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THIN FILLERS IN THE CUBICAL NERVES OF OMEGA-CATEGORIES

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ABSTRACT. It is shown that the cubical nerve of a strict omega-category is a sequence of sets with cubical face operations and distinguished subclasses of thin elements satisfying certain thin filler conditions. It is also shown that a sequence of this type is the cubical nerve of a strict omega-category unique up to isomorphism; the cubical nerve functor is therefore an equivalence of categories. The sequences of sets involved are the analogues of cubical T-complexes appropriate for strict omega-categories. Degeneracies are not required in the definition of these sequences, but can in fact be constructed as thin fillers. The proof of the thin filler conditions uses chain complexes and chain homotopies.

1. Introduction

This paper is concerned with the cubical nerves of strict ω-categories. It has been shown in [1] that these cubical nerves are sequences of sets together with face maps, degeneracies, connections and compositions, subject to various identities, and that the functor taking an ω-category to its cubical nerve is an equivalence of categories. In this paper we give a more conceptual characterisation of cubical nerves: a cubical nerve is a sequence of sets with cubical face operations and distinguished subclasses of thin elements such that certain shells and boxes have unique thin fillers and such that one simple extra condition is satisfied. Here a shell is a configuration like the boundary of a cube, and a box is a configuration like the boundary of a cube with one face removed.

There are similar characterisations of the cubical and simplicial nerves of ω-groupoids in terms of Dakin's T-complexes [5] due to Ashley and Brown and Higgins ([2], [4], [3]), and there is a similar characterisation of the simplicial nerves of ω-categories due to Verity [11] (see also [10]). In the results for ω-groupoids every box or horn has a unique thin filler (a horn is the simplicial analogue of a box) and there are degeneracy operations, but there are no requirements on shells; the resulting structure is called a cubical or simplicial T-complex. In Verity's result also, certain horns have unique thin fillers and there are degeneracies, but there are no requirements on shells. Our result is like Verity's and is different from the results on ω-groupoids because we do not require all boxes to have unique thin fillers. Our result differs from all the previous results because we require thin

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fillers for shells instead of requiring degeneracies. We need thin fillers for shells in order to construct connections (compare the work of Higgins in [6]), and it is more economical to construct degeneracies from shells as well. This method may also be better for extensions to weak ω-categories, where one expects thin fillers to exist but not necessarily to be unique. Degeneracies correspond to identities, and in weak ω-categories one does not necessarily want unique identities, so one may also not want unique degeneracies. Thin fillers for shells could be a suitable alternative.

The main result of this paper, Theorem 2.14, is stated in Section 2. The cubical nerve functor is described in Section 3, with some calculations postponed to Section 6, and the reverse functor is described in Section 4. In Section 5 the functors are shown to be inverse equivalences. The proof of the thin filler conditions in Sections 3 and 6 is along the same lines as the proof for the simplicial case given by Street in [9], but it is simplified by the use of chain complexes as in [7]. Section 7 gives an example showing how the nerve of an ω-category can differ from the nerve of an ω-groupoid.

2. Statement of the main result

Our result concerns sequences of sets with cubical face operations, which we call precubical sets. We define precubical sets in terms of the precubical category, which is an analogue of the simplicial category but without degeneracies.

2.1. DEFINITION. *The* precubical category *is the category with objects* $[0]$, $[1]$, $\ldots$ *indexed by the natural numbers and with a generating set of morphisms*

$$\check{\partial}_1^-, \check{\partial}_1^+, \ldots, \check{\partial}_n^-, \check{\partial}_n^+ \colon [n-1] \to [n]$$

subject to the relations

$$\check{\partial}_j^\beta \check{\partial}_i^\alpha = \check{\partial}_{i+1}^\alpha \check{\partial}_j^\beta \text{ for } i \geq j.$$

A precubical set *is a contravariant functor from the precubical category to sets; equivalently, a precubical set X is a sequence of sets $X_0, X_1, \ldots$ together with* face operations

$$\partial_1^-, \partial_1^+, \ldots, \partial_n^-, \partial_n^+ \colon X_n \to X_{n-1}$$

such that

$$\partial_i^\alpha \partial_j^\beta = \partial_j^\beta \partial_{i+1}^\alpha \text{ for } i \geq j.$$

If X is a precubical set then the members of X_n are called n-cubes. *An operation from n-cubes to m-cubes induced by a morphism in the precubical category is called a* precubical operation. *A* morphism of precubical sets *from X to Y is a sequence of functions from X_n to Y_n commuting with the face operations.*

The precubical operations from n-cubes to $(n-1)$-cubes correspond to the $2n$ faces of dimension $(n-1)$ in a standard geometrical n-cube, and the relations between them correspond to pairwise intersections. More generally, it is well-known that the precubical operations from n-cubes to m-cubes correspond to the m-dimensional faces of a geometrical n-cube, as follows.

2.2. PROPOSITION. *The precubical operations from n-cubes to m-cubes have unique* standard *decompositions*

$$\partial^{\alpha(1)}_{i(1)}\dots\partial^{\alpha(n-m)}_{i(n-m)}$$

with $1\leq i(1)<i(2)<\ldots<i(n-m)\leq n$.

In order to handle thin n-cubes we introduce the following terminology.

2.3. DEFINITION. *A* stratification *on a precubical set X is a sequence of sets $t_1X, t_2X,\dots$ such that t_nX is a subset of X_n. A precubical set X with a stratification t_nX is called a* stratified precubical set, *and the members of t_nX are called* thin n-cubes. *A* morphism of stratified precubical sets *is a morphism of precubical sets taking thin n-cubes to thin n-cubes.*

Note that 0-cubes are never thin.

The cubical analogue of a horn is a configuration of the kind got from the boundary of an n-cube by removing one face, and it is called an n-box. We also need the configuration corresponding to the complete boundary of an n-cube, which is called an n-shell. The precise definitions are as follows.

2.4. DEFINITION. *Let X be a precubical set. Then an n-*shell *s in X is a collection of $(n-1)$-cubes s^α_i, indexed by the $2n$ face operations ∂^α_i on n-cubes, such that*

$$\partial^\alpha_i s^\beta_j=\partial^\beta_j s^\alpha_{i+1}\ \textit{for}\ i\geq j.$$

A filler *for an n-shell s is an n-cube x such that $\partial^\alpha_i x=s^\alpha_i$ for all ∂^α_i.*

2.5. DEFINITION. *Let X be a precubical set and let ∂^γ_k be a face operation on n-cubes. Then an n-*box *b in X opposite ∂^γ_k is a collection of $(n-1)$-cubes b^α_i, indexed by the $(2n-1)$ face operations ∂^α_i on n-cubes other than ∂^γ_k, such that*

$$\partial^\alpha_i b^\beta_j=\partial^\beta_j b^\alpha_{i+1}\ \textit{for}\ i\geq j.$$

A filler *for an n-shell b is an n-cube x such that $\partial^\alpha_i x=b^\alpha_i$ for $\partial^\alpha_i\neq\partial^\gamma_k$.*

If s is an n-shell and

$$\theta=\partial^{\alpha(1)}_{i(1)}\dots\partial^{\alpha(p)}_{i(p)}$$

is a non-identity precubical operation on n-cubes, not necessarily in standard form, then there is a well-defined operation of θ on s given by

$$\theta s=\partial^{\alpha(1)}_{i(1)}\dots\partial^{\alpha(p-1)}_{i(p-1)}s^{\alpha(p)}_{i(p)},$$

and if x is a filler for s then $\theta x=\theta s$. Similar remarks apply to an n-box b opposite ∂^γ_k, except that the operation ∂^γ_k is not defined on b.

In the cubical nerve of an ω-category we will require certain boxes and shells to have unique thin fillers. We will now describe what happens in dimension 2. We regard a

$$\begin{tikzcd}
\partial_1^+\partial_1^- x = \partial_1^-\partial_2^+ x \arrow[r, "\partial_2^+ x"] \arrow[dr, phantom, "\searrow"] & \partial_1^+\partial_2^+ x = \partial_1^+\partial_1^+ x \\
\partial_1^-\partial_1^- x = \partial_1^-\partial_2^- x \arrow[r, "\partial_2^- x"'] \arrow[u, "\partial_1^- x"] & \partial_1^+\partial_2^- x = \partial_1^-\partial_1^+ x \arrow[u, "\partial_1^+ x"']
\end{tikzcd}$$

Figure 1: A 2-cube

$$\begin{tikzcd}
{} \arrow[r, "b_2^+", "="'] & {} \\
{} \arrow[u, dotted, "b_2^-\circ b_1^+"] \arrow[r, "b_2^-"'] & {} \arrow[u, "b_1^+"']
\end{tikzcd}$$

Figure 2: A 2-box opposite ∂_1^- with a unique thin filler

0-cube as an object of a category, and we regard a 1-cube e as a morphism from $\partial_1^- e$ to $\partial_1^+ e$. A 2-cube x can then be viewed as two composites

$$\partial_1^- x \circ \partial_2^+ x, \partial_2^- x \circ \partial_1^+ x \colon \partial_1^-\partial_2^- x \to \partial_1^+\partial_2^+ x,$$

together with some kind of higher morphism or 2-morphism between the two composites; see Figure 1. A 1-cube is thin if it is an identity morphism, and a 2-cube is thin if its 2-morphism is an identity 2-morphism. A 2-shell is like a 2-cube with the 2-morphism omitted, and a 2-box is like a 2-shell with one of the edge morphisms omitted.

