M) = \mathbf{Ab}(M, \mathbb{Q}/\mathbb{Z})$ has the structure of a left A-module: The left A-actions on $C(M)$ is defined by $(a \cdot \psi)(m) = \psi(ma)$. Hence C induces a functor

$$\mathrm{Mod}_A \to ({}_A\mathrm{Mod})^{\mathrm{op}}.$$

We write C_A for this functor. Similarly, if M is a left A-module, then viewing $C(M) = \mathbf{Ab}(M, \mathbb{Q}/\mathbb{Z})$ as a right A-module via $(\psi \cdot a)(m) = \psi(am)$, one has a functor

$${}_AC : {}_B\mathrm{Mod} \to ({}_A\mathrm{Mod})^{\mathrm{op}}.$$

(Note that ${}_{A}C$ could equivalently be defined as the functor

$$C_{A^{\mathrm{op}}} : {}_{A}\mathrm{Mod} = \mathrm{Mod}_{A^{\mathrm{op}}} \to ({}_{A^{op}}\mathrm{Mod})^{\mathrm{op}} = (\mathrm{Mod}_{A})^{\mathrm{op}}).$$

For any $M \in {}_{A}\mathrm{Mod}_{A}$, it is easy to check that the left and right A-module structures on $C(M)$ as defined above are compatible and thus enables us to view $C(M)$ as an A-A-bimodule. So one can introduce a third functor

$${}_{A}\mathrm{Mod}_{A} \to ({}_{A}\mathrm{Mod}_{A})^{\mathrm{op}},\ M \to C(M).$$

We let ${}_{A}C_{A}$ denote this functor.

The following is well known (see, for example, [31]):

7.6. Proposition. *A morphism of right (resp. left) A-modules $f : M \to N$ is pure if and only if the morphism $C_{A}(f) : C_{A}(N) \to C_{A}(M)$ (resp. ${}_{A}C(f) : {}_{A}C(N) \to {}_{A}C(M)$) is a split epimorphism.*

Let A and B be rings. For any (B, A)-bimodule M and any right A-module X, $\mathrm{Mod}_{A}(X, M)$ is a left B-module via $(bf)(x) = b \cdot f(x)$; hence we have a functor

$$\mathrm{Mod}_{A}(-, M) : (\mathrm{Mod}_{A})^{op} \to {}_{B}\mathrm{Mod}.$$

Then the assignment

$$M \to \mathrm{Mod}_{A}(-, M)$$

can be considered in the obvious way as a functor

$$\mathbb{X} : ({}_{B}\mathrm{Mod}_{A}) \to [(\mathrm{Mod}_{A})^{op}, {}_{B}\mathrm{Mod}].$$

(Here $[(\mathrm{Mod}_{A})^{op}, {}_{B}\mathrm{Mod}]$ denotes the category of K-functors from $(\mathrm{Mod}_{A})^{op}$ to ${}_{B}\mathrm{Mod}$), and the following is a simple consequence of the Yoneda lemma.

7.7. Proposition. *The functor $\mathbb{X}$ is full and faithful.*

As a particular case of this proposition, we have:

7.8. Corollary. *A morphism $f : M \to M'$ of (B, A)-bimodules is a split epimorphism if and only if the corresponding natural transformation $\mathbb{X}(f) : \mathbb{X}(M) \to \mathbb{X}(M')$ is.*

In order to proceed, we shall need the following result that gives a sufficient condition on a left adjoint functor with coexact domain to be comonadic.

7.9. Proposition. *Let $\mathcal{A}$ be a coexact category, $\mathcal{B}$ a category admitting cokernel-pairs of split monomorphisms and $F : \mathcal{A} \to \mathcal{B}$ a left adjoint functor. Suppose that there exists a comonadic functor $I : \mathcal{A} \to \mathcal{X}$ with $\mathcal{X}$ Cauchy complete and such that F is I-separable and that F takes I-split monomorphisms into regular monomorphisms, then F is comonadic.*

Proof. Let us first recall that a morphism in $\mathcal{A}$ is an I- *split* monomorphism if its image under I is a split monomorphism in $\mathcal{X}$. Note that the comonodacity of I guarantees that any I-split monomorphism is necessarily a regular monomorphism (see, for example, [29]).

