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# CATEGORICAL REPRESENTATIONS OF CATEGORICAL GROUPS

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ABSTRACT. A representation theory for (strict) categorical groups is constructed. Each categorical group determines a monoidal bicategory of representations. Typically, these bicategories contain representations which are indecomposable but not irreducible. A simple example is computed in explicit detail.

## 1. Introduction

In three-dimensional topology there is a very successful interaction between category theory, topology, algebra and mathematical physics which is reasonably well understood. Namely, monoidal categories play a central role in the construction of invariants of three-manifolds (and knots, links and graphs in three-manifolds), which can be understood using quantum groups and, from a physics perspective, the Chern-Simons functional integral. The monoidal categories determined by the quantum groups are all generalisations of the idea that the representations of a group form a monoidal category.

The corresponding situation for four-manifold topology is less coherently understood and one has the feeling that the current state of knowledge is very far from complete. The complexity of the algebra increases dramatically in increasing dimension (though it might eventually stabilise). Formalisms exist for the application of categorical algebra to four-dimensional topology, for example using Hopf categories [CF], categorical groups [Y-HT] or monoidal 2-categories [CS, BL, M-S]. Since braided monoidal categories are a special type of monoidal 2-category (ones with only one object), then there are examples of the latter construction given by the representation theory of quasi-triangular Hopf algebras. This leads to the construction of the four-manifolds invariants by Crane, Yetter, Broda and Roberts which give information on the homotopy type of the four-manifold [CKY, R, R-EX]. At present it seems that categorical invariants which delve further into the smooth or combinatorial structure of four-manifolds will require different types of examples of monoidal 2-categories.

In this paper we determine a new set of examples of monoidal 2-categories. We show that the categorical representations of a categorical group form a monoidal 2-category, by direct analogy with the way in which the representations of a group form a monoidal category. The categorical definitions are given in section 2 and 3. In section 4, an abstract

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definition of categorical representations, and their morphisms, is given and then unpacked for the case of strict representations. An extended example is calculated in section 4 with explicit matrices, illustrating many of the complexities of more general examples. In section 5 we give a fairly complete characterisation of the one-dimensional categorical representations, outlining a number of examples. Finally in section 6 we make a number of remarks about the structure of the N-dimensional categorical representations, in particular the phenomenon of representations which are indecomposable but not irreducible. These remarks generalise some of the features of the example in section 4.

One particular example of a monoidal category leads to a state-sum model for quantum gravity in three-dimensional space-time [TV, B-O]. The motivation for this work grew out of enquiring whether there is a corresponding model in the more realistic four-dimensional space-time. The first attempts at doing this [BC, DFKR] used a braided monoidal category and suffer from several problems, one of which is that there is no 'data' on the edges of a triangulation of the manifold, which is where one might expect to find the combinatorial version of the metric tensor [REG]. Thus we arrived at the idea of constructing the monoidal 2-category of representations for the example of the categorical Lie group determined by the Lorentz group and its action on the translation group of Minkowski space, generalising the construction of [M-FG]. An early draft of this paper is the reference cited by Crane and Yetter [CY-2G, CY-MC, Y-MC, CSH] who developed the particular example, and the machinery of measurable categories to handle the Lie aspect, much further.

## 2. Categorical groups

By the term 'categorical group' we mean what some authors call a 'strict categorical group'.

2.1. Definition. *A* categorical group *is by definition a group-object in the category of groupoids.*

This means that a categorical group is a groupoid $\mathcal{G}$, with a set of objects $\mathcal{G}_0 \subset \mathcal{G}$, together with functors which implement the group product, $\circ\colon \mathcal{G}\times\mathcal{G} \to \mathcal{G}$, and the inverse ${}^{-1}\colon \mathcal{G} \to \mathcal{G}$, together with an identity object $1 \in \mathcal{G}_0$. These satisfy the usual group laws:

$$a \circ (b \circ c) = (a \circ b) \circ c$$

$$a \circ 1 = 1 \circ a = a$$

$$a \circ a^{-1} = a^{-1} \circ a = 1$$

for all $a$, $b$, $c$ in $\mathcal{G}$. In particular, $\mathcal{G}$ is a strict monoidal category.

2.2. Definition. *A* functorial homomorphism *between two categorical groups is a strict monoidal functor.*

Categorical groups are equivalent to crossed modules of groups. This equivalence, and the basic properties of categorical groups, are explained in [BS]. Here we give a brief outline.

In a categorical group $\mathcal{G}$ with hom-sets $\mathcal{G}(X,Y)$, the categorical composition $f \cdot g$ and the group product $\circ$ are related by the interchange law

$$(f \circ g) \cdot (h \circ k) = (f \cdot h) \circ (g \cdot k).$$

For all categories in this paper the diagrammatic order of composition is used. This means that $f \cdot g$ is defined when the target of $f$ is equal to the source of $g$.

In fact the composition is determined by the product. If $f \in \mathcal{G}(X,Y)$ and $g \in \mathcal{G}(Y,Z)$, then

$$f \cdot g = f \circ 1_{Y^{-1}} \circ g = g \circ 1_{Y^{-1}} \circ f.$$

In particular, $\mathcal{G}(1,1)$ is an abelian group, the composition and product coinciding.

2.3. Definition. *A* crossed module *is a homomorphism of groups*

$$\partial\colon E \to G$$

*together with an action $\triangleright$ of $G$ on $E$ by automorphisms, such that*

$$\partial\big(X \triangleright e\big) = X\big(\partial e\big)X^{-1}$$

$$\big(\partial e\big) \triangleright e' = ee'e^{-1}.$$

*We call $E$ the* principal group *and $G$ the* base group*.*

There is a natural notion of a mapping between crossed modules.

2.4. Definition. *A* homomorphism of crossed modules

$$(E, G, \partial, \triangleright) \to (E', G', \partial', \triangleright')$$

*is given by two vertical homomorphisms*

$$\begin{array}{ccc} E & \xrightarrow{\partial} & G \\ {\scriptstyle F_p}\big\downarrow & & \big\downarrow{\scriptstyle F_b} \\ E' & \xrightarrow{\partial'} & G' \end{array}$$

*which commute and satisfy $F_p(X \triangleright e) = F_b(X) \triangleright' F_p(e)$. We call the latter condition the* action condition*.*

The equivalence with categorical groups is as follows.

2.5. Theorem. *(Verdier) The category of categorical groups and functorial homomorphisms and the category of crossed modules of groups and homomorphisms between them are equivalent.*

The proof is sketched. Given a categorical group, a crossed module is defined by taking the base group $G(\mathcal{G})$ to be the objects,

$$G(\mathcal{G}) = (\mathcal{G}_0, \circ),$$

and the principal group $E(\mathcal{G})$, to be the subset of morphisms which are morphisms from the object 1 (to any object),

$$E(\mathcal{G}) = \bigcup_X \mathcal{G}(1, X),$$

again with the product operation $ab = a \circ b$. The homomorphism $E \to G$ of the crossed module is $e \in \mathcal{G}(1, X) \mapsto X$ and the action is $Y \triangleright e = 1_Y \circ e \circ 1_{Y^{-1}}$.

Conversely, given a crossed module $(E, G, \partial, \triangleright)$, a categorical group is constructed in a canonical way by taking $\mathcal{G}_0 = G$, $\mathcal{G}(X, Y) = \partial^{-1}\big(YX^{-1}\big)$. If $f \in \mathcal{G}(X, Y)$ and $g \in \mathcal{G}(Z, T)$, then the tensor product is defined as

$$f \circ g = f(X \triangleright g) \in \mathcal{G}(XZ, YT)$$

and the composition, for $Y = Z$,

$$f \cdot g = gf \in \mathcal{G}(X, T). \qquad \blacksquare$$

It is worth noting that the definition of a crossed module and of a categorical group makes sense when the groups and groupoids are Lie groups and the equivalence between Lie categorical groups and Lie crossed modules holds in the same way.

2.6. Examples.

1. Let $K$ be a group, then we define $\overline{K}$, the *closure* of $K$, to be the groupoid with one object $\bullet$ and hom-space $\overline{K}(\bullet, \bullet) = K$. If $K$ is abelian, then the group operation also defines a monoidal structure, so that $\overline{K}$ becomes a categorical group. The corresponding crossed module has principal group $E(\overline{K}) = K$ and trivial base group.

2. A categorical group is *connected* if there is a morphism between any pair of objects. The corresponding crossed module $\partial\colon E \to G$ is surjective. This implies that $E$ is a central extension of $G$, and the action of $G$ on $E$ is determined by conjugation in $E$.

3. A categorical group is *skeletal* if there are no morphisms between distinct objects (generalising (1)). In this case the crossed module $\partial\colon E \to G$ has $\partial = 1$ and is determined entirely by the action of $G$ on the abelian group $E$. An example is the *wreath product*, where $G$ is the symmetric group $S_n$ and $E = (\mathbb{C}^*)^n$, with the action of $S_n$ being the permutation of the factors in $E$.

4. A categorical group is *discrete* if there is at most one morphism between any pair of objects. The crossed module $\partial\colon E \to G$ is injective and $E$ is a normal subgroup of $G$. Again the action of $G$ on $E$ is determined by conjugation.