It is now clear that a 2-shell s has a unique thin filler if it is commutative, that is $s_1^- \circ s_2^+ = s_2^- \circ s_1^+$, and that it has no thin filler otherwise. A 2-box therefore has a unique thin filler if it can be extended to a commutative 2-shell in a unique way. For example, a 2-box b opposite ∂_1^- such that b_2^+ is an identity has a unique thin filler x, given by $\partial_1^- x = b_2^- \circ b_1^+$; see Figure 2.

In general a box b opposite ∂_k^γ is to have a unique thin filler if certain of its faces are thin, and we will now explain which faces are involved. We start by considering a class of *extreme precubical operations*, where a precubical operation is called extreme if its standard decomposition $\partial_{i(1)}^{\alpha(1)} \ldots \partial_{i(p)}^{\alpha(p)}$ is such that the signs $(-)^{i(r)-r}\alpha(r)$ are constant. Thus the extreme precubical operations on 2-cubes are

$$\mathrm{id}, \partial_1^-, \partial_1^+, \partial_2^-, \partial_2^+, \partial_1^-\partial_2^-, \partial_1^+\partial_2^+.$$

The non-extreme precubical operations on 2-cubes are $\partial_1^-\partial_2^+$ and $\partial_1^+\partial_2^-$, which are in a sense between $\partial_1^-\partial_2^-$ and $\partial_1^+\partial_2^+$ (see Figure 1). We will say that an extreme precubical operation is *opposite* ∂_k^γ if its standard decomposition has a factor $\partial_k^{-\gamma}$. The extreme precubical operations opposite ∂_k^γ are indexed by the subsets of $\{1,\ldots,n\}$ containing k; for example the extreme precubical operations on 5-cubes opposite ∂_3^- are the composites $\phi'\partial_3^+\phi''$ such that $\phi'\in\{\mathrm{id},\partial_1^-,\partial_1^+\partial_2^+,\partial_2^+\}$ and $\phi''\in\{\mathrm{id},\partial_4^+,\partial_4^+\partial_5^+,\partial_5^-\}$. By omitting the factor $\partial_k^{-\gamma}$ from the standard decomposition for an extreme precubical operation θ opposite ∂_k^γ we get a precubical operation which in a sense joins θ to ∂_k^γ. We say that an operation of this type is *complementary to* ∂_k^γ. The precubical operations complementary to ∂_k^γ are indexed by the subsets of $\{1,\ldots,n\}$ not containing k; for example the precubical operations on 2-cubes complementary to ∂_1^- are ∂_2^+ and id. It turns out that an n-box b opposite ∂_k^γ has a unique thin filler if θb is thin for every non-identity precubical operation on n-cubes complementary to ∂_k^γ. Boxes satisfying this condition will be called *admissible*; they correspond to the admissible horns of [8]. For example, a 2-box b opposite ∂_1^- is admissible if and only if b_2^+ is thin, in which case b has a unique thin filler as in Figure 2. We summarise these ideas concisely as follows.

2.6. Definition. *Let ∂_k^γ be a face operation on n-cubes and let θ be a precubical operation on n-cubes. Then θ is* complementary to ∂_k^γ *if the standard decomposition of θ has no factor ∂_k^- or ∂_k^+ and if the insertion of $\partial_k^{-\gamma}$ produces a standard decomposition*

$$\phi=\partial_{i(1)}^{\alpha(1)}\ldots\partial_{i(p)}^{\alpha(p)}$$

such that the sign $(-)^{i(r)-r}\alpha(r)$ is constant.

2.7. Definition. *In a stratified precubical set a box opposite ∂_k^γ is* admissible *if θb is thin for every non-identity precubical operation θ complementary to ∂_k^γ.*

2.8. Remark. A face operation ∂_l^δ is complementary to ∂_k^γ if and only if $k\neq l$ and $(-)^k\gamma=(-)^l\delta$.

2.9. Example. Let b be an n-box opposite ∂_k^γ such that θb is thin whenever the standard decomposition of θ does not contain ∂_{k-1}^γ, ∂_k^-, ∂_k^+ or ∂_{k+1}^γ. (In the case $k=1$ we take it to be automatically true that the standard decomposition does not have a factor ∂_{k-1}^γ; in the case $k=n$ we take it to be automatically true that the standard decomposition does not have a factor ∂_{k+1}^γ.) Then the standard decomposition of an operation θ such that θb is not thin must have a factor ∂_{k-1}^γ, ∂_k^-, ∂_k^+ or ∂_{k+1}^γ, so it cannot be complementary to ∂_k^γ. Therefore b is admissible.

Next we consider thin fillers for shells. We have already observed that a commutative 2-shell has a unique thin filler, and a similar result holds in general (see Proposition 3.13). In the characterisation, however, we require a shell to have a unique thin filler only when we can prove it to be commutative by using admissible boxes. Shells of this kind will also be called admissible. There are four types of admissible 2-shell s, as shown in Figure 3.

$$
\begin{tikzcd} \cdot \arrow[r, "="] & \cdot \\ \cdot \arrow[u, "x"] \arrow[r, "="'] & \cdot \arrow[u, "x"'] \end{tikzcd}
\qquad
\begin{tikzcd} \cdot \arrow[r, "x"] & \cdot \\ \cdot \arrow[u, "="] \arrow[r, "x"'] & \cdot \arrow[u, "="'] \end{tikzcd}
\qquad
\begin{tikzcd} \cdot \arrow[r, "="] & \cdot \\ \cdot \arrow[u, "x"] \arrow[r, "x"'] & \cdot \arrow[u, "="'] \end{tikzcd}
\qquad
\begin{tikzcd} \cdot \arrow[r, "x"] & \cdot \\ \cdot \arrow[u, "="] \arrow[r, "="'] & \cdot \arrow[u, "x"'] \end{tikzcd}
$$

Figure 3: Admissible 2-shells

In each case there are two equal faces, denoted x in the figure, which correspond to non-complementary face operations, and omitting either of these faces produces an admissible box. In general the definition is as follows.

2.10. Definition. *In a stratified precubical set a shell s is* admissible *if there are distinct non-complementary face operations ∂_k^γ and ∂_l^δ such that $s_k^\gamma = s_l^\delta$ and such the boxes formed by removing s_k^γ or s_l^δ from s are both admissible.*

2.11. Example. As in the first two cases of Figure 3, let s be a shell such that $s_k^- = s_k^+$ for some k and such that θs is thin whenever the standard decomposition of θ does not contain ∂_k^- or ∂_k^+. It follows from Example 2.9 that s is admissible. Shells of this type will be used to construct degeneracies.

2.12. Example. As in the last two cases of Figure 3, let s be a shell such that $s_k^\gamma = s_{k+1}^\gamma$ for some k and γ and such that θs is thin whenever the standard decomposition of θ does not contain ∂_k^γ, ∂_{k+1}^γ or $\partial_k^{-\gamma}\partial_{k+1}^{-\gamma}$. Again it follows from Example 2.9 that s is admissible. Shells of this type will be used to construct connections.

For the admissible 2-box b in Figure 2, note that the additional face $b_2^- \circ b_1^+$ in the thin filler is thin if all the 1-cubes b_i^α are thin. An analogous result holds in general, and it is the final condition characterising nerves of ω-categories. We end up with the following structure.

2.13. Definition. *A* complete stratified precubical set *is a stratified precubical set satisfying the following conditions: every admissible box and every admissible shell has a unique thin filler; if x is the thin filler of an admissible box b opposite ∂_k^γ such that b_i^α is thin for all $\partial_i^\alpha \neq \partial_k^\gamma$, then the additional face $\partial_k^\gamma x$ is thin as well.*

The main result is now as follows.

2.14. Theorem. *The cubical nerve functor is an equivalence between strict ω-categories and complete stratified precubical sets.*

In the remainder of this paper we prove four results, Theorems 3.16, 4.12, 5.4 and 5.5, whose conjunction is equivalent to Theorem 2.14.

2.15. Remark. The only admissible boxes and shells used in the passage from stratified precubical sets to the nerves of ω-categories are those of the types described in Examples 2.9, 2.11 and 2.12. It would therefore be sufficient to use these boxes and shells in the definition of a complete stratified precubical set.