The functor I, being comonadic, is conservative, and it follows from Proposition 3.6 that F is conservative as well. Hence, in view of (the dual of) Theorem 5.1, the desired result will follow if we can show that F preserves those regular monomorphisms whose cokernel-pairs are F-split, and this in turn will follow if we can prove that any such regular monomorphism is I-split. Suppose therefore that $f : a \to a'$ is a regular monomorphism in $\mathcal{A}$ with F-split cokernel-pair $(i_1, i_2 : a' \rightrightarrows a' \sqcup_a a')$. Since $\mathcal{X}$ is assumed to be Cauchy complete, it follows from Proposition 3.7 that the pair (i_1, i_2) is also I-split. But I is comonadic by hypothesis, hence it preserves equalizers of I-split pairs. Since f is a regular monomorphism, it is the equalizer of its cokernel-pair. Thus, the diagram

$$I(a) \xrightarrow{I(f)} I(a') \underset{I(i_2)}{\overset{I(i_1)}{\rightrightarrows}} I(a' \sqcup_a a')$$

is a split equalizer, from which it follows in particular that $I(f)$ is a split monomorphism. Therefore, f is an I-split monomorphism, as needed. ■

7.10. Theorem. *Let $i : B \to A$ be a ring extension. Then the following are equivalent:*

(i) *The functor $-\otimes_B A : \mathrm{Mod}_B \to \mathrm{Mod}_A$ is $\mathcal{C}_B$-separable.*

(ii) *The functor $A\otimes_B - : {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$ is ${}_B\mathcal{C}$-separable.*

(iii) *The morphism ${}_B\mathcal{C}_B(i) : {}_B\mathcal{C}_B(B) \to {}_B\mathcal{C}_B(B)$ is a split epimorphism.*

If these conditions hold, then the functors $-\otimes_B A$ and $A\otimes_B -$ are both comonadic.

Proof. Since the functor $-\otimes_B A : \mathrm{Mod}_B \to \mathrm{Mod}_A$ admits as a right adjoint the functor $\mathrm{Mod}_A(A, -) : \mathrm{Mod}_A \to \mathrm{Mod}_B$, with the unit η being

$$\eta = 1 \otimes_B i : 1 \simeq - \otimes_B B \to - \otimes_B A,$$

to say that $-\otimes_B A$ is $\mathcal{C}_B$-separable is, by Theorem 3.1, to say that the natural transformation

$$\mathcal{C}_B(\eta) : \mathcal{C}_B(- \otimes_B A) \to \mathcal{C}_B(- \otimes_B B)$$

is a split epimorphism. Recalling that for any (B, B)-bimodule X, one has an isomorphism of functors

$$\mathcal{C}_B(- \otimes_B X) \simeq {}_B\mathrm{Mod}(-, {}_BC_B(X)),$$

we see that the above natural transformation is isomorphic to the natural transformation

$${}_B\mathrm{Mod}(-, {}_B\mathcal{C}_B(i)) : {}_B\mathrm{Mod}(-, {}_B\mathcal{C}_B(A)) \to {}_B\mathrm{Mod}(-, {}_B\mathcal{C}_B(B)),$$

which in view of Corollary 7.8 is a split epimorphism if and only if the morphism ${}_B\mathcal{C}_B(i)$ is a split epimorphism in ${}_B\mathrm{Mod}_B$. This proves that (i) and (iii) are equivalent, and the equivalence of (ii) and (iii) can be shown in a similar way, using the fact that the unit of the adjunction $A \otimes_B - \dashv {}_A\mathrm{Mod}(A, -)$ is the natural transformation

$$i \otimes_B - : 1 \simeq B \otimes_B - \longrightarrow A \otimes_B -.$$

Suppose now that the functor $- \otimes_B A$ is $\mathcal{C}_B$-separable. Then, since

- the category Mod_B being abelian is coexact,
- the category Mod_A is complete (and cocomplete),
- the functor $\mathcal{C}_B$ is comonadic; (indeed, the forgetful functor $U_A : \mathrm{Mod}_A \to \mathrm{Ab}$ and the functor $C : \mathrm{Ab} \to (\mathrm{Ab})^{\mathrm{op}}$ are both exact and conservative and since $(U_{A^{\mathrm{op}}})^{\mathrm{op}} \circ C_A = C \circ U_A$, the functor C_A is also exact and conservative. Moreover, it is easy to check that C_A admits as a right adjoint the functor

  $$({}_AC)^{\mathrm{op}} = (C_{A^{\mathrm{op}}})^{\mathrm{op}} : {}_A\mathrm{Mod} \to (\mathrm{Mod}_A)^{\mathrm{op}}.$$

  It follows that the functor C_A is comonadic.)
- $\mathcal{C}_B$-split monomorphisms in Mod_B are exactly the pure monomorphisms of left B-modules (see Proposition 7.6),
- the functor $- \otimes_B A$ takes pure monomorphisms of right B-modules into monomorphisms of right A-modules,

we can apply Proposition 7.9 to conclude that the functor $- \otimes_B A$ is comonadic.

Analogously, (ii) implies that the functor $A \otimes_B -$ is comonadic. ∎

7.11. Theorem. *Let A and B be rings, M a (B, A)-bimodule with M_A finitely generated and projective, $\mathcal{E}_M = \mathrm{Mod}_A(M, M)$ the right endomorphism ring of M and $i_M : B \to \mathcal{E}_M$ the corresponding ring extension. If the morphism ${}_B\mathcal{C}_B(i_M)$ is a split epimorphism, then the induction functors*

$$- \otimes_B M : \mathrm{Mod}_B \to \mathrm{Mod}_A$$

and

$$M^* \otimes_B - : {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$$

are both comonadic.

