Theory and Applications of Categories, Vol. 16, No. 15, 2006, pp. 307–341.

STABLE MEET SEMILATTICE FIBRATIONS AND FREE RESTRICTION CATEGORIES

J.R.B. COCKETT AND XIUZHAN GUO

ABSTRACT. The construction of a free restriction category can be broken into two steps: the construction of a free stable semilattice fibration followed by the construction of a free restriction category for this fibration. Restriction categories produced from such fibrations are "unitary", in a sense which generalizes that from the theory of inverse semigroups. Characterization theorems for unitary restriction categories are derived. The paper ends with an explicit description of the free restriction category on a directed graph.

1. Introduction

A *restriction* on a category **C** is an assignment of a map $\overline{f} : X \to X$ to each map $f : X \to Y$. When a category has a restriction which satisfies the following four axioms:

[R.1] $f\overline{f} = f$ for all map f,

[R.2] $\overline{f}\,\overline{g} = \overline{g}\,\overline{f}$ whenever $\mathrm{dom}(f) = \mathrm{dom}(g)$,

[R.3] $\overline{g\overline{f}} = \overline{g}\,\overline{f}$ whenever $\mathrm{dom}(f) = \mathrm{dom}(g)$,

[R.4] $\overline{g}f = f\overline{gf}$ whenever $\mathrm{cod}(f) = \mathrm{dom}(g)$,

it is called a *restriction category* [3].

A functor $F : \mathbf{C} \to \mathbf{D}$ between two restriction categories is said to be a *restriction* functor if $F(\overline{f}) = \overline{F(f)}$ for any map f in **C**. Restriction categories and restriction functors form a category, denoted by $\mathbf{rCat}_0$. Clearly, there is a forgetful functor to the category of categories, $U_r : \mathbf{rCat}_0 \to \mathbf{Cat}_0$, which forgets restriction structure. U_r has a left adjoint F_r given by the free restriction category on a category which was described in [3].

Also in [3], quite separately, a way to freely add subobjects determined by a fibration was given. It was not realized at that time that the two constructions were actually related. One of the objectives of this paper is to fill out this relationship as, realigned in this way, the free construction factors though the construction which freely adds subobjects specified by a stable meet semilattice fibration.

Breaking the construction of the free restriction category down into two separate steps is useful: for example, this approach was key to understanding the construction of free

Both authors partially supported by NSERC, Canada.
Received by the editors 2005-04-01 and, in revised form, 2006-07-21.
Transmitted by S. Lack. Published on 2006-08-10.
2000 Mathematics Subject Classification: 18D99.
Key words and phrases: Restriction categories, fibrations, semigroups.

range restriction categories [2]. Furthermore, the intermediate step of adding subobjects is of independent interest and leads one to consider the question of how one may characterize the rather special restriction categories which arise from these constructions. This question, it turns out, is a generalization of a question which had already arisen in the study of inverse semigroups (see [8, 11]) for the categories which arise in this manner are "unitary" (or "proper") restriction categories.

This realization led us to seek characterization theorems analogous to those for unitary inverse semigroups. This is the content of the second section below which, incidentally, can be skipped by readers primarily interested in the free construction. While we do provide characterization theorems, any reader familiar with the theory for inverse semigroups will notice that we do not present a parallel to the McAlister triple and, in fact, we do not know whether there is an appropriate generalization of this notion. Instead we use, as our point of generalization, the analogue of an F-inverse semigroup (see [8, 11]) which we propose is a bounded unitary restriction category (see Subsection 3.23). We also found it very natural to cast the development in the language of fibrations which perspective, we feel, is also useful in explaining the significance of the original development of E-unitary inverse semigroups.

For further connections to semigroup theory, see the discussion in Manes [9]. In particular, there he points out that the untyped version of the restriction identities had already been considered in, for example, John Fountain's work [5] and, even more explicitly, in Jackson and Stokes's paper [7].

Restriction categories were introduced to model partial maps and in this regard they are complete in the sense that every restriction category occurs as a full subcategory of a partial map category. There is an extensive literature on partial map categories, see for example [4], [12], [13], and [14], however, much of this literature considers categories with more structure (e.g. partial products) than is being considered here.

That there are free restriction categories is of some interest especially as they have a relatively simple form. We end this paper by describing a particularly simple construction: the free restriction category on a graph. This bears a close relationship to the construction of the free inverse semigroup due to Munn [10]. Free restriction categories on graphs are completely decidable and, thus, allow one to determine what is and is not true in general restriction categories relatively easily.

In Section 2 we introduce stable meet semilattice fibrations and show that each restriction category has functorially associated to it such a fibration. Furthermore, we show how such a fibration can be used to construct a restriction category and that this construction provides a left adjoint to the previous construction. In the next section we characterize the restriction categories which arise from such fibrations and show that this adjunction is both monadic and comonadic. This leads into a discussion of unitary restriction categories. In the last section we note that every category gives rise freely to a stable meet semilattice fibration. Thus the category of stable meet semilattice fibrations acts as a stepping-stone between the category of restriction categories and the category of categories. The composite of the adjoint of this section and the second section produces the Cockett-Lack free restriction category functor.

2. Stable Meet Semilattice Fibrations

Recall that a *stable* homomorphism (that is a *pullback* preserving homomorphism) of meet semilattices preserves *binary meets* but crucially does not necessarily preserve the empty meet (i.e. the top element).

2.1. Definition. A stable meet semilattice fibration *is a fibration* $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ *in which each fiber is a meet semilattice and in which for each map* $f : X \to Y$*, the inverse image functor* $f^* : \delta_{\mathbf{X}}^{-1}(Y) \to \delta_{\mathbf{X}}^{-1}(X)$ *is a stable meet semilattice homomorphism.*

Stable meet semilattice fibrations $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ are equivalently, using the Grothendieck construction [1], those fibrations given by the indexed categories $\mathbf{X}^{\mathrm{op}} \to \mathbf{StabMSLat_0}$, where $\mathbf{StabMSLat_0}$ is the category of meet semilattices with stable homomorphisms.

2.2. Restriction Categories to Stable Meet Semilattice Fibrations. To each restriction category $\mathbf{C}$ one can associate a stable meet semilattice fibration $\partial : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$. The category $\mathbf{r}(\mathbf{C})$ has objects (X, e_X), where e_X is a restriction idempotent over X, namely, a map $e_X : X \to X$ satisfying $\overline{e_X} = e_X$, and maps $f : X \to Y$ such that $e_X = \overline{e_Y f} e_X$ as maps from (X, e_X) to (Y, e_Y). The obvious forgetful functor $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ is a stable meet semilattice fibration as shown in the following lemma:

2.3. Lemma. *If* $\mathbf{C}$ *is a restriction category, then the forgetful functor* $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ *is a stable meet semilattice fibration.*

Proof. Note that each fiber

$$\partial_{\mathbf{C}}^{-1}(X) = \{(X, e_X) \mid e_X : X \to X \text{ is a restriction idempotent on } X\}$$

is a meet semilattice with the order given by $(X, e_X) \le (X, e'_X) \Leftrightarrow e_X = e'_X e_X$, with the binary meet given by $(X, e_X) \wedge (X, e'_X) = (X, e_X e'_X)$, and with $(X, 1_X)$ as the top element.

For any map $f : X \to Y$ in $\mathbf{C}$ and any object $(Y, e_Y) \in \partial_{\mathbf{C}}^{-1}(Y)$, $f : (X, \overline{e_Y f}) \to (Y, e_Y)$ is a map of $\mathbf{r}(\mathbf{C})$ since $\overline{e_Y f}^2 = \overline{e_Y f}$. Moreover, it is straightforward to see that $f : (X, \overline{e_Y f}) \to (Y, e_Y)$ is the cartesian lifting of $f : X \to Y$ at (Y, e_Y). Hence $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ is a fibration.

Obviously, for any map $f : X \to Y$, $f^* : \partial_{\mathbf{C}}^{-1}(Y) \to \partial_{\mathbf{C}}^{-1}(X)$, sending (Y, e_Y) to $(X, \overline{e_Y f})$, the functor introduced by cartesian lifting, is a stable homomorphism since $\overline{e'_Y f}\;\overline{e_Y f} = \overline{e_Y f \overline{e_Y f}} = \overline{e'_Y \overline{e_Y f}} = \overline{e'_Y e_Y f}$. Hence $\partial_{\mathbf{C}} : \mathbf{r}(\mathbf{C}) \to \mathbf{C}$ is a stable meet semilattice fibration. ∎

The category $\mathbf{sFib_0}$ of stable meet semilattice fibrations may be formed as follows:

objects: stable meet semilattice fibrations: $(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$,

maps: a map from $(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$ to $(\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ is a pair (F, F'), where $F : \mathbf{X} \to \mathbf{Y}$ and $F' : \tilde{\mathbf{X}} \to \tilde{\mathbf{Y}}$ are functors such that

$$\begin{tikzcd} \tilde{\mathbf{X}} \arrow[r, "F'"] \arrow[d, "\delta_{\mathbf{X}}"'] & \tilde{\mathbf{Y}} \arrow[d, "\delta_{\mathbf{Y}}"] \\ \mathbf{X} \arrow[r, "F"] & \mathbf{Y} \end{tikzcd}$$

commutes and for any map $f : A \to B$ in $\mathbf{X}$ and any $\sigma, \sigma' \in \delta_{\mathbf{X}}^{-1}(B)$, the following conditions are satisfied:

[sfM.1] $F'(\top_{\delta_{\mathbf{X}}^{-1}(A)}) = \top_{\delta_{\mathbf{Y}}^{-1}(F(A))}$,

[sfM.2] $F'(\sigma \wedge \sigma') = F'(\sigma) \wedge F'(\sigma')$,

[sfM.3] $F'(f^*(\sigma)) = (F(f))^*(F'(\sigma))$,

composition: for any maps $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ and $(G, G') : (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y}) \to (\delta_{\mathbf{Z}} : \tilde{\mathbf{Z}} \to \mathbf{Z})$, $(G, G')(F, F') = (GF, G'F')$,

identities: $1_{(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})} = (1_{\mathbf{X}}, 1_{\tilde{\mathbf{X}}})$.

Note that each map $(F, F') : \delta_{\mathbf{X}} \to \delta_{\mathbf{Y}}$ is a morphism of fibrations in the sense of preserving Cartesian liftings, due to [**sfM.3**], but it also preserves finite limits on the fibers due to [**sfM.1**] and [**sfM.2**].

We can now define a functor $\mathcal{R} : \mathbf{rCat}_0 \to \mathbf{sFib}_0$ by setting $\mathcal{R}(\mathbf{X}) = \partial_{\mathbf{X}} : \mathbf{r}(\mathbf{X}) \to \mathbf{X}$. If $F : \mathbf{X} \to \mathbf{Y}$ is a restriction functor, then we have a functor $\mathbf{r}(F) : \mathbf{r}(\mathbf{X}) \to \mathbf{r}(\mathbf{Y})$ given by sending $f : (A, e_A) \to (B, e_B)$ to $F(f) : (F(A), F(e_A)) \to (F(B), F(e_B))$ and a commutative diagram

$$\begin{tikzcd} \mathbf{r}(\mathbf{X}) \arrow[r, "\mathbf{r}(F)"] \arrow[d, "\partial_{\mathbf{X}}"'] & \mathbf{r}(\mathbf{Y}) \arrow[d, "\partial_{\mathbf{Y}}"] \\ \mathbf{X} \arrow[r, "F"] & \mathbf{Y} \end{tikzcd}$$

2.4. Lemma. *$\mathcal{R} : \mathbf{rCat}_0 \to \mathbf{sFib}_0$, taking $F : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{rCat}_0$ to $(F, \mathbf{r}(F)) : (\partial_{\mathbf{X}} : \mathbf{r}(\mathbf{X}) \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$ in $\mathbf{sFib}_0$, is a functor.*

Proof. We have

$$\begin{aligned} \mathbf{r}(F)(\top_{\partial_{\mathbf{X}}^{-1}(A)}) &= \mathbf{r}(F)(A, 1_A) \\ &= (F(A), F(1_A)) \\ &= (F(A), 1_{F(A)}) \\ &= \top_{\partial_{\mathbf{Y}}^{-1}(F(A))}, \end{aligned}$$

$$\begin{aligned} \mathbf{r}(F)((B, e_B)(B, e'_B)) &= \mathbf{r}(F)(B, e_B e'_B) \\ &= (F(B), F(e_B) \cdot F(e'_B)) \\ &= (F(B), Fe_B)(F(B), F(e'_B)) \\ &= \mathbf{r}(F)(B, e_B) \cdot \mathbf{r}(F)(B, e'_B), \end{aligned}$$

and, for any map $f : A \to B$ in $\mathbf{X}$,

$$\begin{aligned}
\mathbf{r}(F)(f^*(B, e_B)) &= \mathbf{r}(F)(A, \overline{e_B f}) \\
&= (F(A), F(\overline{e_B f})) \\
&= (F(A), \overline{F(e_B) \cdot F(f)}) \\
&= (F(f))^*(F(B), F(e_B)) \\
&= (F(f))^*(\mathbf{r}(F)(B, e_B)).
\end{aligned}$$

Hence the conditions [**sfM.1**], [**sfM.2**], and [**sfM.3**] are satisfied and therefore

$$(F, \mathbf{r}(F)) : (\partial_{\mathbf{X}} : \mathbf{r}(\mathbf{X}) \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$$

is a map in $\mathbf{sFib}_0$.

For any restriction functors $F : \mathbf{X} \to \mathbf{Y}$ and $G : \mathbf{Y} \to \mathbf{Z}$, we have

$$\mathcal{R}(GF) = (GF, \mathbf{r}(GF)) = (GF, \mathbf{r}(G)\mathbf{r}(F)) = (G, \mathbf{r}(G))(F, \mathbf{r}(F)) = \mathcal{R}(G)\mathcal{R}(F).$$

Clearly, $\mathcal{R}(1_{\mathbf{X}}) = (1_{\mathbf{X}}, 1_{\mathbf{r}(\mathbf{X})})$. Hence $\mathcal{R}$ is a functor. ∎

2.5. Stable Meet Semilattice Fibrations to Restriction Categories. Suppose that $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ is a stable meet semilattice fibration, then we can form the category $\mathcal{S}(\delta_{\mathbf{X}})$ with:

objects: $A \in \mathrm{ob}\mathbf{X}$,

maps: $(f, \sigma) : A \to B$, where $f : A \to B$ is a map in $\mathbf{X}$ and $\sigma \in \delta_{\mathbf{X}}^{-1}(A)$ is such that $\sigma \leq f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})$,

composition: For any map $(f, \sigma_1) : A \to B$ and $(g, \sigma_2) : B \to C$, $(g, \sigma_2)(f, \sigma_1) = (gf, \sigma_1 \wedge f^*(\sigma_2))$,

identities: $1_A = (1_A, \top_{\delta_{\mathbf{X}}^{-1}(A)})$.

2.6. Remark. Let $f : A \to B$ be a map in $\mathbf{X}$. The condition that $\sigma \leq f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})$ holds for a map $(f, \sigma) : A \to B$ in $\mathcal{S}(\delta_{\mathbf{X}})$ is to ensure the identity law holds:

$$(1_B, \top_{\delta_{\mathbf{X}}^{-1}(B)})(f, \sigma) = (f, \sigma \wedge f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) = (f, \sigma)$$

and

$$(f, \sigma)(1_A, \top_{\delta_{\mathbf{X}}^{-1}(A)}) = (f, \top_{\delta_{\mathbf{X}}^{-1}(A)} \wedge (1_A)^*(\sigma)) = (f, \sigma).$$

2.7. Proposition. *For each stable meet semilattice fibration $\delta_{\mathbf{X}}$, $\mathcal{S}(\delta_{\mathbf{X}})$ is a restriction category with the restriction given by $\overline{(f, \sigma)} = (1_A, \sigma)$ for any map $(f, \sigma) : A \to B$.*

Proof. Clearly, $(1_A, \sigma) : A \to A$ is a map in $\mathcal{S}(\delta_{\mathbf{X}})$. So it suffices to check that $\overline{(f, \sigma)} = (1_A, \sigma)$ satisfies the four restriction axioms.