5. The *transformation categorical group* of a category $\mathcal{C}$ is defined by

$$\mathcal{G}_0 = \{\text{functorial isomorphisms on } \mathcal{C}\} \text{ and } \mathcal{G} = \{\text{natural isomorphisms}\}.$$

Here we consider, as usual, $\mathcal{G}_0$ to be a subset of $\mathcal{G}$ by identifying a functorial isomorphism with the identity natural isomorphism of it. For example, if $\mathcal{C} = \overline{K}$, then the crossed module corresponding to the transformation categorical group is $K \to \mathrm{Aut}K$, where $\partial$ maps a group element to the corresponding inner automorphism.

## 3. Bicategories

In this section we recall the definitions of 2-dimensional category theory [G, KS]. First we define 2-categories, sometimes called strict 2-categories, and then indicate the changes required to give the weaker notion of bicategories. Finally we discuss monoidal structures on bicategories and the example of 2-Vect.

3.1. DEFINITION. *A* 2-category $\mathcal{C}$ *is given by:*

1. *A set of* objects $\mathcal{C}_0$.

2. *A small category $\mathcal{C}(X,Y)$, for each pair $X, Y \in \mathcal{C}_0$. The set of objects in $\mathcal{C}(X,Y)$ we denote by $\mathcal{C}_1(X,Y)$. The elements of $\mathcal{C}_1(X,Y)$ are called* 1-morphisms. *For each pair $f, g \in \mathcal{C}_1(X,Y)$, we denote the set of morphisms in $\mathcal{C}(X,Y)$ from $f$ to $g$ by $\mathcal{C}_2(f,g)$. The elements of $\mathcal{C}_2(f,g)$ are called* 2-morphisms. *The composition in $\mathcal{C}(X,Y)$ is called the* vertical composition *and denoted by a small dot, e.g. $\mu\cdot\nu$ (or sometimes by simple concatenation without dot).*

3. *A functor*
$$\circ\colon \mathcal{C}(X,Y)\times\mathcal{C}(Y,Z)\to\mathcal{C}(X,Z),$$
*for each triple $X, Y, Z \in \mathcal{C}_0$. Together these are called the* horizontal composition. *The horizontal composition is required to be associative and unital. The last condition means that there is a 1-morphism $1_X \in \mathcal{C}_1(X,X)$, for each $X \in \mathcal{C}_0$, which is a right and left unit for horizontal composition.*

3.2. EXAMPLE. Let $\mathcal{G}$ be a categorical group. Then we define the closure of $\mathcal{G}$, which we denote by $\overline{\mathcal{G}}$, to be the 2-category with one object denoted $\bullet$, such that $\overline{\mathcal{G}}(\bullet,\bullet) = \mathcal{G}$. The horizontal composition is defined by the monoidal structure in $\mathcal{G}$. Note that there is a slightly confusing mixture of subscripts now, because $\overline{\mathcal{G}}_0 = \{\bullet\}$, whereas $\overline{\mathcal{G}}_1 = \overline{\mathcal{G}}_1(\bullet,\bullet) = \mathcal{G}_0$. Unfortunately this renumbering seems unavoidable in this subject and we hope that the context will always avoid confusion in this paper.

Next we recall the definition of 2-functors, pseudonatural transformations and modifications.

3.3. DEFINITION. *Given two 2-categories $\mathcal{C}$ and $\mathcal{D}$, a* 2-functor *between them, $F\colon \mathcal{C} \to \mathcal{D}$, consists of:*

1. *A function $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$.*
2. *A functor $F_1(X,Y)\colon \mathcal{C}(X,Y) \to \mathcal{D}(F(X),F(Y))$, for each pair $X,Y \in \mathcal{C}_0$. These functors are required to be compatible with the horizontal composition and the unit 1-morphisms. By abuse of notation we sometimes denote both $F_0$ and $F_1(X,Y)$ simply by $F$.*

3.4. DEFINITION. *Given two 2-functors between two 2-categories $F,G\colon \mathcal{C} \to \mathcal{D}$, a* pseudonatural transformation *between $F$ and $G$, denoted $h\colon F \Rightarrow G$, consists of:*

1. *A 1-morphism $h(X) \in \mathcal{D}_1(F(X),G(X))$, for each $X \in C_0$.*
2. *For each pair $X,Y \in \mathcal{C}_0$, a natural isomorphism $\tilde{h}\colon F_1(X,Y) \circ h(Y) \to h(X) \circ G_1(X,Y)$, where we consider $F_1(X,Y) \circ h(Y)$ and $h(X) \circ G_1(X,Y)$ both as functors from $\mathcal{C}(X,Y)$ to $\mathcal{D}(F(X),G(Y))$. We require two coherence conditions to be satisfied:*

   *(a)* $\tilde{h}(f \circ g) = (1_{F(f)} \circ \tilde{h}(g)) \cdot (\tilde{h}(f) \circ 1_{G(g)})$.

   *(b)* $\tilde{h}(1_X) = 1_{h(X)}$.

This notion of pseudonatural transformation is almost the same as the notion of *quasi-natural transformation* in [G], which differs in that the 2-cells are not required to be isomorphisms and have the arrows reversed.

3.5. DEFINITION. *Let $F,G,H\colon \mathcal{C} \to \mathcal{D}$ be three 2-functors and let $h\colon F \Rightarrow G$ and $k\colon G \Rightarrow H$ be two pseudonatural transformations. The* horizontal composite *of $h$ and $k$, denoted $h \circ k$, is defined by*

1. *For each $X \in \mathcal{C}_0$, $(h \circ k)(X) = h(X) \circ k(X)$.*
2. *For each pair $X,Y \in \mathcal{C}_0$, $\widetilde{h \circ k} = (\tilde{h} \circ 1_{k(Y)}) \cdot (1_{h(X)} \circ \tilde{k})$.*

3.6. DEFINITION. *Let $F,G\colon \mathcal{C} \to \mathcal{D}$ be two 2-functors and $h,k\colon F \Rightarrow G$ be two pseudonatural transformations. A* modification *$\phi\colon h \Rrightarrow k$ is given by a 2-morphism $\phi(X) \in \mathcal{D}_2(h(X),k(X))$, for each $X \in \mathcal{C}_0$. These 2-morphisms are required to satisfy*

$$\tilde{h}(f) \cdot (\phi(X) \circ 1_{G(f)}) = (1_{F(f)} \circ \phi(Y)) \cdot \tilde{k}(f),$$

*for any $X,Y \in \mathcal{C}_0$ and $f \in \mathcal{C}_1(X,Y)$.*

3.7. Definition. *The* horizontal *and* vertical compositions *of modifications are directly induced by the corresponding compositions in $\mathcal{D}$.*

3.8. Weakenings. The notion of a 2-category can be weakened to the notion of a bicategory, in which the associativity of horizontal composition is not given by an equation between 1-morphisms, but by 2-isomorphisms which are the components of a natural isomorphism called the *1-associator*,

$$\alpha_{f,g,h} \colon f \circ (g \circ h) \Rightarrow (f \circ g) \circ h,$$

which satisfies a certain coherence law. Similarly, the unital nature of the horizontal composition can be weakened by introducing natural isomorphisms [BEN]. However, in all the examples in this paper the unital isomorphisms are all identities. We call this sort of bicategory a *strictly unital bicategory*

The notions of 2-functor, pseudonatural transformation and modification have suitable generalisations to the case of bicategories. One new phenomenon which occurs is that, for the 2-functors, the horizontal composition is no longer preserved exactly, but only up to a family of natural isomorphisms, defined as follows. If $F$ is a 2-functor and $f$ and $g$ composable 1-mophisms in its domain, then the 2-isomorphisms are

$$\tilde{F}_{fg} \colon F(f) \circ F(g) \to F(f \circ g). \tag{1}$$

In general a 2-functor between bicategories involves weakening the condition $F(1_f) = 1_{F(f)}$ to an isomorphism. This notion of 2-functor is called a *homomorphism* in [BEN, STR]. It is also often called a pseudofunctor [KS].

3.9. Definition. *A* strictly unitary homomorphism *is a 2-functor between bicategories for which the isomorphisms $F(1_f) \cong 1_{F(f)}$ are all identity 2-morphisms. If additionally, for all $f, g$, $\tilde{F}_{fg}$ is an identity 2-morphism we call $F$ a* strict homomorphism

The strictly unitary condition is also called normal or normalized in some references. The pseudonatural transformations have a straightforward generalisation to bicategories.

Given two bicategories $\mathcal{C}$ and $\mathcal{D}$, then the following result gives a construction of a new bicategory [STR, GPS].

3.10. Theorem. *The 2-functors from $\mathcal{C}$ to $\mathcal{D}$, together with their pseudonatural transformations and their modifications form a bicategory bicat$(\mathcal{C}, \mathcal{D})$.*

3.11. Monoidal structures. The notion of a monoidal structure on a 2-category is straightforward to define.

3.12. Definition. *A* monoidal 2-category *is a* 2*-category $\mathcal{C}$ together with a* 2*-functor $\boxtimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ the* monoidal product*, which is associative and unital. The latter means that there is a* unit object *$I$ which is a left and right unit for the monoidal product.*

By $\mathcal{C} \times \mathcal{C}$ we mean the cartesian product 2-category. The requirement that $\boxtimes$ be a 2-functor means that the interchange law is satisfied as an identity. This is too strict for the purposes of this paper, as we need to use bicategories rather than 2-categories. This gives the notion of a *monoidal bicategory*, for which the definition is the same as the definition for a monoidal 2-category, but with the appropriate notion of 2-functor for bicategories and pseudonatural transformations which give the associative and unital conditions. This means the 2-functor $\boxtimes\colon \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ carries natural 2-isomorphisms $\widetilde{\boxtimes}$ called the *tensorator*. The monoidal structures in the examples of interest to us are not strictly associative either. There are natural 1-isomorphisms called the *0-associator*. We remark that in our examples the 0-associator is strictly natural and satisfies the pentagon equation on the nose, so all the other coherers in the general definition of a monoidal bicategory besides the ones mentioned above are always trivial in this paper. This entire structure, together with the axioms it obeys, is a special case of the definition of a tricategory given by [GPS], in which the tricategory has only one object.