Simplicial nerves behave in the same way. Indeed Street [9] shows that the simplicial nerve of an ω-category has a large class of admissible horns with unique thin fillers, and Verity [11] shows that simplicial nerves are characterised by a smaller class of complicial horns. Admissible boxes in general correspond to admissible horns, and the admissible boxes of Example 2.9 correspond to complicial horns.

2.16. Remark. One of Dakin's axioms for a cubical or simplicial T-complex ([2], [3], [5]) says that a thin n-cube or n-simplex with all but one of its $(n-1)$-faces thin must have its remaining $(n-1)$-face thin as well. In a complete stratified precubical set we require this condition only when the given thin faces form an admissible box. In the cubical nerve of an ω-category, it is in fact possible for a thin n-cube x to have exactly one non-thin face $\partial_i^\alpha x$; it is even possible for this to happen when x is the thin filler of an admissible box, provided that this box is opposite an operation other than ∂_i^α. An example is given in Section 7. As before, one gets the same behaviour in simplicial nerves.

3. From ω-categories to complete stratified precubical sets

In this section we show that the cubical nerve of an ω-category is a complete stratified precubical set. Recall from [1] that the cubical nerve of an ω-category C is the sequence of sets

$$\hom(\nu I^0, C),\ \hom(\nu I^1, C),\ \ldots,$$

where νI^n is an ω-category associated to the n-cube. We will establish the result by studying the ω-categories νI^n.

We begin by recalling the general theory of ω-categories [8]. An ω-category C is a set with a sequence of compatible partially defined binary composition operations subject to various axioms; in particular each composition operation makes C into the set of morphisms for a category. We will use $\#_0$, $\#_1$, … to denote the composition operations, and we will write $d_p^- x$ and $d_p^+ x$ for the left and right identities of an element x under $\#_p$. We will use the following properties.

3.1. Proposition. *Let C be an ω-category. If $x \in C$ and $p < q$ then*

$$d_p^\alpha d_q^\beta x = d_q^\beta d_p^\alpha x = d_p^\alpha x;$$

if $x \#_q y$ is a composite and $p \neq q$ then

$$d_p^\alpha (x \#_q y) = d_p^\alpha x \#_q d_p^\alpha y;$$

the identities for $\#_p$ form a sub-ω-category $C(p)$ such that $C(p) \subset C(p+1)$.

We will use a construction of νI^n based on chain complexes ([7], Example 3.10). Recall that if K and L are augmented chain complexes, then their tensor product augmented chain complex $K \otimes L$ is given by

$$\begin{aligned}
&(K \otimes L)_q = \bigoplus_{i+j=q} (K_i \otimes L_j),\\
&\partial(x \otimes y) = \partial x \otimes y + (-1)^i x \otimes \partial y \text{ if } x \in K_i,\\
&\epsilon(x \otimes y) = (\epsilon x)(\epsilon y).
\end{aligned}$$

It is convenient to identify cubes with their chain complexes, as follows.

3.2. Definition. *The* standard interval *is the free augmented chain complex I concentrated in degrees* 0 *and* 1 *such that I_1 has basis u_1, such that I_0 has basis $\partial^- u_1, \partial^+ u_1$, such that $\partial u_1 = \partial^+ u_1 - \partial^- u_1$, and such that $\epsilon \partial^- u_1 = \epsilon \partial^+ u_1 = 1$. The* standard n-cube I^n *is the n-fold tensor product $I \otimes \ldots \otimes I$. The n-chain $u_1 \otimes \ldots \otimes u_1$ in I^n is denoted u_n.*

Thus the chain complex I^n is a free chain complex on a basis got by taking n-fold tensor products of the basis elements $u_1, \partial^- u_1, \partial^+ u_1$ for I; in particular I^0 is free on a single zero-dimensional basis element u_0. The elements of the tensor product basis for I^n are called standard basis elements. They correspond to the faces of a geometrical n-cube, or, equivalently, they correspond to the precubical operations. In fact there is an obvious embedding of the precubical category in the category of chain complexes as follows.

3.3. Proposition. *There is a functor from the precubical category to the category of chain complexes given on objects by $[n] \mapsto I^n$ and on morphisms by*

$$\check{\partial}_i^\alpha(x \otimes y) = x \otimes \partial^\alpha u_1 \otimes y \text{ for } x \in I^{i-1} \text{ and } y \in I^{n-i}.$$

The morphisms in the image of this functor are augmentation-preserving and take standard basis elements to standard basis elements.

We will also need subcomplexes corresponding to shells and boxes, as follows.

3.4. Definition. *The* standard n-shell S^n *is the subcomplex of I^n generated by the standard basis elements other than u_n. If σ is an $(n-1)$-dimensional basis element for I^n, then the* standard n-box $B(\sigma)$ *opposite σ is the subcomplex of I^n generated by the standard basis elements other than u_n and σ.*

We give partial orderings to the chain groups in the standard chain complexes by the rule that $x \geq y$ if and only if $x - y$ is a sum of standard basis elements. We then associate ω-categories to these complexes as follows.

3.5. Definition. *Let K be a standard cube, shell or box. The* associated ω-category νK *is the set of double sequences*

$$x = (x_0^-, x_0^+ \mid x_1^-, x_1^+ \mid \ldots),$$

where x_i^- *and* x_i^+ *are* i*-chains in* K *such that*

$$\begin{aligned}
&x_i^- \geq 0,\\
&x_i^+ \geq 0,\\
&\epsilon x_0^- = \epsilon x_0^+ = 1,\\
&x_i^+ - x_i^- = \partial x_{i+1}^- = \partial x_{i+1}^+.
\end{aligned}$$

The left identity $d_p^- x$ *and right identity* $d_p^+ x$ *of the double sequence*

$$x = (x_0^-, x_0^+ \mid x_1^-, x_1^+ \mid \ldots)$$

are given by

$$d_p^\alpha x = (x_0^-, x_0^+ \mid \ldots \mid x_{p-1}^-, x_{p-1}^+ \mid x_p^\alpha, x_p^\alpha \mid 0, 0 \mid \ldots).$$

If $d_p^+ x = d_p^- y = w$*, say, then the composite* $x \#_p y$ *is given by*

$$x \#_p y = x - w + y,$$

where the addition and subtraction are performed termwise.

For example, in νI^2 there are elements

$$a = (\check{\partial}_2^- \check{\partial}_1^- u_0, \check{\partial}_2^+ \check{\partial}_1^- u_0 \mid \check{\partial}_1^- u_1, \check{\partial}_1^- u_1 \mid 0, 0 \mid \ldots)$$

and

$$b = (\check{\partial}_2^+ \check{\partial}_1^- u_0, \check{\partial}_2^+ \check{\partial}_1^+ u_0 \mid \check{\partial}_2^+ u_1, \check{\partial}_2^+ u_1 \mid 0, 0 \mid \ldots)$$

such that

$$d_0^+ a = d_0^- b = (\check{\partial}_2^+ \check{\partial}_1^- u_0, \check{\partial}_2^+ \check{\partial}_1^- u_0 \mid 0, 0 \mid \ldots)$$

and

$$a \#_0 b = (\check{\partial}_2^- \check{\partial}_1^- u_0, \check{\partial}_2^+ \check{\partial}_1^+ u_0 \mid \check{\partial}_1^- u_1 + \check{\partial}_2^+ u_1, \check{\partial}_1^- u_1 + \check{\partial}_2^+ u_1 \mid 0, 0 \mid \ldots).$$

The elements a, b and $a \#_0 b$ correspond to the morphisms $\partial_1^- x$, $\partial_2^+ x$ and $\partial_1^- x \circ \partial_2^+ x$ in Figure 1.

Given a chain x of positive degree we will write $\partial^- x$ and $\partial^+ x$ for the negative and positive parts of ∂x; thus $\partial^- x$ and $\partial^+ x$ are the linear combinations of disjoint families of basis elements with positive integer coefficients such that $\partial x = \partial^+ x - \partial^- x$. This notation is of course consistent with the earlier use of $\partial^- u_1$ and $\partial^+ u_1$. We can now construct some specific double sequences of chains in I^n, which turn out to be members of νI^n, as follows.