[R.1] For any map $(f, \sigma) : A \to B$,

$$(f, \sigma)\overline{(f, \sigma)} = (f, \sigma)(1_A, \sigma) = (f, \sigma \wedge 1_A^*(\sigma)) = (f, \sigma \wedge 1_{\delta_{\mathbf{X}}^{-1}(A)}(\sigma)) = (f, \sigma).$$

[R.2] For any maps $(f, \sigma_1) : A \to B$ and $(g, \sigma_2) : A \to C$,

$$\begin{aligned} \overline{(f, \sigma_1)}\ \overline{(g, \sigma_2)} &= (1_A, \sigma_1)(1_A, \sigma_2) \\ &= (1_A, \sigma_1 \wedge \sigma_2) \\ &= (1_A, \sigma_2 \wedge \sigma_1) \\ &= (1_A, \sigma_2)(1_A, \sigma_1) \\ &= \overline{(g, \sigma_2)}\ \overline{(f, \sigma_1)}. \end{aligned}$$

[R.3] For any maps $(f, \sigma_1) : A \to B$ and $(g, \sigma_2) : A \to C$,

$$\begin{aligned} \overline{(g, \sigma_2)\overline{(f, \sigma_1)}} &= \overline{(g, \sigma_2)(1_A, \sigma_1)} \\ &= \overline{(g, \sigma_1 \wedge 1_A^*(\sigma_2))} \\ &= \overline{(g, \sigma_1 \wedge \sigma_2)} \\ &= (1_A, \sigma_1 \wedge \sigma_2) \\ &= (1_A, \sigma_2)(1_A, \sigma_1) \\ &= \overline{(g, \sigma_2)}\ \overline{(f, \sigma_1)}. \end{aligned}$$

[R.4] For any maps $(f, \sigma_1) : A \to B$ and $(g, \sigma_2) : B \to C$,

$$\overline{(g, \sigma_2)}(f, \sigma_1) = (1_B, \sigma_2)(f, \sigma_1) = (f, \sigma_1 \wedge f^*(\sigma_2)),$$

and

$$\begin{aligned} (f, \sigma_1)\overline{(g, \sigma_2)(f, \sigma_1)} &= (f, \sigma_1)\overline{(gf, \sigma_1 \wedge f^*(\sigma_2))} \\ &= (f, \sigma_1)(1_A, \sigma_1 \wedge f^*(\sigma_2)) \\ &= (f, \sigma_1 \wedge f^*(\sigma_2) \wedge 1_A^*(\sigma_1)) \\ &= (f, \sigma_1 \wedge f^*(\sigma_2)). \end{aligned}$$

Hence $\overline{(g, \sigma_2)}(f, \sigma_1) = (f, \sigma_1)\overline{(g, \sigma_2)(f, \sigma_1)}$.

∎

2.8. EXAMPLES.

1. Suppose that $\mathbf{C}$ is any category, then $\mathcal{S}(1_{\mathbf{C}}) = \mathbf{C}$, which is the restriction category with the trivial restriction structure ($\overline{f} = 1_A$ for each map $f : A \to B$).

2. For each restriction category $\mathbf{C}$, $\mathcal{S}(\partial_{\mathbf{C}})$ is the restriction category with the same objects as $\mathbf{C}$ while a map from A to B in $\mathcal{S}(\partial_{\mathbf{C}})$ is a pair (f, e) with a map $f : A \to B$ in $\mathbf{C}$ and a restriction idempotent $e \leq \overline{f}$ over A in $\mathbf{C}$, the composition is given by $(g, e_B)(f, e_A) = (gf, e_A \wedge \overline{e_B f}) = (gf, \overline{e_B f} e_A)$ for any maps $(f, e_A) : A \to B$ and $(g, e_B) : B \to C$, and the restriction is given by $\overline{(f, e_A)} = (1_A, e_A)$. So $\mathcal{S}(\partial_{\mathbf{C}})$ and $\mathbf{C}$ are different in general.

3. Each meet semilattice L can be viewed as a category with elements of L as objects and with maps $l_1 \to l_2$ given by $l_1 \leq l_2$ so that $L : L \to \mathbf{1}$ is a stable meet semilattice fibration. In this case, $\mathcal{S}(L)$ is the one object category with maps $l : * \to *$ given by elements l of L and with the composition given by $l_1 l_2 = l_1 \wedge l_2$. Note that if we split the restriction idempotents of this category then the subcategory of total maps is precisely L.

Shortly we will see that $\mathcal{S}$, sending $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ to $\mathcal{S}(F, F') : \mathcal{S}(\delta_{\mathbf{X}}) \to \mathcal{S}(\delta_{\mathbf{Y}})$, is a functor, where $\mathcal{S}(F, F') : \mathcal{S}(\delta_{\mathbf{X}}) \to \mathcal{S}(\delta_{\mathbf{Y}})$ is given by sending $(f, \sigma) : A \to B$ to $(F(f), F'(\sigma)) : F(A) \to F(B)$, and, in fact, the left adjoint of $\mathcal{R}$. We shall establish this by exhibiting the universal property of $\mathcal{S}$ (see Lemma 2.11 below).

2.9. THE UNIVERSALITY OF THE CONSTRUCTION $\mathcal{S}$. Let $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ be a stable meet semilattice fibration, $\mathbf{Y}$ a restriction category, and $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$ a map in $\mathbf{sFib}_0$. For any object $A \in \mathbf{X}$ and any object $\sigma \in \delta_{\mathbf{X}}^{-1}(A)$, $F'(\sigma) \in \mathbf{r}(\mathbf{Y})$ can be written as $(F(A), e_\sigma^F)$, where e_σ^F is a restriction idempotent over $F(A)$ in $\mathbf{Y}$. In order to prove the universal property of $\mathcal{S}$, we need the following technical lemma.

2.10. LEMMA. *Let $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ be a stable meet semilattice fibration and $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$ a map in $\mathbf{sFib}_0$. Then*

(*i*) *For any object $A \in \mathbf{X}$ and any $\sigma_1, \sigma_2 \in \delta_{\mathbf{X}}^{-1}(A)$, $e^F_{\top_{\delta_{\mathbf{X}}^{-1}(A)}} = 1_{F(A)}$ and $e^F_{\sigma_1 \wedge \sigma_2} = e^F_\sigma \wedge e^F_{\sigma_2}$;*

(*ii*) *For any map $f : A \to B$ in $\mathbf{X}$ and $\sigma \in \delta_{\mathbf{X}}^{-1}(B)$, $e^F_{f^*(\sigma)} = \overline{e^F_\sigma(F(f))}$;*

(*iii*) *For any map $(f, \sigma) : A \to B$ in $\mathcal{S}(\delta_{\mathbf{X}})$, $e^F_\sigma = \overline{F(f)} e^F_\sigma$.*

PROOF.

(*i*) By [**sfM.1**], $(F(A), e^F_{\top_{\delta_{\mathbf{X}}^{-1}(A)}}) = F'(\top_{\delta_{\mathbf{X}}^{-1}(A)}) = \top_{\delta_{\mathbf{Y}}^{-1}(F(A))} = (F(A), 1_{F(A)})$. Hence $e^F_{\top_{\delta_{\mathbf{X}}^{-1}(A)}} = 1_{F(A)}$. By [**sfM.2**], $(F(A), e^F_{\sigma_1 \wedge \sigma_2}) = F'(\sigma_1 \wedge \sigma_2) = F'(\sigma_1) \wedge F'(\sigma_2) = (F(A), e^F_{\sigma_1})(F(A), e^F_{\sigma_1}) = (F(A), e^F_{\sigma_1} e_{\sigma_1})$. Hence $e^F_{\sigma_1 \wedge \sigma_2} = e^F_{\sigma_1} e^F_{\sigma_1}$.

(*ii*) By [**sfM.3**], $(F(A), e^F_{f^*(\sigma)}) = F'(f^*(\sigma)) = (F(f))^*(F'(\sigma)) = (F(f))^*(F(B), e^F_\sigma) = (F(A), \overline{e^F_\sigma(F(f))})$. Hence $e^F_{f^*(\sigma)} = \overline{e^F_\sigma(F(f))}$.

(*iii*) For any map $(f, \sigma) : A \to B$ in $\mathcal{S}(\delta_{\mathbf{X}})$, since $\sigma \leq f^*(\top_{\delta_{\mathbf{X}}^{-1}(A)})$, we have the following commutative diagram in $\tilde{\mathbf{X}}$:

$$\begin{tikzcd} & f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}) \arrow[dr, "\vartheta_f"] & \\ \sigma \arrow[ur, "\leq"] \arrow[rr] & & \top_{\delta_{\mathbf{X}}^{-1}(B)} \end{tikzcd}$$

where ϑ_f is the cartesian lifting of f at $\top_{\delta^{-1}(B)}$. Applying $\tilde{F}$, we have the following commutative diagram:

$$\begin{tikzcd} & (F(A), e_{f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})}) \arrow[dr, "F(f)"] & \\ (F(A), e_\sigma) \arrow[ur, "1_{F(A)}"] \arrow[rr, "F(f)"] & & (F(B), 1_{F(B)}) \end{tikzcd}$$

Hence $F(f) : (F(A), e^F_\sigma) \to (F(B), 1_{F(B)})$ is a map in $\mathbf{r}(\mathbf{Y})$ and therefore $e^F_\sigma = \overline{F(f)1_{F(B)}e^F_\sigma} = \overline{F(f)e^F_\sigma}$.

■

For a given stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$, we have a functor $I_{\mathbf{X}} : \mathbf{X} \to \mathcal{S}(\delta_{\mathbf{X}})$ by sending $f : A \to B$ to $(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) : A \to B$. Also, we can define $I_{\mathbf{X}}^{\delta_{\mathbf{X}}} : \tilde{\mathbf{X}} \to \mathbf{r}(\mathcal{S}(\delta_{\mathbf{X}}))$ by taking $f : U \to V$ to $(\delta_{\mathbf{X}}(f), (\delta_{\mathbf{X}}(f))^*(\top_{\delta_{\mathbf{X}}^{-1}(\delta_{\mathbf{X}}(V))})) : (\delta_{\mathbf{X}}(U), (1_{\delta_{\mathbf{X}}(U)}, U)) \to (\delta_{\mathbf{X}}(V), (1_{\delta_{\mathbf{X}}(V)}, V))$. Clearly, $\eta = (I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}})$ is a map from $(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$ to $(\partial_{\mathcal{S}(\delta_{\mathbf{X}})} : \mathbf{r}(\mathcal{S}(\delta_{\mathbf{X}})) \to \mathcal{S}(\delta_{\mathbf{X}}))$ in $\mathbf{sFib}_0$. This map turns out to be the unit of the adjunction $\mathcal{S} \dashv \mathcal{R}$.

2.11. Lemma. *Let $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ be a stable meet semilattice fibration and let $\mathbf{Y}$ be a restriction category. If $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\partial_{\mathbf{Y}} : \mathbf{r}(\mathbf{Y}) \to \mathbf{Y})$ is a map in $\mathbf{sFib}_0$, then there is a unique restriction functor $F^{\delta_{\mathbf{X}}} : \mathcal{S}(\delta_{\mathbf{X}}) \to \mathbf{Y}$ sending $(f, \sigma) : A \to B$ to $(F(f))e^F_\sigma : F(A) \to F(B)$ such that*

$$\begin{tikzcd} (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \arrow[r, "(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}})"] \arrow[dr, "(F,F')"'] & \mathcal{R}(\mathcal{S}(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})) \arrow[d, "\mathcal{R}(F^{\delta_{\mathbf{X}}})"] \\ & \mathcal{R}(\mathbf{Y}) \end{tikzcd} \qquad \begin{tikzcd} \mathcal{S}(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \arrow[d, dotted, "\exists!"', "F^{\delta_{\mathbf{X}}}"] \\ \mathbf{Y} \end{tikzcd}$$

commutes, where the restriction idempotent e^F_σ is determined by $F'(\sigma) = (F(A), e^F_\sigma) \in \mathbf{r}(\mathbf{Y})$.

PROOF. Clearly, by [**sfM.1**] and Lemma 2.10 (i),

$$F^{\delta_{\mathbf{X}}}(1_A) = F^{\delta_{\mathbf{X}}}(1_A, \top_{\delta_{\mathbf{X}}^{-1}(A)}) = F(1_A) \cdot e^F_{\top_{\delta_{\mathbf{X}}^{-1}(A)}} = F(1_A) \cdot 1_{F(A)} = 1_{F(A)}.$$

For any maps $(f, \sigma) : A \to B$ and $(g, \sigma') : B \to C$ in $\mathcal{S}(\delta_{\mathbf{X}})$, we have

$$\begin{array}{rcl} F^{\delta_{\mathbf{X}}}((g,\sigma')(f,\sigma)) &=& F^{\delta_{\mathbf{X}}}(gf, \sigma \wedge f^*(\sigma')) \\ &=& F(gf) \cdot e^F_{\sigma \wedge f^*(\sigma')} \\ &=& F(g) \cdot F(f) \cdot e^F_\sigma \wedge e^F_{f^*(\sigma')} \text{ (by Lemma 2.10 } (i)), \end{array}$$

and

$$\begin{array}{rcl} F^{\delta_{\mathbf{X}}}(g,\sigma')F^{\delta_{\mathbf{X}}}(f,\sigma) &=& (F(g) \cdot e^F_{\sigma'})(F(f) \cdot e^F_\sigma) \\ &=& (F(g) \cdot F(f))\overline{e^F_{\sigma'}(F(f))}e^F_\sigma \text{ (by [}\mathbf{R.4}]) \\ &=& (F(g) \cdot F(f))e^F_{f^*(\sigma')}e^F_\sigma \text{ (by Lemma 2.10 } (ii)). \end{array}$$

Hence $F^{\delta_{\mathbf{X}}}((g,\sigma')(f,\sigma)) = F^{\delta_{\mathbf{X}}}(g,\sigma')F^{\delta_{\mathbf{X}}}(f,\sigma)$. Therefore $F^{\delta_{\mathbf{X}}}$ is a functor. Since

$$\begin{array}{rcl} F^{\delta_{\mathbf{X}}}(\overline{(f,\sigma)}) &=& F^{\delta_{\mathbf{X}}}(1_A, \sigma) \\ &=& F(1_A) \cdot e^F_\sigma \\ &=& e^F_\sigma \\ &=& \overline{F(f)}e^F_\sigma \text{ (by Lemma 2.10}(iii)) \\ &=& \overline{(F(f))e^F_\sigma} \text{ (by [}\mathbf{R.3}]) \\ &=& \overline{F^{\delta_{\mathbf{X}}}(f,\sigma)}, \end{array}$$

$F^{\delta_{\mathbf{X}}}$ is a restriction functor. Obviously, $(F^{\delta_{\mathbf{X}}}, \mathbf{r}(F^{\delta_{\mathbf{X}}}))(I_{\mathbf{X}}, I^{\delta_{\mathbf{X}}}_{\mathbf{X}}) = (F, F')$. If $G : \mathcal{S}(\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to \mathbf{Y}$ is a restriction functor such that $(G, \mathbf{r}(G))(I_{\mathbf{X}}, I^{\delta_{\mathbf{X}}}_{\mathbf{X}}) = (F, F')$, then $GI_{\mathbf{X}} = F$ and $\mathbf{r}(G)I^{\delta_{\mathbf{X}}}_{\mathbf{X}} = F'$. Hence for any map $f : A \to B$ in $\mathbf{X}$, G must map A to $F(A)$ and must map $(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) : A \to B$ to $F(f) : F(A) \to F(B)$ and therefore $G(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) = F(f) = F^{\delta_{\mathbf{X}}}(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))$ since $e^F_{f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})} = \overline{F(f)}$. For any map $(f, \sigma) : A \to B$ in $\mathcal{S}(\delta_{\mathbf{X}})$,

$$(F(A), e^F_\sigma) = F'(\sigma) = \mathbf{r}(G)(I^{\delta_{\mathbf{X}}}_{\mathbf{X}})(\sigma) = \mathbf{r}(G)(A, (1_A, \sigma)) = (G(A), G(1_A, \sigma))$$

and so $G(1_A, \sigma) = e^F_\sigma$. Since $(f, \sigma) = (f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))(1_A, \sigma)$ and G is a restriction functor,

$$G(f, \sigma) = G(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))G(1_A, \sigma) = F(f)e_\sigma = F^{\delta_{\mathbf{X}}}(f, \sigma).$$

Then $G = F^{\delta_{\mathbf{X}}}$ and so the uniqueness of $F^{\delta_{\mathbf{X}}}$ follows. ∎

2.12. THEOREM. *There is an adjunction:* $\mathbf{rCat}_0 \underset{\mathcal{R}}{\overset{\mathcal{S}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{sFib}_0$ *with $\mathcal{R}$ and $\mathcal{S}$ faithful functors.*

PROOF. By Lemma 2.11, $\mathcal{S} \dashv \mathcal{R}$. Clearly, the counit ε of $\mathcal{S} \dashv \mathcal{R}$ is given by $\varepsilon_{\mathbf{C}} : \mathcal{S}(\mathcal{R}(\mathbf{C})) \to \mathbf{C}$ sending $(f, e_A) : A \to B$ to $fe_A : A \to B$, where e_A is a restriction idempotent such that $e_A \leq \overline{f}$, for each restriction category $\mathbf{C}$. We define $\lambda_{\mathbf{C}} : \mathbf{C} \to \mathcal{S}(\mathcal{R}(\mathbf{C}))$ by taking $f : A \to B$ to $(f, \overline{f}) : A \to B$. It is easy to check that $\lambda_{\mathbf{C}}$ is a functor such that $\varepsilon_{\mathbf{C}}\lambda_{\mathbf{C}} = 1_{\mathbf{C}}$ in $\mathbf{Cat}_0$. Hence $\varepsilon_{\mathbf{C}}$ is an epic in $\mathbf{rCat}_0$ and therefore is faithful.

