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INVERTING WEAK DIHOMOTOPY EQUIVALENCE USING HOMOTOPY CONTINUOUS FLOW

PHILIPPE GAUCHER

ABSTRACT. A flow is homotopy continuous if it is indefinitely divisible up to S-homotopy. The full subcategory of cofibrant homotopy continuous flows has nice features. Not only it is big enough to contain all dihomotopy types, but also a morphism between them is a weak dihomotopy equivalence if and only if it is invertible up to dihomotopy. Thus, the category of cofibrant homotopy continuous flows provides an implementation of Whitehead's theorem for the full dihomotopy relation, and not only for S-homotopy as in previous works of the author. This fact is not the consequence of the existence of a model structure on the category of flows because it is known that there does not exist any model structure on it whose weak equivalences are exactly the weak dihomotopy equivalences. This fact is an application of a general result for the localization of a model category with respect to a weak factorization system.

Contents

1. Introduction

There are numerous uses of the notion of "category without identities". For recent papers, see for example [30] [26] [32]. An enriched version of this notion, in the sense of [21], over the category of general topological spaces can be found in [31]. By considering "small categories without identities" enriched over the category of compactly generated topological spaces, that is weak Hausdorff k-spaces in the sense of [20], one obtains an object called a *flow* which allows a model categorical treatment of *dihomotopy* (directed homotopy). Indeed, a flow X can model (the time flow of) a higher dimensional automaton [27] [18] [33] as follows. A flow X consists of

1. a set of *states* X^0;

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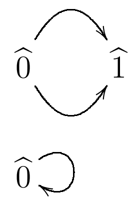

Figure 1: Bad identification of a 1-dimensional empty globe and a loop after a contraction in the direction of time

2. for each pair of states $(\alpha, \beta) \in X^0 \times X^0$, there is a compactly generated topological space $\mathbb{P}_{\alpha,\beta}X$ called the *path space between α and β* representing the concurrency between α and β; each element of $\mathbb{P}_{\alpha,\beta}X$ corresponds to a *non-constant execution path* from α to β; the emptiness of the space $\mathbb{P}_{\alpha,\alpha}X$ for some state α means that there are no loops from α to itself; let
$$\mathbb{P}X = \bigsqcup_{(\alpha,\beta)\in X^0\times X^0} \mathbb{P}_{\alpha,\beta}X.$$
3. for each triple of states $(\alpha, \beta, \gamma) \in X^0 \times X^0 \times X^0$, there is a strictly associative composition law $\mathbb{P}_{\alpha,\beta}X \times \mathbb{P}_{\beta,\gamma}X \longrightarrow \mathbb{P}_{\alpha,\gamma}X$ corresponding to the concatenation of non-constant execution paths.

The main problem to model dihomotopy is that contractions in the direction of time are forbidden. Otherwise in the categorical localization of flows with respect to the dihomotopy equivalences, the relevant geometric information is lost [15] [17]. Here is a very simple example. Take two non-constant execution paths going from one initial state to one final state. If contractions in the direction of time were allowed, then one would find in the same equivalence class a loop (cf. Figure 1): this is not acceptable.

Two kinds of deformations are of interest in the framework of flows. The first one is called *weak S-homotopy equivalence*: it is a morphism of flows $f : X \longrightarrow Y$ such that the set map $f^0 : X^0 \longrightarrow Y^0$ is a bijection and such that the continuous map $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a weak homotopy equivalence. It turns out that there exists a model structure on the category of flows whose weak equivalences are exactly the weak S-homotopy equivalences ([11] and Section 4 of this paper). However, the identifications allowed by the weak S-homotopy equivalences are too rigid. So another kind of weak equivalence is required. The *T-homotopy equivalences* are generated by a set $\mathcal{T}$ of cofibrations obtained by taking the cofibrant replacement of the inclusions of posets [1] of Definition 4.2. This approach of T-homotopy is presented for the first time in [15]. The latter models "refinement of observation". For instance, the inclusion of posets $\{\widehat{0} < \widehat{1}\} \subset \{\widehat{0} < A < \widehat{1}\}$ corresponds to the identification of a directed segment U going from the initial state $\widehat{0}$ to the final state $\widehat{1}$ with the composite $U' * U''$ of two directed segments (cf. Figure 2).

The problem we face can then be presented as follows. We have:

[1] Any poset P can be viewed as a flow in an obvious way: the set of states is the underlying set of P and there is a non-constant execution path from α to β if and only if $\alpha < \beta$. Note that the inequality

$$\widehat{0} \xrightarrow{\quad U \quad} \widehat{1}$$

$$\widehat{0} \xrightarrow{U'} A \xrightarrow{U''} \widehat{1}$$

Figure 2: The simplest example of refinement of observation

1. A model structure on the category of flows **Flow**, called the weak S-homotopy model structure, such that the class of weak equivalences is exactly the class $\mathcal{S}$ of weak S-homotopy equivalences. One wants to invert the weak S-homotopy equivalences because two weakly S-homotopy equivalent flows are equivalent from an observational viewpoint. This model structure provides an implementation of Whitehead's theorem for S-homotopy only.

2. A set of cofibrations $\mathcal{T}$ of *generating T-homotopy equivalences* one would like to invert because these maps model refinement of observation.

3. Three known invariants with respect to weak S-homotopy and T-homotopy: the *underlying homotopy type functor* [12], the *branching homology* and the *merging homology* [14].

4. Every model structure on **Flow** which contains as weak equivalences the class of morphisms $\mathcal{S} \cup \mathcal{T}$, contains weak equivalences which do not preserve the three known invariants [13]. In particular, the category $\mathbf{Flow}[\mathcal{S}_{\mathcal{T}}^{-1}]$ below is not the Quillen homotopy category of a model structure of **Flow**. The left Bousfield localization of the weak S-homotopy model structure with respect to the set of cofibrations $\mathcal{T}$ is therefore not relevant here.

The negative result (4) prevents us from using the machinery of model category on the category **Flow** for understanding the full dihomotopy equivalence relation. There are then several possibilities: reconstructing some pieces of homotopy theory in the framework of flows, finding new categories for studying S-homotopy and T-homotopy, or also relating dihomotopy on **Flow** to other axiomatic presentations of homotopy theory. The possibility which is explored in this paper is the first one.

Indeed, the goal of this work is to prove that it is possible to find a full subcategory of the category of flows which is big enough to contain all dihomotopy types and in which the weak dihomotopy equivalences are exactly the invertible morphisms up to dihomotopy. The main theorem of the paper states as follows (cf. Section 4 for a reminder about flows):

1.1. Theorem. *(Theorem 4.6 and Theorem 4.7) Let J^{gl} be the set of generating trivial cofibrations of the weak S-homotopy model structure of* **Flow**. *Let $\mathcal{T}$ be the set of generating T-homotopy equivalences. Let $\mathbf{Flow}_{cof}$ be the full subcategory of cofibrant flows.*

is strict. Indeed, the ordering of P represents the direction of time and the flow associated with a poset must be loopless.

There exists a full subcategory $\mathbf{Flow}_{cof}^{f,\mathcal{T}}$ *of the category of cofibrant flows* $\mathbf{Flow}_{cof}$*, the one of homotopy continuous flows, a class of morphisms of flows* $\mathcal{S}_{\mathcal{T}}$ *and a congruence* $\sim_{\mathcal{T}}$ *on the morphisms of* $\mathbf{Flow}$ *such that the inclusion functors* $\mathbf{Flow}_{cof}^{f,\mathcal{T}} \subset \mathbf{Flow}_{cof} \subset \mathbf{Flow}$ *induce the equivalences of categories*

$$\mathbf{Flow}_{cof}^{f,\mathcal{T}}/\sim_{\mathcal{T}} \simeq \mathbf{Flow}_{cof}[(\mathcal{S} \cup \mathbf{cof}(J^{gl} \cup T))^{-1}] \simeq \mathbf{Flow}[\mathcal{S}_{\mathcal{T}}^{-1}].$$

Moreover, one has:

1. *The class of morphisms* $\mathcal{S}_{\mathcal{T}}$ *contains the weak S-homotopy equivalences and the morphisms of* $\mathbf{cof}(J^{gl} \cup T)$ *with cofibrant domains.*
2. *Every morphism of* $\mathcal{S}_{\mathcal{T}}$ *preserves the underlying homotopy type, the branching homology and the merging homology.*

We now outline the contents of the paper. The purpose of Section 3 is to give the proof of the theorem above in a more abstract setting. The starting point is a model category $\mathcal{M}$ together with a weak factorization system $(\mathcal{L}, \mathcal{R})$ satisfying some technical conditions which are fulfilled by the weak S-homotopy model structure of **Flow** and by the set of generating T-homotopy equivalences. Several proofs of Section 3 are adaptations of standard proofs [28] [20]. But since the existence of a convenient model structure for $(\mathcal{L}, \mathcal{R})$ is not supposed [2], there are some subtle differences and also new phenomena. The idea of considering the path object construction comes from the reading of Kurz and Rosický's paper [22]. In this paper, Kurz and Rosický have the idea of considering a cylinder object construction with any weak factorization system $(\mathcal{L}, \mathcal{R})$. This allows them to investigate the categorical localization of the underlying category with respect to the class of morphisms $\mathcal{R}$ viewed, morally speaking, as a class of trivial fibrations. The dual situation is explored in this section, with an underlying category which is not only a category but also a model category. The situation described in Section 3 makes one think of the notion of fibration category in the sense of Baues [3]. However, we do not know how to construct a fibration category from the results of Section 3. The path object functor constructed in Section 3 cannot satisfy the whole set of axioms of a P-category in the sense of Baues [3] since the associated homotopy relation is not transitive. In particular, it does not even seem to satisfy the pullback axiom. Next, Section 4 proves the theorem above as an application of Section 3.

Link with the series of papers "T-homotopy and refinement of observation". This paper is independent from the series of papers "T-homotopy and refinement of observation" except for the proof of Theorem 4.7 at the very end of this work in which [15] Theorem 5.2 is used. This paper was written while the author was trying to understand whether the (categorical) localization $\mathbf{Flow}[\mathbf{cof}(\mathcal{T})^{-1}]$ of the category of flows with respect to the T-homotopy equivalences introduced in [15] is locally small. Indeed, the local smallness is not established in the series of papers "T-homotopy and refinement

[2] that is: a model structure such that $\mathcal{L}$ is the class of trivial cofibrations.

of observation". The result we obtain is more subtle. In the "correct" localization, all morphisms of $\mathbf{cof}(J^{gl} \cup \mathcal{T})$ with cofibrant domains are inverted. This is enough for future application in computer science since the real concrete examples are all of them modelled by cofibrant flows. But it is not known whether the other morphisms of $\mathbf{cof}(J^{gl} \cup \mathcal{T})$ are inverted. If this fact should be true, then it would probably be a consequence of the left properness of the weak S-homotopy model structure of **Flow** (which is proved in [16] Theorem 6.4).