3.6. Definition. *Let* σ *be a* p*-dimensional basis element for* I^n*. Then the associated* atom $\langle \sigma \rangle$ *is the double sequence given by*

$$\begin{aligned}
&\langle \sigma \rangle_q^\alpha = (\partial^\alpha)^{p-q} \sigma \text{ for } q \leq p,\\
&\langle \sigma \rangle_q^\alpha = 0 \text{ for } q > p.
\end{aligned}$$

For example

$$\begin{aligned}
\langle u_3\rangle_0^- &= \check\partial_3^-\check\partial_2^-\check\partial_1^- u_0,\\
\langle u_3\rangle_0^+ &= \check\partial_3^+\check\partial_2^+\check\partial_1^+ u_0,\\
\langle u_3\rangle_1^- &= \check\partial_2^-\check\partial_1^- u_1 + \check\partial_3^+\check\partial_1^- u_1 + \check\partial_3^+\check\partial_2^+ u_1,\\
\langle u_3\rangle_1^+ &= \check\partial_3^-\check\partial_2^- u_1 + \check\partial_3^-\check\partial_1^+ u_1 + \check\partial_2^+\check\partial_1^+ u_1,\\
\langle u_3\rangle_2^- &= \check\partial_1^- u_2 + \check\partial_2^+ u_2 + \check\partial_3^- u_2,\\
\langle u_3\rangle_2^+ &= \check\partial_3^+ u_2 + \check\partial_2^- u_2 + \check\partial_1^+ u_2,\\
\langle u_3\rangle_3^- &= \langle u_3\rangle_3^+ = u_3;
\end{aligned}$$

thus the terms of the chains $\langle u_3\rangle_i^\alpha$ correspond to the precubical operations on 3-cubes with standard decompositions $\partial_{i(1)}^{\alpha(1)}\ldots\partial_{i(p)}^{\alpha(p)}$ such that the signs $(-)^{i(r)-r}\alpha(r)$ are constant. These are the precubical operations which were called extreme in Section 2, and one can draw a 3-cube with the chains of $\langle u_3\rangle$ on its extremities; see Figure 4.

Figure 4 also illustrates complementary operations. For example, the operations complementary to ∂_1^- are id, ∂_2^+, ∂_3^-, $\partial_2^+\partial_3^+$. In Figure 4 the face corresponding to id (the 3-cube itself) joins $\check\partial_1^- u_2$ to $\langle u_3\rangle_2^+$; the faces corresponding to ∂_2^+ and ∂_3^- join $\check\partial_1^- u_2$ to $\langle u_3\rangle_1^+$; the face corresponding to $\partial_2^+\partial_3^+$ (the terminal edge in $\langle u_3\rangle_1^-$) joins $\check\partial_1^- u_2$ to $\langle u_3\rangle_0^+$.

The formulae for the chains $\langle u_3\rangle_i^\alpha$ extend to all dimensions. For atoms in general, using the definition of the boundary in a tensor product of chain complexes, we get the following formulae.

3.7. Proposition. *The atoms in νI are given by*

$$\begin{aligned}
\langle u_1\rangle_0^\alpha &= \partial^\alpha u_1,\\
\langle u_1\rangle_1^\alpha &= u_1,\\
\langle u_1\rangle_q^\alpha &= 0 \textit{ for } q>1,\\
\langle \partial^\beta u_1\rangle_0^\alpha &= \partial^\beta u_1,\\
\langle \partial^\beta u_1\rangle_q^\alpha &= 0 \textit{ for } q>0.
\end{aligned}$$

The atoms in νI^n are given by

$$\langle\sigma_1\otimes\ldots\otimes\sigma_n\rangle_q^\alpha = \sum_{i(1)+\ldots+i(n)=q}\langle\sigma_1\rangle_{i(1)}^\alpha\otimes\langle\sigma_2\rangle_{i(2)}^{(-)^{i(1)}\alpha}\otimes\ldots\otimes\langle\sigma_n\rangle_{i(n)}^{(-)^{i(1)+\ldots+i(n-1)}\alpha}.$$

From Proposition 3.7, if σ is a basis element in I^n then $\epsilon\langle\sigma\rangle_0^\alpha = 1$, and it follows that the atoms belong to νI^n. In fact they generate νI^n, and the standard ω-categories have presentations in terms of these generators as follows ([7], Theorem 6.1).

3.8. Theorem. *Let K be a standard cube, shell or box. Then νK is freely generated by the atoms corresponding to the basis elements of K subject to the following relations: if $\langle\sigma\rangle$ is an atom corresponding to a p-dimensional basis element σ, then*

$$d_p^-\langle\sigma\rangle = d_p^+\langle\sigma\rangle = \langle\sigma\rangle;$$

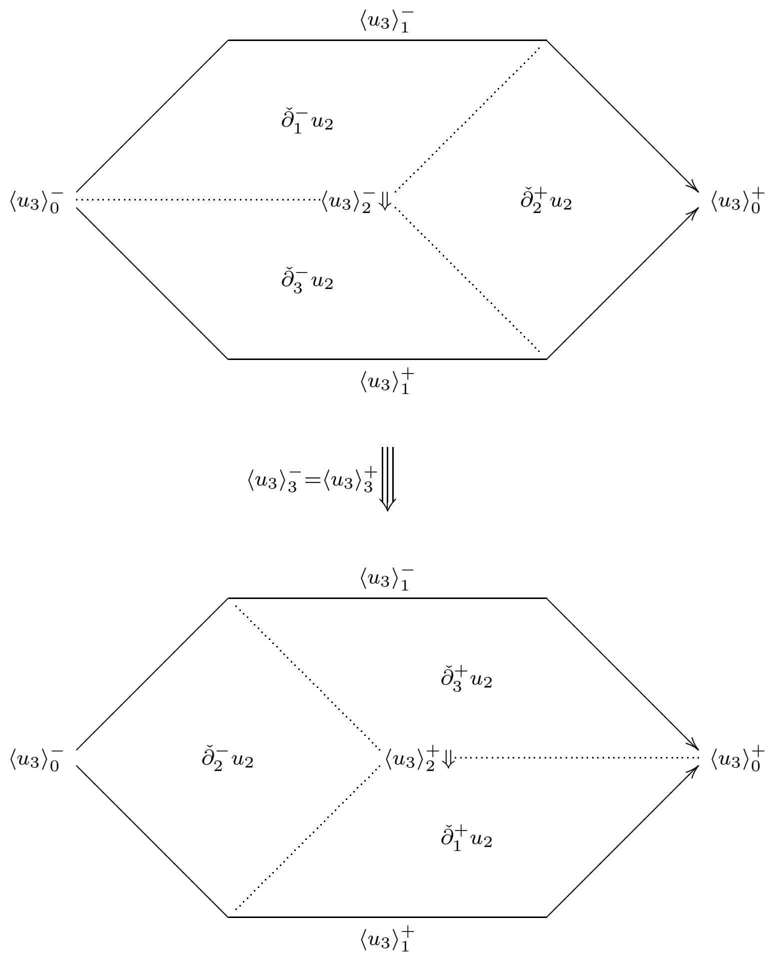

Figure 4: The chains $\langle u_3\rangle_i^\alpha$

if $\langle\sigma\rangle$ is an atom corresponding to a p-dimensional basis element σ with $p > 0$, then $d_{p-1}^-\langle\sigma\rangle = w^-$ and $d_{p-1}^+\langle\sigma\rangle = w^+$, where w^- and w^+ are suitable composites of lower-dimensional atoms.

For example, the presentation of νI^2 is essentially as illustrated in Figure 1. There are generators $\langle\check\partial_2^\beta\check\partial_1^\alpha u_0\rangle$ subject to relations $d_0^\gamma\langle\check\partial_2^\beta\check\partial_1^\alpha u_0\rangle = \langle\check\partial_2^\beta\check\partial_1^\alpha u_0\rangle$; there are generators $\langle\check\partial_1^\alpha u_1\rangle$ and $\langle\check\partial_2^\alpha u_1\rangle$ subject to relations

$$d_1^\gamma\langle\check\partial_i^\alpha u_1\rangle = \langle\check\partial_i^\alpha u_1\rangle,\ d_0^\gamma\langle\check\partial_1^\alpha u_1\rangle = \langle\check\partial_2^\gamma\check\partial_1^\alpha u_0\rangle,\ d_0^\gamma\langle\check\partial_2^\alpha u_1\rangle = \langle\check\partial_2^\alpha\check\partial_1^\gamma u_0\rangle;$$

there is a generator $\langle u_2\rangle$ subject to relations

$$d_2^\gamma\langle u_2\rangle = \langle u_2\rangle,\ d_1^-\langle u_2\rangle = \langle\check\partial_1^- u_1\rangle \#_0\langle\check\partial_2^+ u_1\rangle,\ d_1^+\langle u_2\rangle = \langle\check\partial_2^- u_1\rangle \#_0\langle\check\partial_1^+ u_1\rangle.$$

A more detailed description of the way in which the atoms generate νI^n is as follows ([7], Proposition 5.4).

3.9. Theorem. *Let*

$$x = (x_0^-, x_0^+ \mid x_1^-, x_1^+ \mid \ldots)$$

be an element of νI^n. Then x is an identity for $\#_p$ if and only if $x_q^- = x_q^+ = 0$ for $q > p$. If x is an identity for $\#_p$, then

$$x_p^- = x_p^+ = \sigma_1 + \ldots + \sigma_k$$

for some p-dimensional basis elements σ_i, and x is a composite of the atoms $\langle\sigma_i\rangle$ together with atoms of lower dimension.