Proof. Observe first that if M_A is finitely generated and projective, then so too is ${}_AM^*$, and that the rings $\mathcal{E}_M$ and ${}_{M^*}\mathcal{E}$ are isomorphic. It now follows from Theorem 7.5, its dual and Proposition 7.9 that the functors $- \otimes_B M$ and $M^* \otimes_B -$ are both comonadic. ∎

8. Relation to the descent problem for modules

We begin by describing the descent problem for modules.

Given a ring extension $i : B \to A$, a *(right) descent datum* on a right A-module X is a morphism $\theta : X \to X \otimes_B A$ of right A-modules rendering commutative the following two diagrams

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[dr, equal] & X \otimes_B A \arrow[d, "\lambda_X"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[rr, "\theta"] \arrow[d, "\theta"'] & & X \otimes_B A \arrow[d, "\theta \otimes_B 1_A"] \\ X \otimes_B A \arrow[rr, "\eta_X \otimes_B 1_A"'] & & X \otimes_B A \otimes_B A\,. \end{tikzcd}$$

Here $\eta_X : X \to X \otimes_B A$ is given by $\eta_X(x) = x \otimes_B 1$, while $\lambda_X : X \otimes_B A \to X$ is the right A-module structure on X. We write $\mathfrak{RDes}(i)$ for the category whose objects are pairs (X, θ), where X is a right A-module, and θ is a (right) descent datum on X. Morphisms are homomorphisms of right A-modules compatible in the obvious sense with the descent data. Compositions and identities in $\mathfrak{RDes}(i)$ are induced by those in Mod_A, in the evident manner.

For any right B-module Y, the morphism

$$\eta_{Y \otimes_B A} : Y \otimes_B A \to Y \otimes_B A \otimes_B A$$

is a descent datum on $Y \otimes_B A$, and the assignment

$$Y \longrightarrow (Y \otimes_B A, \eta_{Y \otimes_B A})$$

yields a comparison functor

$$K_i : \mathrm{Mod}_B \to \mathfrak{RDes}(i).$$

$i : B \to A$ is said to be a *right (effective) descent* ring extension if the functor K_i is (an equivalence of categories) full and faithful.

The dual notions are the *left descent datum* on a left B-module, the corresponding category of *left descent data*, $\mathfrak{LDes}(i)$, and the comparison functor

$${}_iK : {}_B\mathrm{Mod} \to \mathfrak{LDes}(i).$$

And one says that $i : B \to A$ is a *left (effective) descent* ring extension if the functor ${}_iK$ is (an equivalence of categories) full and faithful.

The descent problem for modules consists in finding conditions which are either necessary or sufficient in order that a given ring extension be left (or right) effective for descent.

More details on Descent Theory can be found in [23], [24] and [25].

It is not hard to see that the category $\mathfrak{RDes}(i)$ is nothing but the Eilenberg-Moore category of $\mathbf{G}$-coalgebras, where $\mathbf{G}$ is the comonad on Mod_A arising from the adjunction

$-\otimes_B A \dashv \mathrm{Mod}_A(A,-) : \mathrm{Mod}_A \to \mathrm{Mod}_B$. But the functor $\mathrm{Mod}_A(A,-)$, which is (isomorphic to) the restriction of scalars functor, is monadic, so that $\mathrm{Mod}_A \simeq (\mathrm{Mod}_B)^{\mathbf{T}_A}$ (where, recall, $\mathbf{T}_A$ is the monad on Mod_B generated by the adjunction $-\otimes_B A \dashv \mathrm{Mod}_A(A,-)$) and hence $\mathfrak{RDes}(i)$ can be identified with the category $\mathfrak{Des}_{\mathrm{Mod}_B}(\mathbf{T}_A)$. Hence i is a right descent ring extension iff the functor $-\otimes_B A : \mathrm{Mod}_B \to \mathrm{Mod}_A$ is precomonadic, and it is a right effective descent ring extension iff the functor $-\otimes_B A$ is comonadic. Symmetrically, $\mathfrak{LDes}(i) \simeq \mathfrak{Des}_{{}_B\mathrm{Mod}}({}_A\mathbf{T})$, and i is a left descent ring extension iff the functor $A\otimes_B - : {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$ is precomonadic, and it is a left effective descent ring extension iff the functor $A\otimes_B -$ is comonadic.