2. Prerequisites and notations

The initial object (resp. the terminal object) of a category $\mathcal{C}$, if it exists, is denoted by $\varnothing$ (resp. $\mathbf{1}$).

Let $i : A \longrightarrow B$ and $p : X \longrightarrow Y$ be maps in a category $\mathcal{C}$. Then i has the left lifting property (LLP) with respect to p (or p has the right lifting property (RLP) with respect to i) if for every commutative square

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r, "\beta"'] \arrow[ur, dashed, "g"] & Y, \end{tikzcd}$$

there exists a morphism g called a *lift* making both triangles commutative.

Let $\mathcal{C}$ be a cocomplete category. If K is a set of morphisms of $\mathcal{C}$, then the class of morphisms of $\mathcal{C}$ that satisfy the RLP (*right lifting property*) with respect to every morphism of K is denoted by $\mathbf{inj}(K)$ and the class of morphisms of $\mathcal{C}$ that are transfinite compositions of pushouts of elements of K is denoted by $\mathbf{cell}(K)$. Denote by $\mathbf{cof}(K)$ the class of morphisms of $\mathcal{C}$ that satisfy the LLP (*left lifting property*) with respect to every morphism of $\mathbf{inj}(K)$. The cocompleteness of $\mathcal{C}$ implies $\mathbf{cell}(K) \subset \mathbf{cof}(K)$. Moreover, every morphism of $\mathbf{cof}(K)$ is a retract of a morphism of $\mathbf{cell}(K)$ as soon as the domains of K are small relative to $\mathbf{cell}(K)$ ([20] Corollary 2.1.15). An element of $\mathbf{cell}(K)$ is called a *relative K-cell complex*. If X is an object of $\mathcal{C}$, and if the canonical morphism $\varnothing \longrightarrow X$ is a relative K-cell complex, one says that X is a *K-cell complex*.

A *congruence* $\sim$ on a category $\mathcal{C}$ consists of an equivalence relation on the set $\mathcal{C}(X, Y)$ of morphisms from X to Y for every object X and Y of $\mathcal{C}$ such that if $f, g \in \mathcal{C}(X, Y)$, then $f \sim g$ implies $u \circ f \sim u \circ f$ and $f \circ v \sim g \circ v$ for any morphism u and v as soon as $u \circ f$ and $f \circ v$ exist.

Let $\mathcal{C}$ be a cocomplete category with a distinguished set of morphisms I. Then let $\mathbf{cell}(\mathcal{C}, I)$ be the full subcategory of $\mathcal{C}$ consisting of the object X of $\mathcal{C}$ such that the canonical morphism $\varnothing \longrightarrow X$ is an object of $\mathbf{cell}(I)$. In other terms, $\mathbf{cell}(\mathcal{C}, I) = (\varnothing \downarrow \mathcal{C}) \cap \mathbf{cell}(I)$.

It is obviously impossible to read this paper without some familiarity with *model categories*. Possible references for model categories are [20], [19] and [9]. The original reference is [28] but Quillen's axiomatization is not used in this paper. The Hovey's book axiomatization is preferred. If $\mathcal{M}$ is a *cofibrantly generated* model category with set of generating cofibrations I, let $\mathbf{cell}(\mathcal{M}) := \mathbf{cell}(\mathcal{M}, I)$. Any cofibrantly generated model structure $\mathcal{M}$ comes with a *cofibrant replacement functor* $Q : \mathcal{M} \longrightarrow \mathbf{cell}(\mathcal{M})$. For every morphism f of $\mathcal{M}$, the morphism $Q(f)$ is a cofibration, and even an inclusion of subcomplexes. A set K of morphisms of a model category *permits the small object argument* if the domains of the morphisms of K are small relative to $\mathbf{cell}(K)$. For such a set K, one can use the small object argument. The small object argument is recalled in the proof of Proposition 3.18.

In this paper, the notation $\simeq$ means *weak equivalence* or *equivalence of categories*, and the notation $\cong$ means *isomorphism*.

A *partially ordered set* $(P, \leq)$ (or *poset*) is a set equipped with a reflexive antisymmetric and transitive binary relation $\leq$. A poset $(P, \leq)$ is *bounded* if there exist $\widehat{0} \in P$ and $\widehat{1} \in P$ such that $P \subset [\widehat{0}, \widehat{1}]$ and such that $\widehat{0} \neq \widehat{1}$. Let $\widehat{0} = \min P$ (the bottom element) and $\widehat{1} = \max P$ (the top element).

Every poset P, and in particular every ordinal, can be viewed as a small category denoted in the same way: the objects are the elements of P and there exists a morphism from x to y if and only if $x \leq y$. If λ is an ordinal, a λ-*sequence* (or a *transfinite sequence*) in a cocomplete category $\mathcal{C}$ is a colimit-preserving functor X from λ to $\mathcal{C}$. We denote by X_λ the colimit $\varinjlim X$ and the morphism $X_0 \longrightarrow X_\lambda$ is called the *transfinite composition* of the $X_\mu \longrightarrow X_{\mu+1}$.

If $\mathcal{C}$ is a locally small category, and if Σ is a class of morphisms of $\mathcal{C}$, then we denote by $\mathcal{C}[\Sigma^{-1}]$ the *(categorical) localization* of $\mathcal{C}$ with respect to Σ [10] [24] [6]. The category $\mathcal{C}[\Sigma^{-1}]$ is not necessarily locally small. If $\mathcal{M}$ is a model category with class of weak equivalences $\mathcal{W}$, then the localization $\mathcal{M}[\mathcal{W}^{-1}]$ is locally small and it is called the *Quillen homotopy category* of $\mathcal{M}$. It is denoted by $\mathbf{Ho}(\mathcal{M})$.

3. Localizing a model category w.r.t. a weak factorization system

3.1. Definition. *[1] Let $\mathcal{C}$ be a category. A* weak factorization system *is a pair $(\mathcal{L}, \mathcal{R})$ of classes of morphisms of $\mathcal{C}$ such that the class $\mathcal{L}$ is the class of morphisms having the LLP with respect to $\mathcal{R}$, such that the class $\mathcal{R}$ is the class of morphisms having the RLP with respect to $\mathcal{L}$ and such that every morphism of $\mathcal{C}$ factors as a composite $r \circ \ell$ with $\ell \in \mathcal{L}$ and $r \in \mathcal{R}$. The weak factorization system is* functorial *if the factorization $r \circ \ell$ can be made functorial.*

In a weak factorization system $(\mathcal{L}, \mathcal{R})$, the class $\mathcal{L}$ (resp. $\mathcal{R}$) is completely determined by $\mathcal{R}$ (resp. $\mathcal{L}$).

3.2. Definition. *Let $\mathcal{C}$ be a cocomplete category. A weak factorization system $(\mathcal{L}, \mathcal{R})$ is* cofibrantly generated *if there exists a set K of morphisms of $\mathcal{C}$ permitting the small object* *argument such that* $\mathcal{L} = \mathbf{cof}(K)$ *and* $\mathcal{R} = \mathbf{inj}(K)$.

A cofibrantly generated weak factorization system is necessarily functorial. Definition 3.2 appears in [4] in the context of locally presentable category as the notion of *small* weak factorization system.

The data for this section are:

1. *a complete and cocomplete category* $\mathcal{M}$ *equipped with a model structure denoted by* $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W})$ *for respectively the class of cofibrations, of fibrations and of weak equivalences such that the weak factorization system* $(\mathrm{Cof} \cap \mathcal{W}, \mathrm{Fib})$ *is cofibrantly generated : the set of generating trivial cofibrations is denoted by* J.

2. *a cofibrantly generated weak factorization system* $(\mathcal{L}, \mathcal{R})$ *on* $\mathcal{M}$ *satisfying the following property:* $\mathrm{Cof} \cap \mathcal{W} \subset \mathcal{L} \subset \mathrm{Cof}$. *So there exists a set of morphisms* K *such that* $\mathcal{L} = \mathbf{cof}(J \cup K)$ *and* $\mathcal{R} = \mathbf{inj}(J \cup K)$ *and such that* $J \cup K$ *permits the small object argument. Therefore every morphism* f *factors as a composite* $f = \beta(f) \circ \alpha(f)$ *where* $\alpha(f) \in \mathbf{cell}(J \cup K) \subset \mathcal{L}$ *and where* $\beta(f) \in \mathcal{R}$. *The functorial factorization is supposed to be obtained using the small object argument. It is fixed for the whole section.*

3.3. Definition. *Let* X *be an object of* $\mathcal{M}$. *The* path object of X with respect to $\mathcal{L}$ *is the functorial factorization*

$$X \xrightarrow{\alpha(\mathrm{Id}_X, \mathrm{Id}_X)} \mathrm{Path}_{\mathcal{L}}(X) \xrightarrow{\beta(\mathrm{Id}_X, \mathrm{Id}_X)} X \times X$$

of the diagonal morphism $(\mathrm{Id}_X, \mathrm{Id}_X) : X \longrightarrow X \times X$ *by the morphism* $\alpha(\mathrm{Id}_X, \mathrm{Id}_X) : X \longrightarrow \mathrm{Path}_{\mathcal{L}}(X)$ *of* $\mathcal{L}$ *composed with the morphism* $\beta(\mathrm{Id}_X, \mathrm{Id}_X) : \mathrm{Path}_{\mathcal{L}}(X) \longrightarrow X \times X$ *of* $\mathcal{R}$.

3.4. Notation. *Let* $\mathcal{M}_{cof}$ *be the full subcategory of cofibrant objects of* $\mathcal{M}$.

The path object of X with respect to $\mathcal{L}$ is cofibrant as soon as X is cofibrant since the morphism $\alpha(\mathrm{Id}_X, \mathrm{Id}_X) : X \longrightarrow \mathrm{Path}_{\mathcal{L}}(X)$ is a cofibration. So the path object construction yields an endofunctor of $\mathcal{M}_{cof}$.

The reader must notice that we do not assume here that $\alpha(\mathrm{Id}_X, \mathrm{Id}_X)$ is a weak equivalence of any kind, contrary to the usual definition of a path object. As in [22] for the construction of the cylinder functor, we do use the functorial factorization and we do suppose that $\alpha(\mathrm{Id}_X, \mathrm{Id}_X)$ belongs to $\mathcal{L}$. A morphism of $\mathcal{L}$ being an isomorphism of $\mathcal{M}_{cof}[(\mathcal{W} \cup \mathcal{L})^{-1}]$, our condition is stronger than the usual one for the construction of a path object in a model category.