3.10. Example. In νI^3 the element

$$d_2^-\langle u_3\rangle = (\langle u_3\rangle_0^-, \langle u_3\rangle_0^+ \mid \langle u_3\rangle_1^-, \langle u_3\rangle_1^+ \mid \langle u_3\rangle_2^-, \langle u_3\rangle_2^- \mid 0, 0 \mid \ldots),$$

is an identity for $\#_2$ with 2-chain component

$$\langle u_3\rangle_2^- = \check\partial_1^- u_2 + \check\partial_2^+ u_2 + \check\partial_3^- u_2$$

and with decomposition

$$(\langle\check\partial_1^- u_2\rangle \#_0\langle\check\partial_3^+\check\partial_2^+ u_1\rangle) \#_1(\langle\check\partial_3^-\check\partial_1^- u_1\rangle \#_0\langle\check\partial_2^+ u_2\rangle) \#_1(\langle\check\partial_3^- u_2\rangle \#_0\langle\check\partial_2^+\check\partial_1^+ u_1\rangle).$$

There is a corresponding decomposition of the top half of Figure 4, which is shown in Figure 5. Similarly, there is a decomposition of $d_2^+\langle u_3\rangle$ given by

$$(\langle\check\partial_2^-\check\partial_1^- u_1\rangle \#_0\langle\check\partial_3^+ u_2\rangle) \#_1(\langle\check\partial_2^- u_2\rangle \#_0\langle\check\partial_3^+\check\partial_1^+ u_1\rangle) \#_1(\langle\check\partial_3^-\check\partial_2^- u_1\rangle \#_0\langle\check\partial_1^+ u_2\rangle).$$

Recall the functor from the precubical category to the category of chain complexes given in Proposition 3.3. Since the morphisms in the image are augmentation-preserving and take standard basis elements to standard basis elements, they induce morphisms between the ω-categories νI^n; thus there is a functor $[n] \mapsto \nu I^n$ from the precubical category to the category of ω-categories. The functors $\hom(\nu I^n, -)$ therefore produce a functor from ω-categories to precubical sets. In fact we get a functor from ω-categories to stratified precubical sets, as follows.

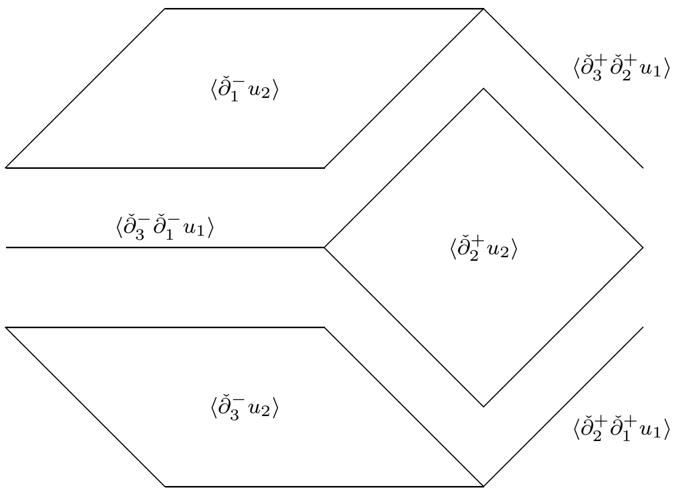

Figure 5: The decomposition of $d_2^-\langle u_3\rangle$

3.11. Definition. *The* cubical nerve *of an ω-category C is the stratified precubical set X given by*

$$X_n = \hom(\nu I^n, C),$$

where an n-cube $x\colon \nu I^n \to C$ with $n > 0$ is thin if $x\langle u_n\rangle$ is an identity for $\#_{n-1}$.

From the structure of the ω-categories associated to the standard complexes (Theorem 3.8), one sees that νS^n is generated by $2n$ copies of νI^{n-1} corresponding to the $(n-1)$-dimensional faces of an n-cube, subject to relations corresponding to their pairwise intersections, and similarly for boxes. This gives the following result.

3.12. Proposition. *Let C be an ω-category. Then the n-shells in the cubical nerve of C correspond to the morphisms $\nu S^n \to C$, and the n-boxes opposite ∂_k^γ correspond to the morphisms $\nu B(\check\partial_k^\gamma u_{n-1}) \to C$. A filler for an n-shell s is an n-cube x such that $x|\nu S^n = s$, and similarly for boxes.*

It remains to show that the cubical nerve of an ω-category has thin fillers satisfying the conditions for a complete stratification. For $n > 0$ it follows from Theorem 3.8 that νI^n is got from νS^n by adjoining a single extra generator $\langle u_n\rangle$ subject to relations on the $d_n^\alpha\langle u_n\rangle$ and the $d_{n-1}^\alpha\langle u_n\rangle$. This gives us the following result on thin fillers for shells.

3.13. Proposition. *Let s be an n-shell with $n > 0$ in the cubical nerve of an ω-category. Then s has a thin filler if and only if $s(d_{n-1}^-\langle u_n\rangle) = s(d_{n-1}^+\langle u_n\rangle)$. If s does have a thin filler, then this thin filler is unique.*

PROOF. From Theorem 3.8, the fillers x of s correspond to elements $x\langle u_n\rangle$ such that

$$\begin{aligned} d_n^- x\langle u_n\rangle &= d_n^+ x\langle u_n\rangle = x\langle u_n\rangle, \\ d_{n-1}^- x\langle u_n\rangle &= s(d_{n-1}^-\langle u_n\rangle), \\ d_{n-1}^+ x\langle u_n\rangle &= s(d_{n-1}^+\langle u_n\rangle). \end{aligned}$$

The filler x is thin if and only if $x\langle u_n\rangle$ is an identity for $\#_{n-1}$; thus x is thin if and only if it satisfies the additional relations

$$d_{n-1}^- x\langle u_n\rangle = d_{n-1}^+ x\langle u_n\rangle = x\langle u_n\rangle.$$

These additional relations actually imply that $d_n^- x\langle u_n\rangle = d_n^+ x\langle u_n\rangle = x\langle u_n\rangle$, because $d_n^\alpha d_{n-1}^\alpha = d_{n-1}^\alpha$ (see Proposition 3.1), so the defining relations for thin fillers reduce to

$$x\langle u_n\rangle = s(d_{n-1}^-\langle u_n\rangle) = s(d_{n-1}^+\langle u_n\rangle).$$

It follows that s has a unique thin filler if $s(d_{n-1}^- u_n) = s(d_{n-1}^+ u_n)$, and that s has no thin filler otherwise. This completes the proof. ∎

In Section 6 we will prove the following result.

3.14. THEOREM. *Let ∂_k^γ be a face operation on n-cubes. Then there is a factorisation*

$$\begin{aligned} &d_{n-1}^{(-)^{k-1}\gamma}\langle u_n\rangle \\ &= A_{n-1}^- \#_{n-2}(A_{n-2}^- \#_{n-3} \ldots (A_1^- \#_0\langle\check{\partial}_k^\gamma u_{n-1}\rangle \#_0 A_1^+) \ldots \#_{n-3} A_{n-2}^+) \#_{n-2} A_{n-1}^+ \end{aligned}$$

in νI^n such that the A_q^α are in $\nu B(\check{\partial}_k^\gamma u_{n-1})$ and such that $b(A_q^-)$ and $b(A_q^+)$ are identities for $\#_{q-1}$ whenever b is an admissible n-box opposite ∂_k^γ.

For example, consider the operation ∂_2^+ on 3-cubes. From Example 3.10 we get

$$d_2^-\langle u_3\rangle = A_2^- \#_1(A_1^- \#_0\langle\check{\partial}_2^+ u_2\rangle \#_0 A_1^+) \#_1 A_2^+,$$

where

$$\begin{aligned} A_2^- &= \langle\check{\partial}_1^- u_2\rangle \#_0\langle\check{\partial}_3^+\check{\partial}_2^+ u_1\rangle, \\ A_2^+ &= \langle\check{\partial}_3^- u_2\rangle \#_0\langle\check{\partial}_2^+\check{\partial}_1^+ u_1\rangle, \\ A_1^- &= \langle\check{\partial}_3^-\check{\partial}_1^- u_1\rangle, \end{aligned}$$

and A_1^+ is an identity for $\#_0$. We see that the atomic factors of A_q^α have dimension at most q; we also see that the q-dimensional factors of the A_q^α correspond to the precubical operations ∂_1^-, ∂_3^- and $\partial_1^-\partial_3^-$, which are the non-identity precubical operations complementary to ∂_2^+. If b is an admissible 3-box opposite ∂_2^+, then $\partial_1^- b$, $\partial_3^- b$ and $\partial_1^-\partial_3^- b$ are thin, so $b\langle\tau\rangle$ is an identity for $\#_{q-1}$ whenever $\langle\tau\rangle$ is an atomic factor in A_q^α, and it follows from Proposition 3.1 that $b(A_q^\alpha)$ is an identity for $\#_{q-1}$ as claimed.