3.13. 2-Vector spaces. In this section we recall the definition, due to Kapranov and Voevodsky [KV], of the monoidal bicategory of 2-vector spaces in the completely coordinatized version.

3.14. Definition. *We define the* monoidal bicategory 2-Vect *as follows:*

1. *2-Vect*$_0 = \mathbb{N}$, *the set of natural numbers including zero.*
2. *For any* $N, M \in$ *2-Vect*$_0$, *we define the category 2-Vect*$(N, M)$ *as follows:*

   i) *The set of objects of 2-Vect*$_1(N, M)$ *consists of all* $N \times M$ *matrices with coefficients in* $\mathbb{N}$.

   ii) *For any* $a, b \in$ *2-Vect*$_1(N, M)$, *the set 2-Vect*$_2(a, b)$ *consists of all* $N \times M$ *matrices whose coefficients are complex matrices such that the* $i, j$*-coefficient has dimension* $a^i_j \times b^i_j$. *For any* $a \in$ *2-Vect*$_1(N, M)$, *we define* $1_a$ *to be the* $N \times M$ *matrix such that* $(1_a)^i_j$ *is the identity matrix of dimension* $a^i_j$ *The* vertical composition *of two composable* 2*-morphisms is defined by componentwise matrix multiplication*
   $$(\mu \cdot \nu)^i_j = \mu^i_j \nu^i_j.$$

3. *The* unit 1-morphism *on an object* $N$, *denoted* $1_N$, *is given by the* $N \times N$ *identity matrix. The* (horizontal) composition *of two* 1*-morphisms is defined by matrix multiplication.*

   *The* horizontal composition *of two composable* 2*-morphisms is defined by*
   $$(\mu \circ \nu)^i_j = \oplus_k \mu^i_k \otimes \nu^k_j.$$

4. *The horizontal composition is not strictly associative, and so there are associativity isomorphisms which carry out the corresponding permutations of bases. In detail,*

*if $f, g, h$ are composable 1-morphisms, then the associativity 2-morphism $\alpha_{f,g,h}$ has components the permutation matrices $[\alpha_{f,g,h}]^i_j$ :*

$$\bigoplus_k \left[\left(\bigoplus_l \left(\mathcal{C}^{f^i_l} \otimes \mathcal{C}^{g^l_k}\right)\right) \otimes \mathcal{C}^{h^k_j}\right] \to \bigoplus_l \left[\mathcal{C}^{f^i_l} \otimes \left(\bigoplus_k \left(\mathcal{C}^{g^l_k} \otimes \mathcal{C}^{h^k_j}\right)\right)\right]$$

*which map any vector of the form*

$$\bigoplus_k \left[\left(\bigoplus_l (x^i_l \otimes y^l_k)\right) \otimes z^k_j\right]$$

*to the corresponding element*

$$\bigoplus_l \left[x^i_l \otimes \left(\bigoplus_k (y^l_k \otimes z^k_j)\right)\right].$$

5. *The* monoidal product *of two objects is defined by*

$$N \boxtimes M = NM.$$

*For two* 1*-morphisms we define*

$$(a \boxtimes b)^{ij}_{kl} = a^i_k b^j_l.$$

*Finally, for two* 2*-morphisms we define*

$$(\mu \boxtimes \nu)^{ij}_{kl} = \mu^i_k \otimes \nu^j_l.$$

*The* unit object *is* 1*.*

*This monoidal product has a tensorator $\widetilde{\boxtimes}$ and an 0-associator which are again given by the corresponding permutations of basis elements.*

6. *The remaining coherers for a monoidal bicategory are trivial.*

So far we have defined the structure of the monoidal 2-category. There are also linear structures in 2-Vect, which we define below. Although we have not defined linear structures in general, we hope that the definitions below are clear.

3.15. Definition. *There are three levels of linear structure in 2-Vect, which we call the* monoidal sum*, the* direct sum *and the* sum *respectively.*

1. *The* monoidal sum *defines a monoidal structure on 2-Vect. For two objects it is defined as*

$$N \boxplus M = N + M,$$

*for two* 1*-morphisms as*

$$a \boxplus b = a \oplus b$$

*and for two* 2*-morphisms as*

$$\mu \boxplus \nu = \mu \oplus \nu.$$

*The* zero object *is* 0*. Unlike* $\widetilde{\boxtimes}$*, which is non-trivial,* $\tilde{\boxplus}$ *is just the identity.*

2. *The* direct sum *defines a monoidal structure on each 2-Vect*$(N, M)$*. The direct sum of two* 1*-morphisms* $a, b$ *in 2-Vect*$(N, M)$ *is defined by*

$$(a \oplus b)^i_j = a^i_j + b^i_j.$$

*The direct sum of two* 2*-morphisms in 2-Vect*$(N, M)$ *is defined by*

$$(\mu \oplus \nu)^i_j = \mu^i_j \oplus \nu^i_j.$$

*The* zero 1-morphism*, denoted* $(0)$*, is given by the* $N \times M$*-dimensional zero matrix.*

3. *The* sum *defines the linear structure in each 2-Vect*$(N, M)$*. For two* 2*-morphisms we define*

$$(\mu + \nu)^i_j = \mu^i_j + \nu^i_j.$$

*The* zero 2-morphism*, denoted* $((0))$*, is given by the matrix all of whose entries are zero matrices of the right size.*

3.16. Definition. *Let* $N \in \mathbb{N}$*. We define the* general linear categorical group $\mathcal{GL}(N) \subset$ *2-Vect*$(N, N)$ *to be the categorical group consisting of all invertible* 1*- and* 2*-morphisms in 2-Vect*$(N, N)$*.*

This definition makes sense because the horizontal composition of invertible 1- and 2-morphisms in 2-Vect is in fact strictly associative, as proved in the following lemma.

3.17. Lemma. *The monoidal category* $\mathcal{GL}(N)$ *is a categorical group, and the associated crossed module of groups is given by*

$$(\mathbb{C}^*)^N \xrightarrow{\ 1\ } S_N,$$

*where* $S_N$ *is the symmetric group on* $N$ *letters and* 1 *the trivial group homomorphism which maps everything to* $1 \in S_N$*. The action of* $S_N$ *on* $(\mathbb{C}^*)^N$ *is given by the permutations of the coordinates.*

Proof. The only invertible 1-morphisms in 2-Vect$(N, N)$ are the permutation matrices. The 1-associator for a triple of permutation matrices is trivial, because it is a set of permutation matrices on 0 or 1 dimensional vector spaces. Hence the horizontal composition in $\mathcal{GL}(N)$ is strictly associative and it forms a categorical group.

Between two permutation matrices there can only be an invertible 2-morphism if they are equal. The reason is that, if the corresponding entries in two permutation matrices are different, then one of them has to be zero.

This shows that 2-Vect$_2(1_N, 1_N)$ has its only non-trivial entries on the diagonal and these are 1-dimensional invertible complex matrices. Hence 2-Vect$_2(1_N, 1_N)$ is isomorphic to $(\mathbb{C}^*)^N$. Theorem 2.5 shows that the action of a permutation matrix $P \in$ 2-Vect$_1(N, N)$ on a diagonal matrix $a \in$ 2-Vect$_2(1_N, 1_N)$ is given by

$$PaP^{-1}.$$

Here we mean the ordinary matrix multiplication, which is what $1_P \otimes a \otimes 1_{P^{-1}}$ amounts to in this case. Thus the element in $S_N$, which corresponds to $P$, acts on the element in $(\mathbb{C}^*)^N$, which corresponds to $a$, by permutation of its coordinates. ∎

## 4. Categorical representations

In this section we give the abstract definitions of categorical representations, the functors between them, the natural transformations between such functors and the monoidal product. These definitions are analogues of Neuchl's [N] definitions for Hopf categories. We also work out a concrete example.

Let $G$ be a group and $\overline{G}$ its closure (defined in Example 1). One can check that a representation of $G$ is precisely a functor $\overline{G} \to$ Vect and an intertwiner precisely a natural transformation between two such functors. This observation motivates the following definition for categorical groups.

The starting point is the selection of a monoidal bicategory as the category in which the categorical group is represented. In this paper we discuss only 2-Vect, although other monoidal bicategories could be used instead (see the remarks in the final section).

Let $\mathcal{G}$ be an arbitrary categorical group and let $\overline{\mathcal{G}}$ be its closure, defined in Ex. 3.2. We first give a conceptual definition.