It is well known (see [11]) that the category $(\mathrm{Mod}_A)_{\mathbf{G}}$ is equivalent to the category of right comodules over the so-called Sweedler's canonical A-coring associated to the ring extension $i : B \to A$. Recall ([45]) that the A-A-bimodule $A \otimes_B A$ is an A-coring with the comultiplication

$$A\otimes_B A \to (A\otimes_B A)\otimes_A (A\otimes_B A) \simeq A\otimes_B A\otimes_B A,\ a_1 \otimes_B a_2 \longrightarrow a_1 \otimes_B 1_B \otimes_B a_2$$

and the counit

$$A\otimes_B A \to A,\ a_1 \otimes_B a_2 \longrightarrow a_1 \cdot a_2.$$

This A-coring is known as a Sweedler's A-coring associated to the ring extension $i : B \to A$. So that, the categories $\mathrm{Mod}^{A\otimes_B A}$ and $(\mathrm{Mod}_A)_{\mathbf{G}}$ are equivalent, and this allows us to identify the functor $K_i : \mathrm{Mod}_B \to \mathfrak{RDes}(i)$ with the functor $K_{\mathbf{G}} : \mathrm{Mod}_B \to \mathrm{Mod}^{A\otimes_B A}$. Similarly, $({}_A\mathrm{Mod})_{\mathbf{G}'} \simeq {}^{A\otimes_B A}\mathrm{Mod}$, where $\mathbf{G}'$ is the comonad on ${}_B\mathrm{Mod}$ generated by the adjunction $A \otimes_B - \dashv {}_A\mathrm{Mod}(A,-) : {}_A\mathrm{Mod} \to {}_B\mathrm{Mod}$, and the functor ${}_iK : {}_B\mathrm{Mod} \to \mathfrak{RDes}(i)$ can be identified with the functor $K_{\mathbf{G}'} : {}_B\mathrm{Mod} \to {}^{A\otimes_B A}\mathrm{Mod}$.

It follows from Theorem 7.10 that

8.1. Theorem. *Let $i : B \to A$ be a ring extension such that the morphism ${}_BC_B(i) : {}_BC_B(A) \to {}_BC_B(B)$ is a split epimorphism. Then i is an effective descent ring extension on both sides.*

After this theorem it is natural to ask:

QUESTION 1: *Are "left effective descent" and "right effective descent" equivalent properties?*

Note that, the question is not fully answered yet. Theorem 8.1 provides only a partial answer to this question.

A more general situation is that where A and B are connected by (B,A)-bimodule M that is finitely generated and projective as a right A-module with a fixed dual basis $e = \sum m_i \otimes_B f_i \in M \otimes_A M^*$. (This situation has been considered by L. El Kaoutit and J. Gómez Torrecillas in [16] and by S. Caenepeel, E. De Groot and J. Vercruyssen in [13].) Then the (A,A)-bimodule $M^* \otimes_B M$ is an A-coring with the comultiplication

$$M^*\otimes_B M \to M^*\otimes_B M\otimes_A M^*\otimes_B M,\ \ f \otimes_B m \longrightarrow f \otimes_B e \otimes_B m$$

and the counit

$$M^*\otimes_B M \to A,\ \ f \otimes_B m \longrightarrow f(m).$$

The A-coring $M^*\otimes_B M$ is known as a comatrix A-coring [16]. Like any A-coring, $M^*\otimes_B M$ determines a comonad $\mathbf{G}_{M^*\otimes_B M} = (G, \epsilon, \delta)$ on Mod_A, where

$$G = -\otimes_A (M^*\otimes_B M),\ \epsilon = -\otimes_A \epsilon_{M^*\otimes_B M},\ \delta = -\otimes_A \delta_{M^*\otimes_B M}.$$

Then the resulting category of coalgebras over this comonad is nothing more than the category of right $M^*\otimes_B M$-comodules $\mathrm{Mod}^{M^*\otimes_B M}$, and one has the comparison functor

$$K_M : \mathrm{Mod}_B \to \mathrm{Mod}^{M^*\otimes_B M},\ K_M(X) = (X \otimes_B M, \sigma_{X\otimes_B M}),$$

where

$$\sigma_{X\otimes_B M} : X\otimes_B M \to X\otimes_B M\otimes_A M^*\otimes_B M$$

is a morphism of right A-modules given by

$$\sigma_{X\otimes_B M}(x\otimes_B m) = x\otimes_B e\otimes_B m.$$

In particular, taking $X = B$ we deduce that the right A-module M is a right $(M^*\otimes_B M)$-comodule with coaction

$$\sigma_M : M \to M\otimes_A M^*\otimes_B M,\ \sigma_M(m) = e\otimes_B m.$$

Given a (B, A)-bimodule M such that it is finitely generated and projective as a right A-module, we say that M_A is of *(effective) descent type* if the functor K_M is (an equivalence of categories) fully faithful; in other words (since $\mathrm{Mod}^{M^*\otimes_B M}$ is (isomorphic to) the Eilenberg-Moore category of $\mathbf{G}_M$-coalgebras) M_A is of (effective) descent type iff the comonad $\mathbf{G}_M$ is. Note that, if $i : B \to A$ is a ring extension, then A_A is of (effective) descent type iff i is a right (effective) descent ring extension, while ${}_AA$ is of (effective) descent type iff i is a left (effective) descent ring extension.

