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SPECTRA OF FINITELY GENERATED BOOLEAN FLOWS

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Abstract.

A **flow** on a compact Hausdorff space X is given by a map $t : X \to X$. The general goal of this paper is to find the "cyclic parts" of such a flow. To do this, we approximate (X, t) by a flow on a Stone space (that is, a totally disconnected, compact Hausdorff space). Such a flow can be examined by analyzing the resulting flow on the Boolean algebra of clopen subsets, using the spectrum defined in our previous paper, *The cyclic spectrum of a Boolean flow TAC 10* 392-419.

In this paper, we describe the cyclic spectrum in terms that do not rely on topos theory. We then compute the cyclic spectrum of any finitely generated Boolean flow. We define when a sheaf of Boolean flows can be regarded as cyclic and find necessary conditions for representing a Boolean flow using the global sections of such a sheaf. In the final section, we define and explore a related spectrum based on minimal subflows of Stone spaces.

1. Introduction

This paper continues the research started in [Kennison, 2002]. The underlying issues we hope to address are illustrated by considering "flows in compact Hausdorff spaces" or maps $t : X \to X$ where X is such a space. Each $x \in X$ has an *orbit* $\{x, t(x), t^2(x), \ldots, t^n(x), \ldots\}$ and we want to know when it is reasonable to say that this orbit is "close to being cyclic". We also want to break X down into its "close-to-cyclic" components. To do this, we approximate X by a Stone space, which has an associated Boolean algebra to which we can apply the cyclic spectrum defined in [Kennison, 2002]. In section 4, we examine ways of computing the cyclic spectrum and give a complete description of it for Boolean flows that arise from symbolic dynamics. Section 5 discusses necessary conditions for cyclic representations. Section 6 considers the "simple spectrum" which is richer than the cyclic spectrum.

We have tried to present this material in a way that is understandable to experts in dynamical systems who are not specialists in category theory. (We do assume some basic category theory, as found in [Johnstone, 1982, pages 15–23]. For further details, [Mac Lane, 1971] is a good reference.) In section 3, we define the cyclic spectrum construction

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without using topos theory. In that section, we review the basic notion of a sheaf over a locale. For details, see the book on Stone spaces, [Johnstone, 1982], which provides a readable treatment of the ideas and techniques used in this paper.

As discussed in section 2, the use of symbolic dynamics allows us to restrict our attention to flows $t : X \to X$ where X is a *Stone space*, which means it is totally disconnected in addition to being compact and Hausdorff. But if X is a Stone space, then X is determined by the Boolean algebra, $\mathrm{Clop}(X)$, of its clopen subsets (where "clopen" subsets are both closed and open). By the **Stone Representation Theorem**, Clop is contravariantly functorial and sets up an equivalence between the category of Stone spaces and the dual of the category of Boolean algebras.

It follows that $t : X \to X$ gives rise to a Boolean homomorphism $\tau : B \to B$ where $B = \mathrm{Clop}(X)$ and $\tau = \mathrm{Clop}(t) = t^{-1}$. Mapping a flow from one category to another is significant because the notion of a cyclic flow depends on the ambient category. We recall the following definition from [Kennison, 2002]. In doing so, we adopt the useful term *iterator* from [Wojtowicz, 2004] and otherwise use the notational conventions adopted in [Kennison, 2002]. So if f and g are morphisms from an object X to an object Y, then $\mathrm{Equ}(f, g)$ is their *equalizer* (if it exists). If $\{A_\alpha\}$ is a family of subobjects of X, then $\bigvee\{A_\alpha\}$ is their supremum (if it exists) in the partially ordered set of subobjects of X.

1.1. Definition. *The pair (X, t) is a* **flow** *in a category C if X is an object of C and $t : X \to X$ is a morphism, called the* **iterator**. *If (X, t) and (Y, s) are flows in C, then a* **flow homomorphism** *is a map $h : X \to Y$ for which $sh = ht$. We let $\mathrm{Flow}(C)$ denote the resulting category of flows in C.*

We say that $(X, t) \in \mathrm{Flow}(C)$ is **cyclic** *if $\bigvee \mathrm{Equ}(\mathrm{Id}_X, t^n)$ exists and is X (the largest subobject of X).*

In listing some examples from [Kennison, 2002], it is convenient to say that if S is a set (possibly with some topological or algebraic structure) and if $t : S \to S$, then $s \in S$ is *periodic* if there exists $n \in \mathbf{N}$ with $t^n(s) = s$.

- A flow (S, t) in Sets is cyclic if and only if every element of S is periodic.
- A flow (X, t) in the category of Stone spaces is cyclic if and only if the periodic elements of X are dense.
- A flow (B, τ) in the category of Boolean algebras is cyclic if and only if every element of B is periodic.
- A flow (X, t) in Stone spaces is "Boolean cyclic" (meaning that $\mathrm{Clop}(X, t)$ is cyclic in Boolean algebras) if and only if the group of profinite integers, $\widehat{Z}$, acts continuously on X in a manner compatible with t. (There is an embedding $\mathbf{N} \subseteq \widehat{Z}$ and an action $\alpha : \widehat{Z} \times X \to X$ is compatible with the action of t if $\alpha(n, x) = t^n(x)$ for all $x \in X$ and all $n \in \mathbf{N}$. Since $\mathbf{N}$ is dense in $\widehat{Z}$, there is at most one such continuous action by $\widehat{Z}$. For details, see [Kennison, 2002]).

- Let $t : S \to S$ be given where S is a set. Then (S, t) is a cyclic flow in the dual of the category of Sets if and only if t is one-to-one.

We are primarily interested in Boolean flows, or flows (B, τ), in the category of Boolean algebras. We sometimes say that "B is a Boolean flow", in which case the iterator (always denoted by τ) is left implicit. Similarly, the iterator for a Stone space will generally be denoted by t. For those interested in pursuing topos theory, we recommend [Johnstone, 1977], [Barr & Wells, 1985] and [Mac Lane & Moerdijk, 1992] while [Johnstone, 2002] is a comprehensive, but readable reference.

2. Symbolic dynamics and flows in Stone spaces

Symbolic dynamics have often been used to show that certain dynamical systems, or flows in topological spaces, are chaotic, as in [Devaney, 1986] and [Preston, 1983]. We will use symbolic dynamics to approximate a flow on a compact Hausdorff space by a flow on a Stone space. An ad hoc process for doing this was used in [Kennison, 2002]; here we are more systematic. Although we will not use this fact, it has been noted in [Lawvere, 1986] and exploited in [Wojtowicz, 2004], that symbolic dynamics is based on the functor from C to $\mathrm{Flow}(C)$ that is right adjoint to the obvious functor from $\mathrm{Flow}(C)$ to C.

2.1. Definition. *Let S be any finite set whose elements will be called "symbols". Then $S^{\mathbf{N}}$ is the Stone space of all sequences $(s_1, s_2, \ldots s_n, \ldots)$ of symbols. Let* $\mathrm{Sym}(S)$ *be the flow consisting of the space $S^{\mathbf{N}}$ together with the "shift map" t as iterator, where $t(s_1, s_2, \ldots s_n, \ldots) = (s_2, s_3, \ldots s_{n+1}, \ldots)$. Then* $\mathrm{Sym}(S)$ *is called the* **symbolic flow** *generated by the symbol set S.*

2.2. Definition. [Method of Symbolic Dynamics]

Let (X, t) be a flow in compact Hausdorff spaces. Let $X = A_1 \cup A_2 \cup \ldots \cup A_n$ represent X as a finite union of closed subsets. (It is not required that the sets $\{A_i\}$ be disjoint, but in practice they have as little overlap as possible.) Let $S = \{1, 2, \ldots n\}$. A sequence $s = (s_1, s_2, \ldots s_n, \ldots)$ in $\mathrm{Sym}(S)$ *is said to be* **compatible** *with $x \in X$ if $t^n(x) \in A_{s_n}$ for all $n \in \mathbf{N}$. We let $\widehat{X}$ denote the set of all sequences in* $\mathrm{Sym}(S)$ *that are compatible with at least one $x \in X$. Then $\widehat{X}$ is readily seen to be a closed subflow of* $\mathrm{Sym}(S)$.

2.3. Remark. It often happens that each $s \in \mathrm{Sym}(S)$ is compatible with at most one $x \in X$ in which case there is an obvious flow map from $\widehat{X}$ to X.

2.4. Definition. *Let (B, τ) be a flow in Boolean algebras. Then a Boolean subalgebra $A \subseteq B$ is a* **subflow** *if $\tau(a) \in A$ whenever $a \in A$.*

We say that (B, τ) is **finitely generated as a flow** *if there is a finite subset $G \subseteq B$ such that if A is a subflow of B with $G \subseteq A$ then $A = B$.*

2.5. Proposition. *Let S be a finite set. Let (X, t) be any closed subflow of* $\mathrm{Sym}(S)$. *Then $(B, \tau) = \mathrm{Clop}(X, t)$ is a finitely generated Boolean flow.*

Proof. We first consider the case where X is all of $\mathrm{Sym}(S)$. For each $n \in \mathbf{N}$, let $\pi_n : \mathrm{Sym}(S) \to S$ be the n^{th} projection, which maps the sequence $s = (s_1, s_2, \ldots s_n, \ldots)$ to s_n. Let $G = \{\pi_1^{-1}(s) \mid s \in S\}$ which is clearly a finite family of clopen subsets of $\mathrm{Sym}(S)$. Note that $\tau^n(\pi_1^{-1}(s)) = \pi_n^{-1}(s)$ so any subflow of B which contains G must also contain all of the subbasic open sets $\pi_n^{-1}(s)$. It must also contain the base of all finite intersections of these sets, and all finite unions of these basic sets. Clearly these finite unions are precisely the clopens of $\mathrm{Sym}(S)$ because a clopen must, by compactness, be a finite union of basic opens.

Now suppose that (X, t) is a closed subflow of $\mathrm{Sym}(S)$. Then, by duality, $\mathrm{Clop}(X, t)$ is a quotient flow of $\mathrm{Clop}(\mathrm{Sym}(S))$ and so $\mathrm{Clop}(X, t)$ is finitely generated because a quotient of a finitely generated algebra is readily seen to be finitely generated. ∎

2.6. Corollary. *The spaces of the form* $\mathrm{Clop}(\widehat{X})$ *are finitely generated Boolean flows.*

3. Review of the cyclic spectrum

The cyclic spectrum of a Boolean flow can be thought of as a kind of "universal cyclic quotient flow". To explain what this means, consider the simpler concept of a "universal quotient flow" of a Boolean flow B. Of course, B does not have a single flow quotient but has a whole "spectrum" of quotients, which can all be written in the form B/I where I varies over the set of "flow ideals" of B (as defined below). The set of these ideals has a natural topology and the union of the quotients B/I forms a sheaf over the space of flow ideals. This sheaf has a universal property, given in Theorem 3.17 below, which justifies calling it the universal quotient flow.

