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EXPONENTIABILITY IN LAX SLICES OF **TOP**

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Abstract.

We consider exponentiable objects in lax slices of **Top** with respect to the specialization order (and its opposite) on a base space B. We begin by showing that the lax slice over B has binary products which are preserved by the forgetful functor to **Top** if and only if B is a meet (respective, join) semilattice in **Top**, and go on to characterize exponentiability over a complete Alexandrov space B.

1. Introduction

Let **Top** denote the category of topological space and continuous maps. Recall that $b \leq c$ in the *specialization order* on a T_0 space B, if whenever U is open in B and $b \in U$, then $c \in U$.

A *lax slice* of **Top** is a category of the form $\mathbf{Top}\diagup\!\!\nearrow B$ defined as follows for any fixed T_0 space B. Objects of $\mathbf{Top}\diagup\!\!\nearrow B$ are continuous maps $p\colon X \longrightarrow B$ and morphisms are triangles

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & \arrow[d, phantom, "\leq"] & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

which commute up to the specialization order on B, i.e., $px \leq qfx$, for all $x \in X$, or equivalently, $f(p^{-1}U) \subseteq q^{-1}U$, for all U open in B.

In [5], Funk used the lax slice over the Sierpinski space **2** to study "homotopy of marked spheres" by identifying $\mathbf{Top}\diagup\!\!\nearrow\mathbf{2}$ with the category whose objects are pairs (X, X_U), where X_U is an open subset of a topological space X, and morphisms $f\colon (X, X_U) \longrightarrow (Y, Y_U)$ are continuous maps $f\colon X \longrightarrow Y$ such that $f(X_U) \subseteq Y_U$. In the preliminaries, he shows that this category has products given by

$$(X, X_U) \times (Y, Y_U) = (X \times Y, X_U \times Y_U)$$

and he characterizes exponentiable objects as follows. A pair (Y, Y_U) is exponentiable if and only if Y is exponentiable in Top and the set

$$Z_U^Y = \{\sigma\colon Y \longrightarrow Z | \sigma(Y_U) \subseteq Z_U\}$$

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is open in the corresponding function space Z^Y, for all pairs (Z, Z_U).

Several natural questions arise. Is there an intrinsic characterization of these exponentiable pairs, i.e., one that depends only on (Y, Y_U) and not directly on the function spaces Z^Y? Are there other lax slices $\mathbf{Top}\nearrow B$ in which the product is preserved by the forgetful functor to **Top**? If so, what are their exponentiable objects? These results would be of interest since lax slices can be used to study other configurations of open subsets of topological spaces. Finally, to study diagrams of closed sets, one can consider the analogous questions for the op-lax slices $\mathbf{Top}\swarrow B$, whose morphisms are diagrams that commute up to $\geq$. In particular, $f\colon X \longrightarrow Y$ is a morphism

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p"'] & \geq & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

if and only if $p^{-1}U \supseteq f^{-1}q^{-1}U$, for all U open in B if and only if $p^{-1}F \subseteq f^{-1}q^{-1}F$, for all F closed in B if and only if $f(p^{-1}F) \subseteq q^{-1}F$, for all F closed in B. For example, $\mathbf{Top}\swarrow\mathbf{2}$ can be identified with the category of pairs (X, X_F), where X_F is a closed subset of X, and morphisms are continuous maps $f\colon X \longrightarrow Y$ such that $f(X_F) \subseteq Y_F$.

The paper proceeds as follows. After a brief review of exponentiability in section two and the introduction of background material in sections three and four, these questions are answered for lax and op-lax slices in sections five and six, respectively.

2. Preliminaries

Let $\mathbf{T}$ be a category with binary products. Recall that an object Y is called *exponentiable* if the functor $- \times Y\colon \mathbf{T} \longrightarrow \mathbf{T}$ has a right adjoint, often denoted by $(\ )^Y$. The category $\mathbf{T}$ is called *cartesian closed* if every object is exponentiable.

When $\mathbf{T} = \mathbf{Top}$, taking X to be a one-point space in the natural bijection

$$\mathbf{Top}(X \times Y, Z) \cong \mathbf{Top}(X, Z^Y)$$

one sees that Z^Y can be identified with the set $\mathbf{Top}(Y, Z)$ of continuous maps from Y to Z, and so the question of exponentiability becomes one of finding suitable topologies on the function spaces Z^Y.

The first exponentiability results in print appear to be the 1945 paper [3] of Fox, where it was shown that a separable metric space is exponentiable if and only if it is locally compact. A complete characterization was achieved by Day and Kelly in their 1970 paper [2], when they proved the functor $- \times Y\colon \mathbf{Top} \longrightarrow \mathbf{Top}$ preserves quotients if and only if the lattice $\mathcal{O}(Y)$ of open sets of Y is a continuous lattice, in the sense of Scott [9], and that a Hausdorff space satisfies this property if and only if it is locally compact. Since $- \times Y$ preserves coproducts in any case, preservation of quotients is necessary and sufficient for exponentiability in **Top**. Note that although sufficiency follows from Freyd's

Special Adjoint Functor Theorem [4], one can construct the exponentials as follows, and use the continuity of $\mathcal{O}(Y)$ to establish the exponentiability adjunction.

Recall that a subset H of $\mathcal{O}(Y)$ is called *Scott-open* if it is upward closed, i.e., $U \in H$ and $U \subseteq V$ implies $V \in H$, and it satisfies the finite union property, $\bigcup_{\alpha\in A} U_\alpha \in H \Rightarrow \bigcup_{\alpha\in F} U_\alpha \in H$, for some finite $F \subseteq A$. Then the sets of the form

$$\langle H, W\rangle = \{\sigma \in Z^Y | \sigma^{-1}(W) \in H\}$$

where H is Scott-open in Y and W is open in Z, generate a topology on Z^Y which agrees with the compact-open topology when Y is locally compact. Moreover, $(\ )^Y : \mathbf{Top} \longrightarrow \mathbf{Top}$ is always a functor, and it is the right adjoint to $- \times Y$ when $\mathcal{O}(Y)$ is a continuous lattice.

A characterization of exponentiability in the strict slices $\mathbf{Top}/B$ first appeared in [7]. Since the product is given by the fiber product there, function spaces are formed relative to the fibers of spaces. Here we need only consider the full function spaces Z^Y, since products in the lax slices we consider below are preserved by the forgetful functor. Note that this excludes all T_1 spaces B for, since the specialization order on any T_1 space B is discrete, the lax and strict slices over B coincide, and so products are clearly not preserved by the forgetful functor, unless B consists of a single point. However, exponentiability in lax slices over T_1 spaces is completely understood as it is just that of the strict slices characterized in [7]. For more on exponentiability in $\mathbf{Top}/B$ and other strict slice categories, the reader is also referred to [8].

Finally, Johnstone's book [6] is a good source for background on posets and lattices, including the specialization order and Scott topology.

3. Products in Lax Slices

In this section, we consider the existence of binary products in $\mathbf{Top}\nearrow B$ and $\mathbf{Top}\swarrow B$. This can be achieved simultaneously if we let $\leq$ denote an arbitrary partial order on B and work in the category $\mathbf{Top}/_{\leq}B$ whose objects are continuous map $p: X \longrightarrow B$ and morphisms are triangles which commute up to $\leq$.

Recall that for a partially-ordered set B, the upward closed subsets form a topology known as the *Alexandrov topology* on B. These spaces, known as *Alexandrov spaces*, are precisely those in which arbitrary intersections of open sets are open [1]. In particular, every *finite* T_0 space is Alexandrov.

Now, if B has the Alexandrov topology relative to $\leq$, or more generally, $\leq$ is the specialization order on B, then $\mathbf{Top}\nearrow B = \mathbf{Top}/_{\leq}B$ and $\mathbf{Top}\swarrow B = \mathbf{Top}/_{\geq}B$.