Theorem 3.19 gives:

8.2. Theorem. *Let A and B be rings, M a (B, A)-bimodule with M_A finitely generated and projective, $\mathcal{E}_M = \mathrm{Mod}_A(M, M)$ the right endomorphism ring of M and $i_M : B \to \mathcal{E}_M$ the corresponding ring extension. Then the functor*

$$K_M : \mathrm{Mod}_B \to \mathrm{Mod}^{M^*\otimes_B M},\ K_M(X) = X\otimes_B M$$

is (an equivalence of categories) fully faithful if and only if the functor

$$K_{\mathcal{E}_M} : \mathrm{Mod}_B \to \mathrm{Mod}^{\mathcal{E}_M\otimes_B \mathcal{E}_M},\ K_{\mathcal{E}}(X) = X\otimes_B \mathcal{E}_M$$

is. Said otherwise, M_A is of (effective) descent type iff i_M is a right (effective) descent ring extension.

In the light of Theorem 7.11, we get from Theorem 8.2 and its dual the following result:

8.3. Theorem. *Let A, B be rings, M a (B,A)-bimodule with M_A finitely generated and projective, $\mathcal{E}_M = \mathrm{Mod}_A(M,M)$ its right endomorphism ring and $i_M : B \to \mathcal{E}_M$ the corresponding ring extension. If the morphism ${}_BC_B(i_M) : {}_BC_B(\mathcal{E}_M) \to {}_BC_B(B)$ is a split epimorphism, then the functors*

$$K_M : \mathrm{Mod}_B \longrightarrow \mathrm{Mod}^{M^* \otimes_B M}$$

and

$$_{M^*}K : {}_B\mathrm{Mod} \longrightarrow {}^{M \otimes_B M^*}\mathrm{Mod}$$

are equivalences of categories. In other words, if ${}_BC_B(i_M)$ is a split epimorphism, then both M_A and ${}_AM^*$ are of effective descent type.

Note that the analogue of QUESTION 1 is

QUESTION 2: *For $M \in {}_B\mathrm{Mod}_A$ with M_A finitely generated and projective, are M_A and ${}_AM^*$ both of effective descent type?*

By Theorem 3.20, the answer to this question is the same as that to QUESTION 1.

It is shown in [17] that the functors

$$-\otimes_{\mathcal{E}_M} M : \mathrm{Mod}_{\mathcal{E}_M} \to \mathrm{Mod}_A$$

and

$$-\otimes_A M^* : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{E}_M}$$

can be lifted to an adjoint pair of functors

$$-\otimes_{\mathcal{E}_M} M \dashv -\otimes_A M^* : \mathrm{Mod}^{M^* \otimes_B M} \to \mathrm{Mod}^{\mathcal{E}_M \otimes_B \mathcal{E}_M} \tag{8.1}$$

for which the diagram

$$\begin{tikzcd}
\mathrm{Mod}^{\mathcal{E}_M \otimes_B \mathcal{E}_M} \arrow[rr, shift left, "-\otimes_{\mathcal{E}_M} M"'] & & \mathrm{Mod}^{M^* \otimes_B M} \arrow[ll, shift left, "-\otimes_A M^*"'] \\
& \mathrm{Mod}_B \arrow[ul, "K_{\mathcal{E}_M}"] \arrow[ur, "K_M"'] &
\end{tikzcd}$$

commutes (up to isomorphism). Symmetrically, one has an adjunction

$$M^* \otimes_{\mathcal{E}_M} - \dashv M \otimes_A - : {}^{M^* \otimes_B M}\mathrm{Mod} \to {}^{\mathcal{E}_M \otimes_B \mathcal{E}_M}\mathrm{Mod} \tag{8.2}$$

for which the diagram

$$\begin{tikzcd}
{}^{\mathcal{E}_M \otimes_B \mathcal{E}_M}\mathrm{Mod} \arrow[rr, shift left, "M^* \otimes_{\mathcal{E}_M} -"'] & & {}^{M^* \otimes_B M}\mathrm{Mod} \arrow[ll, shift left, "M \otimes_A -"'] \\
& {}_B\mathrm{Mod} \arrow[ul, "{}_{\mathcal{E}_M}K"] \arrow[ur, "{}_MK"'] &
\end{tikzcd}$$

is commutative.