3.5. Definition. *An object* X *of* $\mathcal{M}$ *is* fibrant with respect to $\mathcal{L}$ *if the unique morphism* $f_X : X \longrightarrow \mathbf{1}$, *where* $\mathbf{1}$ *is the terminal object of* $\mathcal{M}$, *is an element of* $\mathcal{R}$.

An object which is fibrant with respect to $\mathrm{Cof} \cap \mathcal{W}$ is a fibrant object in the usual sense.

3.6. Notation. *Let $\mathcal{M}^{f,\mathcal{L}}$ be the full subcategory of $\mathcal{M}$ of fibrant objects with respect to $\mathcal{L}$. Let $\mathcal{M}_{cof}^{f,\mathcal{L}}$ be the full subcategory of $\mathcal{M}_{cof}$ of fibrant objects with respect to $\mathcal{L}$.*

If X is fibrant with respect to $\mathcal{L}$, the morphism $X \times X \longrightarrow X \times \mathbf{1} \cong X$ belongs to $\mathcal{R}$. Therefore the composite

$$\mathrm{Path}_{\mathcal{L}}(X) \longrightarrow X \times X \longrightarrow X \longrightarrow \mathbf{1}$$

belongs to $\mathcal{R}$ as well. So the path object $\mathrm{Path}_{\mathcal{L}}(X)$ is also fibrant with respect to $\mathcal{L}$. Thus, the path object construction yields endofunctors of $\mathcal{M}^{f,\mathcal{L}}$ and of $\mathcal{M}_{cof}^{f,\mathcal{L}}$.

If $f : X \longrightarrow Y$ is a morphism of $\mathcal{M}_{cof}^{f,\mathcal{L}}$, then the functorial factorization (α, β) yields a composite a priori in $\mathcal{M}_{cof}$ (since $\mathcal{L} \subset \mathrm{Cof}$)

$$X \xrightarrow{\alpha(f)} Z \xrightarrow{\beta(f)} Y$$

equal to f. The unique morphism $Z \longrightarrow \mathbf{1}$ is equal to the composite $Z \longrightarrow Y \longrightarrow \mathbf{1}$ of two morphisms of $\mathcal{R}$. Therefore Z is fibrant with respect to $\mathcal{L}$ and the functorial weak factorization system $(\mathcal{L}, \mathcal{R})$ restricts to a functorial weak factorization system of $\mathcal{M}_{cof}^{f,\mathcal{L}}$ denoted in the same way.

3.7. Definition. *Let $f, g : X \rightrightarrows Y$ be two morphisms of $\mathcal{M}$. A* right homotopy with respect to $\mathcal{L}$ *from f to g is a morphism $H : X \longrightarrow \mathrm{Path}_{\mathcal{L}}(Y)$ such that*

$$\beta(\mathrm{Id}_Y, \mathrm{Id}_Y) \circ H = (f, g).$$

This situation is denoted by $f \sim^r_{\mathcal{L}} g$.

Note the binary relation $\sim^r_{\mathcal{L}}$ does not depend on the choice of the functorial factorization (α, β). Indeed, with another functorial factorization (α', β'), and the corresponding path object functor $\mathrm{Path}'_{\mathcal{L}}$, one can consider for every object Y of $\mathcal{M}$ the commutative diagram

$$\begin{tikzcd}
Y \arrow[r, equal] \arrow[d] & Y \arrow[d] \\
\mathrm{Path}_{\mathcal{L}}(Y) \arrow[r, dashed, "k"] \arrow[d] & \mathrm{Path}'_{\mathcal{L}}(Y) \arrow[d] \\
Y \times Y \arrow[r, "\mathrm{Id}_Y \times \mathrm{Id}_Y"] & Y \times Y.
\end{tikzcd}$$

The lift k exists since the arrow $Y \longrightarrow \mathrm{Path}_{\mathcal{L}}(Y)$ is in $\mathcal{L}$ and since the arrow $\mathrm{Path}'_{\mathcal{L}}(Y) \longrightarrow Y \times Y$ is in $\mathcal{R}$.

The morphism $\alpha(\mathrm{Id}_Y, \mathrm{Id}_Y) \circ f : X \longrightarrow \mathrm{Path}_{\mathcal{L}} Y$ yields a right homotopy from f to f with respect to $\mathcal{L}$. If $H : X \longrightarrow \mathrm{Path}_{\mathcal{L}}(Y)$ is a right homotopy from f to g with respect to $\mathcal{L}$, then the usual way for obtaining a right homotopy from g to f with respect to $\mathcal{L}$

consists of considering the commutative diagram:

$$\begin{tikzcd} Y \arrow[r, equal] \arrow[d] & Y \arrow[d] \\ \mathrm{Path}_{\mathcal{L}}(Y) \arrow[r, dashed, "k"] \arrow[d] & \mathrm{Path}_{\mathcal{L}}(Y) \arrow[d] \\ Y \times Y \arrow[r, "\tau"] & Y \times Y \end{tikzcd}$$

with $\tau(y,y') = (y',y)$. The existence of the lift k comes from the definition of the path object and of the fact that $(\mathcal{L},\mathcal{R})$ is a weak factorization system. So the binary relation $\sim^r_{\mathcal{L}}$ is reflexive and symmetric.

This relation is not transitive in general. The pair $(\mathcal{R}^{op},\mathcal{L}^{op})$ is a weak factorization system of the opposite category $\mathcal{M}^{op}$ (the model structure of $\mathcal{M}$ is forgotten for this paragraph only). The path object becomes a cylinder object and the binary relation $\sim^r_{\mathcal{L}}$ becomes the homotopy relation of [22]. [22] Example 3.6 gives an example where the homotopy is not transitive. Thus, the opposite category with the opposite weak factorization system gives an example where $\sim^r_{\mathcal{L}}$ is not transitive.

3.8. Notation. *Let us denote by $\sim_{\mathcal{L}}$ the transitive closure of the binary relation $\sim^r_{\mathcal{L}}$.*

3.9. Proposition. *Let X be an object of $\mathcal{M}_{cof}$. Let Y be an object of $\mathcal{M}$. Let $f,g : X \rightrightarrows Y$ be two morphisms between them. Then $f \sim_{\mathrm{Cof}\cap\mathcal{W}} g$ if and only if f and g are right homotopic in the usual sense of model categories.*

Notice that it is crucial in the proof for X to be cofibrant.

Proof. Indeed, two morphisms $f,g : X \rightrightarrows Y$ with X cofibrant are right homotopic in the usual sense if the pair (f,g) is in the transitive closure of the following situation denoted by $f \sim^r g$ (cf. [20] p7):

1. Decompose the diagonal morphism $(\mathrm{Id}_Y,\mathrm{Id}_Y) : Y \longrightarrow Y\times Y$ into a weak equivalence $Y \longrightarrow PY$ of $\mathcal{M}$ followed by a fibration $(p_1,p_2) : PY \longrightarrow Y \times Y$ of $\mathcal{M}$.

2. There exists $H : X \longrightarrow PY$ such that $(p_1,p_2)\circ H = (f,g)$.

Let us factor the weak equivalence $Y \longrightarrow PY$ as a composite $Y \longrightarrow P'Y \longrightarrow PY$ where $Y \longrightarrow P'Y$ is a trivial cofibration and where $P'Y \longrightarrow PY$ is a trivial fibration. Then one can lift the right homotopy $H : X \longrightarrow PY$ to a morphism $\overline{H} : X \longrightarrow P'Y$ since X is cofibrant. But $\overline{H}$ is not yet a right homotopy from f to g with respect to $\mathrm{Cof}\cap\mathcal{W}$ since $P'Y$ is not necessarily the functorial path object $\mathrm{Path}_{\mathrm{Cof}\cap\mathcal{W}}(Y)$! Let us consider the commutative diagram

$$\begin{tikzcd} Y \arrow[rr, equal] \arrow[d] & & Y \arrow[d] \\ P'Y \arrow[rr, dashed, "k"] \arrow[d] & & \mathrm{Path}_{\mathrm{Cof}\cap\mathcal{W}}(Y) \arrow[d] \\ Y\times Y \arrow[rr, "\mathrm{Id}_Y\times\mathrm{Id}_Y"] & & Y\times Y \end{tikzcd}$$

Since the arrow $Y \longrightarrow P'Y$ is a trivial cofibration and since the arrow $\mathrm{Path}_{\mathrm{Cof}\cap\mathcal{W}}(Y) \longrightarrow Y\times Y$ is a fibration, there exists a lift k. Then $k\circ\overline{H}$ is a right homotopy with respect to $\mathrm{Cof}\cap\mathcal{W}$ from f to g.

Conversely, the path object with respect to $\mathrm{Cof}\cap\mathcal{W}$ is a path object in the above sense of model categories. So a right homotopy from f to g with respect to $\mathrm{Cof}\cap\mathcal{W}$ is a right homotopy in the usual sense of model categories. ∎

The following proposition gives a sufficient condition for the binary relation $\sim^r_{\mathcal{L}}$ to be transitive.

3.10. Proposition. *Let us suppose that there exists a model structure* $(\mathrm{Cof}_{\mathcal{L}},\mathrm{Fib}_{\mathcal{L}},\mathcal{W}_{\mathcal{L}})$ *on* $\mathcal{M}$ *such that* $\mathcal{L}=\mathrm{Cof}_{\mathcal{L}}\cap\mathcal{W}_{\mathcal{L}}$ *and such that every cofibrant object of* $\mathcal{M}$ *is a cofibrant object of* $(\mathrm{Cof}_{\mathcal{L}},\mathrm{Fib}_{\mathcal{L}},\mathcal{W}_{\mathcal{L}})$*. Let* X *and* Y *be two objects of* $\mathcal{M}^{f,\mathcal{L}}_{cof}$*. Then the binary relation* $\sim^r_{\mathcal{L}}$ *is an equivalence relation on* $\mathcal{M}^{f,\mathcal{L}}_{cof}(X,Y)$.

Notice that we do not need suppose in the proof of Proposition 3.9 that the weak factorization system $(\mathrm{Cof}\cap\mathcal{W},\mathrm{Fib})$ is cofibrantly generated. So we do not need this hypothesis in the proof of Proposition 3.10.