The cyclic spectrum is also a sheaf, but it might be a sheaf over a "locale", which generalizes the concept of a sheaf over a topological space. The use of locales is suggested by topos theory and allows for a richer spectrum. In what follows, we will quickly outline the theory of sheaves (and sheaves with structure) over locales, construct the cyclic spectrum and then state and prove its universal property. For more details about sheaves, see [Johnstone, 1982, pages 169–180] and for further details, see the references given there.

We note that every Boolean algebra is a ring, with $a + b = (a \wedge \neg b) \vee (b \wedge \neg a)$ and $ab = a \wedge b$. We describe those ideals $I \subseteq B$ for which B/I has a natural flow structure:

3.1. Definition. *If (B, τ) is a Boolean flow, then $I \subseteq B$ is a* **flow ideal** *if it is an ideal such that B/I has a flow structure for which the quotient map $q : B \to B/I$ is a flow homomorphism. It readily follows that $I \subseteq B$ is a flow ideal if and only if:*

- $0 \in I$.
- *If $b \in I$ and $c \leq b$ then $c \in I$.*
- *If $b, c \in I$ then $(b \vee c) \in I$.*
- *If $b \in I$ then $\tau(b) \in I$.*

The flow ideal I is a **cyclic ideal** *of B if B/I is a cyclic flow, which means that for every $b \in B$ there exists $n \in \mathbf{N}$ with $b = \tau^n(b) \pmod{I}$. We say that I is a* **proper flow ideal** *if I is not all of B.*

The set of all flow ideals has a natural topology:

3.2. Definition. *Let $\mathcal{W}$ be the set of all flow ideals of a Boolean flow (B, τ). For each $b \in B$, let $N(b) = \{I \in \mathcal{W} \mid b \in I\}$. Then $N(b) \cap N(c) = N(b \vee c)$ so the family $\{N(b) \mid b \in B\}$ forms the base for a topology on $\mathcal{W}$.*

3.3. Remark. From now on, we assume that (B, τ) is a Boolean flow and that $\mathcal{W}$ is the space of all flow ideals of B with the above topology.

3.4. Proposition. *The space $\mathcal{W}$ of all flow ideals of B is compact (but generally not Hausdorff).*

Proof. Let $\mathcal{U}$ be an ultrafilter on $\mathcal{W}$. Define $I_{\mathcal{U}}$ so that $b \in I_{\mathcal{U}}$ if and only if $N(b) \in \mathcal{U}$. It is readily checked that $I_{\mathcal{U}}$ is a flow ideal of B and $\mathcal{U}$ converges to $I \in \mathcal{W}$ if and only if $I \subseteq I_{\mathcal{U}}$. ∎

In addition to the topological structure on $\mathcal{W}$, there is a natural sheaf B^0 over $\mathcal{W}$. While here we will show there is a natural local homeomorphism from B^0 to $\mathcal{W}$, we will later give a different, but equivalent, definition of "sheaf" in terms of sections.

3.5. Proposition. *(Let $\mathcal{W}$ be the space of all flow ideals of a Boolean flow (B, τ).) Let B^0 be the disjoint union $\bigcup\{B/I \mid I \in \mathcal{W}\}$. Define $p : B^0 \to \mathcal{W}$ so that $B/I = p^{-1}(I)$ for all $I \in \mathcal{W}$. For each $b \in B$ define a map $\widehat{b} : \mathcal{W} \to B^0$ so that $\widehat{b}(I)$ is the image of b under the canonical map $B \to B/I$. We give B^0 the largest topology for which all of the maps $\{\widehat{b} \mid b \in B\}$ are continuous. Then $p : B^0 \to \mathcal{W}$ is a local homeomorphism over $\mathcal{W}$.*

Proof. This is a standard type of argument and the proof is a bit tedious but straightforward. Note that a basic neighborhood of $\widehat{b}(I) \in B/I$ is given by $\widehat{b}[N(c)]$ for $c \in B$. Also note that for $b, c \in B$, the maps $\widehat{b}$ and $\widehat{c}$ coincide on the open set $N(b + c)$. ∎

We note that the maps $\widehat{b}$ in the above proof are examples of "sections". The following definition is useful:

3.6. Definition. *Assume that $p : E \to X$ is a local homeomorphism over X. Suppose $U \subseteq X$ is an open subset. Then a continuous map $g : U \to E$ is a* **section** *over U if $pg = \mathrm{Id}_U$.*

We let $\mathcal{O}(X)$ denote the lattice of all open subsets of X and, for each $U \in \mathcal{O}(X)$ we let $\Gamma(U)$ denote the set of sections over U. We note that if $U, V \in \mathcal{O}(X)$ are given, with $V \subseteq U$, there is a restriction map $\rho^U_V : \Gamma(U) \to \Gamma(V)$.

By a **global section** *we mean a section over the largest open set, X itself. So $\Gamma(X)$, or sometimes, $\Gamma(E)$, denotes the set of all global sections.*

The structure of the sets $\Gamma(U)$ and the restriction maps ρ^U_V determine the sheaf (to within isomorphism).

3.7. Definition. *Let $\mathcal{O}(X)$ be the lattice of all open subsets of a space X. We will say that G is a* **sheaf** *over X if for each $U \in \mathcal{O}(X)$, we have a set $G(U)$ and if whenever $V \subseteq U$, for $U, V \in \mathcal{O}(X)$, there is a* **restriction map** $\rho^U_V : G(U) \to G(V)$ *such that the following conditions are satisfied:*

- **(Restrictions are functorial)** *If $W \subseteq V \subseteq U$ then $\rho^V_W \rho^U_V = \rho^U_W$. Also $\rho^U_U = \mathrm{Id}_{\Gamma(U)}$.*
- **(The Patching Property)** *If $U = \bigcup\{U_\alpha\}$ and if $g_\alpha \in G(U_\alpha)$ is given for each α such that each g_α and g_β have the same restriction to $U_\alpha \cap U_\beta$, then there exists a unique $g \in G(U)$ whose restriction to each U_α is g_α.*

3.8. Proposition. *There is, to within isomorphism, a bijection between sheaves over a space X and local homeomorphisms over X.*

Proof. If $p : E \to X$ is a local homeomorphism, then we can let $G(U)$ denote the set of all sections over U and let ρ^U_V denote the actual restriction of sections over U to sections over V. It is obvious that this yields a sheaf over X as defined above. Conversely, it is well-known that every such sheaf arises from an essentially unique local homeomorphism, for example, see [Johnstone, 1982, page 172]. ■

The concept of a sheaf over X depends only on the lattice $\mathcal{O}(X)$ of all open subsets of X. The definition readily extends to any lattice which has the essential features of $\mathcal{O}(X)$, namely that it is a *frame*:

3.9. Definition. *A* **frame** *is a lattice having arbitrary sups (denoted by $\bigvee\{u_\alpha\}$), which satisfies the distributive law that:*

$$v \wedge \bigvee\{u_\alpha\} = \bigvee\{v \wedge u_\alpha\}$$

It follows that a frame has a largest element, **top**, *denoted by $\top$, which is the sup over the whole lattice, and a smallest element,* **bottom**, *denoted by $\perp$, which is the sup over the empty subset.*

A **frame homomorphism** *from F to G is a map $h : F \to G$ which preserves finite infs and arbitrary sups. In particular, a frame homomorphism preserves $\perp$ and $\top$, which are the sup and inf over the empty subset.*

Clearly, there is a **category of frames**, *whose morphisms are the frame homomorphisms. The* **category of locales** *is the dual of the category of frames. A locale is* **spatial** *if its corresponding frame is of the form $\mathcal{O}(X)$ for a topological space X.*

If X is a topological space, then $\mathcal{O}(X)$ is a frame. Moreover, if $f : X \to Y$ is continuous, then $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$ is a frame homomorphism. If we assume a reasonable separation axiom, known as "soberness" (or perhaps "sobriety"), see [Johnstone, 1982, pages 43–44], the space X is completely determined by the locale $\mathcal{O}(X)$ and the continuous functions $f : X \to Y$ by the frame homomorphisms $f^{-1} : \mathcal{O}(Y) \to \mathcal{O}(X)$. For this reason, we think of locales as generalized (sober) spaces. If X denotes a topological space, we will let X also denote the locale corresponding to the frame $\mathcal{O}(X)$. Nonetheless, we adopt the view, given in [Johnstone, 1982], that locales and frames are the same thing as objects, but differ only when we consider morphisms. If L is a locale, then L is also a frame and the notation $u \in L$ will refer to a member of the frame. (The only exception is the case of the locale associated with a space X. Because of the difference between saying $u \in X$ and $u \in \mathcal{O}(X)$ we usually use X when thinking of the space as a locale and $\mathcal{O}(X)$ for the corresponding frame.)

3.10. Definition. *If L is a locale, then a* **presheaf** *G over L assigns a set $G(u)$ to each $u \in L$ and restriction maps $\rho^u_v : G(u) \to G(v)$ whenever $v \leq u$ which are functorial (meaning that $\rho^v_w \rho^u_v = \rho^u_w$ whenever $w \leq v \leq u$, and $\rho^u_u = \mathrm{Id}_{G(u)}$ for all u).*

A presheaf is a **sheaf** *if it has the* **patching property** *(meaning that if $u = \bigvee\{u_\alpha\}$ and if $g_\alpha \in G(u_\alpha)$ is given for each α such that each g_α and g_β have the same restriction to $u_\alpha \wedge u_\beta$, there then exists a unique $g \in G(u)$ whose restriction to each u_α is g_α). Since $\perp$ is the sup over the empty subset, the patching property implies that, for a sheaf G, the set $G(\perp)$ has exactly one element.*

If G is a sheaf (or a presheaf) then $G(u)$ is called the **set of sections** *over u and $G(\top)$ is the set of* **global sections** *of G.*

Basic definitions for sheaves over locales.

- If G and H are sheaves over L, then a **sheaf morphism** $\theta : G \to H$ is given by functions $\theta_u : G(u) \to H(u)$ which commute with restrictions (i.e. $\rho^u_v \theta_u = \theta_v \rho^u_v$). So, if L is a locale, there is a category $\mathrm{Sh}(L)$ of sheaves over L. (Note that we use the same notation, ρ^u_v, for the restrictions in *any* sheaf.)

- $\theta : G \to H$ is **sheaf monomorphism** if, for all $u \in L$, the function $\theta_u : G(u) \to H(u)$ is one-to-one. Similarly, θ is a **sheaf epimorphism** if, for all $u \in L$, each $h \in H(u)$ can be obtained by patching together sections of the form $\theta_{u_\alpha}(g_\alpha)$ where $u = \bigvee\{U_\alpha\}$.