3.1. Proposition. *The category* $\mathbf{Top}/_{\leq}B$ *has binary products preserved by the forgetful functor to* $\mathbf{Top}$ *if and only if* B *is a topological* $\wedge$*-semilattice, i.e.,* B *is a* $\wedge$*-semilattice with order* $\leq$ *and the function* $\wedge: B \times B \longrightarrow B$ *is continuous.*

Proof. Suppose $\mathbf{Top}/_{\leq}B$ has binary products preserved by the forgetful functor, and let $\wedge = \mathrm{id}_B \times \mathrm{id}_B$, where $\mathrm{id}_B\colon B \longrightarrow B$ is the identity map. Then $\wedge\colon B \times B \longrightarrow B$ is continuous, and the projections $\pi_i\colon B \times B \longrightarrow B$ are morphisms in $\mathbf{Top}/_{\leq}B$, i.e., $\wedge \leq \mathrm{id}_B \circ \pi_i$. Thus, for all $b_1, b_2 \in B$, $b_1 \wedge b_2 \leq b_i$, for $i = 1, 2$. To see that $b_1 \wedge b_2$ is the meet in B, suppose that $a \leq b_i$, for $i = 1, 2$. Since

$$\begin{tikzcd} 1 \arrow[rr, "b_i"] \arrow[dr, "a"'] & \arrow[d, phantom, "\leq"] & B \arrow[dl, "\mathrm{id}_B"] \\ & B & \end{tikzcd}$$

are morphisms in $\mathbf{Top}\nearrow B$, the unique map $\langle b_1, b_2\rangle\colon 1 \longrightarrow B \times B$ satisfies

$$\begin{tikzcd} 1 \arrow[rr, "\langle b_1,b_2\rangle"] \arrow[dr, "a"'] & \arrow[d, phantom, "\leq"] & B \times B \arrow[dl, "\wedge"] \\ & B & \end{tikzcd}$$

and so $a \leq b_1 \wedge b_2$, as required.

For the converse, given $p\colon X \longrightarrow B$ and $q\colon Y \longrightarrow B$, consider

$$p \wedge q\colon X \times Y \xrightarrow{p\times q} B \times B \xrightarrow{\wedge} B$$

Then, given a pair of morphisms,

$$\begin{tikzcd} X \arrow[dr, "p"'] & Z \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "r"] \arrow[dl, phantom, "\geq"] \arrow[dr, phantom, "\leq"] & Y \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

one easily shows that the induced map $\langle f, g\rangle\colon Z \longrightarrow X \times Y$ is a morphism in $\mathbf{Top}/_{\leq}B$, i.e.,

$$\begin{tikzcd} Z \arrow[rr, "\langle f,g\rangle"] \arrow[dr, "r"'] & \arrow[d, phantom, "\leq"] & X \times Y \arrow[dl, "p\wedge q"] \\ & B & \end{tikzcd}$$

and so $p \wedge q$ is the desired product. ∎

One can show that the forgetful functor $\mathbf{Top}/_{\leq}B \longrightarrow \mathbf{Top}$ has a left adjoint if and only if B has a bottom element. Since right adjoints preserve products, we get:

3.2. Corollary. *The following are equivalent for a partial order $\leq$ with a bottom element on a space B.*

(a) $\mathbf{Top}/_{\leq}B$ has binary products.

(b) $\mathbf{Top}/_{\leq}B$ *has binary products preserved by the forgetful functor.*

(c) B *is a topological* $\wedge$*-semilattice (relative to* $\leq$*).*

Taking $\leq$ to be the specialization order and its opposite we get, based on the remark at the beginning of the section, we get the following two corollaries:

3.3. Corollary. *The following are equivalent for a* T_0 *space* B *with a bottom element relative to the specialization order* $\leq$*.*

(a) $\mathbf{Top}{\nearrow}B$ *has binary products.*

(b) $\mathbf{Top}{\nearrow}B$ *has binary products preserved by the forgetful functor.*

(c) B *is a topological* $\wedge$*-semilattice (relative to* $\leq$*).*

3.4. Corollary. *The following are equivalent for a* T_0 *space* B *with a top element relative to the specialization order* $\leq$*.*

(a) $\mathbf{Top}{\swarrow}B$ *has binary products.*

(b) $\mathbf{Top}{\swarrow}B$ *has binary products preserved by the forgetful functor.*

(c) B *is a topological* $\vee$*-semilattice (relative to* $\leq$*).*

Note that we could have stated a more general version of Corollary 3.3 (respectively, 3.4) omitting both (a) and the bottom (respectively, top) assumption, but every finite $\wedge$-semilattice has a bottom (respectively, $\vee$-semilattice has a top), and the infinite ones we consider below are all complete.

4. $\mathbf{Top}{\nearrow}B$ and Open Families

In this section, we introduce a category of B-indexed families of open sets that will be used to characterize exponentiable objects in $\mathbf{Top}{\nearrow}B$.

Given $x \in X$ and $p\colon X \longrightarrow B$, we get a morphism

$$\begin{tikzcd} 1 \arrow[rr, "x"] \arrow[dr, "b"'] & \arrow[d, phantom, "\leq"] & X \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

whenever $b \leq px$. If B is a finite T_0 space, or more generally, any Alexandrov space, then the sets $X_{\hat{b}} = p^{-1}(\uparrow b)$ are open in X. It will be useful to have conditions on this family of open sets that will make it possible to retrieve the original map $p\colon X \longrightarrow B$.

Suppose B is *any* poset with a bottom element $\perp$, and let $\mathbf{OFam}_B$ denote the category whose objects are families $\{X_{\hat{b}}\}_{b \in B}$ of open subsets of a space $X_{\hat{\perp}}$ satisfying

(O1) $b \leq c \Rightarrow X_{\hat{b}} \supseteq X_{\hat{c}}$

(O2) $\forall x \in X_{\hat{\perp}}$, the subposet $\{b \in B | x \in X_{\hat{b}}\}$ of B has a top element $\top_x$

Morphisms of are continuous maps $f\colon X_{\hat{\perp}} \longrightarrow Y_{\hat{\perp}}$ such that $f(X_{\hat{b}}) \subseteq Y_{\hat{b}}$, for all $b \in B$.

For example, if $\mathbf{2}$ denotes the Sierpinski space, then $\mathbf{OFam_2}$ is the category of pairs (X, X_U) used by Funk [5] in his study of marked spheres. If $B = \mathbf{2} \times \mathbf{2}$, then objects of $\mathbf{OFam}_B$ are given by a space X together with a pair of open subsets $X_{(0,1)}$ and $X_{(1,0)}$, and morphisms are continuous maps $f\colon X \longrightarrow Y$ such that $f(X_{(0,1)}) \subseteq Y_{(0,1)}$ and $f(X_{(1,0)}) \subseteq Y_{(1,0)}$. If V is the subposet

$$\begin{tikzcd} (0,1) & & (1,0) \\ & (0,0) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

then $\mathbf{OFam}_V$ is the full subcategory in which the open subsets are disjoint.

We will see, in Proposition 4.1 below, that if B has suprema of nonempty subsets which are bounded above, then (O2) is equivalent to

$$(O2^\star)\quad \forall \{b_\alpha\} \subseteq B, \quad \bigcap X_{\hat{b}_\alpha} = \begin{cases} X_{\hat{b}} & \text{if } \forall \{b_\alpha\} \subseteq B, b = \vee b_\alpha \text{ exists} \\ \emptyset & \text{otherwise} \end{cases}$$

To establish an isomorphism $\mathbf{Top}\nearrow B \cong \mathbf{OFam}_B$, we will consider an alternate view of these families as certain relations, and obtained a bijection on objects from properties that hold in this more general setting. Although the bijection can be established directly in a straightforward manner, this approach sheds light on the role of condition (O2) above.

Given $\{X_{\hat{b}}\}$, let $R = \{(x, b) | x \in X_{\hat{b}}\}$. Using (O1) and the fact that each $X_{\hat{b}}$ is upward closed, being an open subset of $X_{\hat{\perp}}$, it follows that R is an order-ideal (i.e., a downward closed subset) of $X_{\hat{\perp}}^{\mathrm{op}} \times B$.