Proof. By Proposition 3.9 applied to the model structure $(\mathrm{Cof}_{\mathcal{L}},\mathrm{Fib}_{\mathcal{L}},\mathcal{W}_{\mathcal{L}})$, the binary relation $\sim^r_{\mathcal{L}}$ coincides with right homotopy for the model structure $(\mathrm{Cof}_{\mathcal{L}},\mathrm{Fib}_{\mathcal{L}},\mathcal{W}_{\mathcal{L}})$. Since Y is fibrant for the latter model structure, one deduces that $\sim^r_{\mathcal{L}}$ is transitive by [20] Proposition 1.2.5. ∎

3.11. Corollary. *If* $\mathcal{L}$ *is the class of trivial cofibrations of a left Bousfield localization of the model structure of* $\mathcal{M}$*, then the binary relation* $\sim^r_{\mathcal{L}}$ *on the set of morphisms* $\mathcal{M}(X,Y)$ *with* $X\in\mathcal{M}_{cof}$ *and with* Y *fibrant with respect to* $\mathcal{L}$ *is an equivalence relation.*

3.12. Proposition. *(dual to [22] Lemma 3.2) Let* $f,g : X \rightrightarrows Y$ *be two morphisms of* $\mathcal{M}$*. Let* $u : Y \longrightarrow U$ *and* $v : V \longrightarrow X$ *be two other morphisms of* $\mathcal{M}$*. If* $f\sim^r_{\mathcal{L}} g$*, then* $u\circ f\sim^r_{\mathcal{L}} u\circ g$ *and* $f\circ v\sim^r_{\mathcal{L}} g\circ v$*.*

In other terms, the equivalence relation $\sim_{\mathcal{L}}$ defines a congruence in the sense of [24].

Proof. By considering the opposite of the category $\mathcal{M}$, the proof is complete using [22] Lemma 3.2. ∎

The proof of Proposition 3.12 does use the factorization (α, β) and its functoriality. We could avoid using the functoriality since the morphism $\alpha(\mathrm{Id}_Y, \mathrm{Id}_Y) : Y \longrightarrow \mathrm{Path}_{\mathcal{L}}(Y)$ belongs to $\mathcal{L}$ and since the morphism $\beta(\mathrm{Id}_U \times \mathrm{Id}_U) : \mathrm{Path}_{\mathcal{L}}(U) \longrightarrow U \times U$ belongs to $\mathcal{R}$. But anyway, the proof of Proposition 3.12 cannot be adapted to the usual notion of right homotopy. This is once again a difference between our notion of right homotopy and the usual one on model category.

Proposition 3.12 allows to consider the quotients $\mathcal{M}/\!\sim_{\mathcal{L}}$ (resp. $\mathcal{M}_{cof}/\!\sim_{\mathcal{L}}$, $\mathcal{M}^{f,\mathcal{L}}/\!\sim_{\mathcal{L}}$, $\mathcal{M}^{f,\mathcal{L}}_{cof}/\!\sim_{\mathcal{L}}$) of the category $\mathcal{M}$ (resp. $\mathcal{M}_{cof}$, $\mathcal{M}^{f,\mathcal{L}}$ $\mathcal{M}^{f,\mathcal{L}}_{cof}$) by the congruence $\sim_{\mathcal{L}}$. By definition, the objects of $\mathcal{M}/\!\sim_{\mathcal{L}}$ (resp. $\mathcal{M}_{cof}/\!\sim_{\mathcal{L}}$, $\mathcal{M}^{f,\mathcal{L}}/\!\sim_{\mathcal{L}}$, $\mathcal{M}^{f,\mathcal{L}}_{cof}/\!\sim_{\mathcal{L}}$) are the objects of $\mathcal{M}$ (resp. $\mathcal{M}_{cof}$, $\mathcal{M}^{f,\mathcal{L}}$, $\mathcal{M}^{f,\mathcal{L}}_{cof}$), and for any object X and Y of $\mathcal{M}$ (resp. $\mathcal{M}_{cof}$, $\mathcal{M}^{f,\mathcal{L}}$, $\mathcal{M}^{f,\mathcal{L}}_{cof}$), one has $\mathcal{M}/\!\sim_{\mathcal{L}} (X,Y) = \mathcal{M}(X,Y)/\!\sim_{\mathcal{L}}$ (resp. $\mathcal{M}_{cof}/\!\sim_{\mathcal{L}} (X,Y) = \mathcal{M}_{cof}(X,Y)/\!\sim_{\mathcal{L}}$, $\mathcal{M}^{f,\mathcal{L}}/\!\sim_{\mathcal{L}} (X,Y) = \mathcal{M}^{f,\mathcal{L}}(X,Y)/\!\sim_{\mathcal{L}}$, $\mathcal{M}^{f,\mathcal{L}}_{cof}/\!\sim_{\mathcal{L}} (X,Y) = \mathcal{M}^{f,\mathcal{L}}_{cof}(X,Y)/\!\sim_{\mathcal{L}}$). Let

$$[-]_{\mathcal{L}} : \mathcal{M} \longrightarrow \mathcal{M}/\!\sim_{\mathcal{L}}$$

$$[-]_{\mathcal{L}} : \mathcal{M}_{cof} \longrightarrow \mathcal{M}_{cof}/\!\sim_{\mathcal{L}}$$

$$[-]_{\mathcal{L}} : \mathcal{M}^{f,\mathcal{L}} \longrightarrow \mathcal{M}^{f,\mathcal{L}}/\!\sim_{\mathcal{L}}$$

$$[-]_{\mathcal{L}} : \mathcal{M}^{f,\mathcal{L}}_{cof} \longrightarrow \mathcal{M}^{f,\mathcal{L}}_{cof}/\!\sim_{\mathcal{L}}$$

be the canonical functors.

3.13. Proposition. *(dual to [22] Lemma 3.7) Let $f : X \longrightarrow Y$ be a morphism of $\mathcal{M}$ belonging to $\mathcal{L}$. Let us suppose that X is fibrant with respect to $\mathcal{L}$. Then there exists $g : Y \longrightarrow X$ such that $f \circ g \sim_{\mathcal{L}} \mathrm{Id}_Y$ and $g \circ f = \mathrm{Id}_X$.*

Proof. Let us consider the commutative diagram of $\mathcal{M}$

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed, "g"] & \mathbf{1}. \end{tikzcd}$$

Since the left vertical arrow is in $\mathcal{L}$ and since the right vertical arrow is in $\mathcal{R}$ by hypothesis, there exists a lift $g : Y \longrightarrow X$. In other terms, $g \circ f = \mathrm{Id}_X$. The diagram of $\mathcal{M}$

$$\begin{tikzcd}
& & Y \arrow[d, "\alpha(\mathrm{Id}_Y,\mathrm{Id}_Y)"] \\
X \arrow[urr, "f"] \arrow[rr, "\alpha(\mathrm{Id}_Y,\mathrm{Id}_Y)\circ f"'] \arrow[d, "f"'] & & \mathrm{Path}_{\mathcal{L}}(Y) \arrow[d, "\beta(\mathrm{Id}_Y,\mathrm{Id}_Y)"] \\
Y \arrow[urr, dashed, "H"] \arrow[rr, "(f\circ g,\mathrm{Id}_Y)"'] & & Y \times Y
\end{tikzcd}$$

is commutative since

$$\beta(\mathrm{Id}_Y, \mathrm{Id}_Y) \circ \alpha(\mathrm{Id}_Y, \mathrm{Id}_Y) \circ f = (\mathrm{Id}_Y, \mathrm{Id}_Y) \circ f = (f, f) = (f \circ g, \mathrm{Id}_Y) \circ f.$$

Since $f \in \mathcal{L}$ by hypothesis and since $\beta(\mathrm{Id}_Y, \mathrm{Id}_Y) \in \mathcal{R}$, there exists $H : Y \longrightarrow \mathrm{Path}_{\mathcal{L}}(Y)$ preserving the diagram above commutative. The morphism H is by construction a right homotopy from $f \circ g$ to Id_Y with respect to $\mathcal{L}$. ∎

3.14. Proposition. *(almost dual to [22] Theorem 3.9) One has the isomorphism of categories* $\mathcal{M}^{f,\mathcal{L}}_{cof}/\!\sim_{\mathcal{L}} \cong \mathcal{M}^{f,\mathcal{L}}_{cof}[\mathcal{L}^{-1}]$. *In particular, this means that the category* $\mathcal{M}^{f,\mathcal{L}}_{cof}[\mathcal{L}^{-1}]$ *is locally small.*

The proof of Proposition 3.14 also shows the isomorphism of categories

$$\mathcal{M}^{f,\mathcal{L}}/\!\sim_{\mathcal{L}} \cong \mathcal{M}^{f,\mathcal{L}}[\mathcal{L}^{-1}].$$

Proof. We know that the pair $(\mathcal{L}, \mathcal{R})$ restricts to a weak factorization system of $\mathcal{M}^{f,\mathcal{L}}_{cof}$. By considering the opposite category, the proposition is then a consequence of [22] Theorem 3.9. ∎

3.15. Proposition. *(Detecting weak equivalences) A morphism* $f : A \longrightarrow B$ *of* $\mathcal{M}^{f,\mathcal{L}}_{cof}$ *is an isomorphism of* $\mathcal{M}^{f,\mathcal{L}}_{cof}[\mathcal{L}^{-1}]$ *if and only if for every object* X *of* $\mathcal{M}^{f,\mathcal{L}}_{cof}$*, the map*

$$\mathcal{M}(B, X)/\!\sim_{\mathcal{L}} \longrightarrow \mathcal{M}(A, X)/\!\sim_{\mathcal{L}}$$

is bijective.

Note the "opposite" characterization $\mathcal{M}(X, A)/\!\sim_{\mathcal{L}} \longrightarrow \mathcal{M}(X, B)/\!\sim_{\mathcal{L}}$ also holds. The statement of the theorem is chosen for having a characterization as close as possible to the characterization of weak equivalences in a left Bousfield localization.

Proof. The condition means that the map

$$(\mathcal{M}^{f,\mathcal{L}}_{cof}/{\sim_{\mathcal{L}}})(B,X) \longrightarrow (\mathcal{M}^{f,\mathcal{L}}_{cof}/{\sim_{\mathcal{L}}})(A,X)$$

is a bijection. By Yoneda's lemma applied within the locally small category $\mathcal{M}^{f,\mathcal{L}}_{cof}/{\sim_{\mathcal{L}}}$, the condition is equivalent to saying that $f : A \longrightarrow B$ is an isomorphism of $\mathcal{M}^{f,\mathcal{L}}_{cof}/{\sim_{\mathcal{L}}}$. By Proposition 3.14, the condition is equivalent to saying that $f : A \longrightarrow B$ is an isomorphism of $\mathcal{M}^{f,\mathcal{L}}_{cof}[\mathcal{L}^{-1}]$. ∎

3.16. Definition. *Let X be an object of $\mathcal{M}_{cof}$. The* fibrant replacement of X with respect to $\mathcal{L}$ *is the functorial factorization*

$$X \xrightarrow{\alpha(f_X)} R_{\mathcal{L}}(X) \xrightarrow{\beta(f_X)} \mathbf{1}$$

of the unique morphism $f_X : X \longrightarrow \mathbf{1}$.