4.1. Definition. *a) A* categorical representation *of $\mathcal{G}$ is a strictly unitary homomorphism $(R, \tilde{R})\colon \overline{\mathcal{G}} \to$ 2-Vect. We call the non-negative integer $R(\bullet) \in$ 2-Vect$_0$ the* dimension *of the categorical representation.*

*b) A* 1-intertwiner *is a pseudonatural transformation between two categorical representations.*

*c) A* 2-intertwiner *is a modification between two 1-intertwiners with the same source and target.*

Note that Neuchl [N] uses the terms $\mathcal{G}$-functors and $\mathcal{G}$-transformations instead of 1- and 2-intertwiners.

Theorem 3.10 shows that the categorical representations of $\mathcal{G}$, together with the 1- and 2-intertwiners, form a bicategory bicat$(\overline{\mathcal{G}}, \text{2-Vect})$. To give this the structure of a monoidal bicategory, we first promote $\overline{\mathcal{G}}$ and 2-Vect to tricategories and use some general results about those. We consider $\overline{\mathcal{G}}$ as a strict tricategory, denoted $\underline{\overline{\mathcal{G}}}$ by adding only identity 3-morphisms to the existing strict bicategory. For 2-Vect we take the closure $\overline{\text{2-Vect}}$ as the tricategory. By a general result of Gordon, Power and Street [GPS] about tricategories we know that tricat$(\underline{\overline{\mathcal{G}}}, \overline{\text{2-Vect}})$ forms a tricategory. The objects of this tricategory are trihomomorphisms (functors between tricategories). The constant trihomomorphism is the one that sends every 3-morphism to $1_{1_{1_\bullet}}$.

4.2. LEMMA. *$\overline{bicat(\overline{\mathcal{G}}, \textit{2-Vect})}$ is the subtricategory of $tricat(\underline{\overline{\mathcal{G}}}, \overline{\textit{2-Vect}})$ determined by the 1-, 2- and 3-morphisms on the unique constant trihomomorphism.*

PROOF. Just check the diagrams in [GPS]. ∎

Unfortunately Gordon, Power and Street [GPS] do not give explicit definitions of the composition rules for the various morphisms in tricategories. Therefore we spell out the tensor product in bicat$(\overline{\mathcal{G}}, \text{2-Vect})$ below, which corresponds to the horizontal composition in tricat$(\underline{\overline{\mathcal{G}}}, \overline{\text{2-Vect}})$. The first definition can also be found in [ROU].

4.3. LEMMA. *Let $(R, \tilde{R})$ and $(T, \tilde{T})$ be two categorical representations. Then the monoidal product $(R \boxtimes T, \widetilde{R \boxtimes T})$ of $(R, \tilde{R})$ and $(T, \tilde{T})$ is given by*

$$R \boxtimes T(X) = R(X) \boxtimes T(X)$$

*and the following diagram*

$$\begin{array}{ccc}
R(X)R(Y) \boxtimes T(X)T(Y) & \xrightarrow{\tilde{R}_{X,Y} \boxtimes \tilde{T}_{X,Y}} & R(XY) \boxtimes T(XY) = R \boxtimes T(XY) \\
\Big\downarrow \tilde{\boxtimes}_{(R(X),T(X)),(R(Y),T(Y))} & & \Big\uparrow \widetilde{R \boxtimes T}_{X,Y} \\
(R(X) \boxtimes T(X))(R(Y) \boxtimes T(Y)) & = & (R \boxtimes T(X))(R \boxtimes T(Y))
\end{array}$$

4.4. LEMMA. *Let $(h, \tilde{h})\colon R_1 \to R_2$ and $(k, \tilde{k})\colon T_1 \to T_2$ be 1-intertwiners. The monoidal product $(h \boxtimes k, \widetilde{h \boxtimes k})$ is given by*

$$h \boxtimes k(X) = h(X) \boxtimes k(X)$$

*using on the right the monoidal product in 2-Vect, and by the following diagram*

$$\begin{array}{ccc}
R_1 \boxtimes T_1(f) \circ h \boxtimes k(Y) & \xrightarrow{\widetilde{h \boxtimes k}} & (h \boxtimes k(X)) \circ (R_2 \boxtimes T_2(f)) \\
\Big\| & & \Big\| \\
(R_1(f) \boxtimes T_1(f)) \circ (h(Y) \boxtimes k(Y)) & & (h(X) \boxtimes k(X)) \circ (R_2(f) \boxtimes T_2(f)) \\
\tilde{\boxtimes} \Big\downarrow & & \tilde{\boxtimes} \Big\downarrow \\
(R_1(f) \circ h(Y)) \boxtimes (T_1(f) \circ k(Y)) & \xrightarrow{\tilde{h} \boxtimes \tilde{k}} & (h(X) \circ R_2(f)) \boxtimes (k(X) \circ T_2(f))
\end{array}$$

4.5. LEMMA. *Let $\alpha$ and $\beta$ be two 2-intertwiners. The monoidal product is given by*

$$\alpha \boxtimes \beta,$$

*using the monoidal product in 2-Vect.*

There is also a natural way of defining the monoidal sum of two categorical representations.

4.6. Definition. *Let* $(R, \tilde{R})$ *and* $(T, \tilde{T})$ *be two categorical representations. Their* monoidal sum *is defined by*

$$R \boxplus T(X) = R(X) \boxplus T(X),$$

*and*

$$\tilde{R} \boxplus \tilde{T}(X, Y) = \tilde{R}(X, Y) \boxplus \tilde{T}(X, Y).$$

*Note that the latter makes sense, because*

$$(R(X) \circ R(Y)) \boxplus (T(X) \circ T(Y)) = (R(X) \boxplus T(X)) \circ (R(Y) \boxplus T(Y))$$

*holds on the nose, i.e. the 2-isomorphism between both sides of the equation is the identity, as already noticed in definition 3.15.*

Since $\widetilde{\boxplus}$ is trivial, the definition of their monoidal sum of 1- and 2-intertwiners is much simpler than that of their monoidal product.

4.7. Definition. *Let* $(h, \tilde{h})\colon R_1 \to R_2$ *and* $(k, \tilde{k})\colon T_1 \to T_2$ *be* 1*-intertwiners. We define their* monoidal sum, $(h \boxplus k, \widetilde{h \boxplus k})$, *to be*

$$h \boxplus k(X) = h(X) \boxplus k(X)$$

*and*

$$\tilde{h} \boxplus \tilde{k},$$

*using on the right the monoidal sum in 2-Vect.*

4.8. Definition. *Let* $\alpha$ *and* $\beta$ *be two* 2*-intertwiners. We define their* monoidal sum *as*

$$\alpha \boxplus \beta,$$

*using the monoidal product in 2-Vect.*

We do not give a precise definition of the direct sum of 1- and 2-intertwiners and the sum of 2-intertwiners, because we do not need them.

4.9. Strict categorical representations. These definitions can be unpacked by applying the general definitions of bicategories to this particular situation and expressing the result in terms of categorical groups. The categorical representations can be formulated in terms of crossed modules. In this way a strictly unitary homomorphism leads to a weakened notion of morphism between crossed modules, involving a group 2-cocycle on the object group $G$ corresponding to $\tilde{R}$. The extra conditions on this cocycle appear to be very complicated. Therefore we restrict our attention to a subclass of categorical representations in the rest of this paper, which we call *strict categorical representations.*

4.10. Definition. *A* strict categorical representation *is a strict homomorphism* $R\colon \overline{\mathcal{G}} \to$ *2-Vect. This means that* $\tilde{R}_{X,Y}$ *are all identity 2-morphisms.*

Restricting to strict categorical representations still gives a monoidal bicategory, as proved by the following lemma.

4.11. LEMMA. *The monoidal product and the monoidal sum of two strict categorical representations yield strict categorical representations.*

PROOF. The first claim follows from the fact that the top and the left-hand side of the diagram in lemma 4.3 are trivial. Note that $\widetilde{\boxtimes}_{(R(X),T(X)),(R(Y),T(Y))}$ is trivial, because all matrices in the subscript are permutation matrices.

The second claim is obvious as well, because the monoidal sum of two identity 2-morphisms is an identity 2-morphism. ∎

Let us now unpack the definition of a strict categorical representation and 1- and 2-intertwiners.

- A strict categorical representation of $\mathcal{G}$ amounts to a choice of a non-negative integer $N$ and a strict homomorphism $R\colon \overline{\mathcal{G}} \to \overline{\mathcal{GL}(N)}$, or equivalently a functorial homomomorphism between categorical groups $R\colon \mathcal{G} \to \mathcal{GL}(N)$. It can also be described as a homomorphism between the corresponding crossed modules,

$$\begin{array}{ccc} E(\mathcal{G}) & \xrightarrow{\partial} & G(\mathcal{G}) \\ {\scriptstyle R_p}\big\downarrow & & \big\downarrow{\scriptstyle R_b} \\ (\mathbb{C}^*)^N & \xrightarrow{1} & S_N \end{array} \tag{2}$$

  according to theorem 2.5 and lemma 3.17.

- Let $R\colon \mathcal{G} \to \mathcal{GL}(N)$ and $T\colon \mathcal{G} \to \mathcal{GL}(M)$ be two strict categorical representations of $\mathcal{G}$. A 1-intertwiner between them consists of a 1-morphism $h_\bullet \in$ 2-Vect$_1(N,M)$ together with a 2-isomorphism $\tilde{h}(X) \in$ 2-Vect$_2(R(X) \circ h_\bullet, h_\bullet \circ T(X))$, for each $X \in \mathcal{G}_0$.

