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EVERY GROTHENDIECK TOPOS HAS A ONE-WAY SITE

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ABSTRACT. Lawvere has urged a project of characterizing *petit* toposes which have the character of generalized spaces and *gros* toposes which have the character of categories of spaces. Étendues and locally decidable toposes are seemingly petit and have a natural common generalization in sites with all idempotents identities. This note shows every Grothendieck topos has such a site. More, it defines *slanted products* which take any site to an equivalent *one way* site, a site where all endomorphisms are identities. On the other hand subcanonical one-way sites are very special. A site criterion for petit toposes will probably require subcanonical sites.

Lawvere has urged a project of distinguishing which toposes have the character of (generalized) spaces and which the character of categories of spaces (Lawvere 1986). Special cases of such a distinction had been called petit and gros by Giraud and Grothendieck in topology and in étale schemes. Lawvere calls for a general distinction, emphasizing among other things that the gros topos of a point itself amounts to the definition of a branch of geometry. The two should be defined both intrinsically and in terms of their sites. Localic toposes are paradigmatically petit and two results in this direction concern site descriptions for generalizations which are still seemingly petit. Johnstone showed a Grothendieck topos is locally decidable iff it has some small site with all arrows epic (Johnstone 2002, C5.4.4). Kock and Moerdijk proved a topos is an étendue if and only it it has some small site with all arrows monic (Kock & Moerdijk 1991) or (Johnstone 2002, C5.2.5). Lawvere asked about a further generalization: sites **without idempotents** in the sense that all idempotents are identities (equivalently all are epic, or all are monic).

This note shows every Grothendieck topos has such a site, and in fact has a **one way** site, a site where all endomorphisms are identities. On the other hand we notice that subcanonical one-way sites are very special. This suggests that a site criterion for petit toposes will probably require subcanonical sites.

The definitions and proofs are formulated for small categories and Grothendieck toposes over **Set**. But every step through the proof of Theorem 2.1 applies when **Set** is replaced by any elementary topos with a natural number object, reasoning internally to that topos. The proof of Theorem 2.2 uses excluded middle.

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1. Slanted products

The statement above says every presheaf topos $\widehat{\mathbf{C}} = \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ has a topos inclusion to presheaves $\widehat{\mathbf{D}}$ on some one way category $\mathbf{D}$, so the same holds for every subtopos of $\widehat{\mathbf{C}}$. The theorem is much stronger as it shows every small category $\mathbf{C}$ has a **flat cover** by a one way small category constructed from $\mathbf{C}$ in a simple functorial way. In Lawvere's terms a functor $f{:}\mathbf{D}\to\mathbf{C}$ is a **cover** of $\mathbf{C}$ if the induced $f^*{:}\widehat{\mathbf{C}}\to\widehat{\mathbf{D}}$ is full and faithful. It is **flat** if the left adjoint $f_!$ is left exact (Lawvere & Rosebrugh 2003, p. 246). To say f is a flat cover is equivalent to either (1) or (2) and implies (3):

1. $f_! \dashv f^*{:}\widehat{\mathbf{C}}\to\widehat{\mathbf{D}}$ is a topos inclusion.
2. $f_! \dashv f^*$ is a geometric morphism right inverse to $f^* \dashv f_*{:}\widehat{\mathbf{D}}\to\widehat{\mathbf{C}}$ in the direct image sense: $f_* f^* \cong 1_{\widehat{\mathbf{C}}}$.
3. $f^* \dashv f_*{:}\widehat{\mathbf{D}}\to\widehat{\mathbf{C}}$ is an essential connected topos surjection.

Let $\mathbf{P}$ be any one way category. For any small category $\mathbf{C}$ define the **slanted product** $[\mathbf{C}{\times}\mathbf{P}]$ as the largest subcategory of $\mathbf{C}{\times}\mathbf{P}$ such that the projection reflects identity arrows.

$$[\mathbf{C}\times\mathbf{P}] \rightarrowtail \mathbf{C}\times\mathbf{P} \longrightarrow \mathbf{P}$$

It has the same objects as $\mathbf{C}\times\mathbf{P}$ and its arrows are $\langle f,\alpha\rangle{:}\langle A,i\rangle\to\langle B,j\rangle$ where $f{:}A\to B$ in $\mathbf{C}$, and $\alpha{:}i\to j$ in $\mathbf{P}$, and if $i=j$ then $f=1_A$.

Every $[\mathbf{C}\times\mathbf{P}]$ is one way. For $\mathbf{P}{=}\mathbf{1}$ the terminal category, $[\mathbf{C}\times\mathbf{1}]$ is the discrete category $|\mathbf{C}|$. For $\mathbf{P}{=}\mathbf{2}$ the arrow category, $[\mathbf{C}{\times}\mathbf{2}]$ is $\mathbf{C}$ pulled apart just enough to have no compositions: every non-identity arrow goes from some $\langle A,0\rangle$ to some $\langle B,1\rangle$ so that non-identity arrows never compose. The slanted product is close to a device Johnstone uses to replace filtered categories by directed posets ((Johnstone 2002, B2.6.13) refining Deligne's proof (Artin, Grothendieck & Verdier 1972, p. 65)).