Let **Ord** denote the category whose objects are posets and morphisms from X to B are order ideals of $X^{\mathrm{op}} \times B$. The identity on X is the ideal $I_X = \{(x, y) | x \geq y\}$ and composition is given by the usual composition of relations.

Following the custom for relations, morphisms will be denoted $X \longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\; B$. Every order-preserving map $p\colon X \longrightarrow B$ gives rise to a morphism $R_p\colon X \longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\; B$ with a right adjoint $R_p^\sharp\colon B \longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\; X$ defined by

$$R_p = \{(x, b) | px \geq b\} \quad \text{and} \quad R_p^\sharp = \{(b, x) | b \geq px\}$$

The following proposition shows that these are precisely the maps of the bicategory **Ord**, in the sense of [10]. Thus, to establish a relationship between families $\{X_{\hat{b}}\}$ and continuous maps $p\colon X \longrightarrow B$, we would like to determine which families give rise to maps in **Ord**.

Given $R\colon X \longrightarrow\!\!\!\!\!\!\!\!\!+\;\;\; B$ and $x \in X$, let $R[x] = \{b \in B | (x, b) \in R\}$. Then $R[x]$ is a downward closed subset of B and $x \leq y \Rightarrow R[x] \subseteq R[y]$. Similarly, let $R[b] = \{x \in X | (x, b) \in R\}$. Then $R[b]$ is an upward closed subset of X and $b \leq c \Rightarrow R[b] \supseteq R[c]$. Note that if $p\colon X \longrightarrow B$ is an order-preserving map of posets, then $R_p[b] = p^{-1}(\uparrow b)$, and if R is the order ideal related to a family $\{X_{\hat{b}}\}$, then $R[b] = X_{\hat{b}}$ and $R[x] = \{b \in B | x \in X_{\hat{b}}\}$. Note that the latter says that condition (O2) is precisely (c) given below.

4.1. Proposition. *The following are equivalent for* $R: X \nrightarrow B$ *in* **Ord**.

(a) $R = R_p$, *for some order-preserving map* $p: X \longrightarrow B$.

(b) R *has a right adjoint.*

(c) $R[x]$ *has a top element* $\top_x$, *for all* $x \in X$.

Moreover, if B *has suprema of nonempty subsets which are bounded above, then (a) through (c) are equivalent to*

(d) $\forall \{b_\alpha\} \subseteq B, \quad \bigcap R[b_\alpha] = \begin{cases} R[b] & \textit{if } b = \vee b_\alpha \textit{ exists in } B \\ \emptyset & \textit{otherwise} \end{cases}$

Proof. First, $(a) \Rightarrow (b)$ since $R_p \dashv R_p^\sharp$.

For $(b) \Rightarrow (c)$, suppose $R \dashv S$. Since $(x, x) \in I_X$ and $I_X \subseteq S \circ R$, there exists $\top_x \in B$ such that $(x, \top_x) \in R$ and $(\top_x, x) \in S$. Given $b \in R[x]$, since $(\top_x, x) \in S$ and $(x, b) \in R$ implies $(\top_x, b) \in R \circ S \subseteq I_B$, it follows that $\top_x \geq b$, as desired.

For $(c) \Rightarrow (a)$, define $p: X \longrightarrow B$ by $px = \top_x$. Then p is order preserving since $x \leq y \Rightarrow R[x] \subseteq R[y] \Rightarrow \top_x \leq \top_y$, and $R = R_p$ since

$$(x, b) \in R \iff \top_x \geq b \iff px \geq b \iff (x, b) \in R_p$$

where the first "$\Leftarrow$" holds since $(x, \top_x) \in R$ and R is downward closed in $X^{\mathrm{op}} \times B$, and the third "$\Rightarrow$" from the definition of R_p.

It remains to show that $(c) \iff (d)$, when B satisfies the suprema assumption. Suppose (c) holds and $\bigcap R[b_\alpha] \neq \emptyset$, say $x \in \bigcap R[b_\alpha]$. Then for all α, we know $b_\alpha \leq \top_x$ since $b_\alpha \in R[x]$, and so $\vee b_\alpha$ exists and $\vee b_\alpha \leq \top_x$. Since $R[\vee b_\alpha] \supseteq R[\top_x]$, it follows that $x \in R[\vee b_\alpha]$. Thus, $\bigcap R[b_\alpha] \subseteq R[\vee b_\alpha]$, and so $\bigcap R[b_\alpha] = R[\vee b_\alpha]$, as desired.

Conversely, suppose R satisfies (d), and let $x \in X$. Then $t_x = \vee R[x]$ exists by (d), since $\bigcap \{R[b] | b \in R[x]\} \neq \emptyset$, as $x \in R[b]$ for all $b \in R[x]$, and $t_x \in R[x]$ since

$$x \in \bigcap \{R[b] | b \in R[x]\} = R[t_x]$$

Therefore, $R[x]$ has a top element, to complete the proof. ∎

4.2. Corollary. *If* B *is an Alexandrov space on a poset with* $\perp$, *then* $\mathbf{Top} \nearrow B \cong \mathbf{OFam}_B$.

Proof. By Proposition 4.1 and its preceding remarks, families $\{X_{\hat{b}}\}$ correspond to order-preserving maps $p: X_{\hat{\perp}} \longrightarrow B$ such that $R_p[b]$ is open in $X_{\hat{\perp}}$. Since B has the Alexandrov topology and $R_p[b] = p^{-1}(\uparrow b)$, it follows that these are precisely the continuous maps $p: X_{\hat{\perp}} \longrightarrow B$, thus defining a bijection between the objects of $\mathbf{OFam}_B$ and $\mathbf{Top} \nearrow B$. For morphisms, given $p: X \longrightarrow B$ and $q: Y \longrightarrow B$, a continuous map $f: X \longrightarrow Y$ is in $\mathbf{Top} \nearrow B$ if and only if $f\,(p^{-1}(\uparrow b)) \subseteq q^{-1}(\uparrow b)$, for all $b \in B$ if and only if $f(X_{\hat{b}}) \subseteq Y_{\hat{b}}$, for all $b \in B$, and it follows that $\mathbf{Top} \nearrow B \cong \mathbf{OFam}_B$. ∎

4.3. Corollary. *If B is a poset with $\perp$, then $\mathbf{OFam}_B$ has finite products if and only if B is a $\wedge$-semilattice. Moreover, $\{X_{\hat{b}}\} \times \{Y_{\hat{b}}\} = \{X_{\hat{b}} \times Y_{\hat{b}}\}$.*

Proof. This follows directly from Corollaries 3.3 and 4.2. ∎

5. Exponentiability in $\mathbf{Top}/\!\!\nearrow B$

In this section, we characterize exponentiable objects of $\mathbf{Top}/\!\!\nearrow B$ when B is an Alexandrov space on a complete lattice. We begin with the first of four conditions (E1)–(E4) necessary for exponentiability using the presence of $\perp$ to get information about exponentials when they exist. We prove the following in the more general setting of $\mathbf{Top}/_{\leq}B$, so that we can use it when we consider exponentiability in $\mathbf{Top}_{\swarrow}\!/B$.

5.1. Lemma. *If B is a topological $\wedge$-semilattice with $\perp$ and $q\colon Y \longrightarrow B$ is exponentiable in $\mathbf{Top}/_{\leq}B$, then*

(E1) *Y is exponentiable in* **Top**.

Moreover, if $r\colon Z \longrightarrow B$, then $r^q\colon Z^Y \longrightarrow B$ and

$$(r^q)^{-1}(\uparrow b) = \{\sigma \in Z^Y | b \wedge q \leq r\sigma\}$$

for all $b \in B$.