The mapping $X \mapsto R_{\mathcal{L}}(X)$ is functorial and yields a functor from $\mathcal{M}_{cof}$ to $\mathcal{M}^{f,\mathcal{L}}_{cof}$ since the morphism $\alpha(f_X) : X \longrightarrow R_{\mathcal{L}}(X)$ is a cofibration.

3.17. Lemma. *Let λ be a limit ordinal. Let $X : \lambda \longrightarrow \mathcal{M}$ and $Y : \lambda \longrightarrow \mathcal{M}$ be two transfinite sequences. Let $f : X \longrightarrow Y$ be a morphism of transfinite sequences such that for any $\mu < \lambda$, $f_\mu : X_\mu \longrightarrow Y_\mu$ belongs to $\mathcal{L}$. Then $f_\lambda : X_\lambda \longrightarrow Y_\lambda$ belongs to $\mathcal{L}$. Moreover, if for any $\mu < \lambda$, $f_\mu : X_\mu \longrightarrow Y_\mu$ belongs to* $\mathbf{cell}(J \cup K)$*, then $f_\lambda : X_\lambda \longrightarrow Y_\lambda$ belongs to* $\mathbf{cell}(J \cup K)$ *as well.*

Lemma 3.17 and Proposition 3.18 are very close to [19] Proposition 12.4.7. The difference is that we do not suppose here that the underlying model category is cellular.

Proof. Let $T_0 = X_\lambda$. Let us consider the unique transfinite sequence $T : \lambda \longrightarrow \mathcal{M}$ such that one has the pushout diagram

$$\begin{tikzcd} X_\mu \arrow[r, "f_\mu"] \arrow[d] & Y_\mu \arrow[d] \\ T_\mu \arrow[r] & T_{\mu+1} \end{tikzcd}$$

where the left vertical arrow is the composite $X_\mu \longrightarrow X_\lambda \longrightarrow T_\mu$ for any $\mu < \lambda$. Let Z be an object of $\mathcal{M}$ and let $\phi : Y_\lambda \longrightarrow Z$ be a morphism of $\mathcal{M}$. The composite $X_\lambda \longrightarrow Y_\lambda \longrightarrow Z$ together with the composite $Y_0 \longrightarrow Y_\lambda \longrightarrow Z$ yields with the pushout diagram above for $\mu = 0$ a morphism $T_1 \longrightarrow Z$ since f is a morphism of transfinite sequences. And by an immediate transfinite induction, one obtains a morphism $\varinjlim_\mu T_\mu \longrightarrow Z$. So one has the isomorphism $\varinjlim_\mu T_\mu \cong Y_\lambda$ since the two objects of $\mathcal{M}$ satisfy the same universal property. Hence the result since the class of morphisms $\mathcal{L}$ and $\mathbf{cell}(J \cup K)$ are both closed under transfinite composition. ∎

3.18. Proposition. *One has $R_{\mathcal{L}}(\mathcal{L}) \subset \mathcal{L}$, and even $R_{\mathcal{L}}(\mathbf{cell}(J \cup K)) \subset \mathbf{cell}(J \cup K)$.*

Proof. A morphism $f \in \mathbf{cof}(J \cup K)$ is a retract of a morphism $g \in \mathbf{cell}(J \cup K)$ since $J \cup K$ permits the small object argument. And the morphism $R_{\mathbf{cof}(J\cup K)}(f)$ is then a retract of the morphism $R_{\mathbf{cof}(J\cup K)}(g)$. Therefore it suffices to prove that $f \in \mathbf{cell}(J \cup K)$ implies $R_{\mathbf{cof}(J\cup K)}(f) \in \mathbf{cell}(J \cup K)$. The functor $R_{\mathbf{cof}(J\cup K)}$ is obtained by a transfinite construction involving the small object argument. Let $X_0 = X$ and $Y_0 = Y$ and $f = f_0$. For any ordinal λ, let $\overline{Y_\lambda}$ be the object of $\mathcal{M}$ defined by the following commutative diagram:

$$\begin{tikzcd}
X_0 \arrow[r, "f"] \arrow[d] & Y_0 \arrow[r, equal] \arrow[d] & Y_0 \arrow[d] \\
X_\lambda \arrow[r] & \overline{Y_\lambda} \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & Y_\lambda
\end{tikzcd}$$

Let us suppose $f_\lambda : X_\lambda \longrightarrow Y_\lambda$ constructed for some $\lambda \geq 0$ and let us suppose that the morphism $\overline{Y_\lambda} \longrightarrow Y_\lambda$ is an element of $\mathbf{cell}(J \cup K)$. The small object argument consists of considering the sets of commutative squares $\{k \longrightarrow f_{X_\lambda}, k \in J \cup K\}$ and $\{k \longrightarrow f_{Y_\lambda}, k \in J \cup K\}$ where $f_{X_\lambda} : X_\lambda \longrightarrow \mathbf{1}$ and $f_{Y_\lambda} : Y_\lambda \longrightarrow \mathbf{1}$ are the canonical morphisms from respectively X_λ and Y_λ to the terminal object of $\mathcal{M}$. The morphism f_λ allows the identification of $\{k \longrightarrow f_{X_\lambda}, k \in J\cup K\}$ with a subset of $\{k \longrightarrow f_{Y_\lambda}, k \in J\cup K\}$. And the morphism $f_{\lambda+1} : X_{\lambda+1} \longrightarrow Y_{\lambda+1}$ is obtained by the diagram (where the notations $\operatorname{dom}(k)$ and $\operatorname{codom}(k)$ mean respectively domain and codomain of k):

$$\begin{tikzcd}
\bigsqcup_{\{k\longrightarrow f_{X_\lambda}, k\in J\cup K\}} \operatorname{dom}(k) \arrow[r] \arrow[d] & X_\lambda \arrow[r] \arrow[d] & Y_\lambda \arrow[d] \\
\bigsqcup_{\{k\longrightarrow f_{X_\lambda}, k\in J\cup K\}} \operatorname{codom}(k) \arrow[r] & X_{\lambda+1} \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & \overline{Y_\lambda} \arrow[d] \arrow[ul, phantom, "\ulcorner", very near start] \\
& \bigsqcup_{\{k\longrightarrow f_{Y_\lambda}, k\in J\cup K\}\backslash\{k\longrightarrow f_{X_\lambda}, k\in J\cup K\}} \operatorname{dom}(k) \arrow[ur] \arrow[d] & Y_{\lambda+1} \arrow[u, phantom, "\ulcorner", very near start] \\
& \bigsqcup_{\{k\longrightarrow f_{Y_\lambda}, k\in J\cup K\}\backslash\{k\longrightarrow f_{X_\lambda}, k\in J\cup K\}} \operatorname{codom}(k) \arrow[ur] &
\end{tikzcd}$$

Therefore $f_{\lambda+1} : X_{\lambda+1} \longrightarrow Y_{\lambda+1}$ is an element of $\mathbf{cell}(J \cup K)$. The proof is complete with Lemma 3.17. ∎

Note the same kind of argument as the one of Proposition 3.18 leads to the following proposition (worth being noticed, but useless for the sequel):

3.19. Proposition. *One has $\operatorname{Path}_{\mathcal{L}}(\mathcal{L}) \subset \mathcal{L}$, and even $\operatorname{Path}_{\mathcal{L}}(\mathbf{cell}(J\cup K)) \subset \mathbf{cell}(J\cup K)$.*

Proposition 3.18 will be used in particular in the proof of Proposition 3.22 with the functorial weak factorization system $(\mathrm{Cof} \cap \mathcal{W}, \mathrm{Fib})$ and in the proof of Proposition 3.20.

3.20. Proposition. *The inclusion functor $\mathcal{M}_{cof}^{f,\mathcal{L}} \subset \mathcal{M}_{cof}$ induces an equivalence of categories $\mathcal{M}_{cof}^{f,\mathcal{L}}[\mathcal{L}^{-1}] \simeq \mathcal{M}_{cof}[\mathcal{L}^{-1}]$. In particular, this implies that the category $\mathcal{M}_{cof}[\mathcal{L}^{-1}]$ is locally small.*

Proof. Since $R_\mathcal{L}(\mathcal{L}) \subset \mathcal{L}$ by Proposition 3.18, there exists a unique functor $L(R_\mathcal{L})$ making the following diagram commutative:

$$\begin{tikzcd} \mathcal{M}_{cof} \arrow[r, "R_\mathcal{L}"] \arrow[d] & \mathcal{M}_{cof}^{f,\mathcal{L}} \arrow[d] \\ \mathcal{M}_{cof}[\mathcal{L}^{-1}] \arrow[r, dashed, "L(R_\mathcal{L})"] & \mathcal{M}_{cof}^{f,\mathcal{L}}[\mathcal{L}^{-1}]. \end{tikzcd}$$

If $i : \mathcal{M}_{cof}^{f,\mathcal{L}} \longrightarrow \mathcal{M}_{cof}$ is the inclusion functor, then there exists a unique functor $L(i)$ making the following diagram commutative:

$$\begin{tikzcd} \mathcal{M}_{cof}^{f,\mathcal{L}} \arrow[r, "i"] \arrow[d] & \mathcal{M}_{cof} \arrow[d] \\ \mathcal{M}_{cof}^{f,\mathcal{L}}[\mathcal{L}^{-1}] \arrow[r, dashed, "L(i)"] & \mathcal{M}_{cof}[\mathcal{L}^{-1}]. \end{tikzcd}$$

There are two natural transformations $\mu : \mathrm{Id}_{\mathcal{M}_{cof}} \Rightarrow i \circ R_\mathcal{L}$ and $\nu : \mathrm{Id}_{\mathcal{M}_{cof}^{f,\mathcal{L}}} \Rightarrow R_\mathcal{L} \circ i$ such that for any $X \in \mathcal{M}_{cof}$ and any $Y \in \mathcal{M}_{cof}^{f,\mathcal{L}}$, the morphisms $\mu(X)$ and $\nu(Y)$ belong to $\mathcal{L}$. So at the level of localizations, one obtains the isomorphisms of functors $\mathrm{Id}_{\mathcal{M}_{cof}[\mathcal{L}^{-1}]} \cong L(i) \circ L(R_\mathcal{L})$ and $\mathrm{Id}_{\mathcal{M}_{cof}^{f,\mathcal{L}}[\mathcal{L}^{-1}]} \cong L(R_\mathcal{L}) \circ L(i)$. Hence the result. ∎