Assuming Theorem 3.14 we get the following result on admissible boxes.

3.15. Theorem. *Let b be an admissible n-box opposite ∂_k^γ. If s is an n-shell extending b then*

$$s\langle\check{\partial}_k^\gamma u_{n-1}\rangle = s(d_{n-1}^{(-)^{k-1}\gamma}\langle u_n\rangle).$$

If $n \geq 2$ then

$$b(d_{n-2}^\alpha\langle\check{\partial}_k^\gamma u_{n-1}\rangle) = d_{n-2}^\alpha b(d_{n-1}^{(-)^k\gamma}\langle u_n\rangle).$$

(Note here that $b(d_{n-1}^{(-)^k\gamma}\langle u_n\rangle)$ exists because, by Proposition 3.7, $\check{\partial}_k^\gamma u_{n-1}$ is not a term in $\langle u_n\rangle_{n-1}^{(-)^k\gamma}$.)

Proof. Let s be an n-shell extending b and apply s to the factorisation in Theorem 3.14. Since $s(A_q^\beta) = b(A_q^\beta)$ is an identity for $\#_{q-1}$, the factorisation collapses to the equality $s\langle\check{\partial}_k^\gamma u_{n-1}\rangle = s(d_{n-1}^{(-)^{k-1}\gamma}\langle u_n\rangle)$.

Now suppose that $n \geq 2$. By Proposition 3.1 and a collapse like that in the previous paragraph,

$$b(d_{n-2}^\alpha\langle\check{\partial}_k^\gamma u_{n-1}\rangle) = b(d_{n-2}^\alpha d_{n-1}^{(-)^{k-1}\gamma}\langle u_n\rangle).$$

By a further application of Proposition 3.1,

$$b(d_{n-2}^\alpha d_{n-1}^{(-)^{k-1}\gamma}\langle u_n\rangle) = b(d_{n-2}^\alpha d_{n-1}^{(-)^k\gamma}\langle u_n\rangle) = d_{n-2}^\alpha b(d_{n-1}^{(-)^k\gamma}\langle u_n\rangle);$$

therefore $b(d_{n-2}^\alpha\langle\check{\partial}_k^\gamma u_{n-1}\rangle) = d_{n-2}^\alpha b(d_{n-1}^{(-)^k\gamma}\langle u_n\rangle)$ as required. This completes the proof. ∎

We now get the main theorem of this section as follows.

3.16. Theorem. *The cubical nerve of a strict ω-category is a complete stratified precubical set.*

Proof. We will verify the conditions of Definition 2.13.

Let s be an admissible n-shell; we must show that s has a unique thin filler. By the definition of an admissible shell there are distinct non-complementary face operations ∂_k^γ and ∂_l^δ such that $s_k^\gamma = s_l^\delta$ and such that the boxes formed by removing s_k^γ or s_l^δ are admissible. Since ∂_k^γ and ∂_l^δ are not complementary we have $(-)^k\gamma \neq (-)^l\delta$ (see Remark 2.8). Of the expressions $s(d_{n-1}^-\langle u_n\rangle)$ and $s(d_{n-1}^+\langle u_n\rangle)$ it follows from Theorem 3.15 that one is equal to $s\langle\check{\partial}_k^\gamma u_{n-1}\rangle$ and the other is equal to $s\langle\check{\partial}_l^\delta u_{n-1}\rangle$; in other words, one of them is $s_k^\gamma\langle u_{n-1}\rangle$ and the other is $s_l^\delta\langle u_{n-1}\rangle$. But $s_k^\gamma = s_l^\delta$, so $s(d_{n-1}^-\langle u_n\rangle) = s(d_{n-1}^+\langle u_n\rangle)$. By Proposition 3.13, s has a unique thin filler.

Now let b be an admissible n-box opposite ∂_k^γ; we must show that b has a unique thin filler. Because of Proposition 3.13, the thin fillers of b correspond to n-shells s extending b such that

$$s(d_{n-1}^-\langle u_n\rangle) = s(d_{n-1}^+\langle u_n\rangle).$$

Equivalently, by Theorem 3.15, these are the n-shells s extending b such that

$$s\langle\check{\partial}_k^\gamma u_{n-1}\rangle = b(d_{n-1}^{(-)^k\gamma}\langle u_n\rangle).$$

Now it follows from Theorem 3.8 that νS^n is got from $\nu B(\check{\partial}_k^\gamma u_{n-1})$ by adjoining $\langle\check{\partial}_k^\gamma u_{n-1}\rangle$ and imposing certain relations. An n-shell s extending b is therefore uniquely determined by the value of $s\langle\check{\partial}_k^\gamma u_{n-1}\rangle$, and the possible values for $s\langle\check{\partial}_k^\gamma u_{n-1}\rangle$ are given by the following conditions: in all cases, we require

$$d_{n-1}^- s\langle\check{\partial}_k^\gamma u_{n-1}\rangle = d_{n-1}^+ s\langle\check{\partial}_k^\gamma u_{n-1}\rangle = s\langle\check{\partial}_k^\gamma u_{n-1}\rangle;$$

if $n \geq 2$ then we also require

$$d_{n-2}^\alpha s\langle\check{\partial}_k^\gamma u_{n-1}\rangle = b(d_{n-2}^\alpha\langle\check{\partial}_k^\gamma u_{n-1}\rangle).$$

Using Theorem 3.15 in the case $n \geq 2$, we see that these conditions are satisfied when $s\langle\check{\partial}_k^\gamma u_{n-1}\rangle = b(d_{n-1}^{(-)^k\gamma}\langle u_n\rangle)$. Therefore b has a unique thin filler.

Finally let b be an admissible n-box opposite ∂_k^γ such that all the $(n-1)$-cubes b_i^α are thin, and let x be the thin filler of b; we must show that $\partial_k^\gamma x$ is thin. Since x is thin we have

$$x(d_{n-1}^-\langle u_n\rangle) = d_{n-1}^- x\langle u_n\rangle = x\langle u_n\rangle = d_{n-1}^+ x\langle u_n\rangle = x(d_{n-1}^+\langle u_n\rangle),$$

and it then follows from Theorem 3.15 that

$$(\partial_k^\gamma x)\langle u_{n-1}\rangle = x(\check{\partial}_k^\gamma\langle u_{n-1}\rangle) = x(d_{n-1}^{(-)^{k-1}\gamma}\langle u_n\rangle) = x(d_{n-1}^{(-)^k\gamma}\langle u_n\rangle) = b(d_{n-1}^{(-)^k\gamma}\langle u_n\rangle).$$

Since the b_i^α are thin, $b\langle\tau\rangle$ is an identity for $\#_{n-2}$ whenever $\langle\tau\rangle$ is an atom in $\nu B(\check{\partial}_k^\gamma u_{n-1})$. By Theorem 3.8, these atoms generate $\nu B(\check{\partial}_k^\gamma u_{n-1})$, so $b(d_{n-1}^{(-)^k\gamma}\langle u_n\rangle)$ is an identity for $\#_{n-2}$ by Proposition 3.1. Therefore $(\partial_k^\gamma x)\langle u_{n-1}\rangle$ is an identity for $\#_{n-2}$, which means that $\partial_k^\gamma x$ is thin.

This completes the proof. ∎

4. From complete stratified precubical sets to ω-categories

Throughout this section, let X be a complete stratified precubical set. We will show that X is the cubical nerve of an ω-category by constructing degeneracies, connections and compositions with the properties of [1].

The degeneracies are to be operations

$$\epsilon_1, \ldots, \epsilon_n \colon X_{n-1} \to X_n,$$

and we will define $\epsilon_k x$ for $x \in X_{n-1}$ as the unique thin filler of an admissible n-shell; in other words, $\epsilon_k x$ is a thin n-cube with prescribed values for the faces $\partial_i^\alpha \epsilon_k x$. The process is inductive: we define degeneracies on n-cubes in terms of degeneracies on lower-dimensional cubes.