- If $f : L \to M$ is a locale map (i.e. $f : M \to L$ is a frame homomorphism) then the **direct image functor**, $f_* : \mathrm{Sh}(L) \to \mathrm{Sh}(M)$, is defined so that $f_*(G)(v) = G(f(v))$. The **inverse image functor**, $f^* : \mathrm{Sh}(M) \to \mathrm{Sh}(L)$ is the left adjoint of f_*. (A concrete definition of f^* is sketched below, see 3.12.)

- By a **Boolean flow over a locale** L we mean a sheaf $G \in \mathrm{Sh}(L)$ for which each set $G(u)$ has the structure of a Boolean flow such that the restriction maps are flow homomorphisms. If G and H are Boolean flows over L, then a sheaf morphism $\theta : G \to H$ is a **flow morphism over** L if each $\theta_u : G(u) \to H(u)$ is a flow homomorphism.

3.11. Example. [The spatial case] If we regard the topological space X as a locale (corresponding to the frame $\mathcal{O}(X)$) then, as noted above, a sheaf over X is given by a local homeomorphism $p : E \to X$. In this case, the set $E_x = p^{-1}(x)$ is called the "stalk" over $x \in X$.

For sheaves over a spatial locale, given by local homeomorphisms $p : E \to X$ and $q : F \to X$, a sheaf morphism is equivalent to a continuous map $\theta : E \to F$ for which $q\theta = p$. Then θ is a sheaf monomorphism if and only if θ is one-to-one, and a sheaf epimorphism if and only if θ is onto. Moreover $p : E \to X$ is a Boolean flow over X if and only if each stalk E_x has the structure of a Boolean flow such that the Boolean flow operations are continuous. For details, see [Johnstone, 1982, pages 175–176].

3.12. Remark. A concrete definition of f^* for sheaves over locales can be sketched as follows: Given $G \in \mathrm{Sh}(M)$ and a frame homomorphism $f : M \to L$, we first define a presheaf $f^0(G)$ over L so that $f^0(G)(u)$ is the set of all pairs (x, v) with $x \in G(v)$ and $u \leq f(v)$ with the understanding that (x, v) is equivalent to (x', v') if and only if there exists $w \leq v \wedge v'$ with $u \leq f(w)$ such that x, x' have equal restrictions to $G(w)$. As shown in [Johnstone, 1982], every presheaf generates a sheaf, and $f^*(G)$ is the sheaf generated by $f^0(G)$. It can be shown that $f^0(G)$ is a separated presheaf which means that the natural maps from $f^0(G))(u)$ to $f^*(G)(u)$ are one-to-one.

Since there is a natural local homeomorphism $B^0 \to \mathcal{W}$ it follows that B^0 can be regarded as a sheaf over $\mathcal{W}$. Also, B^0 is a Boolean flow over $\mathcal{W}$ in view of 3.11. We want to show that B^0 is a "universal quotient flow" of B, which suggests that there needs to be a quotient map of some kind from B to B^0. But, so far, B and B^0 are in different categories. This is rectified by the following:

3.13. Definition. *The category of* **Boolean flows over locales** *is the category of pairs* (G, L) *where* G *is a Boolean flow over* L*, and with maps* $(\theta, f) : (G, L) \to (H, M)$ *where* $f : M \to L$ *is a locale map (note its direction) and* $\theta : f^*(G) \to H$ *is a flow morphism over* H*. The composition of* $(\theta, f) : (G, L) \to (H, M)$ *with* $(\psi, g) : (H, M) \to (K, N)$ *is* $(\psi g^*(\theta), fg)$*.*

A morphism in this category will be called a **localic flow morphism**

3.14. Notation. We let 1 denote the locale corresponding to the one-point space. Note that as a frame, it is just $\{\perp, \top\}$. If B is a Boolean flow, we can think of B as a Boolean flow over the one-point space (with $B(\top) = B$ and $B(\perp)$ being any one-point set).

If L is a locale, we let γ_L or just γ if there is no danger of confusion, denote the unique locale map from L to 1.

3.15. Definition. *By a* **quotient sheaf** *of a Boolean flow* B*, we mean a localic flow morphism* $(\lambda, \gamma_L) : (B, 1) \to (F, L)$ *for which* λ *is a sheaf epimorphism in* $\mathrm{Sh}(L)$*.*

For example, there is a natural localic flow morphism $(\eta, \gamma_{\mathcal{W}}) : (B, 1) \to (B^0, \mathcal{W})$ which is most easily defined in terms of the stalks (the stalks of $\gamma^*(B)$ are copies of B and the stalk of B^0 over $I \in \mathcal{W}$ is B/I and $\eta_I : B \to B/I$ is the canonical quotient map).

We aim to prove that $(\eta, \gamma_{\mathcal{W}}) : (B, 1) \to (B^0, \mathcal{W})$ is a universal quotient sheaf of B in the sense that any other quotient sheaf factors through it in a nice way. First we need:

3.16. Definition. [**The operation $\|\mathbf{g}=\mathbf{h}\|$**] *If $G \in \mathrm{Sh}(L)$ is a sheaf over L, and if $g, h \in G(u)$ are given for some $u \in L$, then $\|g = h\|$ is defined as the largest $v \subseteq u$ for which $\rho^u_v(g) = \rho^u_v(h)$. Note that:*

$$\|g = h\| = \bigvee\{v_\alpha \mid \rho^u_{v_\alpha}(g) = \rho^u_{v_\alpha}(h)\}$$

We can now prove:

3.17. Theorem. *$B^0 \in \mathrm{Sh}(\mathcal{W})$ is the universal flow quotient of (B, τ) in the sense that if F is a Boolean flow over L, and if $\lambda : \gamma^*(B) \to F$ is an epimorphism in $\mathrm{Sh}(L)$, then there is a unique localic flow morphism $(\overline{\lambda}, m) : (B^0, \mathcal{W}) \to (F, L)$, with $\overline{\lambda}$ an isomorphism, such that the following diagram commutes:*

$$\begin{tikzcd}
(B,1) \arrow[rr, "{(\eta,\gamma_{\mathcal{W}})}"] \arrow[dr, "{(\lambda,\gamma_L)}"'] & & (B^0,\mathcal{W}) \arrow[dl, dashed, "{(\overline{\lambda},m)}"] \\
& (F,L) &
\end{tikzcd}$$

Proof. We need to find a locale map from L to $\mathcal{W}$ or, equivalently, a frame homomorphism $m : \mathcal{O}(\mathcal{W}) \to L$, such that $m^*(B^0)$ is isomorphic to F, where the isomorphism is compatible with the obvious maps from $\gamma^*_L(B)$ to $m^*(B^0)$ and F. We start by establishing some notation. It is clear from the definition of γ^*_L that each $b \in B$ gives rise to a global section $\overline{b}$ of $\gamma^*_L(B)$. (More formally, $\overline{b}$ is the image of b under the unit of adjunction which maps $B \to (\gamma_L)_*\gamma^*_L(B)$. Note that $(\gamma_L)_*\gamma^*_L(B)$ is the set of global sections of $\gamma^*_L(B)$.) Moreover, these sections generate $\gamma^*_L(B)$ in the sense that every section of $\gamma^*_L(B)$ is obtained by patching together various restrictions of global sections of the form $\overline{b}$. So sheaf morphisms on $\gamma^*_L(B)$ are determined by their action on the sections $\overline{b}$ (which also follows from the adjointness). (Note that we could similarly define global sections $\overline{b}$ of $\gamma^*_{\mathcal{W}}(B)$ in which case η would be defined by the condition that it maps $\overline{b}$ to $\widehat{b}$, see 3.5.)

Regardless of how $m : \mathcal{O}(\mathcal{W}) \to F$ is defined, we will have $\gamma_L = \gamma_{\mathcal{W}} m$ so $m^*\gamma^*_{\mathcal{W}} = \gamma^*_L$. We claim that the required flow isomorphism $\overline{\lambda}$, over L, exists if and only if the place where $m^*(\eta)(\overline{b})$ vanishes coincides with the place where $\lambda(\overline{b})$ vanishes. In other words, $\overline{\lambda}$ exists if and only if

$$\|m^*(\eta)(\overline{b}) = 0\| = \|\lambda(\overline{b}) = 0\| \tag{$\star$}$$

(To keep the notation relatively uncluttered we are using $\lambda(\overline{b})$ as an abbreviation of $\lambda_\top(\overline{b})$ and similarly for $m^*(\eta)(\overline{b})$.) It is clear that $(\star)$ is necessary for the existence of the flow isomorphism $\overline{\lambda}$. Sufficiency follows because $m^*(\eta)$ and λ are sheaf epimorphisms so the sections of $m^*(B^0)$ and F are obtained by restricting and patching sections of the form $\overline{b}$. Applying condition $(\star)$ to $b - c$, we see that the images of $\overline{b}$ and $\overline{c}$ in $m^*(B^0)$ coincide when restricted to $u \in L$, if and only if they do so in F. So restrictions of sections can be patched in $m^*(B^0)$ if and only if they can be patched in F, which leads to the desired isomorphism.

But, however m is defined, $\|m^*(\bar{b}) = 0\|$ can readily be shown to be $m(N(b))$, so to conclude the proof we must show that there exists a unique frame homomorphism $m : \mathcal{O}(\mathcal{W}) \to L$ for which $m(N(b)) = \|\lambda(\bar{b}) = 0\|$. Uniqueness follows because the family $\{N(b)|b \in B\}$ is a base for the topology on $\mathcal{W}$ so every $U \in \mathcal{O}(\mathcal{W})$ can be written as $U = \bigcup\{N(b) \mid b \in B_U\}$ for some subset $B_U \subseteq B$. It follows that $m(U)$ must be $\bigvee\{\{\|\lambda(\bar{b}) = 0\| \mid b \in B_U\}$.

Note that m is well-defined provided $\bigvee\{\|\lambda(\bar{b}) = 0\| \mid b \in B_U\}$ depends only on U and not on the choice of B_U. But we may as well assume that $b \in B_U$ and $b \leq c$ imply $b \in B_U$ because closing B_U up under such elements c affects neither $\bigcup\{N(b) \mid b \in B_U\}$ nor $\bigvee\{\|\lambda(\bar{b}) = 0\| \mid b \in B_U\}$. By a similar argument, we may as well assume that $b \in B_U$ if $\tau(b) \in B_U$ or even if $b \vee \tau(b) \in B_U$. If we close B_U under these further operations, then $b \vee \tau(b) \vee \tau^2(b) = (b \vee \tau(b)) \vee \tau(b \vee \tau(b)) \in B_U$ so $b \vee \tau(b) \in B_U$ and $b \in B_U$. By induction, it can then be shown that if $b \vee \tau(b) \vee \ldots \vee \tau^k(b) \in B_U$ then $b \in B_U$. It follows that if $\langle b \rangle \in U$ then $b \in B_U$ where $\langle b \rangle$ is the smallest flow ideal of B which contains b. (See 6.1 and the proof of 6.3 for details.) But now we cannot make B_U any bigger because $b \in B_U$ and $U = \bigcup\{N(b) \mid b \in B_U\}$ readily imply that $\langle b \rangle \in U$. It follows that $m(U)$ is well-defined.