Putting Theorem 8.2, its dual and Theorem 8.3 together, we obtain:

8.4. Proposition. *Let A, B be rings, M a (B,A)-bimodule with M_A finitely generated and projective, $\mathcal{E}_M = \mathrm{Mod}_A(M,M)$ its right endomorphism ring and $i_M : B \to \mathcal{E}_M$ the corresponding ring extension. Then*

(i) *if the functor $-\otimes_B M : \mathrm{Mod}_B \to \mathrm{Mod}_A$ is comonadic, or equivalently, M_A is of effective descent type, then* (8.1) *is an adjoint equivalence;*

(ii) *if the functor $M^*\otimes_B - : {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$ is comonadic, or equivalently, ${}_AM^*$ is of effective descent type, then* (8.2) *is an adjoint equivalence;*

(iii) *if the morphism ${}_B\mathcal{C}_B(i_M) : {}_B\mathcal{C}_B(\mathcal{E}_M) \to {}_B\mathcal{C}_B(B)$ is a split epimorphism, then* (8.1) *and* (8.2) *are both adjoint equivalences.*

Two A-corings Σ and Σ' are said to be *right* (resp. *left*) *Morita-Takeuchi equivalent* if the categories Mod^{Σ} and $\mathrm{Mod}^{\Sigma'}$ (resp. ${}^{\Sigma}\mathrm{Mod}$ and ${}^{\Sigma'}\mathrm{Mod}$) are equivalent. Σ and Σ' are called *Morita-Takeuchi equivalent* if they are both right and left Morita-Takeuchi equivalent.

Using these concepts, Proposition 8.4 can be interpreted as a result about Morita-Takeuchi equivalences as follows:

8.5. Proposition. *Under the assumptions of Proposition 8.4, if*

(i) *the functor $-\otimes_B M : \mathrm{Mod}_B \to \mathrm{Mod}_A$ is comonadic, or equivalently, M_A is of effective descent type, then the A-corings $M^*\otimes_B M$ and $\mathcal{E}_M \otimes_B \mathcal{E}_M$ are right Morita-Takeuchi equivalent;*

(ii) *the functor $M^*\otimes_B - : {}_B\mathrm{Mod} \to {}_A\mathrm{Mod}$ is comonadic, or equivalently, ${}_AM^*$ is of effective descent type, then the A-corings $M^*\otimes_B M$ and $\mathcal{E}_M \otimes_B \mathcal{E}_M$ are left Morita-Takeuchi equivalent;*

(iii) *the morphism ${}_B\mathcal{C}_B(i_M) : {}_B\mathcal{C}_B(\mathcal{E}_M) \to {}_B\mathcal{C}_B(B)$ is a split epimorphism, then the A-corings $M^*\otimes_B M$ and $\mathcal{E}_M \otimes_B \mathcal{E}_M$ are Morita-Takeuchi equivalent.*

It is well known that when a given A-coring Σ is flat as a left A-module, then the category Mod^{Σ} is an (abelian) Grothendieck category (see, for example, [18]). This is a special case of a general result which asserts that if $\mathbf{G}$ is a flat comonad on a category A that is additive (resp. abelian, resp. abelian with small coproducts, resp. an (abelian) Grothendieck category), then the category $\mathcal{A}_{\mathbf{G}}$ has the same property (see, for example, Proposition 6.5 in [30]). Recall [38] that an object a of a Grothendieck category A is a small projective generator when the functor $\mathcal{A}(a,-) : \mathcal{A} \to \mathrm{Mod}_K$ preserves and reflects all small colimits. This is of course equivalent to saying that $\mathcal{A}(a,-)$ is conservative and preserves all small coproducts.

We are now in position to state and prove the following result of L. El Kaoutit and J. Gómez-Torrecillas (see Theorem 3.10 in [17]):

8.6. Proposition. *Let M be a (B,A)-bimodule with M_A finitely generated projective. Define $\mathcal{L}_M = \mathrm{Mod}^{M^*\otimes_B M}((M,\sigma_M),(M,\sigma_M))$ and write $k : B \to \mathcal{L}_M$ for the corresponding ring extension. Then the following assertions are equivalent:*

(i) *$M^*\otimes_B M$ is flat as a left A-module, (M,σ_M) is a small projective generator for $\mathrm{Mod}^{M^*\otimes_B M}$ and k is an isomorphism.*

(ii) *${}_A(M^*\otimes_B M)$ is flat and M_A is of effective descent type.*

(iii) *${}_B M$ is faithfully flat.*

(iv) *${}_A(M^*\otimes_B M)$ is flat and $\mathcal{E}_M$ is a faithfully flat left B-module.*

Proof. That (ii), (iii) and (iv) are equivalent is a direct consequence of Proposition 3.21, since to say that an (R,R)-bimodule X, R being an arbitrary ring, is (faithfully) flat as a left R-module is to say that the functor $-\otimes_R X : \mathrm{Mod}_R \to \mathrm{Mod}_R$ preserves (and reflects) equalizers. We claim that (i) and (ii) are also equivalent. To see this, let us first recall that the comparison functor

$$K_M : \mathrm{Mod}_B \to \mathrm{Mod}^{M^*\otimes_B M},$$

that sends $X \in \mathrm{Mod}_B$ to the $M^*\otimes_B M$-comodule $(X\otimes_B M, \sigma_{X\otimes_B M})$, admits the functor

$$\mathrm{Mod}^{M^*\otimes_B M}((M,\sigma_M),-) : \mathrm{Mod}^{M^*\otimes_B M} \to \mathrm{Mod}_B$$

as a right adjoint.