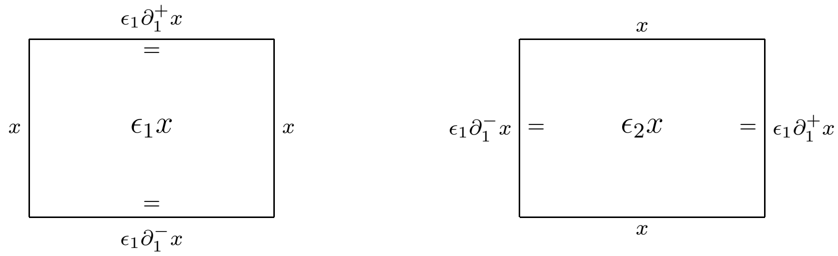

Figure 6: The degeneracies of a 1-cube

4.1. Definition. *The* degeneracies *are the elements $\epsilon_k x$, defined for $k = 1, 2, \ldots, n$ and $x \in X_{n-1}$, such that $\epsilon_k x$ is a thin member of X_n and*

$$\begin{aligned}&\partial_i^\alpha \epsilon_k x = \epsilon_{k-1}\partial_i^\alpha x \text{ for } i < k,\\ &\partial_k^\alpha \epsilon_k x = x,\\ &\partial_i^\alpha \epsilon_k x = \epsilon_k \partial_{i-1}^\alpha x \text{ for } i > k.\end{aligned}$$

The two degeneracies of a 1-cube x are shown in Figure 6, with the thin edges labelled by equality signs. Compare the first two shells in Figure 3.

To justify Definition 4.1, we must show that the prescribed values s_i^α for $\partial_i^\alpha \epsilon_k x$ form an admissible n-shell s. We will in fact show that s is of the type described in Example 2.11, using induction on n. Suppose that there are degeneracies with the required properties on m-cubes for $m < n-1$. Then we get $\partial_i^\alpha s_j^\beta = \partial_j^\beta s_{i+1}^\alpha$ for $i \geq j$ as follows: if $j \leq i < k-1$ then

$$\partial_i^\alpha s_j^\beta = \partial_i^\alpha \epsilon_{k-1}\partial_j^\beta x = \epsilon_{k-2}\partial_i^\alpha \partial_j^\beta x = \epsilon_{k-2}\partial_j^\beta \partial_{i+1}^\alpha x = \partial_j^\beta \epsilon_{k-1}\partial_{i+1}^\alpha x = \partial_j^\beta s_{i+1}^\alpha,$$

if $j \leq i = k-1$ then

$$\partial_i^\alpha s_j^\beta = \partial_i^\alpha \epsilon_{k-1}\partial_j^\beta x = \partial_j^\beta x = \partial_j^\beta s_{i+1}^\alpha,$$

etc. Therefore s is a shell. We also have $s_k^- = s_k^+$. If θ is a non-identity precubical operation on n-cubes whose standard decomposition does not contain ∂_k^- or ∂_k^+, say

$$\theta = (\partial_{i(1)}^{\alpha(1)} \ldots \partial_{i(p)}^{\alpha(p)})(\partial_{j(1)}^{\beta(1)} \ldots \partial_{j(q)}^{\beta(q)})$$

with $p + q > 0$ and with

$$i(1) < \ldots < i(p) < k < j(1) < \ldots < j(q),$$

then

$$\theta s = \epsilon_{k-p}(\partial_{i(1)}^{\alpha(1)} \ldots \partial_{i(p)}^{\alpha(p)})(\partial_{j(1)-1}^{\beta(1)} \ldots \partial_{j(q)-1}^{\beta(q)})x,$$

so θs is thin. It follows from Example 2.11 that s is an admissible shell, as required.

Connections are defined by a similar inductive process, with a similar inductive justification using Example 2.12.

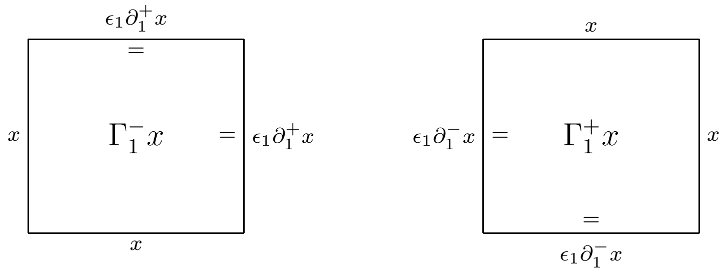

Figure 7: The connections of a 1-cube

4.2. Definition. *The* connections *are the elements* $\Gamma_k^- x$ *and* $\Gamma_k^+ x$*, defined for* $k = 1, 2, \ldots, n$ *and* $x \in X_n$*, such that* $\Gamma_k^\gamma x$ *is a thin member of* X_{n+1} *and*

$$\begin{aligned}
&\partial_i^\alpha \Gamma_k^\gamma x = \Gamma_{k-1}^\gamma \partial_i^\alpha x \text{ for } i < k,\\
&\partial_k^\gamma \Gamma_k^\gamma x = \partial_{k+1}^\gamma \Gamma_k^\gamma x = x,\\
&\partial_k^{-\gamma} \Gamma_k^\gamma x = \partial_{k+1}^{-\gamma} \Gamma_k^\gamma x = \epsilon_k \partial_k^{-\gamma} x,\\
&\partial_i^\alpha \Gamma_k^\gamma x = \Gamma_k^\gamma \partial_{i-1}^\alpha x \text{ for } i > k+1.
\end{aligned}$$

The two connections of a 1-cube x are shown in Figure 7; compare the last two shells in Figure 3.

We will get composites $x \circ_k y$ as the additional faces $\partial_k^- G_k(x, y)$ of thin fillers of admissible boxes opposite ∂_k^-. These thin fillers are called composers, and are again defined inductively. The definitions are as follows.

4.3. Definition. *The* composers *are the elements* $G_k(x, y)$*, defined for* $k = 1, 2, \ldots, n$ *and* $x, y \in X_n$ *with* $\partial_k^+ x = \partial_k^- y$*, such that* $G_k(x, y)$ *is a thin member of* X_{n+1} *and*

$$\begin{aligned}
&\partial_i^\alpha G_k(x, y) = G_{k-1}(\partial_i^\alpha x, \partial_i^\alpha y) \text{ for } i < k,\\
&\partial_k^+ G_k(x, y) = y,\\
&\partial_{k+1}^- G_k(x, y) = x,\\
&\partial_{k+1}^+ G_k(x, y) = \epsilon_k \partial_k^+ y,\\
&\partial_i^\alpha G_k(x, y) = G_k(\partial_{i-1}^\alpha x, \partial_{i-1}^\alpha y) \text{ for } i > k+1.
\end{aligned}$$

The composites *are the elements*

$$x \circ_k y = \partial_k^- G_k(x, y) \in X_n,$$

defined for $k = 1, 2, \ldots, n$ *and* $x, y \in X_n$ *with* $\partial_k^+ x = \partial_k^- y$.

The case $k = n = 1$ is shown in Figure 8; compare Figure 2. Note that $\partial_k^- G_k(x, y)$ is not specified in the definition of $G_k(x, y)$, because we are dealing with a box b opposite ∂_k^-. The justification of this definition is as before; in particular θb is thin if θ is a non-identity

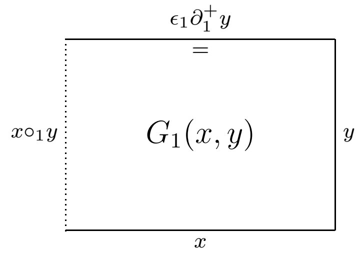

Figure 8: The composer of two 1-cubes

precubical operation whose standard decomposition has no factors ∂_k^-, ∂_k^+ or ∂_{k+1}^-, so b is admissible by Example 2.9.

It remains to verify that X satisfies the axioms for a cubical nerve as given in [1]. We begin with the following observation.

4.4. Proposition. *If $x\circ_k y$ is a composite such that x and y are thin, then $x\circ_k y$ is thin.*

Proof. If x and y are thin n-cubes then all the n-cubes in the admissible box defining $G_k(x,y)$ are thin, so the additional face $x\circ_k y = \partial_k^- G_k(x,y)$ is also thin. ■

4.5. Proposition. *The degeneracies have the property that*

$$\epsilon_k\epsilon_l x = \epsilon_{l+1}\epsilon_k x \textit{ for } k\leq l.$$

Proof. The proof is by induction on n, where x is an n-cube. Using the inductive hypothesis, we find that $\partial_i^\alpha\epsilon_k\epsilon_l x = \partial_i^\alpha\epsilon_{l+1}\epsilon_k x$ for all ∂_i^α. This means that $\epsilon_k\epsilon_l x$ and $\epsilon_{l+1}\epsilon_k x$ are fillers for the same shell s, and in fact they are thin fillers for s. But s is admissible (it is the admissible shell used to define $\epsilon_k\epsilon_l x$), so it has a unique thin filler. Therefore $\epsilon_k\epsilon_l x = \epsilon_{l+1}\epsilon_k x$. ■