We have to show that m is a frame homomorphism. It immediately follows from the definition of m and its independence from the choice of B_U that m preserves arbitrary sups. As for finite infs, we will first prove that $m(\top_{\mathcal{W}}) = \top_L$ (which shows that m preserves the inf over the empty subset). But this follows from:

$$m(\top_{\mathcal{W}}) = m(N(0)) = \|\lambda(\overline{0}) = 0\| = \top_L$$

So, to complete the proof, it suffices, since $N(b \vee c) = N(b) \cap N(c)$, to show that if $\langle b \rangle \in U$ and $\langle c \rangle \in V$ then $\langle b \vee c \rangle \in U \cap V$. But this follows because all open subsets of $\mathcal{W}$ are upwards-closed (meaning that $I \in U$ and $I \subseteq J$ and U open imply $J \in U$.) ∎

3.18. Remark. It follows that, to within isomorphism, the quotient sheaves of B correspond to locale maps into $\mathcal{W}$ where each map $f : L \to \mathcal{W}$ is associated with the quotient $(f^*(B^0), L)$.

To define the cyclic spectrum, we need to know when a Boolean flow over a locale can be regarded as cyclic. For a spatial locale, $\mathcal{O}(X)$, the obvious definition would be that the Boolean flow C over X is cyclic if and only if each stalk, C_x, is a cyclic Boolean flow. However, as often happens, we can find an equivalent definition which does not depend on stalks.

3.19. Definition. *Let G be a Boolean flow over L. Then G is a* **cyclic** *Boolean flow if for every $u \in L$ and every $g \in G(u)$ we have*

$$u = \bigvee\{\|g = \tau^n(g)\| \mid n \in \mathbf{N}\}.$$

Given a Boolean flow over a locale, we can find the largest sublocale for which the "restriction" of the sheaf becomes cyclic. In order to proceed, we need to examine the notion of a sublocale, which corresponds to a frame quotient. Sublocales are best handled in terms of *nuclei*.

3.20. DEFINITION. *Let L be a frame. By a* **nucleus** *on L we mean a function $j : L \to L$ such that for all $u, v \in L$:*

1. $j(u \wedge v) = j(u) \wedge j(v)$
2. $u \leq j(u)$
3. $j(j(u)) = j(u)$

A nucleus is sometimes called a Lawvere-Tierney topology, but this use of the word "topology" can be confusing. Nuclei are useful because:

3.21. PROPOSITION. *There is a bijection between nuclei on a locale L and sublocales of L.*

PROOF. This is given in [Johnstone, 1982, page 49]. A **sublocale** of L is given by an onto frame homomorphism $q : L \to F$, where it is understood that two onto frame homomorphisms represent the same sublocale of L when they induce the same congruence relation on L. (The congruence relation induced by q is the equivalence relation θ_q for which $u\theta_q v$ if and only if $q(u) = q(v)$.)

Given such a frame homomorphism q and given $u \in L$ we define $j(u)$ as the largest element of L for which $q(u) = q(j(u))$. (So $j(u) = \bigvee\{v_\alpha \mid q(u) = q(v_\alpha)\}$.) Then j is a nucleus.

Conversely, given a nucleus j, then we define $u \approx v$ if and only if $j(u) = j(v)$ and let F be the set of equivalence classes $L/\approx$. It can readily be shown that F has a frame structure and is a frame quotient of L. ∎

3.22. NOTATION. Let j be a nucleus on a locale L, then:

- $L_j = \{u \in L \mid u = j(u)\}$ denotes the sublocale (or quotient frame) of L which corresponds to j.
- We let $j : L_j \to L$ denote the locale map associated with the inclusion of the sublocale L_j. (**Caution:** As a frame homomorphism, j maps L **onto** L_j. The inclusion of L_j as a subset of L is generally not a frame homomorphism.)
- Given $G \in \mathrm{Sh}(L)$, the "restriction" of G to the sublocale L_j is $j^*(G)$.
- For j_1 and j_2, nuclei on L, the nucleus j_1 corresponds to the **larger sublocale** if and only if $j_1(u) \leq j_2(u)$ for all $u \in L$. (So the smaller nucleus corresponds to the larger sublocale.)

We can now state:

3.23. Proposition. *Let G be a Boolean flow over the locale L. Then there is a largest sublocale, L_j of L, such that $j^*(G)$, the restriction of G to L_j, is a cyclic Boolean flow.*

Proof. This is a matter of finding the largest sublocale for which certain equations of the form $\bigvee\{u_\alpha\} = u$ become true. In this case, we say that $\{u_\alpha\}$ is to be a "cover" of u. [Johnstone, 1982, pages 57–59] discusses the construction of a sublocale for which given covering families (or "coverages") become sups (in the sense that $\bigvee\{u_\alpha\} = u$ whenever $\{u_\alpha\}$ covers u.) This construction can then be used to find the sublocale for which $u = \bigvee\{\|g = \tau^n(g)\| \mid n \in \mathbf{N}\}$, for all u and all $g \in G(u)$. ■

The cyclic spectrum of a Boolean flow is defined as the restriction of the sheaf B^0 to the largest sublocale of $\mathcal{O}(\mathcal{W})$ for which this restriction is cyclic.

3.24. Definition. *Let (B, τ) be a Boolean flow. Let B^0 be the sheaf defined over the space $\mathcal{W}$ of all flow ideals. Let $j = j_{\mathrm{cyc}}$ be the nucleus which forces B^0 to be cyclic. Let L_{cyc} denote the sublocale $\mathcal{W}_j$ induced by the nucleus j and let $B_{\mathrm{cyc}} = j^*(B^0)$ be the restriction of B^0 to L_{cyc}.*

The **cyclic spectrum** *of B is defined as the sheaf B_{cyc} over the locale L_{cyc}.*

Recall that 1 denotes the one-point space and any Boolean flow can be thought of as a Boolean flow in Sh(1).

3.25. Theorem. *Let B be a Boolean flow and let B_{cyc} be its cyclic spectrum over L_{cyc}. There is a natural localic flow morphism $(\eta', \gamma) : (B, 1) \to (B_{\mathrm{cyc}}, L_{\mathrm{cyc}})$ which has a universal property with respect to maps $(\lambda, \gamma) : (B, 1) \to (C, L)$, where C is a cyclic flow over L and λ is a sheaf epimorphism: such a map (λ, γ) uniquely factors as $(\widehat{\lambda}, h)(\widehat{\eta'}, \gamma)$ through a map $(\widehat{\lambda}, h)$ for which $\widehat{\lambda}$ is an* **isomorphism**.

$$
\begin{tikzcd}
(B,1) \arrow[r, "(\eta,\gamma_{\mathcal{W}})"] \arrow[dr, "(\lambda,\gamma)"'] & (B^0,\mathcal{W}) \arrow[r, "(i,j)"] \arrow[d, dashed, "(\overline{\lambda},m)"] & (B_{\mathrm{cyc}},L_{\mathrm{cyc}}) \arrow[dl, dashed, "(\widehat{\lambda},h)"] \\
& (C,L) &
\end{tikzcd}
$$

Proof. The proof is as suggested by the above diagram. Note that (η', γ) is the composition $(i, j)(\eta, \gamma_{\mathcal{W}})$. The locale map $m : L \to \mathcal{W}$ is determined by Theorem 3.17. We claim that it suffices to show that m maps into the sublocale L_{cyc}, or equivalently that the frame homomorphism $m : \mathcal{O}(\mathcal{W}) \to L$ factors through the frame quotient $j : \mathcal{O}(\mathcal{W}) \to L_{\mathrm{cyc}}$. For if $m = hj$ (as frame homomorphisms) then $h^*(B_{\mathrm{cyc}}) = h^*(j^*(B^0)) = m^*(B^0) \simeq C$.

So we have to show that whenever $j(U) = j(V)$ then $m(U) = m(V)$. It suffices to show that $m(\bigvee \|b + \tau^n(b) = 0\|) = \top$ because the frame congruence associated with j is the smallest for which $\bigvee\{\|b + \tau^n(b) = 0\|\}$ (or, equivalently, for which $\bigvee\{\|b = \tau^n(b)\|\}$ is equated with $\top$). But $\|b = \tau^n(b)\| = N(b + \tau^n(b))$ so, by definition of m, we get:

$$m(\|b = \tau^n(b)\|) = \|\lambda(b + \tau^n(b)) = 0)\| = \|\lambda(b) = \tau^n(\lambda(b))\|$$

and $\bigvee\{\|\lambda(b) = \tau^n(\lambda(b))\| = \top$ as C is cyclic. ■

4. Computing the cyclic spectrum

We first examine when a cyclic spectrum is spatial, that is, a sheaf over a topological space. (In fact it is an open question as to whether this is always the case.) If the spectrum is spatial, we show that it must be a sheaf over the space $\mathcal{W}_{\mathrm{cyc}}$ where:

$$\mathcal{W}_{\mathrm{cyc}} = \{I \in \mathcal{W} \mid \; B/I \text{ is cyclic}\}.$$

and $\mathcal{W}_{\mathrm{cyc}}$ has the topology it inherits as a subspace of $\mathcal{W}$.

Our main application is that the cyclic spectrum of a finitely generated Boolean flow is always spatial, and, for these spaces, we can explicitly compute what the spectrum is.

We conclude this section with a proposition showing that we can always restrict our attention to the "monoflow" ideals. (This was noted in [Kennison, 2002] and here we give a direct proof.)

4.1. PROPOSITION. *As discussed above, $\mathcal{W}$ and $\mathcal{W}_{\mathrm{cyc}}$ are spatial locales, while L_{cyc} is the base locale of the cyclic spectrum. Then:*

(a) $\mathcal{W}_{\mathrm{cyc}} \subseteq L_{\mathrm{cyc}} \subseteq \mathcal{W}$ *where "$\subseteq$" denotes a sublocale (in the obvious way).*

(b) L_{cyc} *is a spatial locale if and only if the inclusion $\mathcal{W}_{\mathrm{cyc}} \subseteq L_{\mathrm{cyc}}$ is an isomorphism.*

(c) *If U, V are open subsets of $\mathcal{W}$ for which $j(U) = j(V)$ then $U \cap \mathcal{W}_{\mathrm{cyc}} = V \cap \mathcal{W}_{\mathrm{cyc}}$.*

(d) L_{cyc} *is spatial if and only if, conversely, $U \cap \mathcal{W}_{\mathrm{cyc}} = V \cap \mathcal{W}_{\mathrm{cyc}}$ implies $j(U) = j(V)$.*

PROOF.