3.21. Proposition. *Let $(\mathcal{L}', \mathcal{R}')$ be another cofibrantly generated weak factorization of $\mathcal{M}$ such that $\mathrm{Cof} \cap \mathcal{W} \subset \mathcal{L}' \subset \mathrm{Cof}$. Let us suppose that $\mathcal{L}' \subset \mathcal{L}$. Then the localization functor $\mathcal{M}_{cof} \longrightarrow \mathcal{M}_{cof}[\mathcal{L}^{-1}]$ factors uniquely as a composite*

$$\mathcal{M}_{cof} \longrightarrow \mathcal{M}_{cof}[\mathcal{L}'^{-1}] \longrightarrow \mathcal{M}_{cof}[\mathcal{L}^{-1}].$$

Proof. One has $\mathcal{L}' \subset \mathcal{L}$. ∎

3.22. Proposition. *The localization functor $L : \mathcal{M}_{cof} \longrightarrow \mathcal{M}_{cof}[\mathcal{L}^{-1}]$ sends the weak equivalences of $\mathcal{M}$ between cofibrant objects to isomorphisms of $\mathcal{M}_{cof}[\mathcal{L}^{-1}]$.*

Proof. The localization functor $L : \mathcal{M}_{cof} \longrightarrow \mathcal{M}_{cof}[\mathcal{L}^{-1}]$ factors uniquely as a composite

$$\mathcal{M}_{cof} \longrightarrow \mathcal{M}_{cof}[(\mathrm{Cof} \cap \mathcal{W})^{-1}] \longrightarrow \mathcal{M}_{cof}[\mathcal{L}^{-1}]$$

by Proposition 3.21 and since $\mathrm{Cof} \cap \mathcal{W} \subset \mathcal{L}$. By Proposition 3.20 and Proposition 3.18 applied to $\mathcal{L} = \mathrm{Cof} \cap \mathcal{W}$, one has the equivalence of categories

$$\mathcal{M}_{cof}[(\mathrm{Cof} \cap \mathcal{W})^{-1}] \simeq \mathcal{M}_{cof}^{f,\mathrm{Cof}\cap\mathcal{W}}[(\mathrm{Cof} \cap \mathcal{W})^{-1}].$$

By Proposition 3.14 applied to $\mathcal{L} = \mathrm{Cof} \cap \mathcal{W}$, one has the isomorphism of categories

$$\mathcal{M}_{cof}^{f,\mathrm{Cof}\cap\mathcal{W}}[(\mathrm{Cof} \cap \mathcal{W})^{-1}] \cong \mathcal{M}_{cof}^{f,\mathrm{Cof}\cap\mathcal{W}}/\sim_{\mathrm{Cof}\cap\mathcal{W}} .$$

Therefore one obtains the equivalence of categories

$$\mathcal{M}_{cof}[(\mathrm{Cof} \cap \mathcal{W})^{-1}] \simeq \mathcal{M}_{cof}^{f,\mathrm{Cof}\cap\mathcal{W}}/\sim_{\mathrm{Cof}\cap\mathcal{W}} .$$

The category $\mathcal{M}_{cof}^{f,\mathrm{Cof}\cap\mathcal{W}}$ is the full subcategory of cofibrant-fibrant objects of $\mathcal{M}$. Since right homotopy with respect to $\mathrm{Cof}\cap\mathcal{W}$ corresponds to the usual notion of right homotopy by Proposition 3.9, one has the equivalence of categories

$$\mathcal{M}_{cof}^{f,\mathrm{Cof}\cap\mathcal{W}}/\sim_{\mathrm{Cof}\cap\mathcal{W}} \simeq \mathbf{Ho}(\mathcal{M})$$

where $\mathbf{Ho}(\mathcal{M}) = \mathcal{M}[\mathcal{W}^{-1}]$. Hence the result. ■

3.23. Proposition. *The categories $\mathcal{M}_{cof}[\mathcal{L}^{-1}]$ and $\mathcal{M}_{cof}[(\mathcal{W} \cup \mathcal{L})^{-1}]$ are isomorphic. In particular, this implies that the category $\mathcal{M}_{cof}[(\mathcal{W} \cup \mathcal{L})^{-1}]$ is locally small.*

Proof. By Proposition 3.22, there exists a unique functor

$$\mathcal{M}_{cof}[(\mathcal{W} \cup \mathcal{L})^{-1}] \longrightarrow \mathcal{M}_{cof}[\mathcal{L}^{-1}]$$

such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{M}_{cof} \arrow[r, equal] \arrow[d] & \mathcal{M}_{cof} \arrow[d] \\
\mathcal{M}_{cof}[(\mathcal{W} \cup \mathcal{L})^{-1}] \arrow[r, dashed] & \mathcal{M}_{cof}[\mathcal{L}^{-1}].
\end{tikzcd}$$

Since $\mathcal{L} \subset \mathcal{W} \cup \mathcal{L}$, there exists a unique functor $\mathcal{M}_{cof}[\mathcal{L}^{-1}] \longrightarrow \mathcal{M}_{cof}[(\mathcal{W} \cup \mathcal{L})^{-1}]$ such that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{M}_{cof} \arrow[r, equal] \arrow[d] & \mathcal{M}_{cof} \arrow[d] \\
\mathcal{M}_{cof}[\mathcal{L}^{-1}] \arrow[r, dashed] & \mathcal{M}_{cof}[(\mathcal{W} \cup \mathcal{L})^{-1}].
\end{tikzcd}$$

Hence the result. ■

In the same way, one can prove the

3.24. Proposition. *The categories $\mathcal{M}_{cof}^{f,\mathcal{L}}[\mathcal{L}^{-1}]$ and $\mathcal{M}_{cof}^{f,\mathcal{L}}[(\mathcal{W} \cup \mathcal{L})^{-1}]$ are isomorphic. In particular, this implies that the category $\mathcal{M}_{cof}^{f,\mathcal{L}}[(\mathcal{W} \cup \mathcal{L})^{-1}]$ is locally small.*

3.25. Notation. *Let*

$$\mathcal{W}_{\mathcal{L}} = \left\{ f : X \longrightarrow Y, \forall Z \in \mathcal{M}_{cof}^{f,\mathcal{L}}, \mathcal{M}(R_{\mathcal{L}}(Q(Y)), Z)/\sim_{\mathcal{L}} \cong \mathcal{M}(R_{\mathcal{L}}(Q(X)), Z)/\sim_{\mathcal{L}} \right\}.$$

3.26. Proposition. *The inclusion functor $\mathcal{M}_{cof} \subset \mathcal{M}$ induces an equivalence of categories $\mathcal{M}_{cof}[(\mathcal{W}\cup\mathcal{L})^{-1}] \simeq \mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}]$. In particular, this implies that the category $\mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}]$ is locally small.*

Proof. Let us consider the composite

$$\mathcal{M} \xrightarrow{Q} \mathcal{M}_{cof} \xrightarrow{L} \mathcal{M}_{cof}[(\mathcal{W}\cup\mathcal{L})^{-1}]$$

where Q is the cofibrant replacement functor of $\mathcal{M}$. By definition of $\mathcal{W}_{\mathcal{L}}$ and by Proposition 3.15, by Proposition 3.20 and by Proposition 3.23, the functor $L \circ Q$ sends the morphisms of $\mathcal{W}_{\mathcal{L}}$ to isomorphisms. Thus, there exists a unique functor $L(Q)$ making the following diagram commutative:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "Q"] \arrow[d] & \mathcal{M}_{cof} \arrow[d] \\
\mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}] \arrow[r, dashed, "L(Q)"] & \mathcal{M}_{cof}[(\mathcal{W}\cup\mathcal{L})^{-1}].
\end{tikzcd}$$

Let $i : \mathcal{M}_{cof} \longrightarrow \mathcal{M}$ be the inclusion functor. Let $f : X \longrightarrow Y \in \mathcal{W}\cup\mathcal{L}$ be a morphism of $\mathcal{M}_{cof}$. Then $Q(f) : Q(X) \longrightarrow Q(Y)$ is still invertible in $\mathcal{M}_{cof}[(\mathcal{W}\cup\mathcal{L})^{-1}]$ since the morphism $Q(X) \longrightarrow X$ and $Q(Y) \longrightarrow Y$ are both weak equivalences of $\mathcal{M}$ between cofibrant objects. So by Proposition 3.20, the morphism $R_{\mathcal{L}}(Q(f)) : R_{\mathcal{L}}(Q(X)) \longrightarrow R_{\mathcal{L}}(Q(Y))$ is invertible in $\mathcal{M}_{cof}^{f,\mathcal{L}}[(\mathcal{W}\cup\mathcal{L})^{-1}] \simeq \mathcal{M}_{cof}^{f,\mathcal{L}}[\mathcal{L}^{-1}]$. So by Proposition 3.15, one deduces that $f \in \mathcal{W}_{\mathcal{L}}$. Thus, there exists a unique functor $L(i)$ making the following diagram commutative:

$$\begin{tikzcd}
\mathcal{M}_{cof} \arrow[r, "i"] \arrow[d] & \mathcal{M} \arrow[d] \\
\mathcal{M}_{cof}[(\mathcal{W}\cup\mathcal{L})^{-1}] \arrow[r, dashed, "L(i)"] & \mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}].
\end{tikzcd}$$

There exist two natural transformations $\mu : Q\circ i \Rightarrow \mathrm{Id}_{\mathcal{M}_{cof}}$ and $\nu : i\circ Q \Rightarrow \mathrm{Id}_{\mathcal{M}}$. If X is an object of $\mathcal{M}_{cof}$, then $\mu(X)$ is a trivial fibration between cofibrant objects, i.e. $\mu(X) \in \mathcal{W}$. So one deduces that $L(\mu(X))$ is an isomorphism of $\mathcal{M}_{cof}[(\mathcal{W}\cup\mathcal{L})^{-1}]$. Therefore one obtains the isomorphism of functors $L(Q) \circ L(i) \cong \mathrm{Id}_{\mathcal{M}_{cof}[(\mathcal{W}\cup\mathcal{L})^{-1}]}$. If Y is an object of $\mathcal{M}$, then $\nu(Y) : Q(Y) \longrightarrow Y$ is a trivial fibration of $\mathcal{M}$. Thus, $Q(\nu(Y)) : Q(Q(Y)) \longrightarrow Q(Y)$ is a trivial cofibration of $\mathcal{M}$ between cofibrant objects. So $Q(\nu(Y)) \in \mathrm{Cof} \cap \mathcal{W} \subset \mathcal{L}$. Since $R_{\mathcal{L}}(\mathcal{L}) \subset \mathcal{L}$, one deduces that $R_{\mathcal{L}}(Q(\nu(Y)))$ is an isomorphism of $\mathcal{M}_{cof}^{f,\mathcal{L}}[\mathcal{L}^{-1}]$. Again by Proposition 3.15, one deduces that $\nu(Y) \in \mathcal{W}_{\mathcal{L}}$ and one obtains the isomorphism of functors $L(i) \circ L(Q) \cong \mathrm{Id}_{\mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}]}$. The proof is complete. ∎