On the other hand, the unit of $\mathcal{S} \dashv \mathcal{R}$ is given by $(I_{\mathbf{X}}, I_{\mathbf{X}}^{\delta_{\mathbf{X}}})$. Clearly, $I_{\mathbf{X}}$ is faithful and each $I_{\mathbf{X}}^{\delta_{\mathbf{X}}}|_{\delta_{\mathbf{X}}}$ is faithful. By Lemma 2.13 below, $\mathcal{S}$ is faithful. ■

2.13. LEMMA. *Let $(F, G) : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ be a morphism of fibrations:*

$$\begin{tikzcd} \tilde{\mathbf{X}} \arrow[r, "G"] \arrow[d, "\delta_{\mathbf{X}}"'] & \tilde{\mathbf{Y}} \arrow[d, "\delta_{\mathbf{Y}}"] \\ \mathbf{X} \arrow[r, "F"] & \mathbf{Y} \end{tikzcd}$$

If F is faithful and for each object X in $\mathbf{X}$, $G|_{\delta_{\mathbf{X}}^{-1}(X)} : \delta_{\mathbf{X}}^{-1}(X) \to \tilde{\mathbf{Y}}$ is faithful, then G is faithful.

PROOF. For any $f, g \in \mathrm{map}_{\tilde{\mathbf{X}}}(U, V)$ with $G(f) = G(g)$, we have $\delta_{\mathbf{Y}}(G(f)) = \delta_{\mathbf{Y}}(G(g))$ which is $F(\delta_{\mathbf{X}}(f)) = F(\delta_{\mathbf{X}}(g))$. Since F is faithful, $\delta_{\mathbf{X}}(f) = \delta_{\mathbf{X}}(g)$. By noticing both Cartesian liftings of $\delta_{\mathbf{X}}(f)$ and $\delta_{\mathbf{X}}(g)$:

$$\begin{tikzcd} & U \arrow[dl, dotted, "f'"'] \arrow[dr, "f"] & \\ W \arrow[rr, "\delta_{\mathbf{X}}(f)^*"'] & & V \end{tikzcd} \qquad \begin{tikzcd} & U \arrow[dl, dotted, "g'"'] \arrow[dr, "g"] & \\ W \arrow[rr, "\delta_{\mathbf{X}}(g)^*"'] & & V \end{tikzcd}$$

where $W = (\delta_{\mathbf{X}}(f))^*(V) = (\delta_{\mathbf{X}}(g))^*(V)$, we have

$$G(f) = G(\delta_{\mathbf{X}}(f)^* f') = (F(\delta_{\mathbf{X}}(f)))^* G(f')$$

and

$$G(g) = G(\delta_{\mathbf{X}}(g)^* g') = (F(\delta_{\mathbf{X}}(g)))^* G(g').$$

Hence $G(f') = G(g')$ and therefore, by the faithfulness of $G|_{\delta_{\mathbf{X}}^{-1}(U)}$, $f' = g'$. Then $f = \delta_{\mathbf{X}}(f)^* f' = \delta_{\mathbf{X}}(g)^* g' = g$ and so G is faithful. ■

3. Restriction Fibered and Unitary Restriction Categories

In Subsection 2.9, we proved that there is an adjunction $\mathcal{S} \dashv \mathcal{R} : \mathbf{sFib}_0 \to \mathbf{rCat}_0$ between the category of stable meet semilattice fibrations and the category of restriction categories and, furthermore, that the functor $\mathcal{S}$ is faithful. This section studies the special properties of restriction categories which arise as the image of this functor. A complete description of these restriction categories is provided both from the fibrational and (in the last subsection) from the unitary point of view. Furthermore we show how to use these descriptions to show that $\mathcal{S}$ is comonadic and $\mathcal{R}$ is monadic.

Restriction categories which arise as $\mathcal{S}(\delta_{\mathbf{X}})$ have the peculiar property that $f = g$ whenever $\overline{f} = \overline{g}$ and $fe = ge$ for some restriction idempotent. A restriction category satisfying this condition is called *unitary*: this directly generalizes the notion of an E-unitary inverse semigroup. These categories are introduced in the last subsection below and given an alternate view (and a generalization) of the main development path we follow using fibrations

In any unitary restriction category the compatibility relation $f \smile g$ ($\Leftrightarrow f\overline{g} = g\overline{f}$) is a congruence, this allows all parallel compatible maps to be viewed as a single "base" map. For a fibration the resulting functor to the category of base maps is just $\varepsilon : \mathcal{S}(\delta_X) \to \mathbf{X}$ and we shall speak of the *base* of a unitary restriction category as the generalization of this situation. In particular, this congruence turns restricted isomorphisms into actual isomorphisms and so it turns a unitary inverse category (a restriction category in which all maps are restricted isomorphisms) into a groupoid. In the one object case this is the observation that the base of an E-unitary inverse monoid is a group which provides an important interface between semigroup theory and group theory. A nice example of this is given by the E-unitary inverse monoid of Möbius functions on the complex plane (described in [8]).

In many unitary restriction categories each equivalence class of compatible maps contains an upper bound. However, the composite of two bounding maps may not necessarily be bounding. In the inverse semigroup literature the corresponding semigroups are called F-unitary: we have chosen to depart slightly from this terminology using instead *bounded unitary*. A bounded unitary restriction category is *strictly* bounded in case the composition of any two bounding maps is itself bounding. Strictly bounded unitary restriction categories are precisely the categories in the image of $\mathcal{S}$.

As mentioned in the introduction of this paper, while we have managed to show that every unitary restriction category can be embedded in a bounded unitary restriction category (which can be constructed from a lax fibration), we have not generalized the notion of McAlister triple – a useful tool of semigroup theory.

3.1. Restriction Fibered Categories. We shall call a restriction category of the form $\mathcal{S}(\delta_{\mathbf{X}})$ which arises from a fibration $\delta_{\mathbf{X}}$ a *restriction fibered* category. The purpose of this subsection is to provide an alternative description of these rather special restriction categories.

We begin with a general fact about restriction categories:

3.2. Lemma.

(i) *Any restriction category is poset-enriched with respect to* $f \leq g \Leftrightarrow f = g\overline{f}$.

(ii) $f \leq g \Rightarrow \overline{f} \leq \overline{g}$.

PROOF.

(i) Clearly, $f = f\overline{f}$ gives $f \leq f$. If $f \leq g$ and $g \leq h$, then $f = g\overline{f}$ and $g = h\overline{g}$ and so, by [**R.3**], $f = g\overline{f} = h\overline{g}\,\overline{f} = h\overline{g\overline{f}} = h\overline{f}$. Hence $f \leq h$. If $f \leq g$ and $g \leq f$, then $f = g\overline{f}$ and $g = f\overline{g}$ and so, by [**R.3**], $\overline{f} = \overline{g\overline{f}} = \overline{g}\,\overline{f} = \overline{f}\,\overline{g} = \overline{f\overline{g}} = \overline{g}$. Hence, by [**R.1**], $f = g\overline{f} = g\overline{g} = g$, as desired. Therefore, $\mathrm{map}_{\mathbf{C}}(A, B)$ is a poset.

Let $f, g : A \to B$ and $f', g' : B \to C$ be maps. If $f \leq g$ and $f' \leq g'$, then $f = g\overline{f}$ and $f' = g'\overline{f'}$. Hence

$$g'g\overline{f'f} = g'g\overline{f'f\overline{f}} = g'g\overline{f}\;\overline{f'f} = g'f\overline{f'f} = g'\overline{f'}f = f'f$$

and therefore $f'f \leq g'g$. Thus, any restriction category is poset-enriched with respect to $f \leq g \Leftrightarrow f = g\overline{f}$.

(ii) $f \leq g$ gives $f = g\overline{f}$. Hence $\overline{f} = \overline{g\overline{f}} = \overline{g}\,\overline{f} = \overline{g}\overline{\overline{f}}$ and therefore $\overline{f} \leq \overline{g}$.

∎

Now, for a pair of objects A, B in a restriction category $\mathbf{C}$, we define

$$\mathrm{map}^{\max}_{\mathbf{C}}(A, B) = \{f \in \mathrm{map}_{\mathbf{C}}(A, B) \mid f \leq h \text{ implies } h = f \text{ in } \mathrm{map}_{\mathbf{C}}(A, B)\}.$$

We shall call a restriction category $\mathbf{C}$ *a lax restriction fibered category* if it satisfies the following condition:

[**M.1**] For any objects A, B and any $f \in \mathrm{map}_{\mathbf{C}}(A, B)$, there is a unique $m_f \in \mathrm{map}^{\max}_{\mathbf{C}}(A, B)$ such that $f \leq m_f$.

A lax restriction fibered category is a *restriction fibered category* if it satisfies in addition

[**M.2**] For any objects A, B, C, $f \in \mathrm{map}^{\max}_{\mathbf{C}}(A, B)$, and $g \in \mathrm{map}^{\max}_{\mathbf{C}}(B, C)$, $gf \in \mathrm{map}^{\max}_{\mathbf{C}}(A, C)$.

Clearly, for any map $f : A \to B$ in a lax fibered restriction category $\mathbf{C}$, $m_{\overline{f}} = 1_A$ since $\overline{f} \leq 1_A$ and $1_A \leq g \Leftrightarrow g = 1_A$ in $\mathrm{map}_{\mathbf{C}}(A, A)$. Since $m_{1_A} = 1_A$, $1_A \in \mathrm{map}^{\max}_{\mathbf{C}}(A, A)$. Note that, for any maps $f : A \to B$ and $g : B \to C$,

$$m_{gf} = m_{m_g m_f},$$

since $f \leq m_f$ and $g \leq m_g$ imply $gf \leq m_g m_f \leq m_{m_g m_f}$ and both m_{gf} and $m_{m_g m_f}$ are maximum above gf.

A restriction functor $F : \mathbf{C} \to \mathbf{D}$ between two lax restriction fibered categories is called a *restriction fibered functor* if it preserves maximal maps, namely, $F(f) \in \mathrm{map}^{\max}_{\mathbf{D}}(F(A), F(B))$ for each $f \in \mathrm{map}^{\max}_{\mathbf{C}}(A, B)$. All lax restriction fibered categories and restriction fibered functors between them form a category, denoted by $\mathbf{lrfCat}_0$. All restriction fibered categories and restriction fibered functors between them form a subcategory of $\mathbf{lrfCat}_0$, denoted by $\mathbf{rfCat}_0$.

3.3. Lemma. *For any stable meet semilattice fibration* $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$*, the restriction category* $\mathcal{S}(\delta_{\mathbf{X}})$ *is a restriction fibered category.*

Proof. For any objects A, B,

$$\mathrm{map}_{\mathcal{S}(\delta_{\mathbf{X}})}(A,B) = \{(f,\sigma) \mid f \in \mathrm{map}_{\mathbf{X}}(A,B) \text{ and } \sigma \in \delta_{\mathbf{X}}^{-1}(A) \text{ with } \sigma \leq f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})\}.$$

and observe that $(f,\sigma) \leq (g,\sigma')$ if and only if $f = g$ and $\sigma \leq \sigma'$. This implies that $(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}))$ is the unique maximal element above (f,σ). So [**M.1**] is satisfied.

For any $(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) \in \mathrm{map}^{\max}_{\mathcal{S}(\delta_{\mathbf{X}})}(A,B)$ and $(g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})) \in \mathrm{map}^{\max}_{\mathcal{S}(\delta_{\mathbf{X}})}(B,C)$, we have

$$\begin{aligned}
(g, g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)}))(f, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)})) &= (gf, f^*(\top_{\delta_{\mathbf{X}}^{-1}(B)}) \wedge f^*g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})) \\
&= (gf, f^*g^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})) \\
&= (gf, (gf)^*(\top_{\delta_{\mathbf{X}}^{-1}(C)})) \\
&\in \mathrm{map}^{\max}_{\mathcal{S}(\delta_{\mathbf{X}})}(A,C).
\end{aligned}$$

Hence [**M.2**] is also satisfied. Thus, $\mathcal{S}(\delta_{\mathbf{X}})$ is a restriction fibered category. ∎

Let $\mathbf{C}$ be a lax restriction fibered category. We define $\mathbf{C}_{\max}$ by following data:

objects: the same as the objects of $\mathbf{C}$,

maps: for any objects A, B, $\mathrm{map}_{\mathbf{C}_{\max}}(A,B) = \mathrm{map}^{\max}_{\mathbf{C}}(A,B)$,

composition: $gf = m_{m_g m_f} = m_{gf}$.

Then $\mathbf{C}_{\max}$ is a category. Now, we define $\tilde{\mathbf{C}}_{\max}$ to be the category given by

objects: (A, e_A), where e_A is a restriction idempotent over A in $\mathbf{C}$,

maps: a map f from (A, e_A) to (B, e_B) is a map $f \in \mathrm{map}^{\max}_{\mathbf{C}}(A,B)$ such that $e_A = \overline{e_B f} e_A$,

composition: the same as in $\mathbf{C}_{\max}$.

Obviously, there is a forgetful functor $\partial_{\mathbf{C}_{\max}} : \tilde{\mathbf{C}}_{\max} \to \mathbf{C}_{\max}$, which forgets restriction idempotents.

3.4. Lemma. *For any restriction fibered category* $\mathbf{C}$*, the forgetful functor* $\partial_{\mathbf{C}_{\max}} : \tilde{\mathbf{C}}_{\max} \to \mathbf{C}_{\max}$ *is a stable meet semilattice fibration.*

Proof. As in Lemma 2.3, For any map $f : A \to B$ in $\mathbf{C}_{\max}$ and any $(B, e_Y) \in \partial^{-1}_{\mathbf{C}_{\max}}(B)$, $f : (A, \overline{e_B f}) \to (B, e_B)$ is the cartesian lifting of a map $f : A \to B$ at (B, e_B). Note that each fiber

$$\partial^{-1}_{\mathbf{C}_{\max}}(A) = \{(A, e_A) \,|\, e_A : A \to A \text{ is a restriction idempotent on } A\}$$

is a meet semilattice with the order given by $(A, e_A) \le (A, e'_A) \Leftrightarrow e_A = e'_A e_A$, with the binary meet given by $(A, e_A) \wedge (A, e'_A) = (A, e_A e'_A)$, and with $(A, 1_A)$ as the top element. Obviously, for any map $f : A \to B$, $f^* : \partial^{-1}_{\mathbf{C}_{\max}}(B) \to \partial^{-1}_{\mathbf{C}_{\max}}(A)$, sending (B, e_B) to $(A, \overline{e_B f})$, is a stable meet semilattice homomorphism. Hence $\partial_{\mathbf{C}_{\max}} : \tilde{\mathbf{C}}_{\max} \to \mathbf{C}_{\max}$ is a stable meet semilattice fibration. ■

3.5. Lemma. *For any restriction fibered category* $\mathbf{C}$, $\mathcal{S}(\partial_{\mathbf{C}_{\max}}) \cong \mathbf{C}$.

Proof. Define the functor $E : \mathcal{S}(\partial_{\mathbf{C}_{\max}}) \to \mathbf{C}$ by sending $(f, (A, e_A)) : A \to B$ to $f e_A : A \to B$. It is easy to check that E is a restriction functor.

For any map $f : A \to B$ in $\mathbf{C}$, by [**M.1**], there is a unique $m_f \in \mathrm{map}_{\max}(A, B)$ such that $f \le m_f$. Then $\overline{f} \le \overline{m_f} = m^*_f(\top_{\partial^{-1}_{\mathbf{C}_{\max}}(B)})$ and so $(m_f, (A, \overline{f})) : A \to B$ is a map in $\mathcal{S}(\partial_{\mathbf{C}_{\max}})$. Now, we define the functor $F : \mathbf{C} \to \mathcal{S}(\partial_{\mathbf{C}_{\max}})$ by sending $f : A \to B$ to $(m_f, (A, \overline{f})) : A \to B$. Note that $F(\overline{f}) = (m_{\overline{f}}, (A, \overline{\overline{f}})) = (1_A, (A, \overline{f})) = \overline{(m_{\overline{f}}, (A, \overline{f}))} = \overline{F(f)}$. Then F is a restriction functor. For any map $f : A \to B$ in $\mathbf{C}$, $(EF)(f) = E(m_f, (A, \overline{f})) = m_f \overline{f} = f$. Hence $EF = 1_{\mathbf{C}}$.