Proof. Given $r\colon Z \longrightarrow B$, let $r^q\colon [q,r] \longrightarrow B$ denote the exponential in $\mathbf{Top}/_{\leq}B$. Since there are bijections

$$X \times Y \xrightarrow{f} Z \longleftrightarrow \begin{tikzcd} X \times Y \arrow[rr, "f"] \arrow[dr, "\perp\wedge q=\perp"'] & \leq & Z \arrow[dl, "r"] \\ & B & \end{tikzcd} \longleftrightarrow \begin{tikzcd} X \arrow[rr, "\hat{f}"] \arrow[dr, "\perp"'] & \leq & {[q,r]} \arrow[dl, "r^q"] \\ & B & \end{tikzcd} \longleftrightarrow X \xrightarrow{\hat{f}} [q,r]$$

which are natural in X, it follows that the functors $X \longmapsto \mathbf{Top}(X \times Y, Z)$ are representable, and so Y is exponentiable in **Top**, $Z^Y = [q,r]$, and $r^q\colon Z^Y \longrightarrow B$. Moreover, since

$$\begin{tikzcd} 1 \arrow[rr, "\sigma"] \arrow[dr, "b"'] & \leq & Z^Y \arrow[dl, "r^q"] \\ & B & \end{tikzcd} \iff \begin{tikzcd} 1 \times Y \arrow[rr, "\sigma"] \arrow[dr, "b\wedge q"'] & \leq & Z \arrow[dl, "r"] \\ & B & \end{tikzcd}$$

it follows that $(r^q)^{-1}(\uparrow b) = \{\sigma \in Z^Y | b \wedge q \leq r\sigma\}$. ∎

Following Corollary 4.2, to define exponentials the r^q when they exist in $\mathbf{Top}\nearrow B$, we will present Z^Y as a family $\{Z^Y_{\hat{b}}\}$ of open subsets of Z^Y. For simplicity, the reference to q and r has been omitted from the notation. Lemma 5.1 tells us how to define $\{Z^Y_{\hat{b}}\}$, when B is any T_0 space with $\wedge$ and $\perp$ relative to specialization order $\leq$.

5.2. Definition. *Given a topological $\wedge$-semilattice B and continuous maps $q\colon Y \longrightarrow B$ and $r\colon Z \longrightarrow B$, let $Z^Y_{\hat{b}} = \{\sigma \in Z^Y | b \wedge q \leq r\sigma\}$.*

In view of the following proposition, it will be necessary to assume that B has all finite infima (i.e., the empty infimum $\top$, as well) if $\mathbf{Top}\nearrow B$ is to have any exponentiable objects. This says that if one is interested in exponentiability in $\mathbf{OFam}_V$, when V is the subspace of $\mathbf{2} \times \mathbf{2}$ considered in section four, then it is necessary to expand to the larger category $\mathbf{OFam}_B$, where $B = \mathbf{2} \times \mathbf{2}$.

5.3. Proposition. *If B is a topological $\wedge$-semilattice such that id_B^q exists in $\mathbf{Top}/_{\leq}B$, for some $q\colon Y \longrightarrow B$, then B has a top element $\top$.*

Proof. Since

$$\begin{tikzcd} 1 \times Y \arrow[rr, "q"] \arrow[dr, "b\wedge q"'] & \leq & B \arrow[dl, "\mathrm{id}_B"] \\ & B & \end{tikzcd} \Longrightarrow \begin{tikzcd} 1 \arrow[rr, "q"] \arrow[dr, "b"'] & \leq & B^Y \arrow[dl, "\mathrm{id}_B^q"] \\ & B & \end{tikzcd}$$

for all b, it follows that $\mathrm{id}_B^q(q)$ is the top element of B. ∎

To show that $Z^Y_{\hat{b}}$ is open in Z^Y, we will assume that B is a complete lattice (an assumption that is already necessary in the finite case), and use Proposition 4.1(d), or equivalently, condition $(O2^\star)$, to prove that $\{Z^Y_{\hat{b}}\}$ is in $\mathbf{OFam}_B$. However, to establish criteria for $(O2^\star)$, it is not necessary to assume that B is complete, only that B has finite infima.

5.4. Lemma. *Suppose B is a $\wedge$-semilattice with $\top$, $c = \bigvee b_\alpha$ exists in B, and B has the Alexandrov topology. Then $q\colon Y \longrightarrow B$ satisfies $\bigcap Z^Y_{\hat{b}_\alpha} = Z^Y_{\hat{c}}$, for all $Z \longrightarrow B$ if and only if*

(E2) $\bigvee(b_\alpha \wedge qy)$ *exists and equals* $(\bigvee b_\alpha) \wedge qy$*, for all $y \in Y$.*

Proof. Suppose $\bigcap Z^Y_{\hat{b}_\alpha} = Z^Y_{\hat{c}}$, for all $r\colon Z \longrightarrow B$. To prove (E2), it suffices to fix $y_0 \in Y$ and show that

$$(\bigvee b_\alpha) \wedge qy_0 \leq d \iff b_\alpha \wedge qy_0 \leq d,\ \forall \alpha$$

for all $d \in B$.

Let $d \in B$, and define $\sigma_d\colon Y \longrightarrow B$ by

$$\sigma_d(y) = \begin{cases} \top & \text{if } qy \not\leq qy_0 \\ d & \text{if } qy \leq qy_0 \end{cases}$$

Then σ_d is clearly continuous, and $b \wedge qy_0 \leq d$ implies $b \wedge qy \leq \sigma_d y$, for all $y \in Y$, since if $qy \not\leq qy_0$, then $\sigma_d y = \top$, and if $qy \leq qy_0$, then $b \wedge qy \leq b \wedge qy_0 \leq d = \sigma_d y$. Thus,

$$b \wedge qy_0 \leq d \iff b \wedge q \leq \sigma_d$$

for all $b \in B$. Taking $r = \mathrm{id}_B$, since $\bigcap B^Y_{\hat{b}_\alpha} = B^Y_{\hat{c}}$, where

$$B^Y_{\hat{b}} = \{\sigma \in B^Y | b \wedge q \leq \sigma\}$$

we see that $(\bigvee b_\alpha) \wedge qy_0 \leq d$ iff $(\bigvee b_\alpha) \wedge q \leq \sigma_d$ iff $c \wedge q \leq \sigma_d$ iff $\sigma_d \in B^Y_{\hat{c}}$ iff $\sigma_d \in \bigcap B^Y_{\hat{b}_\alpha}$ iff $\sigma_d \in B^Y_{\hat{b}_\alpha}$, for all α, iff $b_\alpha \wedge q \leq \sigma_d$, for all α, iff $b_\alpha \wedge qy_0 \leq d$, for all α.

Conversely, suppose (E2) holds and $r: Z \longrightarrow B$. Then $\sigma \in \bigcap Z^Y_{\hat{b}_\alpha}$ iff $\sigma \in Z^Y_{\hat{b}_\alpha}$, for all α, iff $b_\alpha \wedge q \leq r\sigma$, for all α, iff $b_\alpha \wedge qy \leq r\sigma y$, for all $y \in Y$ and for all α, iff $\bigvee(b_\alpha \wedge qy) \leq r\sigma y$, for all $y \in Y$ and for all α, iff $(\bigvee b_\alpha) \wedge qy \leq r\sigma y$, for all $y \in Y$ iff $(\bigvee b_\alpha) \wedge q \leq r\sigma$ iff $c \wedge q \leq r\sigma$ iff $\sigma \in Z^Y_{\hat{c}}$, and so $\bigcap Z^Y_{\hat{b}_\alpha} = Z^Y_{\hat{c}}$. ■

Note that if B is a complete lattice, then (E2) holds if and only if $- \wedge qy$ preserves suprema, for all $y \in Y$ if and only if qy is exponentiable in B, i.e., $- \wedge qy: B \longrightarrow B$ has a right adjoint, for all $y \in Y$.

The following alternate description of $Z^Y_{\hat{b}}$ will be useful in determining when these sets are open in Z^Y.