(a) L_{cyc} is the largest sublocale of $\mathcal{W}$ to which the restriction of B^0 is cyclic. Since the restriction of B^0 to $\mathcal{W}_{\mathrm{cyc}}$ is clearly cyclic, the inclusions follow.

(b) If L_{cyc} is spatial, then the universal property of B_{cyc} shows that the points of L_{cyc} correspond to cyclic quotients of B, and therefore to the points of $\mathcal{W}_{\mathrm{cyc}}$. This guarantees that the inclusion $\mathcal{W}_{\mathrm{cyc}} \subseteq L_{\mathrm{cyc}}$ is an isomorphism.

(c) The sublocales $\mathcal{W}_{\mathrm{cyc}}$ and L_{cyc} are determined by frame quotients of $\mathcal{O}(\mathcal{W})$ hence by equivalence relations (called frame congruences) on $\mathcal{O}(\mathcal{W})$. The subsets U, V are in the frame congruence for L_{cyc} exactly when $j(U) = j(V)$ while they are in the frame congruence for $\mathcal{W}_{\mathrm{cyc}}$ exactly when $U \cap \mathcal{W}_{\mathrm{cyc}} = V \cap \mathcal{W}_{\mathrm{cyc}}$. The result now follows easily.

(d) Follows from the above observations. ■

4.2. Definition. *Let B be a Boolean flow. For each* finite *subset $F \subseteq B$, we say that a flow ideal I of B is* **F-cyclic** *if for every $f \in F$ there exists $n \in \mathbf{N}$ such that $f = \tau^n(f) \pmod I$ (equivalently, that $f + \tau^n(f) \in I$). We let:*

$$\mathcal{W}_{\mathrm{cyc}}(F) = \{I \in \mathcal{W} \mid I \text{ is } F\text{-cyclic}\}$$

4.3. Lemma. *Let $j = j_{\mathrm{cyc}}$ and let $V \in \mathcal{O}(\mathcal{W})$. Then $V = j(V)$ if and only if for every $U \in \mathcal{O}(\mathcal{W})$ we have $U \subseteq V$ whenever there exists a finite $F \subseteq B$ with $U \cap \mathcal{W}_{\mathrm{cyc}}(F) \subseteq V$.*

Proof. Assume that $V = j(V)$ and that $U \cap \mathcal{W}_{\mathrm{cyc}}(F) \subseteq V$ for some finite $F \subseteq B$ and some open $U \subseteq \mathcal{W}$. This implies that:

$$U \cap \bigcap_{f \in F} \left[\bigcup_{n \in \mathbf{N}} N(f + \tau^n(f)) \right] \subseteq V$$

because $I \in \mathcal{W}_{\mathrm{cyc}}(F)$ if and only if $I \in \bigcap_{f\in F} \left[\bigcup_{n\in\mathbf{N}} N(f+\tau^n(f))\right]$. But the nucleus j is defined so that each $\bigcup_{n\in\mathbf{N}} N(f + \tau^n(f))$ is equated with the top element, $\top$, and it follows that $U \subseteq j(V)$.

Conversely, assume that for every finite $F \subseteq B$ and every open $U \subseteq \mathcal{W}$, the condition $U \cap \mathcal{W}_{\mathrm{cyc}}(F) \subseteq V$ implies $U \subseteq V$. We must prove that $V = j(V)$. We define $J : \mathcal{O}(\mathcal{W}) \to \mathcal{O}(\mathcal{W})$ so that:

$$J(W) = \bigcup \{U \mid (\exists \text{ a finite } F \subseteq B) \text{ such that } U \cap \mathcal{W}_{\mathrm{cyc}}(F) \subseteq W\}$$

It is readily shown that $J(W \cap W') = J(W) \cap J(W')$ and $W \subseteq J(W)$, but it is not necessarily the case that $J(J(W)) = J(W)$. However, we can define J^α for every ordinal α so that $J^0 = J$, $J^{\alpha+1} = J(J^\alpha)$ and $J^\alpha(W) = \bigcup\{J^\beta(W) \mid \beta < \alpha\}$, for α a limit ordinal.

It is obvious that for some α, $J^\alpha = J^{\alpha+1}$. So, letting $J' = J^\alpha$ we see that $J'(J'(W)) = J'(W)$ and so J' is readily shown to be a nucleus. By the previous argument, $J'(W) \leq j(W)$. But it is easy to show that J' equates every $\bigcup_{n\in\mathbf{N}} N(f+\tau^n(f))$ with the top element $\top$. By the definition of j, it follows that $j = J'$ and $V = j(V)$ because $V = J'(V)$. ∎

4.4. Remark. Notice that the intersection of the **sets** $\{\mathcal{W}_{\mathrm{cyc}}(F)\}$ is $\mathcal{W}_{\mathrm{cyc}}$, and the **spatial** intersection of the subspaces $\{\mathcal{W}_{\mathrm{cyc}}(F)\}$ is $\mathcal{W}_{\mathrm{cyc}}$. But the **localic** intersection of the sublocales $\{\mathcal{W}_{\mathrm{cyc}}(F)\}$ is the sublocale L_{cyc}. There are examples of families of subspaces of a space with a non-spatial intersection, but it is not clear if this is the case for the family $\{\mathcal{W}_{\mathrm{cyc}}(F)\}$.

We now apply the above results to the case of a finitely generated Boolean flow. First we need some lemmas.

4.5. Lemma. *If the positive integer m is a divisor of n, and if $b \in B$, then $N(b+\tau^m(b)) \subseteq N(b + \tau^n(b))$.*

Proof. Suppose $I \in N(b + \tau^m(b))$ is given. Then $b = \tau^m(b) \pmod I$. But this clearly implies that $b = \tau^n(b) \pmod I$ which implies that $(b+\tau^n(b)) \in I$ and so $I \in N(b+\tau^n(b))$. ∎

4.6. Lemma. *Assume (B, τ) is a Boolean flow which is generated (as a flow) by $G \subseteq B$. If there exist $n \in \mathbf{N}$ such that $\tau^n(g) = g$ for all $g \in G$, then τ^n is the identity on all of B.*

Proof. Let $C \subseteq B$ be the equalizer of τ^n and Id_B. Then C is readily seen to be a subflow which contains G so C is all of B. ∎

4.7. Proposition. *The cyclic spectrum of a finitely generated Boolean flow is always spatial.*

Proof. Let B be a Boolean flow generated by the finite set $G = \{g_1, \ldots g_k\}$. We claim that $\mathcal{W}_{\mathrm{cyc}}(G) = \mathcal{W}_{\mathrm{cyc}}$, which completes the proof in view of Lemma 4.3 and Remark 4.4.

If $I \in \mathcal{W}_{\mathrm{cyc}}(G)$, then for each g_i there exists n_i such that $\tau^{n_i}(g_i) = g_i \pmod I$. By 4.5, applied to B/I, there exists $n \in \mathbf{N}$ (for example the product of the n_i) such that $\tau^n(g_i) = g_i \pmod I$. But B/I is obviously generated by the image of G so, by the above lemma, B/I is cyclic and so $I \in \mathcal{W}_{\mathrm{cyc}}$. ∎

It remains to discuss the topology on $\mathcal{W}_{\mathrm{cyc}}$. First we need:

4.8. Lemma. *If the Boolean flow (B, τ) is finitely generated and satisfies $\tau^n = \mathrm{Id}_B$ for some $n \in \mathbf{N}$, then B is finite.*

Proof. Let $G = \{g_1, \ldots g_k\}$ be a finite set that generates B as a flow. Then it is readily seen that the finite set $\{\tau^i(g_j)\}$ (for $1 \leq i \leq n$ and $1 \leq j \leq k$) generates B as a Boolean algebra, which implies that B is finite. ∎

4.9. Lemma. *Let (B, τ) be a finitely generated Boolean flow, and let I be a cyclic flow ideal of B. Then I is finitely generated as a flow ideal.*

Proof. Let $G = \{g_1, \ldots g_k\}$ generate B as a flow and let I be a cyclic flow ideal of B. Then each g_i becomes cyclic modulo I so there exists $(n_1, \ldots n_k)$ such that $F_0 = \{g_i + \tau^{n_i}(g_i)\} \subseteq I$. Let I_0 be the flow ideal generated by F_0. Then $I_0 \subseteq I$ (as $F_0 \subseteq I$). Also by previous lemmas, 4.6 and 4.8, we see that I_0 is cyclic so B/I_0 is cyclic and finite. Let $q_0 : B \to B/I_0$ and $q : B \to B/I$ be the obvious quotient maps. Since $I_0 \subseteq I$ there exists a flow homomorphism $h : B/I_0 \to B/I$ for which $hq_0 = q$. Let K be the kernel of h. Since B/I_0 is finite, we see that K is finite. For each $x \in K$ choose $b(x) \in q_0^{-1}(x)$, and let $F_1 = \{b(x) \mid x \in K\}$. It readily follows that I is generated (as a flow ideal) by $F_0 \cup F_1$. ∎

4.10. Theorem. *Assume (B, τ) is a finitely generated Boolean flow and let $\mathcal{W}_{\mathrm{cyc}}$ be as above. Then $U \subseteq \mathcal{W}_{\mathrm{cyc}}$ is open if and only if whenever $I \in U$ then $\uparrow (I) \subseteq U$ where $\uparrow (I) = \{J \in \mathcal{W}_{\mathrm{cyc}} \mid I \subseteq J\}$. It follows that $\uparrow (I)$ is the smallest neighborhood of I in $\mathcal{W}_{\mathrm{cyc}}$.*

Proof. We let $N_{\mathrm{cyc}}(b)$ denote $N(b) \cap \mathcal{W}_{\mathrm{cyc}}$. Now, assume that U is open in $\mathcal{W}_{\mathrm{cyc}}$ and that $I \in U$. Then there clearly exists $b \in B$ with $I \in N_{\mathrm{cyc}}(b) \subseteq U$. It is obvious that $\uparrow (I) \subseteq N_{\mathrm{cyc}}(b)$ so $\uparrow (I) \subseteq U$.