(i) $\Longrightarrow$ (ii). We need only to show that under condition (i), M_A is of effective descent type. Since (M,σ_M) is a small projective generator for $\mathrm{Mod}^{M^*\otimes_B M}$, it follows from Mitchell's theorem (see [38]) that the functor $\mathrm{Mod}^{M^*\otimes_B M}((M,\sigma_M),-)$ induces an equivalent between $\mathrm{Mod}^{M^*\otimes_B M}$ and the category of right modules over the ring $\mathcal{L}_M = \mathrm{Mod}^{M^*\otimes_B M}((M,\sigma_M),(M,\sigma_M))$. But, by hypothesis, $\mathcal{L}_M$ is isomorphic to the ring B. It now follows that the functor $\mathrm{Mod}^{M^*\otimes_B M}((M,\sigma_M),-)$ (and hence also its left adjoint K_M) is an equivalence of categories. Thus, M_A is of effective descent type.

(ii) $\Longrightarrow$ (i). Assuming (ii), we have to show that k is an isomorphisms and that (M,σ) is a small projective generator for $\mathrm{Mod}^{M^*\otimes_B M}$. But since M_A is of effective descent type, the adjunction

$$K_A \dashv \mathrm{Mod}^{M^*\otimes_B M}((M,\sigma_M),-) : \mathrm{Mod}^{M^*\otimes_B M} \to \mathrm{Mod}_B$$

is an adjoint equivalence, and in particular the functor K_M is an equivalence of categories. Then the composite

$$B \xrightarrow{\simeq} \mathrm{Mod}_B(B,B) \xrightarrow{(K_M)_{B,B}} \mathrm{Mod}^{M^*\otimes_B M}((M,\sigma_M),(M,\sigma_M)) = \mathcal{L}_M,$$

which is just the morphism $k : B \to \mathcal{L}_M$, is an isomorphism. Moreover, since the functor

$$\mathrm{Mod}^{M^*\otimes_B M}((M,\sigma_M),-) : \mathrm{Mod}^{M^*\otimes_B M} \to \mathrm{Mod}_K$$

is the composite of the equivalence

$$\mathrm{Mod}^{M^*\otimes_B M}((M,\sigma_M),-):\mathrm{Mod}^{M^*\otimes_B M}\to \mathrm{Mod}_B$$

and the forgetful functor

$$\mathrm{Mod}_B \to \mathrm{Mod}_K,$$

and since this latter functor preserves and reflects all small colimits,

$$\mathrm{Mod}^{M^*\otimes_B M}((M,\sigma_M),-):\mathrm{Mod}^{M^*\otimes_B M}\to \mathrm{Mod}_K$$

itself preserves and reflects all small colimits. Therefore, (M,σ_M) is a small projective generator for $\mathrm{Mod}^{M^*\otimes_B M}$.

■

We specialize now to the case where B is a commutative ring. Then any $M \in {}_B\mathrm{Mod}$ is a right B-module with right B-action $m\cdot b = bm$. It is easy to verify that this makes M into (B,B)-bimodule: The compatibility of the left B-action with right one is just the commutativity of B. Thus ${}_B\mathrm{Mod}$ can be viewed as a full subcategory of ${}_B\mathrm{Mod}_B$. Let L denote the canonical embedding ${}_B\mathrm{Mod} \to {}_B\mathrm{Mod}_B$. Similarly, one has a full embedding $R:\mathrm{Mod}_B \to {}_B\mathrm{Mod}_B$. Consider the following diagram

$$\begin{tikzcd}
 & {}_B\mathrm{Mod}_B \arrow[dl, "U_B"'] \arrow[dd, "1"] \arrow[dr, "{}_BU"] & \\
\mathrm{Mod}_B \arrow[dr, "R"'] & & {}_B\mathrm{Mod} \arrow[dl, "L"] \\
 & {}_B\mathrm{Mod}_B &
\end{tikzcd}$$

in which U_B and ${}_BU$ are the evident forgetful functors.

The following result is obvious:

8.7. Lemma. *Let B be a commutative ring. For any (B,B)-bimodule M, the following are equivalent:*

(i) *The left and right B-actions on M coincide, i.e. $bm = mb$ for all $b\in B$ and $m\in M$.*

(ii) $RU_B(M) = M$.

(iii) $L\,{}_BU(M) = M$.

B-B-bimodules satisfying one (and hence all) of the conditions (i), (ii) and (iii) in Lemma 8.7 form a full subcategory of ${}_B\mathrm{Mod}_B$, and we write ${}_B\Im_B$ for this subcategory.