5.5. Proposition. *If $q: Y \longrightarrow B$ and $r: Z \longrightarrow B$ are continuous, then*

$$Z^Y_{\hat{b}} = \{\sigma \in Z^Y | \sigma(Y_{\hat{c}}) \subseteq Z_{\hat{c}}, \forall c \leq b\} = \{\sigma \in Z^Y | Y_{\hat{c}} \subseteq \sigma^{-1}(Z_{\hat{c}}), \forall c \leq b\}$$

where $Y_{\hat{c}} = q^{-1}(\uparrow c)$ and $Z_{\hat{c}} = r^{-1}(\uparrow c)$.

Proof. Suppose $b \wedge q \leq r\sigma$, $c \leq b$, and $y \in Y_{\hat{c}}$. Then $c \leq qy$, and so $c = c \wedge qy \leq b \wedge qy \leq r\sigma y$, and it follows that $\sigma y \in Z_{\hat{c}}$. Thus, $\sigma(Y_{\hat{c}}) \subseteq Z_{\hat{c}}$. Conversely, suppose $\sigma(Y_{\hat{c}}) \subseteq Z_{\hat{c}}$, for all $c \leq b$. To show $b \wedge qy \leq r\sigma y$, for all y, let $y \in Y$ and take $c = b \wedge qy$. Then $y \in Y_{\hat{c}}$. Since $c \leq b$, we know $\sigma(Y_{\hat{c}}) \subseteq Z_{\hat{c}}$, and so $\sigma y \in Z_{\hat{c}}$. Thus, $b \wedge qy = c \leq r\sigma y$, as desired. ■

Suppose B is a complete lattice and $A \subseteq B$, and let $\langle A \rangle$ denote the finite (including empty) $\wedge$ and $\vee$ closure of A in B.

5.6. Lemma. *If B is a complete lattice with the Alexandrov topology and $q: Y \longrightarrow B$ is continuous, then $Z^Y_{\hat{b}}$ is open in Z^Y, for all $b \in B$, $r: Z \longrightarrow B$, if and only if*

(E3) *$q(Y)$ is finite, and*

(E4) *$Y_{\hat{c}}$ is compact, for all $\vee$-irreducible c in $\langle q(Y) \rangle$.*

Proof. Suppose q satisfies (E3) and (E4). Let J denote the set of $\vee$-irreducible elements of $\langle q(Y)\rangle$, and for each $c \in J$, let $H_c = \{U \in \mathcal{O}(Y) | Y_{\hat{c}} \subseteq U\}$. Then

$$\langle H_c, Z_{\hat{c}}\rangle = \{\sigma \in Z^Y | Y_{\hat{c}} \subseteq \sigma^{-1}(Z_{\hat{c}})\}$$

is open in Z^Y, since H_c is Scott-open (by compactness of $Y_{\hat{c}}$) and $Z_{\hat{c}}$ is open in Z. To show that $Z^Y_{\hat{b}}$ is open in Z^Y, since J is finite, it suffices to show that

$$Z^Y_{\hat{b}} = \bigcap_{c \in J \cap \downarrow b} \langle H_c, Z_{\hat{c}}\rangle$$

But, $Z^Y_{\hat{b}} = \bigcap\limits_{c \le b} \langle H_c, Z_{\hat{c}}\rangle$ by Proposition 5.5, and so it suffices to show that

$$\bigcap_{c \in J \cap \downarrow b} \langle H_c, Z_{\hat{c}}\rangle \subseteq \bigcap_{c \le b} \langle H_c, Z_{\hat{c}}\rangle$$

So, suppose $\sigma \in \langle H_c, Z_{\hat{c}}\rangle$, for all $c \in J \cap \downarrow b$, and $c \le b$. If $c \notin \langle q(Y)\rangle$, then $Y_{\hat{c}} \subseteq \sigma^{-1}(Z_{\hat{c}})$, since $Y_{\hat{c}} = \emptyset$. If $c \in \langle q(Y)\rangle$, then writing $c = \vee\{c_j \in J | c_j \le c\}$, we see that

$$Y_{\hat{c}} = \cap Y_{\hat{c}_j} \subseteq \cap \sigma^{-1}(Z_{\hat{c}_j}) = \sigma^{-1}(\cap Z_{\hat{c}_j}) = \sigma^{-1}(Z_{\hat{c}})$$

Thus, $\sigma \in \langle H_c, Z_{\hat{c}}\rangle$, for all $c \le b$, as desired.

Conversely, suppose $Z^Y_{\hat{b}}$ is open in Z^Y, for all $b \in B$, $r\colon Z \longrightarrow B$. To show that $q(Y)$ is finite, take $r = \mathrm{id}_B$ and $b = \top$. Then $q \in B^Y_{\hat{\top}}$, since $\top \wedge q \le q$, and so there exists $H_1, \ldots, H_n$ Scott-open in $\mathcal{O}(Y)$ and finite sets $F_1, \ldots, F_n \subseteq B$ such that

$$q \in \langle H_1, \uparrow F_1\rangle \cap \cdots \cap \langle H_n, \uparrow F_n\rangle \subseteq B^Y_{\hat{\top}}$$

To show that $q(Y)$ is finite, we will show that $q(Y) \subseteq S$, where

$$S = \langle F_1 \cup \cdots \cup F_n\rangle$$

Let $\sigma = fq$, where $f\colon B \longrightarrow B$ is defined by $f(b) = \vee(S \cap \downarrow b)$. Then $f \le \mathrm{id}_B$ and f is order-preserving, hence, continuous. Also, $f(\uparrow s) \subseteq \uparrow s$, for all $s \in S$, since $s \le b \Rightarrow fs \le fb \Rightarrow s \le fb$, as $fs = s$, and so $\uparrow s \subseteq f^{-1}(\uparrow s)$ which gives

$$q^{-1}(\uparrow s) \subseteq q^{-1} f^{-1}(\uparrow s) = \sigma^{-1}(\uparrow s)$$

for all $s \in S$. Thus, for all i, $q^{-1}(\uparrow F_i) \subseteq \sigma^{-1}(\uparrow F_i)$, and it follows that $\sigma^{-1}(\uparrow F_i) \in H_i$, since H_i is Scott-open and $q \in \langle H_i, \uparrow F_i\rangle \Rightarrow q^{-1}(\uparrow F_i) \in H_i$, and so,

$$\sigma \in \bigcap \langle H_i, \uparrow F_i\rangle \subseteq B^Y_{\hat{\top}} \;\Rightarrow\; \top \wedge q \le \sigma \;\Rightarrow\; q \le \sigma \;\Rightarrow\; q \le fq$$

Since $f \le \mathrm{id}_B$, it follows that $q = fq$, and so $qy = fqy \in S$, for all $y \in Y$. Therefore, $q(Y) \subseteq S$, as desired.

Let c be $\vee$-irreducible in $\langle q(Y)\rangle$. To show that $Y_{\hat{c}}$ is compact, we will show that $\uparrow Y_{\hat{c}}$ is Scott-open in $\mathcal{O}(Y)$ by showing $\uparrow Y_{\hat{c}} = \pi^{-1}(B^Y_{\hat{c}})$, for some $\pi\colon \mathcal{O}(Y) \longrightarrow B^Y$. Take $b < c$ maximal in $\langle q(Y)\rangle$, and let $f\colon \mathbf{2} \longrightarrow B$ be the continuous map given by $f(0) = b$ and $f(1) = c$. Then define π to be the composite of $f^Y\colon \mathbf{2}^Y \longrightarrow B^Y$ and the homeomorphism $\mathcal{O}(Y) \cong \mathbf{2}^Y$ given by $U \longmapsto \chi_U$, the characteristic function of U. Then

$$U \in \pi^{-1}(B^Y_{\hat{c}}) \iff c \wedge q \le f\chi_U \iff c \wedge qy \le f\chi_U(y)$$

for all $y \in Y$. Note that if $y \notin Y_{\hat{c}}$, then $c \wedge qy < c$, and so $c \wedge qy \le b$, since $b \le (c\wedge qy)\vee b \le c$, c is $\vee$-irreducible and b is maximal in $\langle q(Y)\rangle \cap \downarrow c$. Thus,

$$c \wedge qy \le \begin{cases} b & \text{if } y \notin Y_{\hat{c}} \\ c & \text{if } y \in Y_{\hat{c}} \end{cases}$$