Conversely, assume $\uparrow (I) \subseteq U$. By the above lemma, there is a finite set F which generates I as a flow ideal. Then $\bigcap\{N_{\mathrm{cyc}}(f) \mid f \in F\}$ is a neighborhood of I, but $J \in \bigcap\{N_{\mathrm{cyc}}(f) \mid f \in F\}$ if and only if $F \subseteq J$ if and only if $I \subseteq J$ so $\bigcap\{N_{\mathrm{cyc}}(f) \mid f \in F\} = \uparrow (I)$, which shows that U is a neighborhood of I. ∎

We conclude this section with two results that may be helpful in computing the cyclic spectrum (of any Boolean flow). It is obvious that $N(b) \subseteq N(\tau(b))$ but $N(b) = N(\tau(b))$ modulo the nucleus $j = j_{\mathrm{cyc}}$ in the sense that:

4.11. Lemma. *Let B be any Boolean flow. Let $j = j_{\mathrm{cyc}}$ be the nucleus on $\mathcal{O}(\mathcal{W})$ associated with the sublocale L_{cyc}. Let $V \in L_{\mathrm{cyc}}$ (so $j(V) = V$) and $b \in B$ be such that $N(b) \subseteq V$. Then $N(\tau^n(b)) \subseteq V$ for all $n \in \mathbf{N}$.*

Proof. We will prove that $N(\tau(b)) \subseteq V$ as the full result then follows by induction. By 4.3, it suffices to show that:

$$N(\tau(b)) \cap \mathcal{W}_{\mathrm{cyc}}(\{b\}) \subseteq V$$

But if $I \in N(\tau(b)) \cap \mathcal{W}_{\mathrm{cyc}}(\{b\})$ then $\tau(b) \in I$ (and therefore $\tau^n(b) \in I$ for all n) and $b = \tau^n(b) \pmod{I}$ for some $n \in \mathbf{N}$ so $b \in I$. But then $I \in N(b) \subseteq V$. ∎

We say that a flow ideal I of B is a **monoflow ideal** if $\tau(b) \in I$ implies $b \in I$. So I is a monoflow ideal if and only if the iterator of B/I is one-to-one if and only if the iterator t of the corresponding flow in Stone spaces is onto. We let $\mathcal{W}_{\mathrm{mono}} \subseteq \mathcal{W}$ be the subspace of all monoflow ideals. In [Kennison, 2002], we constructed the cyclic spectrum starting with $\mathcal{W}_{\mathrm{mono}}$, which was denoted by $\mathcal{V}$ in that paper. Since $\mathcal{W}_{\mathrm{mono}}$ is sometimes considerably simpler than $\mathcal{W}$, it is worth showing that $L_{\mathrm{cyc}} \subseteq \mathcal{W}_{\mathrm{mono}} \subseteq \mathcal{W}$ (which follows by topos theory essentially because every cyclic flow has a one-to-one iterator). Here we give a direct proof:

4.12. Proposition. *Let B be any Boolean flow and let $j = j_{\mathrm{cyc}}$ be the nucleus on $\mathcal{O}(\mathcal{W})$ associated with the sublocale L_{cyc}. Let $V \in L_{\mathrm{cyc}}$ and $U \in \mathcal{O}(\mathcal{W})$ be given. Then $U \cap \mathcal{W}_{\mathrm{mono}} \subseteq V$ if and only if $U \subseteq V$.*

Proof. Clearly, it suffices to assume $U \cap \mathcal{W}_{\mathrm{mono}} \subseteq V$ and $I \in U$ and prove $I \in V$. Since $I \in U$ and U is open, there exists $b \in B$ with $I \in N(b) \subseteq U$. Let $\langle b \rangle$ be the smallest flow ideal of B containing b and let:

$$I_0 = \{c \in B \mid (\exists n \in \mathbf{N})\tau^n(c) \in \langle b \rangle\}$$

It is readily shown that I_0 is a monoflow ideal containing b so $I_0 \in N(b) \cap \mathcal{W}_{\mathrm{mono}} \subseteq U \cap \mathcal{W}_{\mathrm{mono}} \subseteq V$. As V is open, there exists $a \in I_0$ with $I_0 \in N(a) \subseteq V$. By the above lemma, $N(\tau^n(b)) \subseteq V$ for all $n \in \mathbf{N}$. But since $a \in I_0$, there exists $n \in \mathbf{N}$ with $\tau^n(a) \in \langle b \rangle$. Since $\langle b \rangle \subseteq I$ we see that $\tau^n(a) \in I$ so $I \in N(\tau^n(a)) \subseteq V$. ∎

5. Representation results

Suppose the Boolean flow B can be represented as the flow of all global sections of a cyclic Boolean flow over a locale. What does this tell us about the given flow B and the corresponding flow in Stone spaces? What if we only know that B is equivalent to a subflow of the flow of all global sections of a cyclic Boolean flow over a locale? We present some necessary conditions.

5.1. NOTATION. The natural map $\eta' : \gamma^*(B) \to B_{\text{cyc}}$ corresponds, by the adjunction between γ^* and γ_*, to a map $\widehat{\eta} : B \to \gamma_*(B_{\text{cyc}})$. It can be shown that $\gamma_*(B_{\text{cyc}})$ is the set of all global sections of B_{cyc}, so it is often denoted by $\Gamma(B_{\text{cyc}})$.

5.2. DEFINITION. *The Boolean flow B is* **cyclically representable** *if it is isomorphic to the flow of all global sections of a cyclic Boolean flow over a locale.*

We say that B is **cyclically separated** *if it is isomorphic to a subflow of the flow of all global sections of a cyclic Boolean flow over a locale.*

5.3. PROPOSITION. *The Boolean flow B is cyclically separated if and only if $\widehat{\eta} : B \to \gamma_*(B_{\text{cyc}})$, defined in 5.1, is one-to-one.*

PROOF. If $\widehat{\eta}$ is one-to-one, then it is immediate that B is cyclically separated as B_{cyc} is a cyclic flow over L_{cyc}. Conversely, if B is flow isomorphic to a subflow of $\gamma_*(C)$ where C is a cyclic flow over some locale M, then, by adjointness, the map $B \to \gamma_*(C)$ corresponds to a map $\gamma_M^*(B) \to C$. Moreover, it can be shown that the subsheaf of C generated by the image of $\gamma^*(B)$ is a cyclic Boolean flow over M. So, by the universal property of B_{cyc}, Theorem 3.25, the map from $\gamma_M^*(B) \to C$ factors through $\eta' : \gamma^*(B) \to B_{\text{cyc}}$. Evaluating this map at the top element, $\top$, of M, we see that the map $B \to \gamma_*(C)$ factors through $B \to \gamma_*(B_{\text{cyc}})$ which shows that the latter map must be one-to-one. ∎

5.4. REMARK. The analogous proposition with "cyclically representable" instead of "cyclically separated" does not seem to be true (at least the above argument does not extend to that case).

5.5. PROPOSITION. *A Boolean flow is cyclically separated if the intersection of all of its cyclic flow ideals is the zero ideal. The converse holds for finitely generated Boolean flows.*

PROOF. Assume that the intersection of all cyclic flow ideals of B is the zero ideal. Let $B_{\text{cyc}}^{\text{sp}}$ be the restriction of B_{cyc} to the subspace $\mathcal{W}_{\text{cyc}} \subseteq \mathcal{W}$. Then the obvious map $B \to \Gamma(B_{\text{cyc}}^{\text{sp}})$ is one-to-one, which shows that B is cyclically separated.

The converse readily follows from Proposition 5.3, if B is finitely generated, because then $B_{\text{cyc}} = B_{\text{cyc}}^{\text{sp}}$. ∎

5.6. COROLLARY. *Let X be a flow in Stone spaces and let $B = \text{Clop}(X)$. If for every non-empty clopen $b \subseteq X$ there is $x \in b$ with $t^n(x) = x$, then B is cyclically separated.*

Proof. The orbit of such an element x is a closed, cyclic subflow of X and so, by duality, it corresponds to a cyclic quotient of B of the form B/I where I is a cyclic flow ideal. If $x \in b$ then $b \notin I$ so the hypotheses imply that every non-zero member of B is still non-zero modulo at least one cyclic flow ideal. ■

5.7. Example. The flow Clop(Sym(S)) is cyclically separated.

Proof. We will show that this follows from the above corollary. Let b be a non-empty clopen of Sym(S). Recall that sets of the form $\pi_i^{-1}(s)$ (for $i \in \mathbf{N}$ and $s \in S$) form a subbase for the topology of Sym(S). It clearly suffices to assume that b is a basic clopen of the form $b = \bigcap_{1 \le k \le n} \pi_{i_k}^{-1}(s_k)$. We may as well assume that $i_1 < i_2 < \ldots < i_n$. For convenience, we let $m = i_n$. Let $y \in b$ be given and let x be the periodic sequence for which x agrees with y in all coordinates from 1 to m. That is, define x so that $\pi_j(x) = \pi_i(y)$ where $1 \le i \le m$ and $j = i \pmod m$. Then $t^m(x) = x$ and $x \in b$ so the result follows. ■

5.8. Notation. Recall that $\langle b \rangle$ denotes the smallest flow ideal of B containing the element $b \in B$.

5.9. Definition. *The Boolean flow B is* **weakly separated** *if* $\bigcap\{\langle b + \tau^n(b)\rangle\} = \{0\}$ *for all* $b \in B$.

5.10. Proposition. *A cyclically separated Boolean flow is weakly separated.*

Proof. Suppose C is a cyclic flow over a locale L with top element $\top$. Then $C(\top)$ is the Boolean flow of all global sections of C. Let $B \subseteq C(\top)$ be a subflow and let $b, c \in B$ be given. Suppose $c \in \bigcap\{\langle b + \tau^n(b)\rangle\}$ where $\langle b + \tau^n(b)\rangle$ is the flow ideal of B generated by $b + \tau^n(b)$. We have to prove that $c = 0$.

Let K_n be the kernel of $\rho^T_{\|b=\tau^n(b)\|}$ which maps sections over $\top$ to sections over $\|b = \tau^n(b)\|$. Then K_n clearly contains $b + \tau^n(B)$ so $\langle b + \tau^n(b)\rangle \subseteq K_n$. But, since C is cyclic, we see that $\bigvee \|b = \tau^n(b)\| = \top$ and since the global section c becomes 0 when restricted to each $\|b = \tau^n(b)\|$, it follows by the patching property that $c = 0$. ■

5.11. Notation. Recall that $\widehat{Z}$, the profinite integers, is the limit of all finite quotients $\mathbf{Z_n}$ of $\mathbf{Z}$ and all group homomorphisms between them which preserve (the image of) $1 \in \mathbf{Z}$.

For $n \in \mathbf{N}$ we let $q_n : \mathbf{Z} \to \mathbf{Z_n}$ denote the quotient map and let $p_n : \widehat{Z} \to \mathbf{Z_n}$ denote the projection map associated with the limit. Given $\zeta \in \widehat{Z}$ and $k \in \mathbf{N}$ we say that $\zeta = k \pmod n$ if $p_n(\zeta) = q_n(k)$.