  The 1-morphism $h_\bullet$ can be thought of as a vector bundle (with fibres of varying dimension) over the finite set $N \times M$, the cartesian product of the $N$-element set with the $M$-element set. The data above determine an action of the group $\mathcal{G}_0$ on this vector bundle in the following way.

  There is a right action of $X \in \mathcal{G}_0$ on the set $N$, $i \mapsto iR_b(X)$, given by the permutation matrix $R_b(X)$ (acting on the right) and similarly a right action of $\mathcal{G}_0$ on the set $M$ given by $T_b$.

  4.12. LEMMA. *The collection of linear maps*

$$\tilde{h}(X)^i_j\colon {h_\bullet}^{iR_b(X)}_{j} \to {h_\bullet}^{i}_{jT_b(X)^{-1}}$$

  *determines a left action of* $\mathcal{G}_0$ *on the vector bundle* $h_\bullet$.

PROOF. The two coherence conditions in definition 3.4 are

$$\tilde{h}(X \circ Y) = \left(1_{R(X)} \circ \tilde{h}(Y)\right) \cdot \left(\tilde{h}(X) \circ 1_{T(Y)}\right)$$

$$\tilde{h}(1) = 1_{h_\bullet}.$$

Note that the 1-associators in these expressions are trivial because $R$ and $T$ take values in the permutation matrices. Taking the $(i,j)$-th component of $\tilde{h}$ gives

$$\tilde{h}(X \circ Y)^i_j = \tilde{h}(Y)^{iR_b(X)}_j \cdot \tilde{h}(X)^i_{jT_b(Y)^{-1}},$$

which is the condition for a left action. This covers the left action of $\mathcal{G}_0$ on $N \times M$ given by $(iR_b(X), j) \mapsto (i, jT_b(X)^{-1})$. ■

The remaining condition satisfied by the 1-intertwiner is the naturality condition. This is that the diagram

$$\begin{array}{ccc} R(X) \circ h_\bullet & \xrightarrow{\tilde{h}(X)} & h_\bullet \circ T(X) \\ {\scriptstyle R(f)\circ 1_{h_\bullet}} \big\downarrow & & \big\downarrow {\scriptstyle 1_{h_\bullet}\circ T(f)} \\ R(Y) \circ h_\bullet & \xrightarrow{\tilde{h}(Y)} & h_\bullet \circ T(Y) \end{array} \tag{3}$$

commutes.

Since $\tilde{h}(X)$ is invertible, this implies the equation on 1-morphisms

$$R(X) \circ h_\bullet = h_\bullet \circ T(X). \tag{4}$$

Also restricting naturality to the crossed module data ($X = 1$), gives the equation

$$(R(e) \circ 1_{h_\bullet}) \cdot \tilde{h}(\partial e) = 1_{h_\bullet} \circ T(e) \tag{5}$$

for $e \in E(\mathcal{G})$.

It is possible to show that this last condition is actually sufficient to recover all of (3).

4.13. LEMMA. *For intertwiners of strict categorical representations, the naturality condition (3) follows from condition (5) for all $e\colon 1 \to Y$ and the coherence condition 2(a) of definition 3.4.*

Proof. The naturality condition follows from the following computation. Suppose $f\colon X \to X \circ Y$ is an arbitrary morphism in $\mathcal{G}_1$. We can write $f = 1_X \circ e$, where $e\colon 1 \to Y$. Then

$$
\begin{aligned}
&(R(f) \circ 1_{h_\bullet}) \cdot \tilde{h}(X \circ Y) \\
&\qquad = \left(1_{R(X)} \circ R(e) \circ 1_{h_\bullet}\right) \cdot \left((1_{R(X)} \circ \tilde{h}(Y)\right) \cdot \left(\tilde{h}(X) \circ 1_{T(Y)}\right)
\end{aligned}
$$

(using 2(a) of definition 3.4)

$$
\begin{aligned}
&= \left(1_{R(X)} \circ \left((R(e) \circ 1_{h_\bullet}) \cdot \tilde{h}(Y)\right)\right) \cdot \left(\tilde{h}(X) \circ 1_{T(Y)}\right) \\
&= \left(1_{R(X)} \circ 1_{h_\bullet} \circ T(e)\right) \cdot \left(\tilde{h}(X) \circ 1_{T(Y)}\right)
\end{aligned}
$$

(using (5))

$$
\begin{aligned}
&= \tilde{h}(X) \circ T(e) \\
&= \left(\tilde{h}(X) \circ 1_1\right) \cdot \left(1_{h_\bullet} \circ 1_{T(X)} \circ T(e)\right) \\
&= \tilde{h}(X) \cdot (1_{h_\bullet} \circ T(f)) \, .
\end{aligned}
$$

In the computation, all 1-associators which occur are the identity. ∎

- Let $h = (h_\bullet, \tilde{h}), k = (k_\bullet, \tilde{k})\colon R \to T$ be 1-intertwiners. A 2-intertwiner between them consists of a single 2-morphism $\phi \in$ 2-$\mathrm{Vect}_2(h_\bullet, k_\bullet)$.

  The condition that this satisfies is that for each $X \in \mathcal{G}_0$ the following diagram commutes:

$$
\begin{array}{ccc}
R(X) \circ h_\bullet & \xrightarrow{\tilde{h}(X)} & h_\bullet \circ T(X) \\
{\scriptstyle 1_{R(X)} \circ \phi}\downarrow & & \downarrow{\scriptstyle \phi \circ 1_{T(X)}} \\
R(X) \circ k_\bullet & \xrightarrow{\tilde{k}(X)} & k_\bullet \circ T(X)
\end{array}
\tag{6}
$$

4.14. Example.

4.15. Convention. From now on a categorical representation will always mean a *strict* categorical representation.

In this section we work out the categorical representations of a concrete example of a finite skeletal categorical group. Recall that the crossed modules corresponding to skeletal categorical groups are determined by a group $G$ and an abelian group $E$ on which $G$ acts by automorphisms. The simplest example is $G = C_2$, the cyclic group with two elements $\pm 1$, and $E = C_3 = \{1, x, x^{-1}\}$, with the non-trivial action of $C_2$ on $C_3$, $-1 \triangleright x = x^{-1}$. We call this categorical group $\mathcal{G}(2,3)$. Another way of looking at this example is by defining the total space of all the morphisms

$$\mathcal{G}(2,3) =$$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \begin{pmatrix} x^{-1} & 0 \\ 0 & x \end{pmatrix}, \begin{pmatrix} 0 & x \\ x^{-1} & 0 \end{pmatrix}, \begin{pmatrix} 0 & x^{-1} \\ x & 0 \end{pmatrix} \right\}.$$

The source and target of each morphism is determined by the place of the non-zero coefficients of the corresponding matrix, e.g. the sources and targets of the first two matrices are equal to 1 and $-1$ respectively. In this matrix notation the monoidal product, corresponding to the horizontal composition in $\overline{\mathcal{G}(2,3)}$ and denoted by $\circ$, is defined by matrix multiplication (giving the dihedral group $D_3$) and the composition, corresponding to the vertical composition in $\overline{\mathcal{G}(2,3)}$ and denoted by simple concatenation, by coefficientwise multiplication.

The constructions will be carried out in this example. The main features of the general case are apparent in this example; some comments on the generalisations are given at the end of the section.

We classify all 1- and 2-dimensional categorical representations of $\mathcal{G}(2,3)$.

1. $\mathcal{V}(1)$. This is the identity representation defined by $R_b(\pm 1) = 1 \in S_1$ and $R_p(x) = 1 \in \mathbb{C}^*$. It is the only 1-dimensional categorical representation, due to the following argument. Obviously $R_b(\pm 1) = 1 \in S_1$ has to hold. By (2) we see that $R(x) = \xi(x)$, where $\xi$ is a complex group character on $C_3$. By the action condition in definition 2.4 we see that $\xi(x^{-1}) = \xi(x)$, so $\xi$ has to be the trivial character and $R_p(x) = 1$.

2. $\mathcal{V}(2)$. This is the trivial 2-dimensional categorical representation defined by $R_b(\pm 1) = 1 \in S_2$ and $R_p(x) = (1,1) \in (\mathbb{C}^*)^2$. As in the 1-dimensional case, the action condition forces $R_p$ to be trivial if $R_b$ is trivial. Below we show that $\mathcal{V}(2)$ is isomorphic to $\mathcal{V}(1) \boxplus \mathcal{V}(1)$.

3. $\mathcal{V}(2)_\xi$. These are the non-trivial 2-dimensional categorical representations, where $R_b(\pm 1) = \pm 1 \in S_2$ and $R_p$ is determined by one complex group character, $\xi$, on $C_3$.

   By (2) we see that $R_p(x) = (\xi(x), \psi(x))$, where $\xi$ and $\psi$ are both complex group characters on $C_3$. The action condition now becomes $(\xi(x)^{-1}, \psi(x)^{-1}) = (\psi(x), \xi(x))$, so we have $\xi = \psi^{-1}$. There are no further restriction on $\xi$. A nice way of picturing the strict homomorphisms $R$ of these representations is by using the matrix definition of $\mathcal{G}(2,3)$:

$$R\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R\left(\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\right) = \begin{pmatrix} \xi(x) & 0 \\ 0 & \xi(x)^{-1} \end{pmatrix}.$$

   The image of the other endomorphisms is obtained via horizontal composition.