3.27. Theorem. *(Whitehead's theorem for the localization of a model category with respect to a weak factorization system) The inclusion functors $\mathcal{M}_{cof}^{f,\mathcal{L}} \subset \mathcal{M}_{cof} \subset \mathcal{M}$ induce the equivalences of categories*

$$\mathcal{M}_{cof}^{f,\mathcal{L}}/\sim_{\mathcal{L}} \simeq \mathcal{M}_{cof}[(\mathcal{W} \cup \mathcal{L})^{-1}] \simeq \mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}].$$

The functor $\mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}] \longrightarrow \mathcal{M}_{cof}^{f,\mathcal{L}}/\sim_{\mathcal{L}}$ is given by the cofibrant-fibrant w.r.t. $\mathcal{L}$ functor $R_{\mathcal{L}} \circ Q : \mathcal{M} \longrightarrow \mathcal{M}_{cof}^{f,\mathcal{L}}$. Moreover, the localization functor $\mathcal{M} \longrightarrow \mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}]$ factors uniquely as a composite

$$\mathcal{M} \longrightarrow \mathcal{M}[\mathcal{W}^{-1}] \longrightarrow \mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}].$$

The equivalence of categories $\mathcal{M}_{cof}[(\mathcal{W} \cup \mathcal{L})^{-1}] \simeq \mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}]$ shows that up to weak equivalence and up to the 2-out-of-3 axiom, a morphism of $\mathcal{W}_{\mathcal{L}}$ is a morphism of $\mathcal{W} \cup \mathcal{L}$ between cofibrant objects of $\mathcal{M}$. This means that the class of morphisms $\mathcal{W}_{\mathcal{L}}$ is not too big.

Proof. The equivalence of categories $\mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}] \simeq \mathcal{M}_{cof}[(\mathcal{W} \cup \mathcal{L})^{-1}]$ is given by Proposition 3.26. By Proposition 3.26, the functor $\mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}] \longrightarrow \mathcal{M}_{cof}[(\mathcal{W} \cup \mathcal{L})^{-1}]$ is induced by the cofibrant replacement functor $Q : \mathcal{M} \longrightarrow \mathcal{M}_{cof}$. Therefore every morphism of $\mathcal{W}$ is in $\mathcal{W}_{\mathcal{L}}$. At last, one has

$$\begin{aligned}
&\mathcal{M}_{cof}[(\mathcal{W} \cup \mathcal{L})^{-1}] && \\
&\cong \mathcal{M}_{cof}[\mathcal{L}^{-1}] && \text{by Proposition 3.23} \\
&\simeq \mathcal{M}_{cof}^{f,\mathcal{L}}[\mathcal{L}^{-1}] && \text{by Proposition 3.20} \\
&\simeq \mathcal{M}_{cof}^{f,\mathcal{L}}/\sim_{\mathcal{L}} && \text{by Proposition 3.14.}
\end{aligned}$$

∎

Theorem 3.27 says that the category $\mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}]$ inverts all weak equivalences of $\mathcal{M}$ and all morphisms of $\mathcal{L}$ with cofibrant domains. We do not know if all morphisms of $\mathcal{L}$ (and not only the ones with cofibrant domain) are inverted in the category $\mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}]$. But there is a kind of reciprocal statement:

3.28. Proposition. *Let us suppose that $\mathcal{M}$ is left proper. Let f be a cofibration of $\mathcal{M}$ such that $Q(f) \in \mathcal{L}$. Then $f \in \mathcal{L}$.*

Proof. Let $p \in \mathcal{R}$. Since $\mathrm{Cof} \cap \mathcal{W} \subset \mathcal{L}$, the morphism p is a fibration of $\mathcal{M}$. By hypothesis, p satisfies the RLP with respect to $Q(f)$. Since f is a cofibration and by [19] Proposition 13.2.1, one deduces that p satisfies the RLP with respect to f. So $f \in \mathcal{L}$. ∎

Before treating the case of T-homotopy equivalences in Section 4, let us give some examples of the situation explored in this section.

Example 1. Let $\mathcal{M}$ be a cofibrantly generated model category with set of generating cofibrations I and with set of generating trivial cofibrations J and with class of weak equivalences $\mathcal{W}$. Let $(\mathcal{L},\mathcal{R}) = (\mathbf{cof}(J), \mathbf{inj}(J))$. The main theorem gives the equivalences of categories

$$\mathcal{M}_{cof}^{fib}/\sim_{\mathbf{cof}(J)} \simeq \mathcal{M}_{cof}[\mathbf{cof}(J)^{-1}] \simeq \mathcal{M}[\mathcal{W}_{\mathbf{cof}(J)}^{-1}]$$

where $\mathcal{M}_{cof}^{fib}$ is the full subcategory of cofibrant-fibrant objects. One can directly check that $\mathcal{W}_{\mathbf{cof}(J)}^{-1} = \mathcal{W}$. This is not surprising since the category $\mathcal{M}_{cof}^{fib}/\sim_{\mathbf{cof}(J)}$ is the category of cofibrant-fibrant objects of $\mathcal{M}$ up to homotopy.

Example 2. Let $\mathcal{M}$ be a cofibrantly generated model category with set of generating cofibrations I and with set of generating trivial cofibrations J and with class of weak equivalences $\mathcal{W}$. Then one can consider the model structure $(\underline{\mathrm{All}}, \underline{\mathrm{All}}, \underline{\mathrm{Iso}})$ where all morphisms are a cofibration and a fibration and where the weak equivalences are the isomorphisms. Indeed, one has $(\underline{\mathrm{Iso}}, \underline{\mathrm{All}}) = (\mathbf{cof}(\mathrm{Id}_\varnothing), \mathbf{inj}(\mathrm{Id}_\varnothing))$. The main theorem applied with the latter model structure and with the weak factorization system $(\mathcal{L},\mathcal{R}) = (\mathbf{cof}(J), \mathbf{inj}(J))$ gives the equivalences of categories

$$\mathcal{M}^{fib}/\sim_{\mathbf{cof}(J)} \simeq \mathcal{M}[\mathbf{cof}(J)^{-1}] \simeq \mathcal{M}[\underline{\mathrm{Iso}}_{\mathbf{cof}(J)}^{-1}]$$

where $\mathcal{M}^{fib}$ is the full subcategory of fibrant objects, where $\sim_{\mathbf{cof}(J)}$ is a congruence on the morphisms of the full subcategory of fibrant objects of $\mathcal{M}$. The functor $R_{\mathbf{cof}(J)}$ is a fibrant replacement functor of $\mathcal{M}$ and the functor Q is a cofibrant replacement functor of the model structure $(\underline{\mathrm{All}}, \underline{\mathrm{All}}, \underline{\mathrm{Iso}})$. That is one can suppose that $Q = \mathrm{Id}_{\mathcal{M}}$.

Example 3. Let $\mathcal{M}$ be a combinatorial model category (in the sense of Jeff Smith), that is a cofibrantly generated model category such that the underlying category is locally presentable [2]. Let J be the set of generating trivial cofibrations. Then for any set K of morphisms of $\mathcal{M}$, the pair $(\mathbf{cof}(J \cup K), \mathbf{inj}(J \cup K))$ is a cofibrantly generated weak factorization system by [4] Proposition 1.3. Then the main theorem of this section applies. One obtains the equivalences of categories

$$\mathcal{M}_{cof}^{f,\mathbf{cof}(J\cup K)}/\sim_{\mathbf{cof}(J\cup K)} \simeq \mathcal{M}_{cof}[(\mathcal{W} \cup \mathbf{cof}(J \cup K))^{-1}] \simeq \mathcal{M}[\mathcal{W}_{\mathbf{cof}(J\cup K)}^{-1}].$$

Assume Vopěnka's principle ([2] chapter 6). If $\mathcal{M}$ is left proper, then the left Bousfield localization $L_K\mathcal{M}$ of the model category $\mathcal{M}$ with respect to the set of morphisms K exists by a theorem of Jeff Smith proved in [4] Theorem 1.7 and in [29] Theorem 2.2. The category $\mathcal{M}[\mathcal{W}_{\mathbf{cof}(J\cup K)}^{-1}]$ is not necessarily equivalent to the Quillen homotopy category $\mathbf{Ho}(L_K\mathcal{M})$ of this Bousfield localization. But all morphisms inverted by $\mathcal{M}[\mathcal{W}_{\mathbf{cof}(J\cup K)}^{-1}]$ are inverted by the Bousfield localization. In other terms, the functor $\mathcal{M} \longrightarrow \mathbf{Ho}(L_K\mathcal{M})$ factors uniquely as a composite $\mathcal{M} \longrightarrow \mathcal{M}[\mathcal{W}_{\mathbf{cof}(J\cup K)}^{-1}] \longrightarrow \mathbf{Ho}(L_K\mathcal{M})$.

Example 4. Let $\mathcal{M}$ be a model category with model structure denoted by $(\mathrm{Cof}, \mathrm{Fib}, \mathcal{W})$ for respectively the class of cofibrations, of fibrations and of weak equivalences such that the weak factorization system $(\mathrm{Cof} \cap \mathcal{W}, \mathrm{Fib})$ is cofibrantly generated : the set of generating trivial cofibrations is denoted by J. Let $(\mathcal{L}, \mathcal{R})$ be a cofibrantly generated weak factorization system such that $\mathrm{Cof} \cap \mathcal{W} \subset \mathcal{L} \subset \mathrm{Cof}$. Let us suppose that the left Bousfield localization $L_{\mathcal{L}}\mathcal{M}$ of $\mathcal{M}$ with respect to $\mathcal{L}$ exists and let us suppose that $\mathcal{L}$ is the class of trivial cofibrations of this Bousfield localization. One obtains the equivalences of categories

$$\mathcal{M}_{cof}^{f,\mathcal{L}}/\sim_{\mathcal{L}} \simeq \mathcal{M}_{cof}[(\mathcal{W} \cup \mathcal{L})^{-1}] \simeq \mathcal{M}[\mathcal{W}_{\mathcal{L}}^{-1}].$$

The category $\mathcal{M}_{cof}^{f,\mathcal{L}}$ is the full subcategory of $\mathcal{M}$ containing the cofibrant-fibrant object of $L_{\mathcal{L}}\mathcal{M}$. The congruence $\sim_{\mathcal{L}}$ is the usual notion of homotopy in $\mathcal{M}$ ([19] Proposition 3.5.3). Then the category $\mathcal{M}_{cof}^{f,\mathcal{L}}/\sim_{\mathcal{L}}$ is equivalent to the Quillen homotopy category $\mathbf{Ho}(L_{\mathcal{L}}\mathcal{M})$.