For any poset $\mathbf{P}$ and small category $\mathbf{C}$ the projection $\mathbf{p}{:}[\mathbf{C}\times\mathbf{P}]\to\mathbf{C}$ induces $\mathbf{p}^*{:}\widehat{\mathbf{C}}\to [\mathbf{C}\times\mathbf{P}]\widehat{\ }$. If $\mathbf{P}$ is connected and not discrete then $\mathbf{p}$ is a cover:

1.1. Lemma. *If $\mathbf{P}$ is connected then $\mathbf{p}^*$ is full. If $\mathbf{P}$ has any non-identity arrow $i\to j$ then $\mathbf{p}^*$ is faithful:*

Proof. For any natural transformation $\tau'{:}\mathbf{p}^*F\to\mathbf{p}^*G$, object A of $\mathbf{C}$, and arrow $\alpha{:}i\to j$ in $\mathbf{P}$ the naturality square

$$\begin{tikzcd}
\langle A,j\rangle & \mathbf{p}^*F\langle A,j\rangle \arrow[r, "\tau'_{\langle A,j\rangle}"] \arrow[d] & \mathbf{p}^*G\langle A,j\rangle \arrow[d] \\
\langle A,i\rangle \arrow[u, "\langle 1_A,\alpha\rangle"] & \mathbf{p}^*F\langle A,i\rangle \arrow[r, "\tau'_{\langle A,i\rangle}"'] & \mathbf{p}^*G\langle A,i\rangle
\end{tikzcd}$$

has identity arrows on the sides. If $\mathbf{P}$ is connected then for all objects i,j of $\mathbf{P}$ the components $\tau'_{\langle A,i\rangle} = \tau'_{\langle A,j\rangle}$ are equal, and $\tau' = \mathbf{p}^*\tau$ for the obvious $\tau{:}F \to G$. If $\mathbf{P}$ has any non-identity arrow then the projection $\mathbf{p}$ is onto. ∎

1.2. Lemma. *If* $\mathbf{P}$ *is cofiltered (Johnstone 2002, B2.6) and downward unbounded (each element has an element below it) then* $\mathbf{p}$ *is flat.*

Proof. $\mathbf{p}_!$ is given by colimits of contravariant diagrams over comma categories. For each presheaf F on $[\mathbf{C}\times\mathbf{P}]$ and object A of $\mathbf{C}$:

$$(\mathbf{p}_! F)(A) \;=\; \varinjlim_{(A\downarrow\mathbf{p})} \; F(B,j)$$

But if $\mathbf{P}$ is cofiltered and downward unbounded then for each A the functor $\mathbf{P}\to(A\downarrow\mathbf{p})$ defined by $i \mapsto \langle A,i\rangle$ is initial. So $p_!$ is given by filtered colimits

$$(\mathbf{p}_! F)(A) \;=\; \varinjlim_{i\in\mathbf{P}} \; F(A,i)$$

∎

2. The theorems

2.1. Theorem. *Every small category* $\mathbf{C}$ *has a small flat one way cover.*

Proof. Take $[\mathbf{C}\times\mathbf{P}]$ for any downward unbounded cofiltered poset $\mathbf{P}$. E.g. the dual to the usual order on the natural numbers. ∎

Every category $\mathbf{C}$ has a cover by a category with no compositions, namely $[\mathbf{C}\times\mathbf{2}]$. But $\mathbf{C}$ has no flat cover by such a category unless $\mathbf{C}$ is equivalent to such a category:

2.2. Theorem. *If* $\mathbf{q}{:}\mathbf{D}\to\mathbf{C}$ *is a flat cover where* $\mathbf{D}$ *has no compositions, then the skeleton of* $\mathbf{C}$ *has no compositions.*

Proof. Given any topology on a category $\mathbf{D}$ with no compositions we can remove all objects with empty covers to get an equivalent site where only maximal sieves cover. So without loss of generality we may assume $\mathbf{q}^*{:}\widehat{\mathbf{C}}\to\widehat{\mathbf{D}}$ is an equivalence. Then, since $\mathbf{D}$ is its own Karoubi completion, $\mathbf{q}{:}\mathbf{D}\to\mathbf{C}$ is the skeleton of $\mathbf{C}$ (Johnstone 2002, A1.1.10). ∎

3. Subcanonical sites

Subcanonical sites are much more special. For a germane proof consider toposes satisfying the **Nullstellensatz**: Every non-empty object A has at least one point $x{:}1\to A$. Many toposes in geometry have this including the usual models of synthetic differential geometry and the gros étale topos over any algebraically closed field. These paradigms of gros toposes have no subcanonical one way sites nor even subcanonical sites without idempotents:

Observation. **Set** is the only nontrivial Grothendieck topos $\mathbf{E}$ with Nullstellensatz and a subcanonical site without idempotents.

Proof. Every subcanonical site for **E** is a full subcategory of **E**. By Nullstellensatz every non-empty object A of **E** has at least one idempotent

$$A \longrightarrow 1 \xrightarrow{x} A$$

A full subcategory where all idempotents are identities has all objects either empty or terminal. Up to equivalence it is either empty, a singleton category, or the arrow category $0 \to 1$. Only the second case has a subcanonical topology with a nontrivial topos with Nullstellensatz, and that topos is **Set**. ∎

Without attempting a precise definition we can plausibly associate gros toposes with Nullstellensatz and petit toposes with subcanonical sites without indempotents. The category **Set** naturally is both as the gros topos of discrete spaces and the petit topos of sheaves on a one point space. But this association with petit toposes would fail if not limited to subcanonical sites.

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