Since

$$f\chi_U(y) = \begin{cases} b & \text{if } y \notin U \\ c & \text{if } y \in U \end{cases}$$

it follows that

$$U \in \pi^{-1}(B^Y_{\hat{c}}) \iff Y_{\hat{c}} \subseteq U \iff U \in \uparrow Y_{\hat{c}}$$

and so $\uparrow Y_{\hat{c}} = \pi^{-1}(B^Y_{\hat{c}})$, as desired. ∎

5.7. Theorem. *Suppose B is a complete lattice with the Alexandrov topology. Then $q\colon Y \longrightarrow B$ is exponentiable in* **Top**$\nearrow B$ *if and only if*

(E1) *Y is exponentiable in* **Top**,

(E2) *qy is exponentiable in B, i.e., $- \wedge qy$ preserves $\bigvee$, for all $y \in Y$,*

(E3) *$q(Y)$ is finite, and*

(E4) *$Y_{\hat{c}}$ is compact, for all $\vee$-irreducible c in $\langle q(Y)\rangle$.*

Proof. Suppose $q\colon Y \longrightarrow B$ is exponentiable. Applying Lemma 5.1, we see that (E1) holds, and for all $r\colon Z \longrightarrow B$, we have $r^q\colon Z^Y \longrightarrow B$ and the sets $Z^Y_{\hat{b}}$ defined in 5.2 agree with those in the family of open sets corresponding to r^q via Corollary 4.2. Then (E2) follows from Lemma 5.4, since the family satisfies $(O2^\star)$, and properties (E3) and (E4) follow from Lemma 5.6, since $Z^Y_{\hat{b}}$ is open in Z^Y, for all $b \in B$.

Conversely, suppose q satisfies (E1)–(E4). Given $r\colon Z \longrightarrow B$, define

$$Z^Y_{\hat{b}} = \{\sigma \in Z^Y | b \wedge q \le r\sigma\}$$

Then $Z^Y_{\hat{\perp}} = Z^Y$ and $\{Z^Y_{\hat{b}}\}$ is in $\mathbf{OFam}_B$ by Lemmas 5.4 and 5.6, since (E2)–(E4) hold, and so we get a continuous map $r^q\colon Z^Y \longrightarrow B$. Since Y is exponentiable, we have the usual bijection between $f\colon X \times Y \longrightarrow Z$ and $\hat{f}\colon X \longrightarrow Z^Y$. It remains to show that

$$\begin{tikzcd} X \times Y \arrow[rr, "f"] \arrow[dr, "p\wedge q"'] & \le & Z \arrow[dl, "r"] \\ & B & \end{tikzcd} \iff \begin{tikzcd} X \arrow[rr, "\hat{f}"] \arrow[dr, "p"'] & \le & Z^Y \arrow[dl, "q^r"] \\ & B & \end{tikzcd}$$

But, $p \wedge q \leq rf$ iff $px \wedge qy \leq rf(x,y)$, for all (x,y), iff $px \wedge q \leq r\hat{f}x$, for all x, iff $\hat{f}x \in Z^Y_{\widehat{px}}$, for all x, iff $px \leq r^q(\hat{f}x)$, for all x, iff $p \leq r^q\hat{f}$, where the third "iff" follows from the definition of $Z^Y_{\hat{b}}$ and the fourth from the definition of r^q via the family $\{Z^Y_{\hat{b}}\}$. Therefore, $q\colon Y \longrightarrow B$ is exponentiable in $\mathbf{Top}\nearrow B$. ∎

Since every finite T_0 space is an Alexandrov space, applying Propositions 3.3 and 5.3, we get:

5.8. Corollary. *Suppose B is a finite T_0 space with a bottom element $\perp$ relative to the specialization order. Then $\mathbf{Top}\nearrow B$ has finite products if and only if B is a topological $\wedge$-semilattice. Moreover, $q\colon Y \longrightarrow B$ is exponentiable in $\mathbf{Top}\nearrow B$ if and only if*

(a) B has a top element $\top$,

(b) Y is exponentiable in $\mathbf{Top}$,

(c) qy is exponentiable in B, i.e., $- \wedge qy$ preserves $\bigvee$, for all $y \in Y$, and

(d) $Y_{\hat{c}}$ is compact, for all $\vee$-irreducible $c \in B$.

Taking B to be the Sierpinski space $\mathbf{2}$, the desired intrinsic characterization of exponentials alluded to in the introduction follows.

5.9. Corollary. *(Y, Y_U) is exponentiable in $\mathbf{Top}\nearrow\mathbf{2}$ if and only if Y_U is compact and Y is exponentiable in $\mathbf{Top}$.*

6. Exponentiability in $\mathbf{Top}\swarrow B$

To characterize exponentiable objects in $\mathbf{Top}\swarrow B$, we define a category of B-indexed families of closed sets, and proceed as before. Note that we consider closed rather than open sets here since the *opposite* of the specialization order gives us information about the sets $\downarrow b$.

Given $x \in X$ and $p\colon X \longrightarrow B$, we get a morphism

$$\begin{tikzcd} 1 \arrow[rr, "x"] \arrow[dr, "b"'] & \arrow[d, phantom, "\geq"] & X \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

whenever $b \geq px$. If B is a finite T_0 space, or more generally, any Alexandrov space, then the sets $X_{\check{b}} = p^{-1}(\downarrow b)$ are closed in X. As before, we would like to have conditions on the family of sets $X_{\check{b}}$ that will make it possible to retrieve the original map $p\colon X \longrightarrow B$, but we need an additional assumption here since the sets $B \backslash \downarrow b$ do not necessarily generate the topology on B (in the sense that one cannot represent every open set as a union of a *finite* intersection of these open sets).

Suppose B is *any* poset with a top element $\top$, and let $\mathbf{CFam}_B$ denote the category whose objects are families $\{X_{\check{b}}\}_{b\in B}$ of closed subsets of a space $X_{\check{\top}}$ satisfying

(C1) $b \leq c \Rightarrow X_{\check{b}} \subseteq X_{\check{c}}$

(C2) $\forall x \in X_{\check{\top}}$, the subposet $\{b \in B | x \in X_{\check{b}}\}$ of B has a bottom element $\perp_x$

Morphisms of $\mathbf{CFam}_B$ are continuous maps $f: X_{\check{\top}} \longrightarrow Y_{\check{\top}}$ such that $f(X_{\check{b}}) \subseteq Y_{\check{b}}$, for all $b \in B$.

For example, if $B = \mathbf{2} \times \mathbf{2}$, then objects of $\mathbf{CFam}_B$ are given by a space X together with a pair of closed subsets $X_{(0,1)}$ and $X_{(1,0)}$, and morphisms $f: X \longrightarrow Y$ such that $f(X_{(0,1)}) \subseteq Y_{(0,1)}$ and $f(X_{(1,0)}) \subseteq Y_{(1,0)}$.

Given $\{X_{\check{b}}\}$, let $R^\sharp = \{(b, x) | x \in X_{\check{b}}\}$. Using (C1) and the fact that each $X_{\check{b}}$ is downward closed, being a closed subset of $X_{\check{\top}}$, it follows that $R^\sharp: B \longrightarrow\!\!\!\!\!\!\!\!\!\!+\;\;\; X$ is a relation in **Ord**. Thus, by the dual of Proposition 4.1, if B has infima of nonempty subsets which are bounded below, then (C2) is equivalent to

$$(C2^\star) \quad \bigcap X_{\check{b}_\alpha} = \begin{cases} X_{\check{b}} & \text{if } b = \wedge b_\alpha \text{ exists} \\ \\ \emptyset & \text{otherwise} \end{cases} \qquad \forall \{b_\alpha\} \subseteq B$$

If $p: X \longrightarrow B$ is an order-preserving map of posets, then $R_p^\sharp[b] = p^{-1}(\downarrow b)$, and if $R^\sharp$ is the order ideal related to a family $\{X_{\check{b}}\}$, then $R^\sharp[b] = X_{\check{b}}$ and $R^\sharp[x] = \{b \in B | x \in X_{\check{b}}\}$. The latter says that condition (C2) is precisely the dual of Proposition 4.1(c), and we get the following corollary to Proposition 4.1.