Next we study all the 1-intertwiners between these categorical representations. In Lemma 4.12 we showed that these can be seen as vector bundles with a left action of $\mathcal{G}_0$. In particular, equation (4) holds. The naturality condition simplifies a bit further in this example. Let $(h_\bullet, \tilde{h})$ be a 1-intertwiner between two categorical representations $R$ and $T$. By lemma 4.13 the naturality condition reduces to the equation

$$R(e) \circ 1_{h_\bullet} = 1_{h_\bullet} \circ T(e), \tag{7}$$

for any $e \in E(\mathcal{G}(2,3))$. Note that this shows that the action on the vector bundle does not have to satisfy any additional conditions.

First we study the 1-intertwiners between categorical representations of the same dimension.

1. $\mathcal{C}(\mathcal{V}(1), \mathcal{V}(1))$: A 1-intertwiner, in this case, is given by $h_\bullet = (n)$, i.e. a 1-dimensional matrix with a non-negative integer coefficient, and an $n$-dimensional representation of $C_2$ denoted by $\tilde{h}$. The naturality condition for 1-endomorphisms does not impose any restrictions in this case, as one can easily check.

2. $\mathcal{C}(\mathcal{V}(2), \mathcal{V}(2))$: A 1-intertwiner is given by

$$h_\bullet = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix},$$

   with $n_i \in \mathbb{N}$ for $i = 1, 2, 3, 4$, and $\tilde{h}$. The naturality condition (7) does not impose any restrictions in this case, because $R$ is trivial. Just as in the one-dimensional case we see that $\tilde{h}$ defines a representation of $C_2$ on $\mathbb{C}^{n_i}$, for all $i = 1, 2, 3, 4$.

3. $\mathcal{C}(\mathcal{V}(2)_\xi, \mathcal{V}(2)_\psi)$: In this case a 1-intertwiner is given by

$$h_\bullet = \begin{pmatrix} n_1 & n_2 \\ n_3 & n_4 \end{pmatrix}$$

   and $\tilde{h}$. The naturality condition imposes restrictions on $h$. We need consider only equations (4) and (7). First, let $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This gives

$$\begin{aligned} \begin{pmatrix} 1_{n_2} & 1_{n_1} \\ 1_{n_4} & 1_{n_3} \end{pmatrix} &= \begin{pmatrix} 1_{n_1} & 1_{n_2} \\ 1_{n_3} & 1_{n_4} \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \circ \begin{pmatrix} 1_{n_1} & 1_{n_2} \\ 1_{n_3} & 1_{n_4} \end{pmatrix} \\ &= \begin{pmatrix} 1_{n_3} & 1_{n_4} \\ 1_{n_1} & 1_{n_2} \end{pmatrix}. \end{aligned} \tag{8}$$

   This shows that $n_1 = n_4 = n$ and $n_2 = n_3 = m$.

Now consider $e = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$ in equation (7). In the same way, this gives

$$\begin{pmatrix} \xi(x)1_n & \xi(x)1_m \\ \xi(x)^{-1}1_m & \xi(x)^{-1}1_n \end{pmatrix} = \begin{pmatrix} \psi(x)1_n & \psi(x)^{-1}1_m \\ \psi(x)1_m & \psi(x)^{-1}1_n \end{pmatrix}.$$

Therefore there are three possible cases: a) $\psi = \xi \not\equiv 1$, b) $\psi = \xi^{-1} \not\equiv 1$ and c) $\psi = \xi \equiv 1$.

a) Since the character group of $C_3$ is isomorphic to $C_3$ we see that $\xi \neq \xi^{-1}$. Equation (8) holds if and only if $n \in \mathbb{N}$ is arbitrary, and $m = 0$. Thus we have

$$h_\bullet = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}.$$

The condition that $\tilde{h}$ is an action on the vector bundle defined by $h$ implies that we have

$$\tilde{h}(1) = \begin{pmatrix} 1_n & 0 \\ 0 & 1_n \end{pmatrix} \quad \text{and} \quad \tilde{h}(-1) = \begin{pmatrix} 0 & A \\ A^{-1} & 0 \end{pmatrix},$$

where $A \in \mathrm{GL}(n, \mathbb{C})$ is arbitrary.

b) Equation (8) holds if and only if $n = 0$, and $m \in \mathbb{N}$ is arbitrary. Thus we have

$$h_\bullet = \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}.$$

Again we can determine $\tilde{h}$ explicitly:

$$\tilde{h}(1) = \begin{pmatrix} 0 & 1_m \\ 1_m & 0 \end{pmatrix} \quad \text{and} \quad \tilde{h}(-1) = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix},$$

where $A \in \mathrm{GL}(m, \mathbb{C})$ is arbitrary. Taking $m = 1$ shows that $\mathcal{V}(2)_\xi$ and $\mathcal{V}(2)_{\xi^{-1}}$ are isomorphic.

c) Equation (8) holds with no restriction on $n$ and $m$. Thus we have

$$h_\bullet = \begin{pmatrix} n & m \\ m & n \end{pmatrix}.$$

Obviously we can decompose this matrix as

$$\begin{pmatrix} n & m \\ m & n \end{pmatrix} = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix} \oplus \begin{pmatrix} 0 & m \\ m & 0 \end{pmatrix}.$$

As in the previous two cases we see that

$$\tilde{h}(1) = \begin{pmatrix} 1_n & 1_m \\ 1_m & 1_n \end{pmatrix} \quad \text{and} \quad \tilde{h}(-1) = \begin{pmatrix} A & B \\ B^{-1} & A^{-1} \end{pmatrix},$$

with arbitrary $A \in \mathrm{GL}(n, \mathbb{C})$ and $B \in \mathrm{GL}(m, \mathbb{C})$.

Finally let us describe the 1-intertwiners between categorical representations of different type.

1. Let $(h_\bullet, \tilde{h})\colon \mathcal{V}(1) \to \mathcal{V}(2)$ be a 1-intertwiner. Then $h_\bullet$ has the form $h_\bullet = \begin{pmatrix} n & m \end{pmatrix}$, where $n, m \in \mathbb{N}$. The naturality condition does not impose any restrictions because both categorical representations are trivial. As before, $\tilde{h}$ simply defines two representations of $C_2$ of dimensions $n$ and $m$ respectively.

   Analogously we see that any 1-intertwiner $(h_\bullet, \tilde{h})\colon \mathcal{V}(2) \to \mathcal{V}(1)$ is of the form
$$h_\bullet = \begin{pmatrix} n \\ m \end{pmatrix},$$
   with $\tilde{h}$ defining two representations of $C_2$ again. Taking $n = 1, m = 0$ and $n = 0, m = 1$ respectively, and $\tilde{h}$ the trivial representation in both cases, shows that $\mathcal{V}(2)$ is isomorphic to $\mathcal{V}(1) \boxplus \mathcal{V}(1)$.

2. Let $(h_\bullet, \tilde{h})\colon \mathcal{V}(1) \to \mathcal{V}(2)_\xi$ be a 1-intertwiner. Again $h_\bullet$ has the form $h_\bullet = \begin{pmatrix} n & m \end{pmatrix}$. The naturality condition now imposes the following restriction:
$$\begin{pmatrix} 1_n & 1_m \end{pmatrix} \circ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1_m & 1_n \end{pmatrix} = \begin{pmatrix} 1_n & 1_m \end{pmatrix}.$$
   This holds if and only if $n = m$. Taking $e = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}$, we see that $\xi$ has to be equal to 1. Just as before $\tilde{h}$ defines an action on a vector bundle over 2 with fibre $\mathbb{C}^n$.

   Likewise non-zero 1-intertwiners $(h_\bullet, \tilde{h})\colon \mathcal{V}(2)_\xi \to \mathcal{V}(1)$ can be seen to exist if and only if $\xi$ is trivial and the intertwiners are the transposes of the previous ones.

3. Let $(h_\bullet, \tilde{h})\colon \mathcal{V}(2) \to \mathcal{V}(2)_\xi$ be a 1-intertwiner. We already know that $\mathcal{V}(2) \cong \mathcal{V}(1) \boxplus \mathcal{V}(1)$. Therefore this case reduces to the direct sum of the previous case. Again by transposition we get the classification of all 1-intertwiners between $\mathcal{V}(2)_\xi$ and $\mathcal{V}(2)$.

The 2-intertwiners are very easy to describe. Given two 1-intertwiners between two categorical representations, we know that we can interpret them as homogeneous vector bundles by the above results. A 2-intertwiner between them can then be interpreted as a bundle map between these homogeneous vector bundles which commutes with the actions of $C_2$. This interpretation follows immediately from the above and diagram (6).

Let us now have a look at the monoidal product of the above categorical representations.

1. Clearly we have $\mathcal{V}(1) \boxtimes \mathcal{V} \cong \mathcal{V}$, for any categorical representation $\mathcal{V}$. Because we also know that $\mathcal{V}(2) \cong \mathcal{V}(1) \boxplus \mathcal{V}(1)$, we see that $\mathcal{V}(2) \boxtimes \mathcal{V} \cong \mathcal{V} \boxplus \mathcal{V}$.