On the other hand, for any map $(f, (A, \sigma_A)) : A \to B$ in $\mathcal{S}(\partial_{\mathbf{C}_{\max}})$, $\sigma_A \le f^*(\top_{\partial^{-1}_{\mathbf{C}_{\max}}(B)}) = \overline{f}$. Then $\sigma_A = \overline{f}\sigma_A = \overline{f e_A}$ and so $f e_A = f\overline{f e_A}$ which gives $f e_A \le f$. Since $f \in \mathrm{map}^{\max}_{\mathbf{C}}(A, B)$, we have $m_{f e_A} = f$. Hence $(FE)(f, (A, e_A)) = F(f e_A) = (m_{f e_A}, (A, \overline{f e_A})) = (f, (A, e_A))$, and therefore $FE = 1_{\mathcal{S}(\partial_{\mathbf{C}_{\max}})}$. Thus, $\mathcal{S}(\partial_{\mathbf{C}_{\max}}) \cong \mathbf{C}$. ■

Lemmas 3.3 and 3.5 immediately yield the following proposition.

3.6. Proposition. $\mathbf{C}$ *is a restriction fibered category if and only if there is a stable meet semilattice fibration* $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ *such that* $\mathbf{C} \cong \mathcal{S}(\delta_{\mathbf{X}})$.

Now let's look at which restriction functors between restriction fibered categories are in the image of $\mathcal{S}$.

3.7. Proposition. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be two restriction fibered categories and* $F : \mathbf{C} \to \mathbf{D}$ *a functor. Then* F *is of form* $\mathcal{S}(G, H)$ *for some map* (G, H) *in* **sFib** *if and only if* F *is a restriction fibered functor.*

Proof. If $F : \mathbf{C} \to \mathbf{D}$ is a restriction fibered functor, then we have two functors $F_1 : \mathbf{C}_{\max} \to \mathbf{D}_{\max}$ and $F_2 : \tilde{\mathbf{C}}_{\max} \to \tilde{\mathbf{D}}_{\max}$, where F_1 is the restriction of F on the max maps and F_2 is the functor introduced by F_1 on the restriction idempotents. By Lemma 3.5, we have

$$\mathbf{C} \cong \mathcal{S}(\partial_{\mathbf{C}_{\max}}) \text{ and } \mathbf{D} \cong \mathcal{S}(\partial_{\mathbf{D}_{\max}})$$

and a commutative diagram:

$$\begin{tikzcd} \tilde{\mathbf{C}}_{\max} \arrow[r, "F_2"] \arrow[d, "\partial_{\mathbf{C}}"'] & \tilde{\mathbf{D}}_{\max} \arrow[d, "\partial_{\mathbf{D}}"] \\ \mathbf{C}_{\max} \arrow[r, "F_1"] & \mathbf{D}_{\max} \end{tikzcd}$$

Now, it is easy to check that $(F_1, F_2) : \partial_{\mathbf{C}_{\max}} \to \partial_{\mathbf{C}_{\max}}$ is a map in $\mathbf{sFib}_0$ and $\mathcal{S}(F_1, F_2) \cong F$.

Conversely, if $F \cong \mathcal{S}(G, F)$ for some map $(G, H) : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ in $\mathbf{sFib}_0$, then, for each $(f, \sigma) \in \max_{\mathcal{S}(\delta_{\mathbf{X}})}(A, B)$, $\mathcal{S}(G, H)(f, \sigma) = (G(f), H(\sigma)) \in \max_{\mathcal{S}(\delta_{\mathbf{Y}})}(G(A), G(B))$. Hence $\mathcal{S}(G, H) : \mathcal{S}(\delta_{\mathbf{X}}) \to \mathcal{S}(\delta_{\mathbf{Y}})$ is a restriction fibered functor. ∎

By Propositions 3.6 and 3.7, we have immediately:

3.8. Theorem. $\mathcal{S} : \mathbf{sFib}_0 \to \mathbf{rfCat}_0$ *is an equivalence of categories.*

3.9. Monadicity of $\mathcal{R}$ and Comonadicity of $\mathcal{S}$. Theorem 3.8 gives us a characterization of the image of $\mathcal{S}$. A natural question, therefore, is: what does the image of $\mathcal{R}$ look like? The image of $\mathcal{R}$ was characterized in [6] as the restriction fibrations:

3.10. Definition. *A stable meet semilattice fibration* $\delta : \mathbf{D} \to \mathbf{C}$ *is called a* restriction fibration *if for each object* $E \in \mathbf{D}$*, there is a map* $\varepsilon_E : \delta(E) \to \delta(E)$ *in* $\mathbf{C}$ *satisfying*

[rF.1] $\varepsilon_{\top_{\delta^{-1}(\delta(E))}} = 1_{\delta(E)}$,

[rF.2] $\varepsilon_E^*(\top_{\delta^{-1}(\delta(E))}) = E$, *where* $\vartheta_{\varepsilon_E} : \varepsilon_E^*(\top_{\delta^{-1}(\delta(E))}) \to \top_{\delta^{-1}(\delta(E))}$ *is the cartesian lifting of* ε_E *at* $\top_{\delta^{-1}(\delta(E))}$,

[rF.3] $\varepsilon_E \varepsilon_{E'} = \varepsilon_{E \wedge E'}$,

[rF.4] $\varepsilon_F f = f \varepsilon_{f^*(F)}$,

for all map $f : \delta(E) \to Y$ *in* $\mathbf{C}$*,* $E' \in \delta^{-1}(\delta(E))$*, and* $F \in \delta^{-1}(Y)$*.*

A map from a restriction fibration $(\delta : \mathbf{D} \to \mathbf{C})$ to a restriction fibration $(\delta' : \mathbf{D}' \to \mathbf{C}')$ is given by a pair (F, F'), where $F : \mathbf{C} \to \mathbf{C}'$ and $F' : \mathbf{D} \to \mathbf{D}'$ are functors such that

$$\begin{tikzcd} \mathbf{D} \arrow[r, "F'"] \arrow[d, "\delta"'] & \mathbf{D}' \arrow[d, "\delta'"] \\ \mathbf{C} \arrow[r, "F"] & \mathbf{C}' \end{tikzcd}$$

commutes and for any map $f : X \to Y$ in $\mathbf{C}$ and any $E \in \delta^{-1}(X), W \in \delta^{-1}(Y)$, the following conditions are satisfied:

[pR.1] $F'(\top_{\delta^{-1}(X)}) = \top_{(\delta')^{-1}(F(X))}$,

[pR.2] $F(\varepsilon_E) = \varepsilon_{F'(E)}$,

[pR.3] $F'(f^*(W)) = (F(f))^*(F'(W))$.

We have:

3.11. Theorem. *If $\delta : \mathbf{D} \to \mathbf{C}$ is a restriction fibration, then it is isomorphic to $\mathcal{R}(\mathbf{C})$, where $\mathbf{C}$ is the restriction category with the restriction $\overline{f} = \varepsilon_{f^*(\top)}$, called the restriction category induced by $\delta : \mathbf{D} \to \mathbf{C}$. Furthermore, a restriction fibration map $(F, F') : (\delta : \mathbf{D} \to \mathbf{C}) \to (\delta' : \mathbf{D}' \to \mathbf{C}')$ induces a restriction functor $F : \mathbf{C} \to \mathbf{C}'$.*

Sketch of Proof: First of all, it is easy to check that if $\delta : \mathbf{D} \to \mathbf{C}$ is a restriction fibration then $\mathbf{C}$ is the restriction category with the restriction $\overline{f} = \varepsilon_{f^*(\top)}$ and that for a given restriction category $\mathbf{X}$ the forgetful functor $\partial_{\mathbf{X}} : \mathbf{r}(\mathbf{X}) \to \mathbf{X}$ is a restriction fibration.

Let $\delta : \mathbf{D} \to \mathbf{C}$ be a restriction fibration and $\mathbf{C}$ the restriction category with the restriction structure induced by δ. Then we have a restriction fibration map $(1_{\mathbf{C}}, 1_\delta) : \delta \to \partial_{\mathbf{C}}$, where $1_\delta : \mathbf{D} \to \mathbf{r}(\mathbf{C})$ is given by sending $f : D_1 \to D_2$ to $\delta(f) : (\delta(D_1), \varepsilon_{E_1}) \to (\delta(D_2), \varepsilon_{E_2})$. On the other hand, we have a restriction fibration map $(1_{\mathbf{C}}, G) : \partial_{\mathbf{C}} \to \delta$, where $G : \mathbf{r}(\mathbf{C}) \to \mathbf{D}$ is given by sending $f : (C_1, \varepsilon_{E_1}) \to (C_2, \varepsilon_{E_2})$ in $\mathbf{r}(\mathbf{C})$ to $\text{lift}(f) = \vartheta_{f,E_2} \circ \text{leq} : E_1 \to E_2$, here ϑ_{f,E_2} is the cartesian lifting of f at E_2 and leq is the map $E_1 \le f^*(E_2)$. Now it is routine to verify that $(1_{\mathbf{C}}, 1_\delta)(1_{\mathbf{C}}, G) = 1_{\partial_{\mathbf{C}}}$ and $(1_{\mathbf{C}}, G)(1_{\mathbf{C}}, 1_\delta) = 1_\delta$. Hence $\delta \cong \partial_{\mathbf{C}}$.

If $(F, F') : (\delta : \mathbf{D} \to \mathbf{C}) \to (\delta' : \mathbf{D}' \to \mathbf{C}')$ is a restriction fibration map, then, clearly, $F : \mathbf{C} \to \mathbf{C}'$ is a restriction functor since $F(\overline{f}) = F(\varepsilon_{f^*(\top_{\delta^{-1}(Y)})}) = \varepsilon_{(F(f))^*(\top_{(\delta')^{-1}(FY)})} = \overline{F(f)}$. $\Box$

The above result can be used to show the monadicity and comonadicity of the adjunction $\begin{tikzcd} \mathbf{rCat}_0 \arrow[r, shift right, "\mathcal{R}"'] \arrow[r, phantom, "\perp"] & \mathbf{sFib}_0 \arrow[l, shift right, "\mathcal{S}"'] \end{tikzcd}$. We have the very strong result:

3.12. Theorem. *$\mathcal{R}$ is monadic and $\mathcal{S}$ is comonadic.*

To prove that $\mathcal{R}$ is monadic, we use a technical lemma which states that a retraction of a restriction fibration in $\mathbf{sFib}_0$ is a restriction fibration.

3.13. Lemma. *Let $\delta_i : \mathbf{D}_i \to \mathbf{C}_i$ be stable meet semilattice fibrations such that $(F, F')(S, S') = 1_{\delta_2}$ for some maps $(F, F') : \delta_1 \to \delta_2$ and $(S, S') : \delta_2 \to \delta_1$ in $\mathbf{sFib}_0$. If δ_1 is a restriction fibration, then so is δ_2.*

Proof. Suppose that δ_1 is the restriction fibration with the map ε_U for each object U in $\delta_1^{-1}(O)$. It is clear that each fiber $\delta_2^{-1}(X)$ is a stable meet semilattice in which $E_1 \le E_2$ if and only if there is a map from E_1 to E_2. In order to prove that δ_2 is a restriction fibration with the map $\varepsilon'_E = F(\varepsilon_{S'(E)})$, it suffices to check the four conditions [**rF.1**], [**rF.2**], [**rF.3**], and [**rF.4**]. For any map $f : X \to Y$ in $\mathbf{C}_2$, any $E, E' \in \delta_2^{-1}(X)$ and any $I \in \delta_2^{-1}(Y)$, we have:

[rF.1]

$$\begin{aligned}
\varepsilon'_{\top_{\delta_2^{-1}(X)}} &= F(\varepsilon_{S'(\top_{\delta_2^{-1}(X)})}) \\
&= F(\varepsilon_{\top_{\delta_2^{-1}(T(X))}}) \text{ (by [}\mathbf{sfM.1}\text{])} \\
&= F(1_{T(X)}) \text{ (by [}\mathbf{rF.1}\text{])} \\
&= 1_{FT(X)} \\
&= 1_X.
\end{aligned}$$

[rF.2]

$$\begin{aligned}
\varepsilon'^{*}_{E}(\top_{\delta_2^{-1}(X)}) &= (F(\varepsilon_{S'(E)}))^{*}(\top_{\delta_2^{-1}(X)}) \\
&= (F(\varepsilon_{S'(E)}))^{*}(F'\top_{\delta_1^{-1}(SX)}) \text{ (by [}\mathbf{sfM.1}\text{])} \\
&= F'(\varepsilon^{*}_{S'(E)})(\top_{\delta_1^{-1}(SX)}) \text{ (by [}\mathbf{sfM.3}\text{])} \\
&= F'(S'(E)) \text{ (by [}\mathbf{rF.2}\text{])} \\
&= E.
\end{aligned}$$

[rF.3] $\varepsilon'_{E}e'_{E'} = F(\varepsilon_{S'(E)})F(\varepsilon_{S'(E')}) = F(\varepsilon_{S'(E)}\varepsilon_{S'(E')}) = F(\varepsilon_{S'(E)\wedge S'(E')}) = F(\varepsilon_{S'(E\wedge E')}) = \varepsilon'_{E\wedge E'}$.

[rF.4]

$$\begin{aligned}
\varepsilon'_{I}f &= F(\varepsilon_{S'(I)})f \\
&= F(\varepsilon_{S'(I)}T(f)) \\
&= F(T(f)\varepsilon_{T(f)^{*}(S'(I))}) \text{ (by [}\mathbf{rF.4}\text{])} \\
&= fF(\varepsilon_{T(f)^{*}(S'(I))}) \\
&= fF(\varepsilon_{S'(f^{*}(I))}) \text{ (by [}\mathbf{sfM.3}\text{])} \\
&= f\varepsilon'_{f^{*}(I)}.
\end{aligned}$$

■

Proof of Theorem 3.12. To show that $\mathcal{R}$ is monadic, we need to prove that $\mathcal{R}$ reflects isomorphisms and that $\mathbf{rCat}_0$ has coequalizers of reflexive $\mathcal{R}$-split coequalizer pairs and $\mathcal{R}$ preserves them. Let $F : \mathbf{X} \to \mathbf{Y}$ be a restriction functor such that $\mathcal{R}(F) : \mathcal{R}(\mathbf{X}) \to \mathcal{R}(\mathbf{Y})$ is an isomorphism in $\mathbf{sFib}_0$. Then there is a map $(N, N') : \mathcal{R}(\mathbf{Y}) \to \mathcal{R}(\mathbf{X})$ such that $(N, N')\mathcal{R}(F) = 1_{\mathcal{R}(\mathbf{X})}$ and $\mathcal{R}(F)(N, N') = 1_{\mathcal{R}(\mathbf{Y})}$. Thus, $NF = 1_{\mathbf{X}}$ and $FN = 1_{\mathbf{Y}}$. For any map $f : A \to B$ in $\mathbf{Y}$, we have $N(\overline{f}) = N(\overline{F(N(f))}) = N(F(\overline{(N(f))})) = \overline{N(f)}$. Hence N is a restriction functor and therefore N is an isomorphism in $\mathbf{rCat}_0$. Thus, $\mathcal{R}$ reflects isomorphisms. By Corollary 2.4 in [3], $\mathbf{rCat}_0$ is cocomplete. Suppose that $F, G : \mathbf{X} \to \mathbf{Y}$

is a reflexive $\mathcal{R}$-split coequalizer pair and that $\partial_{\mathbf{X}} \underset{(G,\mathbf{r}(G))}{\overset{(F,\mathbf{r}(F))}{\rightrightarrows}} \partial_{\mathbf{Y}} \xrightarrow{(U,U')} \delta_{\mathbf{U}}$ is the split coequalizer diagram in $\mathbf{sFib}_0$. Let $\mathbf{X} \underset{G}{\overset{F}{\rightrightarrows}} \mathbf{Y} \xrightarrow{C} \mathbf{C}$ be a coequalizer diagram in $\mathbf{rCat}_0$. Then monadicity of $\mathcal{R}$ amounts to checking that $\partial_{\mathbf{X}} \underset{(G,\mathbf{r}(G))}{\overset{(F,\mathbf{r}(F))}{\rightrightarrows}} \partial_{\mathbf{Y}} \xrightarrow{(C,\mathbf{r}(C))} \partial_{\mathbf{C}}$ is a coequalizer diagram in $\mathbf{sFib}_0$. Since $\delta_{\mathbf{U}}$ is a retraction of a restriction fibration $\partial_{\mathbf{Y}}$, by Lemma 3.13 $\delta_{\mathbf{U}}$ is a restriction fibration and so $\delta_{\mathbf{U}} \cong \partial_{\mathbf{Z}}$ for some restriction category $\mathbf{Z}$. Note that $\mathbf{X} \underset{G}{\overset{F}{\rightrightarrows}} \mathbf{Y} \xrightarrow{U} \mathbf{Z}$ is a split coequalizer diagram in $\mathbf{Cat}_0$. Since the forgetful functor $\mathbf{rCat}_0 \to \mathbf{Cat}_0$ is monadic (Theorem 2.28, [3]), $\mathbf{C} \cong \mathbf{Z}$, as desired.