4. Application : homotopy continuous flow and Whitehead's theorem

The category **Top** of *compactly generated topological spaces* (i.e. of weak Hausdorff k-spaces) is complete, cocomplete and cartesian closed (more details for this kind of topological spaces in [7, 25], the appendix of [23] and also the preliminaries of [11]). For the sequel, all topological spaces will be supposed to be compactly generated. A *compact space* is always Hausdorff. The category **Top** is equipped with the unique model structure having the *weak homotopy equivalences* as weak equivalences and having the *Serre fibrations* [3] as fibrations [20].

As already described in the introduction, the time flow of a higher dimensional automaton is encoded in an object called a *flow*. The category **Flow** is equipped with the unique model structure such that [11]:

- The weak equivalences are the *weak S-homotopy equivalences*, i.e. the morphisms of flows $f : X \longrightarrow Y$ such that $f^0 : X^0 \longrightarrow Y^0$ is a bijection and such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a weak homotopy equivalence.
- The fibrations are the morphisms of flows $f : X \longrightarrow Y$ such that $\mathbb{P}f : \mathbb{P}X \longrightarrow \mathbb{P}Y$ is a Serre fibration.

This model structure is cofibrantly generated. The set of generating cofibrations is the set $I_+^{gl} = I^{gl} \cup \{R, C\}$ with

$$I^{gl} = \{\mathrm{Glob}(\mathbf{S}^{n-1}) \subset \mathrm{Glob}(\mathbf{D}^n), n \geq 0\}$$

where $\mathbf{D}^n$ is the n-dimensional disk, where $\mathbf{S}^{n-1}$ is the $(n-1)$-dimensional sphere, where R and C are the set maps $R : \{0,1\} \longrightarrow \{0\}$ and $C : \varnothing \longrightarrow \{0\}$ and where for any topological space Z, the flow $\mathrm{Glob}(Z)$ is the flow defined by $\mathrm{Glob}(Z)^0 = \{\widehat{0}, \widehat{1}\}$, $\mathbb{P}\mathrm{Glob}(Z) = Z$, $s = \widehat{0}$ and $t = \widehat{1}$, and a trivial composition law. The set of generating trivial cofibrations is

$$J^{gl} = \{\mathrm{Glob}(\mathbf{D}^n \times \{0\}) \subset \mathrm{Glob}(\mathbf{D}^n \times [0,1]), n \geq 0\}.$$

[3]that is a continuous map having the RLP with respect to the inclusion $\mathbf{D}^n \times 0 \subset \mathbf{D}^n \times [0,1]$ for all $n \geq 0$ where $\mathbf{D}^n$ is the n-dimensional disk.

The weak S-homotopy model structure of **Flow** has some similarity with the model structure on the category of small simplicial categories (with identities !) constructed in [5]. The weak equivalences (resp. the fibrations) of the latter look like the weak equivalences (resp. the fibrations) of the model structure of **Flow** with an additional condition. The weak S-homotopy model structure of **Flow** has also some similarity with the model structure on the category of small simplicial categories (with identities again !) on a *fixed* set of objects O constructed in [8]. For the latter, the set maps $R : \{0,1\} \longrightarrow \{0\}$ and $C : \varnothing \longrightarrow \{0\}$ are not used since the set of objects is fixed.

4.1. DEFINITION. *A flow X is* loopless *if for any $\alpha \in X^0$, the space $\mathbb{P}_{\alpha,\alpha}X$ is empty.*

Recall that a flow is a small category without identities morphisms enriched over a category of topological spaces. So the preceding definition is meaningful.

A poset $(P, \leq)$ can be identified with a loopless flow having P as set of states and such that there exists a non-constant execution path from x to y if and only if $x < y$. The corresponding flow is still denoted by P. This defines a functor from the full subcategory of posets whose morphisms are the strictly increasing maps to the full subcategory of loopless flows. The category of finite bounded posets is essentially small. Let us choose a small subcategory of representatives.

4.2. DEFINITION. *[15] Let $\mathcal{T}$ be the set of cofibrations $Q(f) : Q(P_1) \longrightarrow Q(P_2)$ such that $f : P_1 \longrightarrow P_2$ is a morphism of posets satisfying the following conditions:*

1. *The posets P_1 and P_2 are finite and bounded.*
2. *The morphism of posets $f : P_1 \longrightarrow P_2$ is one-to-one; in particular, if x and y are two elements of P_1 with $x < y$, then $f(x) < f(y)$.*
3. *One has $f(\min P_1) = \min P_2 = \widehat{0}$ and $f(\max P_1) = \max P_2 = \widehat{1}$.*
4. *The posets P_1 and P_2 are objects of the chosen small subcategory of representatives of the category of finite bounded posets.*

The set $\mathcal{T}$ is called the set of generating T-homotopy equivalences*.*

The set $\mathcal{T}$ is introduced in [15] for modelling T-homotopy as a refinement of observation. By now, this is the best known definition of T-homotopy.

4.3. DEFINITION. *A flow X is* homotopy continuous *if the unique morphism of flows $f_X : X \longrightarrow \mathbf{1}$ belongs to $\mathbf{inj}(J^{gl} \cup \mathcal{T})$.*

Notice that $\mathbf{inj}(J^{gl} \cup \mathcal{T}) = \mathbf{inj}(\mathcal{T})$ because all flows are fibrant for the weak S-homotopy model structure of **Flow**.

Let X be a homotopy continuous flow. Then, for instance, consider the unique morphism $Q(f) : Q(\{\widehat{0} < \widehat{1}\}) \longrightarrow Q(\{\widehat{0} < A < \widehat{1})\}$ of $\mathcal{T}$. For any commutative square

$$\begin{tikzcd}
Q(\{\widehat{0} < \widehat{1}\}) \arrow[r, "\phi"] \arrow[d] & X \\
Q(\{\widehat{0} < A < \widehat{1}\}) \arrow[ur, dashed, "k"] &
\end{tikzcd}$$

there exists $k : Q(\{\widehat{0} < A < \widehat{1}\}) \longrightarrow X$ making the triangle commutative. Therefore, the existence of k ensures that any directed segment of X can always be divided up to S-homotopy. Roughly speaking, a flow X is homotopy continuous if it is indefinitely divisible up to S-homotopy.

4.4. Proposition. *The pair $(\mathbf{cof}(J^{gl} \cup \mathcal{T}), \mathbf{inj}(J^{gl} \cup \mathcal{T}))$ is a cofibrantly generated weak factorization system. Moreover, it satisfies the conditions of Section 1, that is:*

$$\mathbf{cof}(J^{gl}) \subset \mathbf{cof}(J^{gl} \cup \mathcal{T}) \subset \mathbf{cof}(I^{gl}_+).$$

Proof. For every $g \in J^{gl} \cup \mathcal{T}$, the continuous map $\mathbb{P}g$ is a closed inclusion of topological spaces. Therefore by [11] Proposition 11.5 and by [20] Theorem 2.1.14, the small object argument applies. ∎

4.5. Notation.

$$\begin{aligned}
&\mathcal{S}_{\mathcal{T}} = \mathcal{S}_{\mathbf{cof}(J^{gl}\cup\mathcal{T})},\\
&R_{\mathcal{T}} = R_{\mathbf{cof}(J^{gl}\cup\mathcal{T})},\\
&\sim_{\mathcal{T}} = \sim_{\mathbf{cof}(J^{gl}\cup\mathcal{T})},\\
&\mathbf{Flow}^{f,\mathcal{T}}_{cof} = \mathbf{Flow}^{f,\mathbf{cof}(J^{gl}\cup\mathcal{T})}_{cof}.
\end{aligned}$$

We can now apply Theorem 3.27 to obtain the theorem:

4.6. Theorem. *The inclusion functors $\mathbf{Flow}^{f,\mathcal{T}}_{cof} \subset \mathbf{Flow}_{cof} \subset \mathbf{Flow}$ induce the equivalences of categories*

$$\mathbf{Flow}^{f,\mathcal{T}}_{cof}/\sim_{\mathcal{T}} \simeq \mathbf{Flow}_{cof}[(\mathcal{S} \cup \mathbf{cof}(J^{gl} \cup T))^{-1}] \simeq \mathbf{Flow}[\mathcal{S}_{\mathcal{T}}^{-1}].$$

It remains to check the invariance of the underlying homotopy type and of the branching and merging homology theories:

4.7. Theorem. *A morphism of $\mathcal{S}_{\mathcal{T}}$ preserves the underlying homotopy type and the branching and merging homology theories.*

Proof. It has been already noticed above that up to weak S-homotopy and up to the 2-out-of-3 axiom, a morphism of $\mathcal{S}_{\mathcal{T}}$ is a morphism of $\mathcal{S} \cup \mathbf{cof}(J^{gl} \cup \mathcal{T})$ between cofibrant flows. Formally, let $f : A \longrightarrow B$ be an element of $\mathcal{S}_{\mathcal{T}}$. Then consider the commutative diagram:

$$\begin{tikzcd} Q(A) \arrow[r, "\simeq"] \arrow[d, "Q(f)"'] & A \arrow[d, "f"] \\ Q(B) \arrow[r, "\simeq"] & B \end{tikzcd}$$

The morphism $Q(f)$ is an isomorphism of $\mathbf{Flow}_{cof}[(\mathcal{S} \cup \mathbf{cof}(J^{gl} \cup T))^{-1}]$. Therefore it preserves the underlying homotopy type and the branching and merging homology theories because any morphism of $\mathcal{S}$ preserves these invariants by [12] Proposition VII.2.5 and by [14] Corollary 6.5 and Corollary A.11, and because any morphism of $\mathbf{cof}(J^{gl} \cup \mathcal{T})$ preserves these invariants by [15] Theorem 5.2. The morphisms $Q(A) \longrightarrow A$ and $Q(B) \longrightarrow B$ are weak S-homotopy equivalences. So both preserve the underlying homotopy type [12] Proposition VII.2.5 and the branching and merging homology theories [14] Corollary 6.5 and Corollary A.11. Hence the result. ∎

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Preuves Programmes et Systèmes
Université Paris 7–Denis Diderot
Case 7014
2 Place Jussieu
75251 PARIS Cedex 05
France
Email: `gaucher@pps.jussieu.fr`
`http://www.pps.jussieu.fr/~gaucher/`

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