2. We now study $\mathcal{V}(2)_\xi \boxtimes \mathcal{V}(2)_\psi$. The easiest way to understand this tensor product is by looking at

$$\begin{pmatrix} \xi(x) & 0 \\ 0 & \xi(x)^{-1} \end{pmatrix} \boxtimes \begin{pmatrix} \psi(x) & 0 \\ 0 & \psi(x)^{-1} \end{pmatrix} =$$

$$\begin{pmatrix} \xi(x)\psi(x) & 0 & 0 & 0 \\ 0 & \xi(x)\psi(x)^{-1} & 0 & 0 \\ 0 & 0 & \xi(x)^{-1}\psi(x) & 0 \\ 0 & 0 & 0 & \xi(x)^{-1}\psi(x)^{-1} \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & \xi(x) \\ \xi(x)^{-1} & 0 \end{pmatrix} \boxtimes \begin{pmatrix} 0 & \psi(x) \\ \psi(x)^{-1} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 & \xi(x)\psi(x) \\ 0 & 0 & \xi(x)\psi(x)^{-1} & 0 \\ 0 & \xi(x)^{-1}\psi(x) & 0 & 0 \\ \xi(x)^{-1}\psi(x)^{-1} & 0 & 0 & 0 \end{pmatrix}.$$

It is now easy to check that

$$h_\bullet = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\tilde{h} = \begin{pmatrix} (1) & 0 & 0 & 0 \\ 0 & 0 & 0 & (1) \\ 0 & (1) & 0 & 0 \\ 0 & 0 & (1) & 0 \end{pmatrix}$$

define an invertible 1-intertwiner

$$\mathcal{V}(2)_\xi \boxtimes \mathcal{V}(2)_\psi \to \mathcal{V}(2)_{\xi\psi} \boxplus \mathcal{V}(2)_{\xi\psi^{-1}}.$$

Note that $\xi\psi$ or $\xi\psi^{-1}$ is trivial, for any choice of $\xi$ and $\psi$.

## 5. Categorical characters

In this section we study the 1-dimensional (strict) categorical representations of an arbitrary categorical group $\mathcal{G}$.

5.1. Theorem. *(a) A one-dimensional categorical representation $R$ of $\mathcal{G}$ is completely determined by a group character $\xi_R$ on $E = E(\mathcal{G})$ which is invariant under the action of $G = G(\mathcal{G})$.*

*(b) A 1-intertwiner $(h_\bullet, \tilde{h})$ between two one-dimensional categorical representations $R$ and $T$ is either zero or given by a representation $\tilde{h}\colon G \to GL(h_\bullet)$ such that*

$$\tilde{h}(X) = \xi_R(e)^{-1}\xi_T(e), \tag{9}$$

*for $X = \partial e$ and any $e \in E$. The right-hand side of (9) should be read as a scalar matrix of the right size for the equation to make sense. In particular there exists no non-zero 1-intertwiner if the restrictions of $\xi_R$ and $\xi_T$ to $\ker \partial \subset E$ are different. If $R = T$ then $\tilde{h}$ has to be trivial on $\partial(E)$. The composition of two 1-intertwiners corresponds to the tensor product of the two respective representations of $G$.*

*(c) Suppose we have two 1-intertwiners between the same pair of categorical representations, then a 2-intertwiner between them is given by an ordinary intertwiner between the corresponding representations. The horizontal composition of two 2-intertwiners corresponds to the tensor product of the respective ordinary intertwiners, whereas the vertical composition of two 2-intertwiners corresponds to the ordinary (matrix) product of the two respective intertwiners.*

Proof. (a) The character is $\xi_R(e) = R(e)$ restricted to $e \in E$. The first claim follows immediately from diagram (2).

(b) Now consider a 1-intertwiner $(h_\bullet, \tilde{h})$ between $R$ and $T$. In this particular case $h_\bullet \in \mathrm{Vect}_1(1,1) \cong \mathbb{N}$, so we can identify $h$ with a natural number. Just as in the previous section we see that $\tilde{h}$ defines a representation of $G$ on $\mathbb{C}^{h_\bullet}$. Condition (9) expresses the naturality condition for $h$ and can be read off from diagram (5).

The composite of two 1-intertwiners is given by the tensor product of the respective representations of $G$, say $\tilde{h}_1$ and $\tilde{h}_2$, because, in the case of one-dimensional categorical representations, we have (see definition 3.5)

$$\Big((\tilde{h}_1) \circ 1\Big) \cdot \Big(1 \circ (\tilde{h}_2)\Big) = (\tilde{h}_1 \otimes \tilde{h}_2),$$

where the 1's denote the identity matrices of the right size.

(c) This follows directly from condition (6). ∎

We now apply our results to the examples in Sect. 2.

5.2. Example. Let $\mathcal{G}$ be a categorical group and $(E, G, \partial, \triangleright)$ the corresponding crossed module of groups. A one-dimensional categorical representation is determined by a $G$-invariant group character on $E$, say $\xi$, and is denoted by $\mathcal{V}_\xi$. We determine the Hom-categories between an arbitrary pair $\mathcal{V}_\xi$ and $\mathcal{V}_\psi$. Let $(h_\bullet, \tilde{h})$ be an arbitrary 1-intertwiner between $\mathcal{V}_\xi$ and $\mathcal{V}_\psi$.

1. Suppose $\mathcal{G}$ is connected. Then theorem 5.1 says that $\tilde{h}$ is a scalar representation of $G$, which is completely determined by $\psi\xi^{-1}$. Therefore, for any two characters $\xi$ and

$\psi$ which coincide on the kernel of $\partial$, there is exactly one 1-intertwiner between $\mathcal{V}_\xi$ and $\mathcal{V}_\psi$. The 2-intertwiners are just the ordinary intertwiners between these scalar representations of $G$.

2. Suppose $\mathcal{G}$ is skeletal. Let Rep($G$) be the monoidal category of representations of $G$. Then $\mathrm{Hom}(\mathcal{V}_\xi, \mathcal{V}_\psi) = \mathrm{Rep}(G)$, if $\xi = \psi$, and zero otherwise.

3. Suppose $\mathcal{G}$ is discrete, so that $E$ is a normal subgroup of $G$. Note that in this case $\xi$ (and $\phi$) have to be characters which are constant on each conjugacy class of $E$. Theorem 5.1 says that the restriction of $\tilde{h}$ to $E$ is the scalar representation of $G$ determined by $\psi\xi^{-1}$. A concrete example of some interest is the case where $E = \{\pm 1\} \subset SU(2) = G$. We have two characters on $E$, namely the trivial one, say $\xi$, and the inclusion $\{\pm 1\} \subset \mathbb{C}^*$, say $\psi$. Now an easy exercise reveals that we have

$$Hom(\xi, \xi) = Hom(\psi, \psi) = Rep(SU(2))_{even} = Rep(SO(3))$$

and

$$Hom(\xi, \psi) = Hom(\psi, \xi) = Rep(SU(2))_{odd},$$

where the last expression denotes the subcategory of $Rep(SU(2))$ generated by the odd spins only.

## 6. Concluding remarks

In this section we give a rather incomplete sketch of some features of the categorical representations of general categorical groups and also make some remarks about possible generalizations of our constructions.

As already proved, a (strict) categorical representation corresponds precisely to a homomorphism of crossed modules

$$\begin{array}{ccc} E(\mathcal{G}) & \xrightarrow{\partial} & G(\mathcal{G}) \\ {\scriptstyle R_p}\big\downarrow & & \big\downarrow{\scriptstyle R_b} \\ (\mathbb{C}^*)^N & \xrightarrow{1} & S_N. \end{array} \tag{10}$$

This leads to the following concrete description of a categorical representation.

6.1. Lemma. *An $N$-dimensional categorical representation of $\mathcal{G}$ consists of a group homomorphism $R_b\colon G(\mathcal{G}) = \mathcal{G}_0 \to S_N$ whose kernel contains the image of $\partial$, together with $N$ group characters $\xi_1, \ldots, \xi_n$ on $E = E(\mathcal{G})$ satisfying*

$$\xi_i(X \triangleright e) = \xi_{R_b(X)i}(e), \tag{11}$$

*for any $X \in G(\mathcal{G})$, $e \in E$ and $i = 1, \ldots, N$. Here $R_b(X)$ denotes the left action on the set of $N$ elements.*

PROOF. Both $R_b$ and $R_p$ in (10) are group homomorphisms. Clearly $R_p$ is a group homomorphism if and only if it defines $N$ group characters. Also (10) is commutative if and only if the kernel of $R_b$ contains the image of $\partial$. Finally, (11) is equivalent to the action condition on $(R_b, R_p)$. ∎

6.2. REMARK. Note that (11) implies that each $\xi_i$ has to be invariant under the action of $\ker(R_b)$, i.e.

$$\xi_i(X \triangleright e) = \xi_i(e),$$

for any $X \in \ker(R_b)$ and $e \in E$.

6.3. DEFINITION. *Let $R$ be a categorical representation.*

*a) $R$ is called* decomposable *if there are two non-zero categorical representations $S$ and $T$ such that $R \cong S \boxplus T$ holds. $R$ is called* indecomposable *if it is not decomposable.*

*b) $R$ is called* reducible *if there exist a categorical representation $S$ of dimension less than $R$ and two 1-intertwiners $h\colon S \to R$ and $h'\colon R \to S$ such that $h \circ h'\colon S \to S$ is isomorphic to the identity 1-intertwiner on $S$. $R$ is called* irreducible *if it is not reducible.*

Note that any irreducible categorical representation is indecomposable as well, but the converse is false as the following example shows.