On the other hand, for the comonadicity of $\mathcal{S}$, we first note that $\mathcal{S}$ reflects isomorphisms. For any map $(F, F') : \delta_{\mathbf{X}} \to \delta_{\mathbf{Y}}$ in $\mathbf{sFib}_0$ such that $\mathcal{S}(F, F') : \mathcal{S}(\delta_{\mathbf{X}}) \to \mathcal{S}(\delta_{\mathbf{Y}})$ is an isomorphism in $\mathbf{rCat}_0$, there is a restriction functor $M : \mathcal{S}(\delta_{\mathbf{Y}}) \to \mathcal{S}(\delta_{\mathbf{X}})$ such that $M\mathcal{S}(F, F') = 1_{\mathcal{S}(\delta_{\mathbf{X}})}$ and $\mathcal{S}(F, F')M = 1_{\mathcal{S}(\delta_{\mathbf{Y}})}$. Clearly, M preserves max maps so that M is a map in $\mathbf{rfCat}_0$. Hence (F, F') is an isomorphism in $\mathbf{sFib}_0$ since $\mathcal{S} : \mathbf{sFib}_0 \to \mathbf{rfCat}_0$ is an equivalence of categories.

For any maps $(F, F'), (G, G') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ in $\mathbf{sFib}_0$, the equalizer of (F, F') and (G, G') is given by $(I, I') : (\delta_{\mathbf{E}} : \tilde{\mathbf{E}} \to \mathbf{E}) \to (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X})$, where $I' : \tilde{\mathbf{E}} \to \tilde{\mathbf{X}}$ and $I : \mathbf{E} \to \mathbf{X}$ are equalizers of $F', G' : \tilde{\mathbf{X}} \to \tilde{\mathbf{Y}}$ and $F, G : \mathbf{X} \to \mathbf{Y}$ in $\mathbf{Cat}_0$, respectively. So $\mathbf{sFib}_0$ has equalizers.

Suppose that $(F, F'), (G, G') : \delta_{\mathbf{X}} \to \delta_{\mathbf{Y}}$ is a reflexive $\mathcal{S}$-split equalizer pair in $\mathbf{sFib}_0$ and that $\mathbf{M} \overset{M}{\hookrightarrow} \mathcal{S}(\delta_{\mathbf{X}}) \underset{\mathcal{S}(G,G')}{\overset{\mathcal{S}(F,F')}{\rightrightarrows}} \mathcal{S}(\delta_{\mathbf{Y}})$ is the split equalizer diagram in $\mathbf{rCat}_0$. Then $\mathbf{M}$ a restriction fibered category and M preserves max maps so that the last equalizer diagram is an equalizer diagram in $\mathbf{rfCat}_0$. Let $\delta_{\mathbf{E}} \xrightarrow{(E,E')} \delta_{\mathbf{X}} \underset{(G,G')}{\overset{(F,F')}{\rightrightarrows}} \delta_{\mathbf{Y}}$ be an equalizer diagram in $\mathbf{sFib}_0$. Then $\mathcal{S}(\delta_{\mathbf{E}}) \xrightarrow{\mathcal{S}(E,E')} \mathcal{S}(\delta_{\mathbf{X}}) \underset{\mathcal{S}(G,G')}{\overset{\mathcal{S}(F,F')}{\rightrightarrows}} \mathcal{S}(\delta_{\mathbf{Y}})$ is an equalizer diagram in $\mathbf{rfCat}_0$ since $\mathcal{S} : \mathbf{sFib}_0 \to \mathbf{rfCat}_0$ is an equivalence of categories. Hence $\mathbf{M} \cong \mathcal{S}(\delta_{\mathbf{E}})$ and therefore $\mathcal{S}(\delta_{\mathbf{E}}) \xrightarrow{\mathcal{S}(E,E')} \mathcal{S}(\delta_{\mathbf{X}}) \underset{\mathcal{S}(G,G')}{\overset{\mathcal{S}(F,F')}{\rightrightarrows}} \mathcal{S}(\delta_{\mathbf{Y}})$ is an equalizer diagram in $\mathbf{rCat}_0$. ∎

3.14. Lax Restriction Fibered Categories. Our next aim is to characterize lax restriction fibered categories. As shown in Subsection 3.1, restriction fibered categories are essentially the same as the stable meet semilattice fibrations. We now show that the lax restriction fibered categories have an analogous description via lax fibrations. In particular, we show something even more general (whose ramifications we still do not fully understand) that a lax restriction functor into the dual of stable meet semilattices can be used as the basis of a construction analogous to $\mathcal{S}$. This construction in particular generates all the lax fibered restriction categories.

A graph map between two restriction categories is called a *lax restriction semi-functor* in case:

[lr.1] $\overline{F(x)} \leq F(\overline{x})$,

[lr.2] $F(y)F(x) = F(yx)\overline{F(y)F(x)}$.

[lr.3] $x \leq y \Rightarrow F(x) \leq F(y)$

A *lax restriction functor* also requires that $1_{F(A)} \leq F(1_A)$, and this forces $F(1_A) = 1_{F(A)}$ as we now see:

3.15. Lemma.

(*i*) *If F is a lax restriction semi-functor then F preserves restriction idempotents, that is* $\overline{F(\overline{f})} = F(\overline{f})$.

(*ii*) *The condition* [**lr.3**] *can be replaced by*

[lr.3a] $F(y\overline{x}) = F(y)F(\overline{x})$.

(*iii*) *If a lax restriction semi-functor is a lax functor, that is* $1_{F(A)} \leq F(1_A)$*, then* $F(1_A) = 1_{F(A)}$.

Proof.

(*i*) $\overline{F(\overline{x})} = F(\overline{x})$ follows from [**lr.1**] as $\overline{F(\overline{x})} \leq F(\overline{x})$ so that $\overline{F(\overline{x})} = F(\overline{x})\overline{F(\overline{x})} = F(\overline{x})$.

(*ii*) Suppose that [**lr.3**] holds. Then $F(y\overline{x}) \leq F(y)$ implies $F(y\overline{x}) = F(y)\overline{F(y\overline{x})} \leq F(y)F(\overline{y\overline{x}}) \leq F(y)F(\overline{x})$. Furthermore, [**lr.2**] gives $F(y)F(\overline{x}) \leq F(y\overline{x})$. So that $F(y)F(\overline{x}) = F(y\overline{x})$ and [**lr.3a**] follows. Conversely, assume [**lr.3a**]. Then $x \leq y$ gives $x = y\overline{x}$ and so $F(x) = F(x)\overline{F(x)} = F(y\overline{x})\overline{F(x)} = F(y)F(\overline{x})\overline{F(x)} = F(y)\overline{F(x)}$. Hence $F(x) \leq F(y)$.

(*iii*) If $1 \leq F(1)$ then F is normal since $F(1) = \overline{F(1)} \leq 1$.

∎

The category of stable meet semilattices opposite, $\mathbf{StabMSLat}_0^{\rm op}$, is a restriction category. The (co)restriction of a map $f : X \to Y$ in $\mathbf{StabMSLat}_0$ is given by setting $\overline{f}(y) = f(\top_X) \wedge y$.

Given a lax restriction functor $F : \mathbf{C} \to \mathbf{StabMSLat}_0^{\rm op}$, we may construct $\mathcal{L}(F)$ as follows:

objects: $A \in \mathrm{ob}\mathbf{C}$,

maps: $(f, e) : A \to B$, where $f : A \to B$ is a map in $\mathbf{C}$ and $e \in F(A)$ with $e \leq F(f)(\top_{F(B)})$,

composition: For any map $(f, e) : A \to B$ and $(g, e') : B \to C$, $(g, e')(f, e) = (gf, e \wedge F(f)(e'))$,

identities: $1_A = (1_A, \top_{F(A)})$,

restriction: $\overline{(f, e)} = (\overline{f}, e)$.

3.16. Lemma. *$\mathcal{L}(F)$ is a restriction category.*

Proof. Since $\top_{F(A)} = F(1_A)(\top_{F(A)})$, the identities are well-defined. If $e \leq F(f)(\top_{F(B)})$ and $e' \leq F(g)(\top_{F(C)})$, then

$$e \wedge F(f)(e') \leq F(f)(F(g)(\top_{F(C)})) \leq F(gf)(\top_{F(C)})$$

and so the composition is well-defined too.

For any map $(f, e) : A \to B$ in $\mathcal{L}(F)$, we must show that $\overline{(f, e)} = (\overline{f}, e)$ is well-defined. That is we need $e \leq F(\overline{f})(\top_{F(A)})$. It suffices to show that $F(\overline{f})(e) = e$ (which will be useful later):

$$\begin{array}{rcl} e & \leq & F(f)(\top_{F(B)}) \\ & = & F(f\overline{f})(\top_{F(B)}) \\ & = & \overline{F(\overline{f})}(F(f)(\top_{F(B)})) \ \text{ (by [}\mathbf{lr.3a}\text{])} \\ & = & F(\overline{f})(\top_{F(A)}) \wedge F(f)(\top_{F(B)}) \\ & \leq & F(\overline{f})(\top_{F(A)}) \end{array}$$

so $e = F(\overline{f})(\top_{F(A)}) \wedge e = F(\overline{f})(e)$.

Note that

$$(1_B, \top_{F(B)})(f, e) = (f, e \wedge F(f)(\top_{F(B)})) = (f, e)$$

and

$$(f, e)(1_A, \top_{F(A)}) = (f, \top_{F(A)} \wedge F(1_A)(e)) = (f, F(1_A)(e)) = (f, e).$$

Then the identity laws hold.

For any maps $(f,e) : A \to B$, $(g,e') : B \to C$, and $(h,e'') : C \to D$ in $\mathcal{L}(F)$, Since $F(f)F(g) \leq F(gf)$, we have $F(f)F(g)(e'') = \overline{F(f)F(g)}F(gf)(e'') = F(f)F(g)(\top_{F(C)}) \wedge F(gf)(e'')$. Then

$$\begin{aligned}
\Big((h,e'')(g,e')\Big)(f,e) &= \Big(hg, e' \wedge F(g)(e'')\Big)(f,e) \\
&= \Big((hg)f, e \wedge F(f)(e') \wedge F(f)F(g)(e'')\Big) \\
&= \Big(h(gf), e \wedge F(f)(e') \wedge F(f)F(g)(\top_{F(C)}) \wedge F(gf)(e'')\Big) \\
&= \Big(h(gf), e \wedge F(f)(e') \wedge F(gf)(e'')\Big) \\
&= (h,e'')\Big(gf, e \wedge F(f)(e')\Big) \\
&= (h,e'')\Big((g,e')(f,e)\Big)
\end{aligned}$$

and so the associative law holds. Thus, $\mathcal{L}(F)$ is a category.

To prove that $\mathcal{L}(F)$ is restriction category with $\overline{(f,e)} = (\overline{f}, e)$, we need to check the four restriction axioms.

[R.1] $(f,e)\overline{(f,e)} = (f,e)(\overline{f},e) = \Big(f, e \wedge F(\overline{f})(e)\Big) = (f,e)$.

[R.2] $\overline{(f_1,e_1)}\ \overline{(f_2,e_2)} = (\overline{f_1},e_1)(\overline{f_2},e_2) = (\overline{f_1}\ \overline{f_2}, e_2 \wedge F(\overline{f_2})e_1) = (\overline{f_2}\ \overline{f_1}, e_2 \wedge \overline{F(\overline{f_2})}e_1) = \Big(\overline{f_2}\ \overline{f_1}, e_2 \wedge F(\overline{f_2})(\top) \wedge e_1\Big) = (1, e_2 \wedge e_1) = \overline{(f_2,e_2)}\ \overline{(f_1,e_1)}$.

[R.3] $\overline{(g,e')\overline{(f,e)}} = \overline{(g,e')(\overline{f},e)} = \overline{\Big(g\overline{f}, e \wedge F(\overline{f})(e')\Big)} = \Big(\overline{g\overline{f}}, e \wedge F(\overline{f})(e')\Big) = (\overline{g}\overline{f}, e \wedge e') = \overline{(g,e')}\ \overline{(f,e)}$.

[R.4]

$$\begin{aligned}
\overline{(g,e')}(f,e) &= (\overline{g},e')(f,e) \\
&= \Big(\overline{g}f, e \wedge F(f)(e')\Big) \\
&= \Big(f\overline{gf}, e \wedge F(f)(e') \wedge F(\overline{g})(F(\overline{f})(e))\Big) \quad \text{(since } F(\overline{f})(e) = e) \\
&= \Big(f\overline{gf}, e \wedge F(f)(e') \wedge F(\overline{f}\ \overline{gf})(e)\Big) \\
&= \Big(f\overline{gf}, e \wedge F(f)(e') \wedge F(\overline{gf})(F(\overline{f})(e))\Big) \\
&= \Big(\Big(f\overline{gf}, e \wedge F(f)(e') \wedge F(\overline{gf})(e)\Big) \\
&= (f,e)\Big(\overline{gf}, e \wedge F(f)(e')\Big) \\
&= (f,e)\overline{(g,e')(f,e)}.
\end{aligned}$$

Hence $\mathcal{L}(F)$ is restriction category with $\overline{(f,e)} = (\overline{f}, e)$. ∎

3.17. Remarks.

1. Given a lax restriction functor $F : \mathbf{C} \to \mathbf{StabMSLat}_0^{\mathrm{op}}$, we obtain an obvious forgetful restriction functor $\tilde{F} : \mathcal{L}(F) \to \mathbf{C}$ which sends $(f, e) : A \to B$ to $f : A \to B$. This functor is induced by a congruence on $\mathcal{L}(F) : (f, e) \sim (g, e') \Leftrightarrow f = g$, which has the property that congruent maps are "compatible". See Subsection 3.23 below for the link to unitary restriction categories.

2. Given a lax restriction semi-functor $F : \mathbf{X} \to \mathbf{Y}$, we can construct a lax restriction functor $e(F) : \mathbf{X} \to \mathrm{im}(F)$, sending $f : X \to Y$ to $F(1_Y)F(f) : X \to Y$ and a faithful restriction semi-functor $m(F) : \mathrm{im}(F) \to \mathbf{Y}$, sending $g : X \to Y$ to $g : F(X) \to F(Y)$:

$$\begin{tikzcd} \mathbf{X} \arrow[rr, "F"] \arrow[rr, bend right=15, "\widehat{F}"'] \arrow[rr, phantom, shift right=1, "\leq"] \arrow[dr, "e(F)"'] & & \mathbf{Y} \\ & \mathrm{im}(F) \arrow[ur, "m(F)"'] & \end{tikzcd}$$

where $\widehat{F} = m(F) \circ e(F) \leq F$ as $m(F)(e(F)(f)) = F(1_Y)F(f)$ and $\mathrm{im}(F)$ is the following category:

objects: $X \in \mathrm{ob}\mathbf{X}$,

maps: maps from X to Y are given by maps from $F(X)$ to $F(Y)$ in $\mathbf{Y}$ which can be written in the form

$$F(1_Y)F(f_1)\cdots F(f_n)\prod_i \overline{F(1)F(g_{m_i i})\cdots F(g_{1i})},$$

where $g_{1i}\cdots g_{m_i i} : X \to Z_i$ and $f_1 \cdots f_n : X \to Y$ are maps in $\mathbf{X}$,

identities: $1_X = F(1_X)$,

composition and restriction: same as in $\mathbf{Y}$.

The semi-functor $m(F)$ can then be further factorized as a restriction functor to $K_r(\mathbf{Y})$ followed by the couniversal (idempotent forgetting) restriction semi-functor $K_r(\mathbf{Y}) \to \mathbf{Y}$. If $\mathbf{Y}$ is split, which is the case when $\mathbf{Y} = \mathbf{StabMSLat}_0^{\mathrm{op}}$, then this gives a lax restriction functor to $\mathbf{Y}$ itself. Thus, this allows the construction above to be performed on any lax restriction semi-functor with $F = \widehat{F}$.