6.1. Corollary. *If B a poset with $\top$ and the sets $B \backslash \downarrow b$ generate the Alexandrov topology on B, then* $\mathbf{Top}_{\swarrow}/B \cong \mathbf{CFam}_B$.

Proof. By the dual of Proposition 4.1, families $\{X_{\check{b}}\}$ correspond to order-preserving maps $p: X_{\check{\top}} \longrightarrow B$ such that $R_p^\sharp[b]$ is closed in $X_{\check{\top}}$. Since the sets $B \backslash \downarrow b$ generate the topology on B and $R_p^\sharp[b] = p^{-1}(\downarrow b)$, it follows that these are precisely the continuous maps $p: X_{\check{\top}} \longrightarrow B$, thus defining a bijection between the objects of $\mathbf{CFam}_B$ and $\mathbf{Top}_{\swarrow}/B$. For morphisms, given $p: X \longrightarrow B$ and $q: Y \longrightarrow B$, a map $f: X \longrightarrow Y$ is in $\mathbf{Top}_{\swarrow}/B$ if and only if

$$f\left(p^{-1}(\downarrow b)\right) \subseteq q^{-1}(\downarrow b), \quad \text{for all } b \in B$$

if and only if $f(X_{\check{b}}) \subseteq Y_{\check{b}}$, for all $b \in B$, and so $\mathbf{Top}_{\swarrow}/B \cong \mathbf{CFam}_B$. ■

6.2. Corollary. *If B is as in Corollary 6.1, then* $\mathbf{CFam}_B$ *has finite products if and only if B is a $\vee$-semilattice. Moreover, $\{X_{\check{b}}\} \times \{Y_{\check{b}}\} = \{X_{\check{b}} \times Y_{\check{b}}\}$.*

Proof. This follows directly from Corollaries 3.4 and 6.1. ■

Following Corollary 6.1, to define exponentials r^q when they exist in $\mathbf{Top}_{\swarrow}/B$, we will present Z^Y as a family $\{Z_{\check{b}}^Y\}$ of closed subsets of Z^Y. The dual of Lemma 5.1 tells us how to define $\{Z_{\check{b}}^Y\}$, when B is any T_0 space with $\vee$ and $\top$ relative to specialization order $\leq$.

6.3. Definition. *Given a topological $\vee$-semilattice B and continuous maps $q: Y \longrightarrow B$ and $r: Z \longrightarrow B$, let $Z_{\check{b}}^Y = \{\sigma \in Z^Y | b \vee q \geq r\sigma\}$.*

6.4. Theorem. *Suppose B is a complete lattice and the sets $B \setminus {\downarrow} b$ generate the Alexandrov topology on B. Then $q: Y \longrightarrow B$ is exponentiable in $\mathbf{Top}\swarrow B$ if and only if*

(E1) *Y is exponentiable in* **Top** *and*

(E2) *qy is exponentiable in B^{op}, i.e., $- \vee qy$ preserves $\bigwedge$, for all $y \in Y$.*

Proof. Suppose $q: Y \longrightarrow B$ is exponentiable in $\mathbf{Top} \swarrow B$. Applying the dual of Lemma 5.1, we see that (E1) holds, and for all $r: Z \longrightarrow B$, we have $r^q: Z^Y \longrightarrow B$ and the sets $Z^Y_{\check{b}}$ defined in 6.3 agree with those in the family of closed sets corresponding to r^q via Corollary 6.1. In addition, (E2) follows from the dual of Lemma 5.4, since the family satisfies $(C2^\star)$.

Conversely, suppose q satisfies (E1) and (E2). Given $r: Z \longrightarrow B$, define

$$Z^Y_{\check{b}} = \{\sigma \in Z^Y | b \vee q \geq r\sigma\}$$

Then $Z^Y_{\check{\top}} = Z^Y$ and $\{Z^Y_{\check{b}}\}$ satisfies (C1) and (C2) by definition and the dual of Lemmas 5.4. To see that $Z^Y_{\check{b}}$ is closed in Z^Y, we will show that its complement is open. Since

$$Z^Y_{\check{b}} = \{\sigma \in Z^Y | Y_{\check{c}} \subseteq \sigma^{-1}(Z_{\check{c}}), \forall c \geq b\}$$

by the dual of Proposition 5.5, it follows that

$$Z^Y \setminus Z^Y_{\check{b}} = \bigcup_{b \leq c} \langle H_c, Z \setminus Z_{\check{c}} \rangle$$

where $H_c = \{U \in \mathcal{O}(Y) | U \not\subseteq Y \setminus Y_{\check{c}}\}$. Since H_c is Scott-open, each $\langle H_c, Z \setminus Z_{\check{c}} \rangle$ is open in Z^Y, and so $Z^Y \setminus Z^Y_{\check{b}}$ is open, as desired. Finally, the proof that

$$\begin{tikzcd} X \times Y \arrow[rr, "f"] \arrow[dr, "p \vee q"'] & \arrow[d, phantom, "\geq"] & Z \arrow[dl, "r"] \\ & B & \end{tikzcd} \iff \begin{tikzcd} X \arrow[rr, "\hat{f}"] \arrow[dr, "p"'] & \arrow[d, phantom, "\geq"] & Z^Y \arrow[dl, "q^r"] \\ & B & \end{tikzcd}$$

is dual to that of Theorem 5.7. Therefore, $q: Y \longrightarrow B$ is exponentiable in $\mathbf{Top}\nearrow B$. ∎

Since every finite T_0 space is an Alexandrov space, applying Proposition 3.4, and the dual of Proposition 5.3, we get:

6.5. Corollary. *Suppose B is a finite T_0 space with a top element $\top$ relative to the specialization order. Then $\mathbf{Top}\swarrow B$ has finite products if and only if B is a topological $\vee$-semilattice. Moreover, $q: Y \longrightarrow B$ is exponentiable in $\mathbf{Top}\swarrow B$ if and only if*

(a) B has a bottom element $\perp$,

(b) Y is exponentiable in **Top***, and*

(c) qy is exponentiable in B^{op}, i.e., $-\vee qy$ preserves $\bigwedge$, for all $y \in Y$.

Thus, if B is a finite T_0 space which is a co-Heyting algebra relative to the specialization order (so that $-\vee b$ preserves $\bigwedge$, for all $b \in B$), then $q\colon Y \longrightarrow B$ is exponentiable in $\mathbf{Top}\swarrow B$ if and only if Y is exponentiable in **Top**. There are also infinite spaces satisfying the latter, for example, take B to be the negative integers with the natural order and a bottom adjoined. In fact:

6.6. Corollary. *Suppose B^{op} is a spatial locale. Then the following are equivalent for the Alexandrov space on the poset B.*

(a) B is Noetherian

(b) $q\colon Y \longrightarrow B$ is exponentiable in $\mathbf{Top}\swarrow B$ $\iff$ Y is exponentiable in **Top**

Proof. Suppose B^{op} is a spatial locale and B is Noetherian. Then every element is exponentiable in B^{op}. Since

$$\uparrow c = \bigcap_{b \in B \setminus \uparrow c} B \setminus \downarrow b$$

to apply Theorem 6.4, it suffices to show that the set of maximal elements of $B \setminus \uparrow c$ is finite. Since B^{op} is a spatial, we know $B^{\mathrm{op}} = \mathcal{O}(X)$, for some sober space X, and so it suffices to show that set $\mathcal{M}$ of minimal (nonempty) elements of $\{U \in \mathcal{O}(X) | U \not\subseteq V\}$ is finite, for all $V \in \mathcal{O}(X)$.