6.4. EXAMPLE. In the example $\mathcal{G}(2,3)$ of the previous section, we saw that $\mathcal{V}(2) \cong \mathcal{V}(1) \boxplus \mathcal{V}(1)$ is decomposable and $\mathcal{V}(2)_\xi$ is irreducible for $\xi \neq 1$. For $\xi = 1$, we see that $\mathcal{V}(2)_1$ is indecomposable, but reducible. The lemma below shows that all categorical representations of $\mathcal{G}(2,3)$ of dimension greater than two are decomposable.

6.5. REMARK. The appearance of indecomposable reducible categorical representations is of course due to the fact that the 1-morphisms in 2-Vect are matrices with only non-negative integer entries. For example, to decompose the non-trivial 2-dimensional representation $R_b\colon C_2 \to S_2$ as a representation of groups, one uses an intertwiner with negative and fractional entries. Consequently the monoidal 2-category of categorical representations of $\mathcal{G}(2,3)$ is not semi-simple (see [M-S] for the precise definition of semi-simplicity). Therefore it is not clear if it can be used for the construction of topological state-sums, because semi-simplicity is an essential ingredient in the proof of topological invariance of these state-sums. Note that one cannot simply ignore the indecomposable reducible categorical representations, because, as we showed in the previous section, one of the summands in the decomposition of $\mathcal{V}(2)_\xi \boxtimes \mathcal{V}(2)_\psi$ is equal to $\mathcal{V}(2)_1$, for any choice of $\xi$ and $\psi$. It seems that this problem also exists in Crane and Yetter's generalization of categorical representations [CY-MC].

There are two elementary results about indecomposable categorical representations that we decided to include in these remarks because they are very easy to prove.

6.6. Lemma. *An $N$-dimensional categorical representation, $R$, is indecomposable if and only if the action of $R_b(G(\mathcal{G}))$ on $\{1,\dots,N\}$ is transitive.*

Proof. Suppose we have $R \cong S \boxplus T$ and $\dim S = K$ and $\dim T = N-K$. Clearly we can write $R_b \cong S_b \oplus T_b$, where $\oplus$ means the composite of $S_b \times T_b$ and the canonical map between the symmetric groups $S_K \times S_{N-K} \to S_{K+(N-K)} = S_N$, and $\cong$ means equal up to conjugation by a fixed permutation. Thus we see that the action of $R_b(G(\mathcal{G}))$ on $\{1,\dots,N\}$ is not transitive.

Conversely, suppose that $\{1,\dots,N\}$ can be written as the union of two non-empty subsets $A$ and $B$ which are both invariant under the action of $R_b(G(\mathcal{G}))$. By reordering we may assume that $A = \{1,\dots,K\}$ and $B = \{K+1,\dots,N\}$. The restrictions of $R_b(G(\mathcal{G}))$ to $A$ and $B$ respectively yield a decomposition $R_b = S_b \oplus T_b$. Take $S_p = (\xi_1,\dots,\xi_K)$ and $T_p = (\xi_{K+1},\dots,\xi_N)$. Then condition (11) shows that $(S_b, S_p)$ and $(T_b, T_p)$ are both categorical representations and $R = S \oplus T$. Note that $\cong$ had become the identity here because of our assumption that $A = \{1,\dots,K\}$ and $B = \{K+1,\dots,N\}$, which corresponds to the choice of a fixed permutation. ∎

6.7. Lemma. *If $R$ is indecomposable, then it is completely determined by $R_b$ and just one character on $E(\mathcal{G})$ which is invariant under the action of* $\ker(R_b)$.

Proof. Suppose $R$ is indecomposable. Let $i \neq j \in \{1,\dots,N\}$ be arbitrary. By assumption there exists an $X \in G(\mathcal{G})$ such that $R_b(X)i = j$. By (11) we have $\xi_i(X \triangleright f) = \xi_j(f)$. Thus, if you fix $\xi_i$, then $\xi_j$ is uniquely determined. Since $i, j$ were arbitrary, this shows that one character determines uniquely all the others. We already remarked that any such character has to be invariant under $\ker(R_b)$. ∎

6.8. Remark. Given a subgroup $H \subset G = G(\mathcal{G})$ of index $N$, the action of $G$ on $G/H$ by left (or right) multiplication is transitive. Given an ordering on the elements of $G/H$ we thus obtain a group homomorphism $R_b \colon G \to S_N$ such that $R_b(G)$ acts transitively on $\{1,\dots,N\}$. Another choice of ordering leads to a conjugate group homomorphism. For any $X \in G$, the construction above applied to $XHX^{-1}$ yields the homomorphism $XR_bX^{-1}$. It is easy to check that the converse is also true: given $R_b$ satisfying the above condition, the kernel of $R_b$ has index $N$ in $G$ and any group homomorphism $G \to S_N$, determined by an ordering on $G/\ker(R_b)$, is conjugate to $R_b$. This sets up a bijective correspondence between conjugacy classes of group homomorphisms $G \to S_N$ and conjugacy classes of subgroups of $G$ of index $N$.

There is quite some literature on the theory of permutation representations of finite groups. In this theory the building blocks are the transitive permutation representations. The key observation about them, from the point of view of representation theory, is that Mackey's theory of induced representations carries over to the context of transitive permutation representations without problems [LLC, LLBC]. This allows for a complete description of the decomposition of the tensor product of two transitive permutation representations into a direct sum of transitive permutation representations, for example. Clearly categorical representations, as defined in this paper, are a generalization of permutation representations, the indecomposable ones being the generalizations of the transitive permutation representations, and one could probably generalize some of the techniques used for the latter to study categorical representations.

Although we have not worked out all the details about general categorical representations, we can say some more about the categorical representations of skeletal categorical groups $\mathcal{G}$, of which $\mathcal{G}(2,3)$ is a very simple example. Despite its simplicity the case of $\mathcal{G}(2,3)$ reveals the general features: an indecomposable categorical representation $\mathcal{V}$ corresponds to an orbit, $G(\mathcal{G})\xi$, in the set of characters of $E(\mathcal{G})$, and a homomorphism $G(\mathcal{G}) \to S_N$ corresponding to a subgroup $1 \subseteq H \subseteq G(\mathcal{G})_\xi$, where $G(\mathcal{G})_\xi$ is the stabilizer of $\xi$ in $G(\mathcal{G})$. $\mathcal{V}$ is irreducible if and only $H = G(\mathcal{G})_\xi$. 1-Intertwiners between two categorical representations can be interpreted as homogeneous vector bundles on the cartesian product of the two orbits in the spaces of characters of $E(\mathcal{G})$. 2-Intertwiners can be interpreted as maps between homogeneous bundles. The decomposition of the monoidal product of two indecomposable categorical represention $\mathcal{V}_1$ and $\mathcal{V}_2$ can be obtained by looking at the decomposition into orbits of the cartesian product of the two orbits, corresponding to $\mathcal{V}_1$ and $\mathcal{V}_2$ respectively, and can, in general, contain indecomposable reducible categorical representations.

A short remark on the fact that we have only worked out examples of strict categorical representations is in place. If $\mathcal{G}$ is skeletal, then nothing terribly interesting happens when considering general categorical representations. Each $N$-dimensional indecomposable categorical representations is determined by a homomorphism of crossed modules, just as in the strict case, together with an additional group 2-cocycle on $G(\mathcal{G})$ with values in $(\mathbb{C}^*)^N$. Because we have only considered strict units, the 2-cocycles have to be normalized, but that is the only restriction. The 1-intertwiners between two indecomposable categorical representations $\mathcal{V}_1$ and $\mathcal{V}_2$ become projective homogeneous vector bundles on the cartesian product of the orbits corresponding to $\mathcal{V}_1$ and $\mathcal{V}_2$, with a projective action of $G(\mathcal{G})$ which fails to be an ordinary action by the quotient of the 2-cocycles of $\mathcal{V}_1$ and $\mathcal{V}_2$. The 2-intertwiners are just maps between these vector bundles which intertwine the projective actions. However, for general $\mathcal{G}$ the categorical representations are not so easy to describe. We tried to weaken the notion of homomorphism between crossed modules by introducing a 2-cocycle, but we found that this 2-cocycle has to satisfy an additional independent equation. Because the interpretation of this equation is not clear, we decided to work out the strict categorical representations only.

Finally we want to comment on possible generalizations of our framework. One can consider generalizations of the notion of categorical representation and generalizations of the notion of categorical group. As an example of the former one could consider monoidal bicategories other than 2-Vect. For example, one could consider Crane and Yetter's monoidal bicategory of measurable categories [CY-MC]. As they remark in their introduction, this allows for more interesting categorical representations of categorical Lie groups because the base group can be represented in more general topological symmetry groups than $S_N$. However, as we remarked above, indecomposable reducible categorical representations seem to appear in this setting as well, which puts its applicability for state-sums at risk. Somehow the discreteness of the non-negative integer entries in the

1-morphisms in 2-Vect has not been solved completely in this new setting.

If one only considers 1-dimensional categorical representations, then, of course, no indecomposable reducible categorical representations appear, e.g. the last example of the previous section yields a semi-simple monoidal 2-category and could be used for the construction of topological state-sums of 3- and 4-dimensional manifolds. We have not worked out what these invariants are, but possibly they are connected to Yetter's [Y-EX] and Roberts [R-EX] refined invariants. One idea for a generalization would be to replace Vect by a more interesting braided monoidal category, such as the ones appearing in the representation theory of quantum groups.

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