The remainder of this section, will be concerned with lax restriction functors $F : \mathbf{C} \to \mathbf{StabMSLat}_0^{\mathrm{op}}$ where $\mathbf{C}$ is a mere category (so we regard $\mathbf{C}$ as being a restriction category in which all maps are total). In this case, $\mathcal{L}(F)$ is lax restriction fibered category with $\overline{(f, e)} = (1, e)$ and with $m_{(f,e)} = (f, F(f)(\top_{F(B)}))$ for any map $(f, e) : A \to B$. As $\mathbf{C}$ is a mere category, we do not need the condition [**lr.3**]. This means such a lax functor may be described more concretely as follows:

[lax.1] For each object A in $\mathbf{C}$, $F(1_A) = 1_{F(A)}$,

[lax.2] For any maps $f : A \to B$ and $g : B \to C$, and element $c \in F(C)$, $F(f)(F(g)(\top_{F(C)})) \wedge F(gf)(c) = F(f)F(g)(c)$.

Notice that the condition $\overline{F(x)} \leq F(\overline{x})$ reduces to $\overline{F(x)} \leq 1$, which is always true as all x are total. The condition [**lr.2**] translates straight into the condition [**lax.2**]. Notice also we may equivalently express [**lax.2**] as:

[lax.2a] For any maps $f : A \to B$ and $g : B \to C$, and elements $c \in F(C)$ and $b \in F(B)$ with $b \leq F(g)(\top_{F(C)})$, we have $F(f)(b) \wedge F(gf)(c) = F(f)(b) \wedge F(f)F(g)(c)$.

We shall use this form in the proofs below.

A *transformation* between two lax functors $F : \mathbf{X} \to \mathbf{StabMSLat}_0^{\mathrm{op}}$ and $G : \mathbf{Y} \to \mathbf{StabMSLat}_0^{\mathrm{op}}$ is a pair (W, ω), where $W : \mathbf{X} \to \mathbf{Y}$ is a functor and $\omega : GW \to F$ is a transformation such that each ω_A preserves the top element: $\omega_A(\top_{F(A)}) = \top_{GW(A)}$.

All lax functors to $\mathbf{StabMSLat}_0^{\mathrm{op}}$ and transformations between them form a category, denoted by $\mathbf{laxFun}_0$.

Let $F : \mathbf{C} \to \mathbf{StabMSLat}_0^{\mathrm{op}}$ and $G : \mathbf{D} \to \mathbf{StabMSLat}_0^{\mathrm{op}}$ be two lax functors and $(W, \omega) : F \to G$ is a map in $\mathbf{laxFun}_0$. Then we define a restriction fibered function $\mathcal{L}(W, \omega) : \mathcal{L}(F) \to \mathcal{L}(G)$ given by taking $(f, e) : A \to B$ to $(W(f), \omega_A(e)) : W(A) \to W(B)$, here, $\mathcal{L}(W, \omega)$ preserving identities follows from each ω_A preserving the top element and $\mathcal{L}(W, \omega)$ preserving composition follows from the naturality of ω. Clearly, $\mathcal{L}(W, \omega)$ preserves restriction and maximal maps. So, we have a functor $\mathcal{L} : \mathbf{laxFun}_0 \to \mathbf{lrfCat}_0$.

Let $\mathbf{C}$ be a lax restriction fibered category. Then we have a category $\mathbf{C}_{\max}$ and a functor $M : \mathbf{C} \to \mathbf{C}_{\max}$ given by taking $f : A \to B$ to $m_f : A \to B$. On the other hand, $M : \mathbf{C} \to \mathbf{C}_{\max}$ gives arise to, as the next lemma will show, a lax functor $\mathcal{M}(\mathbf{C}) : \mathbf{C}_{\max} \to \mathbf{StabMSLat}_0^{\mathrm{op}}$ given by sending each $f : A \to B$ in $\mathbf{C}_{\max}$ to $f^* : E(B) \to E(A); e \mapsto \overline{ef}$ in $\mathbf{StabMSLat}_0$ (where $E(B)$ and $E(A)$ are the meet semilattice of all restriction idempotents over B and A respectively).

3.18. Lemma. *For any lax restriction fibered category $\mathbf{C}$, $\mathcal{M}(\mathbf{C}) : \mathbf{C}_{\max} \to \mathbf{StabMSLat}_0^{\mathrm{op}}$, as above, is a lax functor.*

Proof. We need to check the conditions [**lax.1**] and [**lax.2**].

[lax.1] Clearly, $\mathcal{M}(\mathbf{C})(1_A) = 1_{E(A)} = 1_{\mathcal{M}(\mathbf{C})(A)}$.

[lax.2] For any maps $f : A \to B$, $g : B \to C$ in $\mathbf{C}_{\max}$, and any restriction idempotent e'' on C,we have: $\mathcal{M}(\mathbf{C})(f)(\mathcal{M}(\mathbf{C})(g)(\top)) \wedge \mathcal{M}(\mathbf{C})(gf)(e'') = \overline{\overline{g}f}\,\overline{e''m_{gf}} = \overline{e''m_{gf}}\,\overline{gf} = \overline{e''m_{gf}\overline{gf}} = \overline{e''gf}$ since $gf \leq m_{gf}$. Finally, $\overline{e''gf} = \overline{\overline{e''g}f} = \mathcal{M}(\mathbf{C})(f)\Big(\mathcal{M}(\mathbf{C})(g)(e'')\Big)$ so that [**lax.2**] follows.

Hence $\mathcal{M}(\mathbf{C}) : \mathbf{C}_{\max} \to \mathbf{StabMSLat}_0^{\mathrm{op}}$ is a lax functor. ∎

If $F : \mathbf{C} \to \mathbf{D}$ is a restriction fibered functor, then we have a map $(F_{\mathcal{M}}\omega^F) : \mathcal{M}(\mathbf{C}) \to \mathcal{M}(\mathbf{D})$ in $\mathbf{laxFun}_0$, where $\omega^F_A : E(A) \to E(F(A))$ is given by $\omega^F_A(e) = F(e)$. So, we have a functor $\mathcal{M} : \mathbf{lrfCat} \to \mathbf{laxFun}_0$, which sends $F : \mathbf{C} \to \mathbf{D}$ to $(F_{\mathcal{M}}, \omega^F) : \mathcal{M}(\mathbf{C}) \to \mathcal{M}(\mathbf{D})$.

3.19. Lemma. *For any lax restriction fibered category* $\mathbf{C}$*,* $\mathcal{L}(\mathcal{M}(\mathbf{C})) \cong \mathbf{C}$*.*

Proof. $\mathcal{L}(\mathcal{M}(\mathbf{C}))$ is the category with the same objects as $\mathbf{C}$ and with maps $(f, e) : A \to B$, where $f : A \to B$ is a maximal map in $\mathbf{C}$ and e is an idempotent over A such that $e \leq \overline{f}$. Define functors $S : \mathcal{L}(\mathcal{M}(\mathbf{C})) \to \mathbf{C}$ by $S(f, e) = fe$ and $T : \mathbf{C} \to \mathcal{L}(\mathcal{M}(\mathbf{C}))$ by $T(f) = (m_f, \overline{f})$. Clearly,

$$ST(f) = S(m_f, \overline{f}) = m_f\overline{f} = f$$

and

$$TS(f, e) = T(fe) = (m_{fe}, \overline{fe}) = (f, \overline{f}e) = (f, e).$$

Hence $\mathcal{L}(\mathcal{M}(\mathbf{C})) \cong \mathbf{C}$. ∎

3.20. Lemma. *For a given lax functor* $F : \mathbf{C} \to \mathbf{StabMSLat}_0^{\mathrm{op}}$*,* $\mathcal{M}(\mathcal{L}(F)) \cong F$*.*

Proof. For a given lax functor $F : \mathbf{C} \to \mathbf{StabMSLat}_0^{\mathrm{op}}$, $\mathcal{L}(F)$ is the lax restriction fibered category with the same objects as $\mathbf{C}$, with maps $(f, e) : A \to B, e \in F(A)$, with the maximal maps: $m_{(f,e)} = (f, \top_{F(A)})$, and with the restriction given by $\overline{(f, e)} = (1_A, e)$. $(\mathcal{L}(F))_{\max}$ is the category with the same objects as $\mathbf{C}$, with the maps $(f, \top_{F(A)}) : A \to B$ and with the composition $(g, \top_{F(B)})(f, \top_{F(A)}) = (gf, \top_{F(A)})$. Clearly, $(\mathcal{L}(F))_{\max} \cong \mathbf{C}$ and $\mathcal{M}(\mathcal{L}(F)) : (\mathcal{L}(F))_{\max} \to \mathbf{StabMSLat}_0^{\mathrm{op}}$ is the lax functor given by sending $(f, \top_{F(A)}) : A \to B$ to $(f, \top_{F(A)})^* : E(B) \to E(A)$ in $\mathbf{StabMSLat}_0$, where $E(B) = \{(1_B, e) \,|\, e \in F(B)\}$ and $(f, \top_{F(A)})^*(1_B, e) = \overline{(1_B, e)(f, \top_{F(A)})} = \overline{(f, \top_{F(A)} \wedge F(f)e)} = (1_A, F(f)e)$. Hence $\mathcal{M}(\mathcal{L}(F)) \cong F$ with the obvious isomorphisms. ∎

Combining Lemmas 3.19 and 3.20 yields immediately:

3.21. Theorem. $\mathcal{L} : \mathbf{laxFun}_0 \to \mathbf{lrfCat}_0$ *is an equivalence of categories.*

3.22. Remark. Since each stable meet semilattice fibration gives rise to a lax functor by the Grothendieck construction, we have the following commutative diagram:

$$\begin{tikzcd} \mathbf{sFib}_0 \arrow[r, "\mathcal{S}"] \arrow[d, hook] & \mathbf{rfCat}_0 \arrow[d, hook] \\ \mathbf{laxFun}_0 \arrow[r, "\mathcal{L}"] & \mathbf{lrfCat}_0 \end{tikzcd}$$

3.23. Unitary Restriction Categories. We now show how this whole development can be approached from a completely different perspective, as was outlined in the introduction.

A restriction category is *unitary* (or *proper*) [1] if for any maps f, g and any restriction idempotent e, $fe = ge$ and $\overline{f} = \overline{g}$ imply $f = g$. All unitary restriction categories and

[1]John Fountain pointed out to the first author that historically the correct term is actually "proper" as the unitary condition (as originally defined) was not the one given above but $xy = x \Rightarrow y = y^2$. For inverse categories, this is equivalent to our notion, but for restriction categories (and Fountain's left ample semigroups) it is not.

restriction functors between them form a subcategory of $\mathbf{rCat}_0$, denoted by $\mathbf{urCat}_0$. Our first observation is:

3.24. Proposition. *Lax restriction fibered categories are unitary.*

Proof. Let $\mathbf{C}$ be a lax restriction fibered category and suppose that f, g are two maps in $\mathbf{C}$ such that $\overline{f} = \overline{g}$ and $fe = ge$ for some restriction idempotent e. Now $fe \leq f \leq m_f$ and $ge \leq g \leq m_g$; so since $fe = ge$ we have $m_f = m_g$, but this means $f = m_f\overline{f} = m_g\overline{g} = g$. ∎

Not only is a restriction category a partial order enriched category with the order $\leq$ on each hom-set given by

$$f \leq g \Leftrightarrow f = g\overline{f}$$

but also a compatibility relation $\smile$ enriched category, where $\smile$ is given by

$$f \smile g \Leftrightarrow g\overline{f} = f\overline{g}.$$

A set S of maps is *compatible* if for any $s, s' \in S$, $s \smile s'$. Clearly, each subset of a compatible set is also compatible.

An equivalence relation $\sim$ on the class of maps in a restriction category $\mathbf{X}$ is a *restriction congruence* if for any maps $x : X \to A$, $f : A \to B$, $g : A \to B$, $y : B \to Y$ in $\mathbf{X}$,

(i) $f \sim g \Leftrightarrow yfx \sim ygx$,

(ii) $f \sim g \Rightarrow \overline{f} \sim \overline{g}$.

A unitary restriction category can be characterized by how the relation $\smile$ behaves:

3.25. Lemma. *Let* $\mathbf{X}$ *be a restriction category then the following are equivalent:*

(i) $\mathbf{X}$ *is unitary.*

(ii) $f \smile g$ *if and only if there is a restriction idempotent* e *such that* $fe = ge$.

(iii) $\smile$ *is a restriction congruence on* $\mathbf{X}$.

Proof. (i) $\Rightarrow$ (ii) If $f \smile g$, then $f\overline{g} = g\overline{f}$ and so $fe = ge$ with $e = \overline{f\overline{g}}$.

Conversely, if $fe = ge$ for some restriction idempotent e, then $f\overline{g}\,\overline{f}\overline{g}e = g\overline{f}\,\overline{f}\overline{g}e$. Since $\overline{f\overline{g}} = \overline{f}\overline{g} = \overline{g\overline{f}}$ and $\mathbf{X}$ is unitary, we have $f\overline{g} = g\overline{f}$. Hence $f \smile g$.

(ii) $\Rightarrow$ (iii) ¿From (ii) it is easy to see that $\smile$ is an equivalence relation on parallel maps. Reflexivity and symmetry are immediate while transitivity follows by composing the mediating restrictions. The additional two requirement of a congruence are, in fact, always true for the compatibility relation of any restriction category. In a unitary restriction category this can proved more simply. If $f \smile g$, then there is a mediating restriction idempotent e such that $fe = ge$ and so $xfey = xgey$, which implies that $xfy\overline{ey} = xgy\overline{ey}$ whence, by (ii), $xfy \smile xgy$. On the other hand, $f \smile g \Rightarrow f\overline{g} = g\overline{f} \Rightarrow \overline{f\overline{g}} = \overline{g\overline{f}} \Rightarrow \overline{f} \smile \overline{g}$. Thus, $\smile$ is a restriction congruence on $\mathbf{X}$.

(iii) $\Rightarrow$ (i) Let f, g be maps in $\mathbf{X}$ such that $fe = ge$ and $\overline{f} = \overline{g}$ for some restriction idempotent e. Then $f \smile fe = ge \smile g$ and so, by (iii), $f \smile g$. Hence $f\overline{g} = g\overline{f}$ and therefore $f = f\overline{f} = f\overline{g} = g\overline{f} = g\overline{g} = g$. Thus, $\mathbf{X}$ is unitary, as desired. ∎

This result means that for each unitary restriction category $\mathbf{X}$ there is a *base functor* $B : \mathbf{X} \to \mathbf{X}/\smile$. A map $f : X \to Y$ in a restriction category is a restricted isomorphism in case there is a (necessarily unique) $f^{(-1)} : Y \to X$ such that $\overline{f} = f^{(-1)}f$ and $\overline{f^{(-1)}} = ff^{(-1)}$ (see [3]). As restriction idempotents are exactly the endo-maps which are compatible with the identity map we immediately have:

3.26. Corollary.

(*i*) *In a unitary restriction category, if $f : X \to Y$ is a restricted isomorphism then $B(f)$ is an isomorphism.*

(*ii*) *Any unitary inverse category $\mathbf{X}$ has $\mathbf{X}/\smile$ a groupoid.*

An *inverse category* is a restriction category in which *all* maps are restricted isomorphisms (see [3]). This direct relation between unitary inverse monoids and groups is one reason why this theory has particular importance in semigroup theory.

We shall say a unitary restriction category is *bounded* in case each equivalence class of compatible maps has a maximal element (called the *bound*). A unitary restriction category is said to be *strictly bounded* in case the composite of any two bounds is itself a bound. Clearly any bounded unitary restriction category satisfies [**M.1**] and, as we have seen above, any lax restriction fibered category is unitary so the two notions are the same. Similarly, to say a unitary restriction category is strictly bounded is precisely to say [**M.2**] is satisfied:

3.27. Corollary.

(*i*) *A bounded unitary restriction category is precisely a lax restriction fibered category.*

(*ii*) *A strictly bounded unitary restriction category is precisely a restriction fibered category.*

This therefore gives an alternative, and perhaps more gentle, nomenclature for these categories.