From the above lemma one obtains:

8.8. Corollary. *For a morphism $f : M \to N$ in ${}_B\Im_B$, the following are equivalent:*

(i) *f is a split epimorphism (in ${}_B\Im_B$ or in ${}_B\mathrm{Mod}_B$ − it makes no difference).*

(ii) *f, viewed as a morphism of right B-modules, is a split epimorphism, or equivalently, $U_B(f)$ is a split epimorphism (in Mod_B).*

(iii) *f, viewed as a morphism of left B-modules, is a split epimorphism, or equivalently, ${}_BU(f)$ is a split epimorphism (in ${}_B\mathrm{Mod}$).*

A straightforward calculation shows:

8.9. Proposition. *The functor*

$${}_BC_B : {}_BMod_B \to ({}_BMod_B)^{op}$$

restricts to a functor

$${}_B\Im_B \to ({}_B\Im_B)^{op}.$$

In other words, $f \in {}_B\Im_B$ implies ${}_BC_B(f) \in {}_B\Im_B$, for all morphisms $f \in {}_B\Im_B$.

Combining this with Proposition 7.6, we conclude that:

8.10. Proposition. *For a morphism $f \in {}_B\Im_B$, the following are equivalent:*

(i) *f, viewed as a morphism of left B-modules, is pure;*

(ii) *f, viewed as a morphism of right B-modules, is pure.*

We can now state and prove the following result:

8.11. Theorem. *(cf.* Theorem 2.7 in [13]) *Let A be a ring, B be a commutative ring. If $M \in {}_B\mathrm{Mod}_A$ with M_A finitely generated and projective is such that the ring extension $i_M : B \to \mathcal{E}_M = \mathrm{Mod}_A(M, M)$ factorizes through the center of $\mathcal{E}_M$, then the following are equivalent:*

(i) *i_M is pure in ${}_B\mathrm{Mod}$;*

(ii) *i_M is pure in Mod_B;*

(iii) *${}_BC(i_M)$ is a split epimorphism;*

(iv) *$C_B(i_M)$ is a split epimorphism;*

(v) *${}_BC_B(i_M)$ is a split epimorphism;*

(vi) *i_M is a right (effective) descent ring extension;*

(vii) *i_M is a left (effective) descent ring extension;*

(viii) *M_A is of (effective) descent type;*

(ix) *${}_A M^*$ is of (effective) descent type;*

(x) *the monad $\mathbf{T}_{\mathcal{E}_M}$ on Mod_B is of (effective) descent type;*

(xi) *the monad ${}_{\mathcal{E}_M}\mathbf{T}$ on ${}_B\mathrm{Mod}$ is of (effective) descent type;*

(xii) *the monad $\mathbf{T}_M$ on Mod_B is of (effective) descent type;*

(xiii) *the monad ${}_{M^*}\mathbf{T}$ on ${}_B\mathrm{Mod}$ is of (effective) descent type;*

(xiv) *the functor $K_{\mathcal{E}_M} : \mathrm{Mod}_B \longrightarrow \mathrm{Mod}^{\mathcal{E}_M \otimes_B \mathcal{E}_M}$ is an equivalence of categories;*

(xv) *the functor ${}_{\mathcal{E}_M}K : {}_B\mathrm{Mod} \longrightarrow {}^{\mathcal{E}_M \otimes_B \mathcal{E}_M}\mathrm{Mod}$ is an equivalence of categories;*

(xvi) *the functor $K_M : \mathrm{Mod}_B \longrightarrow \mathrm{Mod}^{M^* \otimes_B M}$ is an equivalence of categories;*

(xvii) *the functor ${}_{M^*}K : {}_B\mathrm{Mod} \longrightarrow {}^{M \otimes_B M^*}\mathrm{Mod}$ is an equivalence of categories;*

Proof. We only observe that, since the center of the B-algebra A can equivalently be defined as the algebra of (A, A)-bimodule endomorphisms of A, to say that i_M factorizes through the center of A is to say that $A \in {}_B\Im_B$, and since clearly $B \in {}_B\Im_B$, i_M is a morphism in ${}_B\Im_B$.

■

Combining the last result and Proposition 8.4((iii)), we obtain the following result:

8.12. Proposition. *In the situation of Theorem 8.11, if $i_M : B \to \mathcal{E}_M$ is pure as either left or right B-module, then* (8.1) *and* (8.2) *are both adjoint equivalences.*

As a special case of the Theorem 8.11, we obtain the following result of J. -P. Olivier [41] and A. Joyal and M. Tierney (unpublished, but see [35]) (note that, after Proposition 8.10, we are at liberty to drop the adjective "left" and "right" when we talk about purity of morphisms lying in ${}_B\Im_B$.):

8.13. Theorem. *Let $i : B \to A$ be a morphism of commutative rings. Then i is an effective descent morphism for modules if and only if it is pure as a morphism of B-modules.*

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