If C is a cyclic Boolean flow over a locale L, then for each $\zeta \in \widehat{Z}$ we define a map $\tau^\zeta : C \to C$ so that for $c \in C(u)$, we have $\tau^\zeta(c) = \tau^k(c)$ when restricted to $\|c = \tau^n(c)\|$ and where $\zeta = k \pmod n$. Then, as shown in [Kennison, 2002], this defines an action of $\widehat{Z}$ of C. It follows that $\widehat{Z}$ acts on the set of global sections of C. This suggests the following definition:

5.12. Definition. *Let B be a Boolean flow. Then an action $\alpha : \widehat{Z} \times B \to B$ is a* **regular action** *by $\widehat{Z}$ if $\alpha(\zeta, b) = \tau^k(b) \pmod{\langle b + \tau^n(b)\rangle}$ whenever there exists n, k such that $\zeta = k \pmod n$.*

5.13. PROPOSITION.

1. *A weakly separated Boolean flow admits at most one regular action by $\widehat{Z}$.*

2. *A cyclically representable Boolean flow admits a regular action by $\widehat{Z}$.*

3. *Let B, C be weakly separated Boolean flows which admit regular actions by $\widehat{Z}$. Then any flow homomorphism from B to C preserves the action by $\widehat{Z}$.*

PROOF.

1. If α is such a regular action, then $\alpha(\zeta, b)$ is determined modulo $\langle b + \tau^n(b)\rangle$ for each $n \in \mathbf{N}$ and, since B is weakly separated, any two elements agreeing modulo these ideals must coincide.

2. This follows from [Kennison, 2002]. (Also see the discussion preceding the above definition). It can readily be verified that the map $\alpha : \widehat{Z} \times B \to B$ is compatible with the group operation on $\widehat{Z}$ by proving the required identities modulo the ideals $\langle b + \tau^n(b)\rangle$.

3. This follows because a flow homomorphism h maps $\langle b+\tau^n(b)\rangle$ onto $\langle h(b)+\tau^n(h(b))\rangle$ and the actions are determined modulo these ideals. ∎

5.14. NOTATION. If $\alpha : \widehat{Z} \times B \to B$ is a regular action on B and if B is weakly separated, then we denote $\alpha(\zeta, b)$ by $\tau^\zeta(b)$. This entails no danger of confusion, in view of:

5.15. PROPOSITION. *A regular action on a weakly separated Boolean flow extends the action of τ in the sense that if $k \in \mathbf{N} \subseteq \widehat{Z}$ then $\alpha(k, b) = \tau^k(b)$.*

PROOF. Again this follows by considering $\alpha(k, b)$ and $\tau^k(b)$ modulo each of the ideals $\langle b + \tau^n(b)\rangle$. ∎

It follows that, for weakly separated Boolean flows which admit regular actions, we can talk about transfinite iterations, τ^ζ of τ, and flow homomorphisms will preserve them. If $B = \mathrm{Clop}(X)$, then these flow homomorphisms τ^ζ correspond, by duality, to continuous, transfinite iterations t^ζ of t. (While each map t^ζ is continuous, the action $\alpha : \widehat{Z} \times X \to X$ need not be continuous, which means that X need not be Boolean cyclic.)

5.16. PROPOSITION. *A cyclically representable Boolean flow has an iterator τ which is one-to-one and onto.*

PROOF. Since $-1 \in \widehat{Z}$, we can define τ^{-1} which is then an inverse for τ. ∎

5.17. Example. The Boolean flow $\mathrm{Clop}(\mathrm{Sym}(S))$ is not cyclically representable, as τ is not onto (because the shift map, $t : \mathrm{Sym}(S) \to \mathrm{Sym}(S)$ is not one-to-one).

Suppose X is a flow in Stone spaces such that $B = \mathrm{Clop}(X)$ is cyclically separated but not cyclically representable. Then, by Proposition 5.3, the map $B \to \gamma_*(B_{\mathrm{cyc}}) = \Gamma(B_{\mathrm{cyc}})$ is one-to-one but not onto. If $\widehat{X}$ is the flow in Stone spaces dual to the Boolean flow $\Gamma(B_{\mathrm{cyc}})$, then we have a map $\widehat{X} \to X$ which is onto but not one-to-one. So $\widehat{X}$ is a kind of flow preserving cover of X such that the iterator $\widehat{t}$ of $\widehat{X}$ is one-to-one and onto and the maps $\widehat{t^n}$ can naturally be extended to maps $\widehat{t^\zeta}$ for all $\zeta \in \widehat{Z}$. One goal of our work is to describe these spaces $\widehat{X}$.

6. The Simple spectrum

It follows from Theorem 4.10 that, for finitely generated Boolean flows, the space of maximal cyclic ideals is a discrete subspace of $\mathcal{W}_{\mathrm{cyc}}$. But, as we will see, the space of *all* maximal flow ideals need not be discrete, not even in the finitely generated case. These observations suggest that a more interesting spectrum would use all maximal flow ideals (not just the cyclic ones). Our first step is to characterize flows of the form B/M, where M is a maximal flow ideal, in a manner that extends usefully to Boolean flows over locales, and this is done in Definition 6.5.

6.1. Notation. Let B be a Boolean flow with iterator τ. We let τ^0 denote the identity map on B. If $b \in B$ and $k \in \mathbf{N}$, we let:

$$\mathbf{k}\text{-}\mathrm{Exp}(\mathbf{b}) = \bigvee_{\mathbf{0 \le i \le k}} \tau^{\mathbf{i}}(\mathbf{b})$$

Finally, we recall that $\langle b \rangle$ denotes the smallest flow ideal of B that contains b.

6.2. Definition. *A Boolean flow B is* **simple** *if it has precisely two flow ideals (which must then be B itself and $\{0\}$ and, moreover, these ideals must be different, so B must be* **non-trivial***, meaning that it satisfies $0 \neq 1$).*

A flow X in Stone spaces is **minimal** *if it has precisely two closed subflows (which must be X itself and the empty subset and, moreover, these subflows must be distinct, so X must be non-empty).*

A flow ideal I of B is **maximal** *if it is a maximal element of the set of all* proper *flow ideals of B.*

6.3. Lemma.

1. *The Boolean flow B is simple if and only if B is isomorphic to a flow of the form $\mathrm{Clop}(X)$ where X is a minimal flow in Stone spaces.*

2. *A flow ideal $I \subseteq B$ is maximal if and only if B/I is simple.*

3. *The non-trivial Boolean flow B is simple if and only if for every non-zero $b \in B$ there exists $k \in \mathbf{N}$ for which k-$\mathrm{Exp}(b) = 1$.*

4. *If X is a flow in Stone spaces, then X is minimal if and only if for every $x \in X$ the "orbit" $\{t^n(x) \mid n \geq 0\}$ is dense in X (recall that $t^0 = \mathrm{Id}_X$).*

PROOF.

1. This follows from the duality between Boolean algebras and Stone spaces.

2. Obvious.

3. Since b is non-zero and B is simple, the flow ideal $\langle b \rangle$ must be all of B, so we must have $1 \in \langle b \rangle$. But it can readily be shown that:

$$\langle b \rangle = \{a \in B \mid \exists k \in \mathbf{N}\ a \leq k\text{-}\mathrm{Exp}(b)\}$$

The result follows from this observation.

4. Obvious. ∎

6.4. EXAMPLE. Let X be the unit circle with its usual compact, connected topology and let $t : X \to X$ rotate X through λ radians, where λ is an irrational multiple of 2π. Then it is well-known that every orbit of X is dense so X is a minimal flow in the category of compact Hausdorff spaces. We can use symbolic dynamics, 2.2, to approximate this flow by a flow on a Stone space, by, for example, letting A_0, A_1 be closed semi-circles of X which overlap at exactly two points. The same type of analysis that shows that X is minimal also shows that the flow in Stone spaces, given by symbolic dynamics, is a closed minimal subflow of $\mathrm{Sym}\{0, 1\}$. Alternatively, we could prove this by using the characterization of such minimal flows given below, in 6.14.

6.5. DEFINITION. *Let E be a Boolean flow over the locale L. Then E is a* **simple Boolean flow over L** *if for every $u \in L$, with $u \neq \bot$, we have:*

- *The Boolean algebra $E(u)$ is non-trivial (that is, $0 \neq 1$ in $E(u)$).*
- *For all $g \in E(u)$ we have $\left[\|g = 0\| \vee \bigvee_{k \in \mathbf{N}}\{\|k\text{-}\mathrm{Exp}(g) = 1\|\}\right] = u$.*

6.6. REMARK. It follows from 6.3, that a Boolean flow over a spatial locale is simple if and only if every stalk is a simple Boolean flow.

6.7. PROPOSITION. *Let E be a Boolean flow over a locale L. Then there is a largest sublocale L_j of L for which the restriction of E to L_j is a simple flow over L.*

PROOF. Again this is a matter of defining the largest sublocale, or smallest nucleus, for which certain equations of the form $\bigvee\{u_\alpha\} = u$ become true, and this can be accomplished by requiring that certain families become "coverages" (see [Johnstone, 1982, pages 57–59]). For example, whenever $E(u)$ is a trivial Boolean algebra (with $0 = 1$) then we let u be covered by the empty family.

Similarly, for all $u \in L$ and all $g \in E(u)$, we require that the family $\{\|g = 0\|\} \cup \{\|k\text{-Exp}(g)\| \mid k \in \mathbf{N}\}$ cover u. ∎

6.8. DEFINITION. *Let (B, τ) be a Boolean flow and let B^0 be the sheaf over the space of all flow ideals of B, given in 3.5 . We define the* **simple spectrum** *of (B, τ) as B_{sim}, the restriction of B^0 to the largest sublocale $L_{\mathrm{sim}} \subseteq \mathcal{W}$ for which this restriction becomes simple.*

6.9. THEOREM. *Let (B, τ) be a Boolean flow in* Sh(1)*. Then B_{sim} is a simple flow over L_{sim}. There is a localic flow morphism $(\widehat{\eta}, \gamma) : (B, 1) \to (B_{\mathrm{sim}}, L_{\mathrm{sim}})$ which has a universal property with respect to maps $(\lambda, \gamma) : (B, 1) \to (S, L)$, where S is a simple flow in* Sh(L) *and λ is a sheaf epimorphism: such a map, (λ, γ) uniquely factors as $(\widehat{\lambda}, h)(\widehat{\eta}, \gamma)$ through a map $(\widehat{\lambda}, h)$ for which $\widehat{\lambda}$ is an* **isomorphism***.*

PROOF. The same type of argument as was used for 3.25 applies here. ∎

In what follows, we explore some features of the simple spectrum of Clop(Sym(S)), where S is a finite set of symbols. Our results, however, are considerably less complete than the results we obtained for the cyclic spectra of such spaces.