Clearly any restriction subcategory of a unitary restriction category is necessarily unitary. Our purpose now is to prove that every unitary restriction category occurs as a subcategory of a bounded unitary restriction category. This result allows us to replace the base of the fibration $\mathbf{X}_{\max}$ by $\mathbf{X}/\smile$ in the unitary case. However, we still need to construct for any unitary category a lax fibered or strictly bounded unitary extension. To bridge this gap we need to briefly introduce join restriction categories:

3.28. Definition. *A restriction category $\mathbf{C}$ is called a* join restriction category *if for each pair of objects A and B and each compatible subset $S \subseteq \mathbf{C}(A, B)$ there is $\bigvee_{s\in S} s : A \to B$ such that*

- *$\bigvee_{s\in S} s$ is the join with respect to the partial order $\leq$ on $\mathbf{C}(A, B)$,*
- *$\overline{\bigvee_{s\in S} s} = \bigvee_{s\in S} \overline{s}$, and for any $f \in \mathbf{C}(B, Y)$ and $g \in \mathbf{C}(X, A)$,*
- *$f(\bigvee_{s\in S} s) = \bigvee_{s\in S}(fs)$,*

- $(\bigvee_{s\in S} s)g = \bigvee_{s\in S}(sg)$.

For any subset S of a poset $(X, \leq)$, write

$$\downarrow S = \{x \in X \mid \exists s \in S \text{ such that } x \leq s\}.$$

We say a subset S of a poset $(X, \leq)$ is *down closed* if $\downarrow S = S$. The operator $\downarrow(\;)$ is a closure operator.

Given any restriction category $\mathbf{X}$, we construct a category $\widehat{\mathbf{X}}$ with

objects: $X \in \mathbf{X}$;

maps: $S : A \to B$ are given by nonempty, down closed, and compatible sets $S \subseteq \mathbf{X}(A, B)$;

identities: $1_A = \downarrow\{1_A\} = \{e | e = \overline{e} : A \to A \text{ in } \mathbf{X}\}$;

composition: for any maps $S : A \to B$ and $T : B \to C$ in $\widehat{\mathbf{X}}$, $TS = \{ts | s \in S,\ t \in T\}$;

restriction: $\overline{S} = \{\overline{s} | s \in S\}$;

join: $\bigvee_{i\in\Gamma} S_i = \bigcup_{i\in\Gamma} S_i$, where $\{S_i\}_{i\in\Gamma}$ is a compatible set in $\widehat{\mathbf{X}}$.

For any maps $S : A \to B$ and $T : B \to C$ in $\widehat{\mathbf{X}}$, $TS = \{ts | s \in S,\ t \in T\}$ is down closed as if $f \leq ts$ then $f = t \cdot s\overline{f}$ but $s\overline{f} \in S, t \in T$. To show the join in $\widehat{\mathbf{X}}$ is well-defined, the following observation is crucial:

3.29. Lemma. *For any map $S, T : A \to B$ in $\widehat{\mathbf{X}}$, $S \smile T$ in $\widehat{\mathbf{X}}$ if and only if $s \smile t$ in $\mathbf{X}$ for all $s \in S$ and $t \in T$.*

Proof. If $S \smile T$ in $\widehat{\mathbf{X}}$, then $S\overline{T} = T\overline{S}$. For any $s \in S$ and $t \in T$, since $s\overline{t} \in S\overline{T} = T\overline{S}$, there are $s' \in S, t' \in T$ such that $s\overline{t} \leq t'\overline{s'}$. Hence, by using $t \smile t'$, that is $t\overline{t'} = t'\overline{t}$,

$$s\overline{t} = t'\overline{s'}\,\overline{s\overline{t}} = t'\overline{s'}\,\overline{s}\overline{t} = t'\overline{t}\;\overline{s'}\overline{s} = t\overline{t'}\;\overline{s'}\overline{s} = t\overline{s}(\overline{t'}\;\overline{s'}) \leq t\overline{s}.$$

By symmetry, we have $t\overline{s} \leq s\overline{t}$, $t\overline{s} = s\overline{t}$ and $s \smile t$.

Conversely, if for any $s \in S, t \in T$, $s \smile t$, then $s\overline{t} = \overline{s}t$ and so

$$S\overline{T} = \{s\overline{t} \mid s \in S, t \in T\} = \{t\overline{s} \mid s \in S, t \in T\} = T\overline{S}.$$

Thus, $S \smile T$ in $\widehat{\mathbf{X}}$, as desired. ∎

It now is straightforward to verify that $\widehat{\mathbf{X}}$ is indeed a join restriction category and that $\mathbf{X}$ can be embedded into $\widehat{\mathbf{X}}$ with the faithful functor $I : \mathbf{X} \to \widehat{\mathbf{X}}$ given by sending $f : A \to B$ to $\downarrow\{f\} : A \to B$. Moreover, when $\mathbf{X}$ is a unitary restriction category $\widehat{\mathbf{X}}$ is lax fibered as shown in the following proposition.

3.30. PROPOSITION. *If $\mathbf{X}$ is a unitary restriction category, then $\widehat{\mathbf{X}}$ is a bounded unitary restriction category and $\widehat{\mathbf{X}}_{\mathsf{max}} = \mathbf{X}/\smile$*

PROOF. For any map $S : A \to B$ in $\widehat{\mathbf{X}}$, write

$$\mathcal{M}_S = \{\text{down closed and compatible } X \subseteq \mathbf{X}(A, B) \text{ with } S \subseteq X\}.$$

If $X_1, X_2 \in \mathcal{M}_S$, then, for each $x_1 \in X_1$, $x_2 \in X_2$, $x_1 \smile s$ and $x_2 \smile s$ for some $s \in S$ and so $x_1 \smile x_2$. Hence $X_1 \smile X_2$ and therefore $\mathcal{M}_S$ is compatible in $\widehat{\mathbf{X}}$. Thus $\bigvee_{X\in\mathcal{M}_S} X = \bigcup_{X\in\mathcal{M}_S} X$ exists in $\widehat{\mathbf{X}}$. Now it is routine to check that $m_S = \bigcup_{X\in\mathcal{M}_S} X$. So [**M.1**] is satisfied. Hence $\widehat{\mathbf{X}}$ is a bounded unitary restriction category.

For the last observation a maximal map in $\widehat{\mathbf{X}}$ is a maximal compatible set which in a unitary restriction category is a $\smile$-equivalence class. ■

3.31. THEOREM. *A restriction category is unitary if and only if it is a restriction subcategory of a lax restriction fibered category (or a bounded unitary category).*

PROOF.

($\Leftarrow$) By Proposition 3.24.

($\Rightarrow$) Let $\mathbf{X}$ be a unitary restriction category. Since the functor $I : \mathbf{X} \to \widehat{\mathbf{X}}$, given by sending $f : A \to B$ to $\downarrow\{f\} : A \to B$, is faithful, $\mathbf{X}$ is a subcategory of $\widehat{\mathbf{X}}$. But $\widehat{\mathbf{X}}$ is lax fibered by Proposition 3.30. Hence $\mathbf{X}$ is a restriction subcategory of a lax restriction fibered category $\widehat{\mathbf{X}}$.

■

4. Free Restriction Categories

In this section, we return to the consider the construction of a free restriction category from a category. We start by filling in a missing step: namely generating a free stable semilattice fibration from a category. The section ends with an explicit description of the free restriction category on a graph.

4.1. THE FREE STABLE MEET SEMILATTICE FIBRATION ON A CATEGORY. The objective of this subsection is to provide a left adjoint to $U_f : \mathbf{sFib}_0 \to \mathbf{Cat}_0$, the base functor, which sends $(F, F') : (\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}) \to (\delta_{\mathbf{Y}} : \tilde{\mathbf{Y}} \to \mathbf{Y})$ to $F : \mathbf{X} \to \mathbf{Y}$.

Each category gives rise to a canonical stable meet semilattice fibration. In order to see this, we first recall Cockett-Lack's *closure operator* $\Downarrow(\ )$. Let $\mathbf{C}$ be a category,

$K = \{f_i : X \to Z_i \mid i \in I\}$ a set of maps with domain X, and $g : Y \to X$ a map. Then we write Kg for the set $\{f_i g \mid i \in I\}$, and $\Downarrow(K)$ for the set $\{f : X \to Z \mid uf = f_i \text{ for some } i \in I \text{ and some } u : Z \to Z_i\}$. Suppose that K and L are sets of maps with domain X. Clearly, if $K \subseteq L$ then $\Downarrow(K) \subseteq \Downarrow(L)$.

4.2. LEMMA. *For any category, $\Downarrow(\ )$ is a Kuratowski closure operator on the maps with domain X. Namely, if K, K_1, and K_2 are sets of maps with domain X, then*

$$\Downarrow(\emptyset) = \emptyset, \Downarrow(K_1 \cup K_2) = \Downarrow(K_1) \cup \Downarrow(K_2), K \subseteq \Downarrow(K), \Downarrow(\Downarrow(K)) = \Downarrow(K).$$

PROOF. See [3]. ∎

To form the category $\mathbf{s}(\mathbf{C})$ below, we need:

4.3. LEMMA. *If $\Downarrow(K) = \Downarrow(K')$, then $\Downarrow(Kf) = \Downarrow(K'f)$.*

PROOF. If $x \in \Downarrow(Kf)$, then $ux = kf$ for some u and $k \in K$. As certainly $k \in \Downarrow(K)$ there is a v such that $vk \in K'$ but then $vux = vkf$ so that $x \in \Downarrow(K'f)$. Hence $\Downarrow(Kf) \subseteq \Downarrow(K'f)$. Similarly, $\Downarrow(K'f) \subseteq \Downarrow(Kf)$. Therefore, $\Downarrow(Kf) = \Downarrow(K'f)$. ∎

Remark. In particular, by Lemma 4.3, $\Downarrow((\Downarrow(K))f) = \Downarrow(Kf)$ and so $\Downarrow(f) : Y/\mathbf{C} \to X/\mathbf{C}$ given by $h \mapsto hf$, is a continuous function, corresponding to topology provided by $\Downarrow(\ )$. Hence $\Downarrow(\ ) : \mathbf{C}^{\mathrm{op}} \to \mathbf{Top}$ is a functor.

Now, we form $\mathbf{s}(\mathbf{C})$ by the following data:

objects: $(X, \Downarrow\{x_1, \cdots, x_m\})$, where $X \in \mathrm{ob}(\mathbf{C})$ and $\{x_1, \cdots, x_m\} \subseteq \mathrm{ob}(X/\mathbf{C})$,

maps: a map from $(X, \Downarrow\{x_1, \cdots, x_m\})$ to $(Y, \Downarrow\{y_1, \cdots, y_n\})$ is a map $f : X \to Y$ in $\mathbf{C}$ such that $\Downarrow\{y_1 f, \cdots, y_n f\} \subseteq \Downarrow\{x_1, \cdots, x_m\}$,

composition and **identities** are formed as in $\mathbf{C}$.

By Lemma 4.3, $\mathbf{s}(\mathbf{C})$ is a category. Clearly, there is a forgetful functor $\Delta_{\mathbf{C}} : \mathbf{s}(\mathbf{C}) \to \mathbf{C}$, which forgets the sieves $\Downarrow(K)$.

4.4. LEMMA. *Let $\mathbf{C}$ be a category. Then the forgetful functor $\Delta_{\mathbf{C}} : \mathbf{s}(\mathbf{C}) \to \mathbf{C}$ is a stable meet semilattice fibration.*

PROOF. For any map $f : X \to Y$ in $\mathbf{C}$ and any object $(Y, \Downarrow\{y_1, \cdots, y_n\}) \in \Delta_{\mathbf{C}}^{-1}(Y)$, clearly $f : (X, \Downarrow\{y_1 f, \cdots, y_n f\}) \to (Y, \Downarrow\{y_1, \cdots, y_n\})$ is a map of $\mathbf{s}(\mathbf{C})$. Moreover,

$$f : (X, \Downarrow\{y_1 f, \cdots, y_n f\}) \to (Y, \Downarrow\{y_1, \cdots, y_n\})$$

is the cartesian lifting of a map $f : X \to Y$ at $(Y, \Downarrow\{y_1, \cdots, y_n\})$. Hence $\Delta_{\mathbf{C}} : \mathbf{s}(\mathbf{C}) \to \mathbf{C}$ is a fibration. Note that each fiber

$$\Delta_{\mathbf{C}}^{-1}(X) = \{(X, \Downarrow\{x_1, \cdots, x_m\}) \mid \{x_1, \cdots, x_m\} \subseteq \mathrm{map}(X/\mathbf{C})\}$$

is a meet semilattice with the order given by

$$(X, \Downarrow\{x_1, \cdots, x_m\}) \le (X, \Downarrow\{x'_1, \cdots, x'_l\}) \Leftrightarrow \Downarrow\{x'_1, \cdots, x'_l\} \subseteq \Downarrow\{x_1, \cdots, x_m\},$$

with the binary meet given by

$$(X, \Downarrow\{x_1, \cdots, x_m\}) \wedge (X, \Downarrow\{x'_1, \cdots, x'_l\}) = (X, \Downarrow\{x_1, \cdots, x_m, x'_1, \cdots, x'_l\}),$$

and with $(X, \Downarrow\{1_X\})$ as the top element. Clearly, for any map $f : X \to Y$ in $\mathbf{C}$, $f^* : \Delta_{\mathbf{C}}^{-1}(Y) \to \Delta_{\mathbf{C}}^{-1}(X)$, which takes $(Y, \Downarrow\{y_1, \cdots, y_n\})$ to $(X, \Downarrow\{y_1 f, \cdots, y_n f\})$, is a stable meet semilattice homomorphism. Therefore, $\Delta_{\mathbf{C}} : \mathbf{s}(\mathbf{C}) \to \mathbf{C}$ is a stable meet semilattice fibration. ∎

In order to prove the universal property of U_f, we need:

4.5. Lemma. *For any stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ and any functor $G : \mathbf{C} \to \mathbf{X}$, there is a unique functor $G' : \mathbf{s}(\mathbf{C}) \to \tilde{\mathbf{X}}$ such that $(G, G') : \Delta_{\mathbf{C}} \to \delta_{\mathbf{X}}$ is a map in $\mathbf{sFib}_0$.*

Proof. For any map $f : (C, \Downarrow\{c_1, \cdots, c_m\}) \to (D, \Downarrow\{d_1, \cdots, d_n\})$ in $\mathbf{s}(\mathbf{C})$, where $c_i : C \to X_i$ and $d_j : D \to Y_j$ are maps in $\mathbf{C}$, $i = 1, \cdots, m, j = 1, \cdots, n$, we have $\Downarrow\{d_1 f, \cdots, d_n f\} \subseteq \Downarrow\{c_1, \cdots, c_m\}$ and the following commutative diagram:

$$\begin{tikzcd}
 & (C, \Downarrow\{c_1, \cdots, c_m\}) \arrow[dl, "1_C"'] \arrow[dr, "f"] & \\
(C, \Downarrow\{d_1 f, \cdots, d_n f\}) \arrow[rr, "f"] & & (D, \Downarrow\{d_1, \cdots, d_n\})
\end{tikzcd}$$

where $(C, \Downarrow\{d_1 f, \cdots, d_n f\}) = f^*(D, \Downarrow\{d_1, \cdots, d_n\})$. Now we define the functor $G' : \mathbf{s}(\mathbf{C}) \to \tilde{\mathbf{X}}$ by sending

$$f : (C, \Downarrow\{c_1, \cdots, c_m\}) \to (D, \Downarrow\{d_1, \cdots, d_n\})$$

to

$$\vartheta_{G(f)} \le : \wedge_{i=1}^m (G(c_i))^* \top_{\delta_{\mathbf{X}}^{-1}(G(X_i))} \to \wedge_{j=1}^n (G(d_j))^* \top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))},$$

which is given by the following commutative diagram in $\tilde{\mathbf{X}}$:

$$\begin{tikzcd}
 & \wedge_{i=1}^m (G(c_i))^* \top_{\delta_{\mathbf{X}}^{-1}(G(X_i))} \arrow[dl, "\le"'] \arrow[dr, "G'(f)"] & \\
(G(f))^*(\wedge_{j=1}^n (G(d_j))^* \top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))}) \arrow[rr, "\vartheta_{G(f)}"] & & \wedge_{j=1}^n (G(d_j))^* \top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))}
\end{tikzcd}$$

It is routine to check that $(G, G') : \Delta_{\mathbf{C}} \to \delta_{\mathbf{X}}$ satisfies the conditions [**sfM.1**], [**sfM.2**], and [**sfM.3**] and it is a map in $\mathbf{sFib}_0$.