Suppose $U_1, U_2, \ldots$ are distinct elements of $\mathcal{M}$. Since $U_1 \not\subseteq V$, there exists $x_1 \in U_1$ such that $x_1 \notin V$. Then $x_1 \notin U_2$, since $U_1 \cap U_2 \subseteq V$ (by minimality of $U_2 \not\subseteq V$). Since $U_2 \not\subseteq V$, there exists $x_2 \in U_2$ such that $x_2 \notin V$. Then $x_1, x_2 \notin U_3$, since $U_1 \cap U_3, U_2 \cap U_3 \subseteq V$ (by minimality of $U_3 \not\subseteq V$). Continuing we get $x_1, x_2, \ldots \notin V$ such that $x_1, \ldots x_{n-1} \notin U_n$ and $x_n \in U_n$, and so

$$x_n \notin \overline{\{x_1, x_2, \ldots x_{n-1}\}}$$

giving a descending sequence

$$X \setminus \overline{\{x_1\}} \supseteq X \setminus \overline{\{x_1, x_2\}} \supseteq \ldots \supseteq X \setminus \overline{\{x_1, x_2, \ldots x_n\}} \supseteq \ldots$$

of proper subsets, contradicting that $B = \mathcal{O}(X)^{\mathrm{op}}$ is Noetherian.

For the converse, we will show that if B is not Noetherian, say

$$y_1 < y_2 < y_3 \ldots$$

in B, then $Y = \{y_1, y_2, \ldots\}$ is exponentiable in **Top** but the inclusion $j\colon Y \longrightarrow B$ is not exponentiable in $\mathbf{Top}\swarrow B$. Exponentiability of Y is clear since the Alexandrov topology on Y is locally compact. So, assume for contradiction that j is exponentiable in $\mathbf{Top}\swarrow B$, and consider $\mathrm{id}_B^j\colon B^Y \longrightarrow B$.

Define $\sigma\colon Y \longrightarrow B$ by $\sigma(y_n) = y_{n+1}$, and let $b = \mathrm{id}_B^j(\sigma)$. Then $\sigma \in (\mathrm{id}_B^j)^{-1}(\uparrow b)$, and so there exist Scott-open sets $H_1, \ldots, H_k$ of $\mathcal{O}(Y)$ and open sets $W_1, \ldots, W_k$ of B such that

$$\sigma \in \langle H_1, W_1 \rangle \cap \ldots \cap \langle H_k, W_k \rangle \subseteq (\mathrm{id}_B^j)^{-1}(\uparrow b)$$

and $\langle H_i, W_i\rangle \neq B^Y$, for any i. Note that $\sigma^{-1}(W_i) \neq \emptyset$ for otherwise $\emptyset \in H_i$ making $\langle H_i, W_i\rangle = B^Y$.

First, we show that we can assume

$$H_1 = H_{n_1} = \{U \in \mathcal{O}(Y) | y_{n_1} \in U\} \quad \text{and} \quad W_1 = \uparrow y_{n_1+1}$$

where n_1 is the least integer such that $y_{n_1} \in \sigma^{-1}(W_1)$. Since $\sigma^{-1}(W_1)$ is open in $Y = \{y_1, y_2, \ldots\}$, the minimality of n_1 implies that $\sigma^{-1}(W_1) = \uparrow y_{n_1}$. Since $\sigma(y_{n_1}) = y_{n_1+1}$, we know $\uparrow y_{n_1+1} \subseteq W_1$ and $\sigma \in \langle H_{n_1}, \uparrow y_{n_1+1}\rangle$. Thus, $\langle H_{n_1}, \uparrow y_{n_1+1}\rangle \subseteq \langle H_1, W_1\rangle$, since $\tau^{-1}(\uparrow y_{n_1+1}) \in H_{n_1} \Rightarrow y_{n_1} \in \tau^{-1}(\uparrow y_{n_1+1}) \Rightarrow y_{n_1} \in \tau^{-1}(W_1) \Rightarrow \uparrow y_{n_1} \subseteq \tau^{-1}(W_1) \Rightarrow \sigma^{-1}(W_1) \subseteq \tau^{-1}(W_1) \Rightarrow \tau^{-1}(W_1) \in H_1$, as H_1 is Scott-open and $\sigma^{-1}(W_1) \in H_1$. Similarly, for $i = 2, \ldots, k$, we can assume

$$H_i = H_{n_i} = \{U \in \mathcal{O}(Y) | y_{n_i} \in U\} \quad \text{and} \quad W_1 = \uparrow y_{n_i+1}$$

for some $y_i \in Y$.

Thus, without loss of generality, we have $n_1 < \ldots < n_k$ such that

$$\sigma \in \langle H_{n_1}, \uparrow y_{n_1+1}\rangle \cap \ldots \cap \langle H_{n_k}, \uparrow y_{n_k+1}\rangle \subseteq (\mathrm{id}_B^j)^{-1}(\uparrow b)$$

Let $N = n_k$ and define $\tau\colon Y \longrightarrow B$ by

$$\tau(y_n) = \begin{cases} y_{n+1} & \text{if } n \leq N \\ y_{N+1} & \text{otherwise} \end{cases}$$

Then τ is continuous and

$$\tau \in \langle H_{n_1}, \uparrow y_{n_1+1}\rangle \cap \ldots \cap \langle H_{n_k}, \uparrow y_{n_k+1}\rangle \subseteq (\mathrm{id}_B^j)^{-1}(\uparrow b)$$

and so $\mathrm{id}_B^j(\tau) \geq b$. Now, $y_{N+1} \vee j \geq \tau$, since $y_{N+1} \vee j(y_n) \geq y_{N+1} \geq \tau(y_n)$, for all n. Thus, $y_{N+1} \geq \mathrm{id}_B^j(\tau) \geq b$. Since $b = \mathrm{id}_B^j(\sigma)$, we know that $b \vee j \geq \sigma$, Thus, $b \vee j(y_{N+1}) \geq \sigma(y_{N+1})$, or equivalently, $b \vee y_{N+1} \geq y_{N+2}$. But, the latter contradicts $y_{N+1} < y_{N+2}$, since $y_{N+1} \geq b$ would imply $y_{N+1} = b \vee y_{N+1} \geq y_{N+2}$. Therefore, the inclusion j in B is not exponentiable in $\mathbf{Top}_{\swarrow}B$. ■

7. Concluding Remarks

As noted in the introduction, this work began as a search for an intrinsic characterization of exponentiability in $\mathbf{Top}\nearrow B$ and $\mathbf{Top}_{\swarrow}B$ when the category has finite products preserved by the forgetful functor and B is a finite T_0 space. It soon became apparent that the results could be extended to infinite Alexandrov spaces by adding an additional condition to obtain exponentiability in $\mathbf{Top}\nearrow B$ and assuming that B has the topology generated by the sets $B\backslash{\downarrow}b$ for $\mathbf{Top}_{\swarrow}B$. In the process of writing this paper, it became clear that some of the results for $\mathbf{Top}\nearrow B$ were dual to those in $\mathbf{Top}_{\swarrow}B$, and did not require a separate proof, if they were cast in $\mathbf{Top}/_{\leq}B$. Although the paper covers more than was initially intended, many new questions arise with the relaxation of the finiteness assumption and the introduction of $\mathbf{Top}/_{\leq}B$. Here are a few.

Does the characterization of exponentiability in $\mathbf{Top}_{\swarrow}B$ extend to all Alexandrov spaces without the assumption that the sets $B\backslash{\downarrow}b$ generate the topology on B? What can be said about exponentiability in $\mathbf{Top}/_{\leq}B$ when B does not have the Alexandrov topology on $\leq$, or when $\leq$ is not the specialization order or its opposite? What happens if we relax the assumption that the forgetful functor preserves products?

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