6.10. NOTATION. From now until the end of this section, we assume that S is a given finite set whose elements are called "symbols". Also:

- A **string** of length n is an n-tuple $(s_1, \ldots, s_n)$ of symbols.
- If $x \in \mathrm{Sym}(S)$ is a given sequence of symbols and if s is a string of length n, then s is a **substring of x at position p** (for $p \geq 0$) if $s_i = x_{p+i}$ for $i = 1, \ldots, n$. We further say that an **initial substring** is a substring at position 0.
- Let s be a string of length n. By $\pi^{-1}(s)$, we mean the clopen subset of Sym(S) of all x having s as an initial substring. Note that:

$$\pi^{-1}(s) = \pi_1^{-1}(s_1) \cap \ldots \cap \pi_n^{-1}(s_n).$$

6.11. LEMMA. *Sets of the form $\pi^{-1}(s)$ form a base for the topology on* Sym(S)*.*

PROOF. Recall that Sym(S) has the product topology so sets of the form $\pi_n^{-1}(s_n)$ form a subbase. So if $x \in U \subseteq \mathrm{Sym}(S)$ is given with U open, there exists a finite intersection of these subbasic sets such that:

$$x \in \pi_{n_1}^{-1}(a_n) \cap \ldots \cap \pi_{n_k}^{-1}(a_k) \subseteq U.$$

We may as well assume that $n_1 < n_2 < \ldots < n_k$. It is readily seen that if s is the initial substring of x of length n_k, then:

$$x \in \pi^{-1}(s) \subseteq \pi_{n_1}^{-1}(a_n) \cap \ldots \cap \pi_{n_k}^{-1}(a_k) \subseteq U.$$

and the result follows. ∎

6.12. Definition. *Let $x \in \mathrm{Sym}(S)$ be given and let s be a string. Then:*

- *We say that s* **never appears** *in x if s is not a substring of x at position p for any $p \geq 0$.*
- *We say that s appears* **k-frequently** *in x if for all $q \in \mathbf{N}$ there exists p with $q \leq p \leq (q + k)$ such that s is a substring of x at position p.*

Let A be a closed subflow of $\mathrm{Sym}(S)$. Then we say that:

- *s is of* **type 0** *with respect to A if s never appears in any member of A.*
- *s is of* **type k***, for $k > 0$, with respect to A, if s appears k-frequently in every member of A.*

6.13. Notation. Let X be a flow in Stone spaces and let $B = \mathrm{Clop}(X)$ be the corresponding Boolean flow. By the Stone Duality Theorem, flow quotients of B correspond to closed subflows of X. So each ideal $I \subseteq B$ corresponds to a flow quotient of B which corresponds to a closed subflow $A \subseteq X$. We call A the **closed subflow corresponding to the flow ideal I**. Similarly, I is the **flow ideal corresponding to the closed subflow A**. The relation between I and A is indicated by:

- Given A, then $I = \{b \mid b \cap A = \emptyset\}$.
- Given I, then $A = \bigcap\{\neg b \mid b \in I\}$.

6.14. Proposition. *Let A be a closed, non-empty subflow of $\mathrm{Sym}(S)$. Then A is a minimal subflow if and only if for every string s there exists $k \in \mathbf{N} \cup \{0\}$ such that s is of type k with respect to A.*

Proof. Let I be the flow ideal corresponding to the closed subflow A. We first assume that A is minimal, so I is a maximal flow ideal of $B = \mathrm{Clop}(\mathrm{Sym}(S))$. Let s be any string and let $b = \pi^{-1}(s)$. By Lemma 6.3, either $b \in I$ or there exists k such that k-$\mathrm{Exp}(b) = 1 \pmod{I}$. If $b \in I$, then $b \cap A = \emptyset$ so s can never be an initial substring of any $a \in A$. Moreover, s cannot be a substring at position p in any $a \in A$ for s would then be an initial substring of $t^p(a) \in A$. So s never appears in any member of S, and s is of type 0 with respect to A. On the other hand, suppose that k-$\mathrm{Exp}(b) = 1 \pmod{I}$. This means that $A \subseteq k$-$\mathrm{Exp}(b)$ so for every $a \in A$, we see that s is an initial substring of $t^i(a)$ for some i with $0 \leq i \leq k$. This, in turn, means that s is a substring of any $a \in A$

at position p where $0 \le p \le k$. If we apply this to $t^q(a) \in A$, we readily deduce that s appears k-frequently in a so s is of type k.

Conversely, assume that each string s is of type k with respect to A for some $k \in \mathbf{N} \cup \{0\}$. We claim that if $x, y \in A$ are given, then y is in the closure of the orbit of x, which implies that A is minimal. Let $\pi^{-1}(s)$ be a basic neighborhood of y. Then s appears as an initial substring of y, so there exists k such that s appears k-frequently in x. So s is a substring at some position p of x and this implies that $t^p(x) \in \pi^{-1}(s)$, the given neighborhood of y. ∎

6.15. Definition. *A* **type assignment function** *is a function which assigns a type $\nu(s) \in \mathbf{N} \cup \{0\}$ to each string s. We say that $x \in \mathrm{Sym}(S)$* **satisfies** *ν if whenever $\nu(s) = 0$ then s never appears in x and whenever $\nu(s) = k$ then s appears k-frequently in x.*

A type assignment function is **consistent** *if it is satisfied by at least one $x \in \mathrm{Sym}(S)$.*

6.16. Lemma. *If ν is a consistent type assignment function and if A is the set of all $x \in \mathrm{Sym}(S)$ which satisfy ν, then A is a closed minimal subflow.*

Proof. It is obvious that if $x \in \mathrm{Sym}(S)$ satisfies ν, then so does $t(x)$, so A is a subflow. To show that A is closed, suppose y is in the closure of A. We claim that y satisfies ν. Let s be a string with $\nu(s) = 0$. Then s cannot appear as a substring of y at any position, because, if so, the set of all $z \in \mathrm{Sym}(S)$ having s appear at that position would form a neighborhood of y which misses A. Similarly, assume that s is a string with $\nu(s) = k > 0$. If s does not appear k-frequently in y, then there exists a substring r of y of length greater than k in which s never starts to appear. But then $\nu(r) = 0$ as no member of $\mathrm{Sym}(S)$ in which r appears can have s appear k-frequently. So y has a substring, r, with $\nu(r) = 0$ which, as shown in the previous case, contradicts the fact that y is in the closure of A.

It follows that A is closed minimal subflow in view of the previous lemma. ∎

We next want to show that the maximal cyclic ideals of $B = \mathrm{Clop}(\mathrm{Sym}(S))$ are dense in the family of all proper flow ideals (in the topology on $\mathcal{W}$). We need some lemmas and notation:

6.17. Lemma. *The clopen subsets of* $\mathrm{Sym}(S)$ *are precisely the finite unions of sets of the form $\pi^{-1}(s)$ where s is a string. (Note that the empty union is a finite union.)*

Proof. We have previously seen sets of the form $\pi^{-1}(s)$ are a base. So each clopen $b \subseteq \mathrm{Sym}(S)$, being open, is a union of such basic open sets. But, being compact, b is a finite union of such sets. ∎

6.18. Lemma. *Every non-empty closed subflow of any flow on a Stone space contains a closed minimal subflow.*

Proof. Zorn's Lemma. ∎

6.19. NOTATION. We use the following notation for working with strings:

- We have defined substring of a sequence and can extend this to a **substring of a string** in the obvious way. So $r = (r_1, \dots, r_m)$ is a substring of $s = (s_1, \dots s_k)$ if there exists $q \geq 0$ such that $r_i = s_{q+i}$ for $1 \leq i \leq m$ (which implies that $q + m \leq k$).
- If $r = (r_1, \dots, r_m)$ and $s = (s_1, \dots s_k)$ are strings, then the **concatenation** of r and s, denoted by $r * s$, is the obvious string
$$r * s = (r_1, \dots, r_m, s_1, \dots s_k)$$
.
- If $s = (s_1, \dots s_k)$ is any string, then the **infinite concatenation**
$$s^\infty = s * s * s * \dots$$
is the sequence $x \in \mathrm{Sym}(S)$ which has s as a substring at position p for $p = 0, k, 2k, \dots nk \dots$. It follows that $t^k(x) = x$.

6.20. LEMMA. *Let A be a non-empty, closed subflow of* $\mathrm{Sym}(S)$ *and let b be a clopen subset of* $\mathrm{Sym}(S)$ *such that $b \cap A = \emptyset$. Then there exists $y \in \mathrm{Sym}(S)$ and $p \in \mathbf{N}$ such that $t^p(y) = y$ and the entire orbit of y is disjoint from b.*

PROOF. We may as well assume that A is a closed minimal subflow of $\mathrm{Sym}(S)$. By Lemma 6.17, we may write b as a finite union of sets of the form $\pi^{-1}(s(1))$ where $s(1), s(2), \dots s(n)$ are strings. This means that no member of A contains $s(i)$ as a substring for $1 \leq i \leq n$. Let $x \in A$ be given and let u be an initial substring of x of a length m which exceeds the length of each string $s(i)$ used in the representation of b. Find an integer p, exceeding the length of u such that u reappears as a substring of x at position p (which is clearly possible as u must be a string of type k for $k > 0$). Let r be the initial substring of x of length p. Let $r * r$ be the concatenation of r with itself. By the choice of p, the first $p + m$ members of $r * r$ is an initial substring of x. It readily follows that none of the strings $s(i)$ is a substring (at any position) of $r * r$, nor, therefore, of r^∞. Then $y = r^\infty$ is the required member of $\mathrm{Sym}(S)$. ■

6.21. PROPOSITION. *Let $(B, \tau) = \mathrm{Clop}(\mathrm{Sym}(S))$ and let $\mathcal{W}$ be the space of flow ideals of B. Let $\mathcal{W}_{\text{prop}}$ be the subspace of proper flow ideals and $\mathcal{W}_{\text{max-cyc}}$ the subspace of maximal cyclic flow ideals. Then $\mathcal{W}_{\text{max-cyc}}$ is dense in $\mathcal{W}_{\text{prop}}$.*

PROOF. Let $I \in \mathcal{W}_{\text{prop}}$ be given and let $N(b)$ be a basic neighborhood of I where $b \in I$ is clopen. We claim that there exists a maximal cyclic ideal $J \in \mathcal{W}_{\text{max-cyc}}$ with $J \in N(b)$.

Let A be the closed subflow of $\mathrm{Sym}(S)$ which corresponds to the ideal I. Then $b \in I$ means that $b \cap A = \emptyset$. By the above lemma, there exists $y \in \mathrm{Sym}(S)$ with $t^p(y) = y$ such that b is disjoint from the orbit of y. Since this orbit is minimal and cyclic and finite, it is a closed subflow which corresponds to a maximal cyclic ideal J. Since b is disjoint from the orbit of y, we see that $b \in J$ and so $J \in N(b)$. ■

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