Assume that $G'' : \mathbf{s}(\mathbf{C}) \to \tilde{\mathbf{X}}$ is a functor such that $(G, G'') : \Delta_{\mathbf{C}} \to \delta_{\mathbf{X}}$ is a map in $\mathbf{sFib}_0$. Let $f : (C, \Downarrow\{c_1, \cdots, c_m\}) \to (D, \Downarrow\{d_1, \cdots, d_n\})$ be a map in $\mathbf{s}(\mathbf{C})$. By [**sfM.1**], $G''(C, \Downarrow\{1_C\}) = G''(\top_{\Delta_{\mathbf{C}}^{-1}(C)}) = \top_{\delta_{\mathbf{X}}^{-1}(G(C))}$. By [**sfM.3**],

$$G''(C, \Downarrow\{c_1\}) = G''(c_1^*(X_1, \Downarrow\{1_{X_1}\})) = (G(c_1))^*(G''(X_1, \Downarrow\{1_{X_1}\})) = (G(c_1))^*\top_{\delta_{\mathbf{X}}^{-1}(G(X_1))}.$$

By [**sfM.2**],

$$\begin{aligned} G''(C, \Downarrow\{c_1, \cdots, c_m\}) &= G''(\wedge_{i=1}^m (C, \Downarrow\{c_i\})) \\ &= \wedge_{i=1}^m G''(C, \Downarrow\{c_i\}) \\ &= \wedge_{i=1}^m (G(c_i))^*\top_{\delta_{\mathbf{X}}^{-1}(G(X_i))}. \end{aligned}$$

Hence $G''(C, \Downarrow\{c_1, \cdots, c_m\}) = G'(C, \Downarrow\{c_1, \cdots, c_m\})$. Similarly,

$$G''(D, \Downarrow\{d_1, \cdots, d_n\}) = G'(D, \Downarrow\{d_1, \cdots, d_n\}).$$

Since

$$\begin{tikzcd} \mathbf{s}(\mathbf{C}) \arrow[r, "G''"] \arrow[d, "\Delta_{\mathbf{C}}"'] & \tilde{\mathbf{X}} \arrow[d, "\delta_{\mathbf{X}}"] \\ \mathbf{C} \arrow[r, "G"'] & \mathbf{X} \end{tikzcd}$$

commutes, for any map $f : (C, \Downarrow\{c_1, \cdots, c_m\}) \to (D, \Downarrow\{d_1, \cdots, d_n\})$ in $\mathbf{s}(\mathbf{C})$, we have $\delta_{\mathbf{X}} G''(f) = G\Delta_{\mathbf{C}}(f) = G(f)$. Hence there is a unique map

$$h : \wedge_{j=1}^n (G(d_j f))^*\top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))} \to \wedge_{j=1}^n (G(d_j f))^*\top_{\delta_{\mathbf{X}}^{-1}(G(Y_j))}$$

in $\delta_{\mathbf{X}}^{-1}(G(C))$ such that $\delta_{\mathbf{X}}(h) = 1_{G(C)}$ and $\vartheta_{G(f)} h = G''(f)$. Since $\delta_{\mathbf{X}}$ is a stable meet semilattice fibration, $h = \leq$. Then, by the definition of $G'(f)$, $G''(f) = \vartheta_{G(f)} h = \vartheta_{G(f)} \leq = G'(f)$ and so the uniqueness of G' follows. ∎

Let $F_f(\mathbf{C}) = \Delta_{\mathbf{C}}$ for each category $\mathbf{C}$. Now we are ready to prove $F_f \dashv U_f$.

4.6. Theorem. *There is an adjunction:* $\begin{tikzcd} \mathbf{sFib}_0 \arrow[r, bend right, "U_f"'] \arrow[r, phantom, "\perp"] & \mathbf{Cat}_0 \arrow[l, bend right, "F_f"'] \end{tikzcd}$ *with the identity unit* $\eta_{\mathbf{C}} = 1_{\mathbf{C}}$.

Proof. For any category $\mathbf{C}$, clearly $U_f F_f(\mathbf{C}) = \mathbf{C}$. For any stable meet semilattice fibration $\delta_{\mathbf{X}} : \tilde{\mathbf{X}} \to \mathbf{X}$ and any functor $G : \mathbf{C} \to U_f(\delta_{\mathbf{C}})$, by Lemma 4.5 we have a unique map $G^* = (G, G') : \Delta_{\mathbf{C}} \to \delta_{\mathbf{X}}$ in $\mathbf{sFib}_0$ such that

$$\begin{tikzcd} \mathbf{C} \arrow[r, "\eta_{\mathbf{C}} = 1_{\mathbf{C}}"] \arrow[dr, "G"'] & U_f F_f(\mathbf{C}) \arrow[d, "U_f(G^*)"] & F_f(\mathbf{C}) \arrow[d, dotted, "\exists! \, G^*"] \\ & U_f(\delta_{\mathbf{X}}) & \delta_{\mathbf{X}} \end{tikzcd}$$

commutes. Hence $F_f \dashv U_f$ with the unit $\eta_{\mathbf{C}} = 1_{\mathbf{C}}$. ∎

4.7. The Free Restriction Category on a Category. Recall that there is an evident forgetful functor $U_r : \mathbf{rCat}_0 \to \mathbf{Cat}_0$, which forgets restriction structures. In [3], Cockett and Lack gave the explicit description of the free restriction categories over categories. In this subsection, as an application of Theorems 2.12 and 4.6, we shall reproduce Cockett-Lack's free restriction category $F_r(\mathbf{C})$ using the free stable meet semilattice fibration $\Delta_{\mathbf{C}}$ for any given category $\mathbf{C}$.

By Theorem 2.12, there is an adjunction: $\mathbf{rCat}_0 \underset{\mathcal{R}}{\overset{\mathcal{S}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{sFib}_0$ with a faithful functor $\mathcal{R}$.

By Theorem 4.6, there is an adjunction: $\mathbf{sFib}_0 \underset{U_f}{\overset{F_f}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathbf{Cat}_0$ with the identity unit $\eta_{\mathbf{C}} = 1_{\mathbf{C}}$.

Given Theorem 2.12 and Theorem 4.6, we define F_r to be the composite $\mathcal{S}F_f : \mathbf{Cat}_0 \to \mathbf{rCat}_0$ and U_r to be the composite $U_f\mathcal{R} : \mathbf{rCat}_0 \to \mathbf{Cat}_0$ which clearly just forgets the restriction structure. So define $F_r \dashv U_r$ as adjoints composite.

Explicitly, for a given category $\mathbf{C}$, $F_r(\mathbf{C})$ is the category with

- the same objects as $\mathbf{C}$,
- the map from C to D being a pair of $(f, \Downarrow(K))$, where $f : C \to D$ is a map of $\mathbf{C}$ and K is a set of maps in $\mathbf{C}$ with domain C such that $f \in \Downarrow(K)$ and $|K| < +\infty$,
- the composition given by

$$(g, \Downarrow(L))(f, \Downarrow(K)) = (gf, \Downarrow(\Downarrow(K) \cup (\Downarrow f)\Downarrow(L))) = (gf, \Downarrow(K \cup Lf)),$$

- the identities given by $1_C = (1_C, \Downarrow\{1_C\})$.

$F_r(\mathbf{C})$ is precisely Cockett-Lack's free restriction category over $\mathbf{C}$ as described in [3].

4.8. Theorem. [Cockett-Lack [3]] $U_r : \mathbf{rCat}_0 \to \mathbf{Cat}_0$ *has a left adjoint.*

The free category generated by a graph gives $F_c \dashv U_c : \mathbf{Cat}_0 \to \mathbf{Graph}$. This, combining with Theorems 2.12 and 4.6, yields immediately the following theorem.

4.9. Theorem. *There is a sequence of adjunctions:*

$$\begin{tikzcd} \mathbf{Graph} \arrow[r, shift left, "F_c"] & \mathbf{Cat}_0 \arrow[l, shift left, "U_c"] \arrow[r, shift left, "F_f"] & \mathbf{sFib}_0 \arrow[l, shift left, "U_f"] \arrow[r, shift left, "\mathcal{S}"] & \mathbf{rCat}_0 \arrow[l, shift left, "\mathcal{R}"] \end{tikzcd}$$

Theorem 2.28 [3] states that $U_f\mathcal{R} : \mathbf{rCat}_0 \to \mathbf{Cat}_0$ is monadic. In Subsection 3.1, we noticed that $\mathcal{R}$ is monadic. We remark here, however, that U_f is certainly not monadic since $U_fF_f = 1_{\mathbf{Cat}_0}$.

4.10. The Free Restriction Category on a Graph. By Theorem 4.9, $U_{rc} = U_c U_f \mathcal{R}$: $\mathbf{rCat}_0 \to \mathbf{Graph}$ has a left adjoint $F_{rc} = \mathcal{S} F_f F_c$ which is the free restriction category over a graph. In this subsection, we provide an explicit description of this construction. This construction is of some special interest as it is a predecessor to Munn's description of the free inverse semigroup [10].

Given any graph G, $F_c(G)$ is the category with the nodes of G as objects, with paths, p, from node A to node B giving the maps $(A, p, B) : A \to B$, and with the empty path on A giving the identity $(A, [\,], A) : A \to A$. Composition is given by concatenation of paths: $(B, q, C)(A, p, B) = (A, qp, C)$.

Applying F_f to $F_c(G)$ gives the free stable meet semilattice fibration over $F_c(G)$ and applying $\mathcal{S}$ to this gives the free restriction category $F_{rc}(G)$ generated by graph G. Explicitly, $F_{rc}(G)$ is the restriction category with:

objects: nodes of G,

maps: a map from A to B is given by a pair (s, S) of a finite prefix closed subset S of paths starting at A and a path $s \in S$ ending at B. Explicitly this means pairs (s, S) with $S \subseteq_f F_c(G)(A, -)$ with $s \in S$ and $s : A \to B$, where S is prefix closed in the sense that $pq \in S \Rightarrow p \in S$.

composition: $(t, T)(s, S) = (ts, S \cup Ts)$ (Note that $S \cup Ts$ is prefixed closed as if for any $p \in S \cup Ts$ and $p' \preceq p$ then $p \in S$ or $p \in Ts$ and so in both cases $p' \in S \cup Ts$ since both S and T are prefixed closed.),

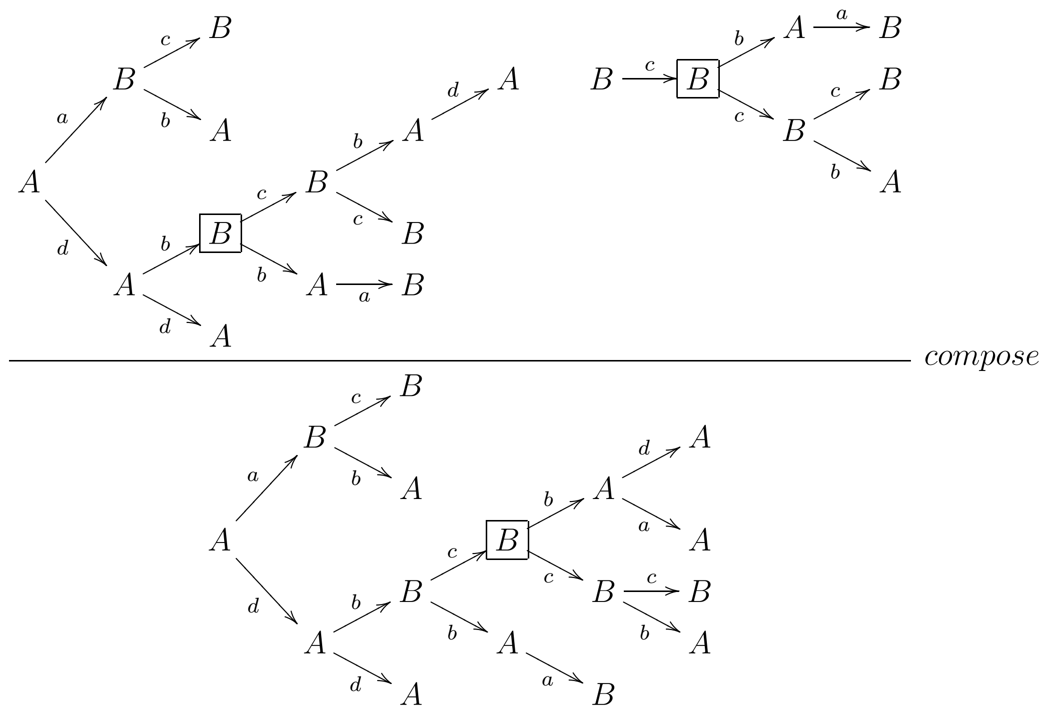
identities: $1_A = ((A, [\,], A), \{(A, [\,], A)\})$.

The maps, the composition, and restriction of $F_{rc}(G)$ can be described graphically. One can present the maps as a directed trees with a selected node together with a tree map into the graph G. The map starts at the graph node associated with the root of the tree and ends at the graph node associated with the selected node. The composition is given by "gluing" together at the selected node of the first and the root of second in a obvious way and removing repeated branches with the second selected node selected. The identity on a node A is given by the smallest tree starting at A: $\boxed{A}$. The restriction is given by making the selected node the root.

For example, if G is the graph given by

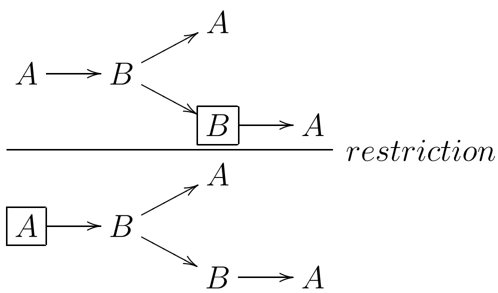
$$\begin{tikzcd} A \arrow[loop left, "d"] \arrow[r, bend left, "a"] & B \arrow[l, bend left, "b"] \arrow[loop right, "c"] \end{tikzcd}$$

then an example of a composition of a map from A to B and a map from B to B in the free restriction category generated by G is:



where the tree with selected node B at top-left is a map from A to B in the free restriction category, the tree at top-right is a map from B to B, and the result of the composition is below the line.

The restriction can be displayed as:



where the top is a map from A to B and its restriction, with the selected node move to the root is below the line.

Notice that in a restriction category generated by a graph, the only total map are the identity maps $(X, [\,], X)$. Thus the only monics are the identities: this is in contrast to the free category (or path category) in which all maps are monic.

References

[1] M. Barr and C. Wells, *Category Theory for Computing Science, the Electronic Supplement*, 1995

[2] J.R.B. Cockett and Xiuzhan Guo, *Introduction to range restriction categories*, in progress.

[3] J.R.B. Cockett and S. Lack, *Restriction categories I: Categories of partial maps*, Theoret. Comput. Science **270**(2002), 223-259.

[4] R.A. Di Paola and A. Heller, *Dominical categories: recursion theory without elements*, J. Symbolic Logic **52**(1987), 595-635.

[5] John Fountain, *Adequate semigroups*, Proceedings of the Edinburgh Mathematical Society **22**(1979), 113-125.

[6] Xiuzhan Guo, *Ranges, Restrictions, Partial Maps, and Fibrations*, Thesis, University of Calgary, 2004.

[7] M. Jackson and T. Stokes, *An invitation to C-semigroups*, Semigroup Form **62**(2001), 279-310.

[8] Mark Lawson, *Inverse Semigroups*, World Scientific (1998)

[9] E.G. Manes, *Guarded and banded semigroups*, preprint, 2004.

[10] W.D. Munn, *Free inverse semigroups*, Proc. London Math. Soc. (3)**29**(1974), 385-404.

[11] Mario Petrich, *Inverse semigroups*, John Wiley (1984)

[12] E.P. Robinson and G. Rosolini, *Categories of partial maps*, Infom. and Comput. 79(1988), 94-130.

[13] G. Rosolini, *Domains and dominical categories*, Riv. Mat. Univ. Parma 11(1985), 387-397.

[14] G. Rosolini, *Continuity and effectiveness in topoi*, D. Phil. Thesis, Oxford University, 1986.

Department of Computer Science
University of Calgary
2500 University Drive NW
Calgary, Alberta T2N 1N4
Canada
Email: `robin@cpsc.ucalgary.ca`
`guox@cpsc.ucalgary.ca`

This article may be accessed via WWW at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/16/15/16-15.{dvi,ps,pdf}`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.
The method of distribution of the journal is via the Internet tools WWW/ftp. The journal is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION. Individual subscribers receive (by e-mail) abstracts of articles as they are published. Full text of published articles is available in .dvi, Postscript and PDF. Details will be e-mailed to new subscribers. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, rrosebrugh@mta.ca.

INFORMATION FOR AUTHORS. The typesetting language of the journal is TEX, and LATEX2e is the preferred flavour. TEX source of articles for publication should be submitted by e-mail directly to an appropriate Editor. They are listed below. Please obtain detailed information on submission format and style files from the journal's WWW server at http://www.tac.mta.ca/tac/. You may also write to tac@mta.ca to receive details by e-mail.