4.6. Proposition. *The connections have the following properties*:

$$\begin{aligned}
&\Gamma_k^\gamma\epsilon_l x = \epsilon_{l+1}\Gamma_k^\gamma x \textit{ for } k<l,\\
&\Gamma_k^\gamma\epsilon_k x = \epsilon_{k+1}\epsilon_k x,\\
&\Gamma_k^\gamma\epsilon_l x = \epsilon_l\Gamma_{k-1}^\gamma x \textit{ for } k>l,\\
&\Gamma_k^\gamma\Gamma_l^\delta x = \Gamma_{l+1}^\delta\Gamma_k^\gamma x \textit{ for } k<l,\\
&\Gamma_k^\gamma\Gamma_k^\gamma x = \Gamma_{k+1}^\gamma\Gamma_k^\gamma x.
\end{aligned}$$

Proof. Similar. ■

4.7. Proposition. *If $x\circ_k y$ is defined, then*

$$\begin{aligned}
&\partial_i^\alpha(x\circ_k y) = \partial_i^\alpha x\circ_{k-1}\partial_i^\alpha y \textit{ for } i<k,\\
&\partial_k^-(x\circ_k y) = \partial_k^- x,\\
&\partial_k^+(x\circ_k y) = \partial_k^+ y,\\
&\partial_i^\alpha(x\circ_k y) = \partial_i^\alpha x\circ_k\partial_i^\alpha y \textit{ for } i>k.
\end{aligned}$$

PROOF. This follows straightforwardly from the definition: if $i < k$ then $\partial_i^\alpha(x \circ_k y) = \partial_i^\alpha x \circ_{k-1} \partial_i^\alpha y$ because

$$\partial_i^\alpha \partial_k^- G_k(x, y) = \partial_{k-1}^- \partial_i^\alpha G_k(x, y) = \partial_{k-1}^- G_{k-1}(\partial_i^\alpha x, \partial_i^\alpha y),$$

etc. ■

4.8. PROPOSITION. *The composites have the properties*

$$\begin{aligned}
&\epsilon_k \partial_k^- x \circ_k x = x = x \circ_k \epsilon_k \partial_k^+ x,\\
&\Gamma_k^+ x \circ_k \Gamma_k^- x = \epsilon_{k+1} x,\\
&\Gamma_k^+ x \circ_{k+1} \Gamma_k^- x = \epsilon_k x.
\end{aligned}$$

PROOF. There are composers $G_k(\epsilon_k \partial_k^- x, x)$ and $G_k(x, \epsilon_k \partial_k^+ x)$ because

$$\partial_k^+ \epsilon_k \partial_k^- x = \partial_k^- x, \qquad \partial_k^+ x = \partial_k^- \epsilon_k \partial_k^+ x.$$

An inductive argument shows that $\partial_i^\alpha G_k(\epsilon_k \partial_k^- x, x) = \partial_i^\alpha \epsilon_k x$ for $\partial_i^\alpha \neq \partial_k^-$, so that $\epsilon_k x$ is a thin filler for the admissible box whose unique thin filler is $G_k(\epsilon_k \partial_k^- x, x)$. Therefore $G_k(\epsilon_k \partial_k^- x, x) = \epsilon_k x$ (compare Figures 6 and 8). By a similar argument, $G_k(x, \epsilon_k \partial_k^+ x) = \Gamma_k^- x$ (compare Figures 7 and 8). Applying ∂_k^- now gives $\epsilon_k \partial_k^- x \circ_k x = x$ and $x \circ_k \epsilon_k \partial_k^+ x = x$.

It is clear that $\Gamma_k^+ x \circ_k \Gamma_k^- x$ exists. From Proposition 4.4 it is thin, and an inductive argument shows that it is a filler for the shell whose unique thin filler is $\epsilon_{k+1} x$. Therefore $\Gamma_k^+ x \circ_k \Gamma_k^- x = \epsilon_{k+1} x$. Similarly $\Gamma_k^+ x \circ_{k+1} \Gamma_k^- x = \epsilon_k x$. ■

4.9. PROPOSITION. *If $x \circ_k y$ is defined then*

$$\begin{aligned}
&\epsilon_j(x \circ_k y) = \epsilon_j x \circ_{k+1} \epsilon_j y \text{ for } j \leq k,\\
&\epsilon_j(x \circ_k y) = \epsilon_j x \circ_k \epsilon_j y \text{ for } j > k,\\
&\Gamma_j^\beta(x \circ_k y) = \Gamma_j^\beta x \circ_{k+1} \Gamma_j^\beta y \text{ for } j < k,\\
&\Gamma_k^-(x \circ_k y) = (\Gamma_k^- x \circ_k \epsilon_{k+1} y) \circ_{k+1} \Gamma_k^- y = (\Gamma_k^- x \circ_{k+1} \epsilon_k y) \circ_k \Gamma_k^- y,\\
&\Gamma_k^+(x \circ_k y) = \Gamma_k^+ x \circ_k (\epsilon_k x \circ_{k+1} \Gamma_k^+ y) = \Gamma_k^+ x \circ_{k+1} (\epsilon_{k+1} x \circ_k \Gamma_k^+ y),\\
&\Gamma_j^\beta(x \circ_k y) = \Gamma_j^\beta x \circ_k \Gamma_j^\beta y \text{ for } j > k.
\end{aligned}$$

PROOF. In each equality the first expression is defined as the unique thin filler of some shell and the other expressions are well-defined and thin. It therefore suffices to show that the other expressions are also fillers for the appropriate shells, and this is done by inductive arguments. ■

4.10. PROPOSITION. *If $k \neq l$, then*

$$(x \circ_k y) \circ_l (z \circ_k w) = (x \circ_l z) \circ_k (y \circ_l w)$$

whenever both sides are defined.

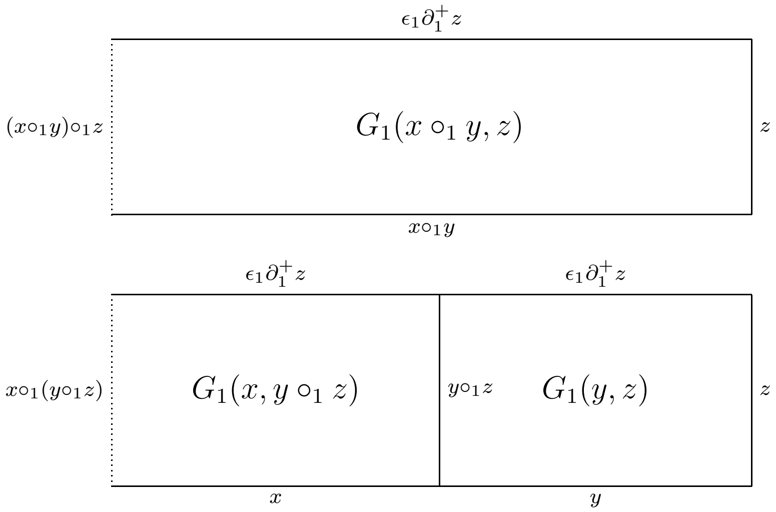

Figure 9: The equality $G_1(x \circ_1 y, z) = G_1(x, y) \circ_1 G_1(x, y \circ_1 z)$

PROOF. For definiteness, suppose that $k > l$. By an inductive argument one shows that

$$G_k(x, y) \circ_l G_k(z, w) = G_k(x \circ_l z, y \circ_l w);$$

indeed the composite on the left exists and is a thin filler for the box opposite ∂_k^- whose unique thin filler is the expression on the right. The result then follows by applying ∂_k^- to both sides. ■

4.11. PROPOSITION. *If $x, y, z \in X_n$ are such that $\partial_k^+ x = \partial_k^- y$ and $\partial_k^+ y = \partial_k^- z$, then*

$$(x \circ_k y) \circ_k z = x \circ_k (y \circ_k z).$$

PROOF. We first show that

$$G_k(x \circ_k y, z) = G_k(x, y \circ_k z) \circ_k G_k(y, z)$$

by the usual inductive argument: the expression on the right exists and is a thin filler for the admissible box opposite ∂_k^- whose unique thin filler is the expression on the left. The case $k = n = 1$ is shown in Figure 9. We then get $\partial_k^- G_k(x \circ_k y, z) = \partial_k^- G_k(x, y \circ_k z)$, which means that $(x \circ_k y) \circ_k z = x \circ_k (y \circ_k z)$ as required. ■

We have now verified all the axioms of [1], so we have proved the following result.

4.12. THEOREM. *If X is a complete stratified precubical set then the induced degeneracies, connections and compositions make X into the cubical nerve of an ω-category.*

5. The equivalence

We have shown that an ω-category nerve structure on a precubical set induces a complete stratification (Theorem 3.16) and that a complete stratification induces an ω-category nerve structure (Theorem 4.12). In this section we complete the proof of Theorem 2.14 by showing that the two processes are mutually inverse.

We begin by recalling some properties of nerves from [1].

5.1. Proposition. *Let X be the cubical nerve of an ω-category C, so that X has degeneracies, connections and compositions and an induced complete stratification. Then degeneracies and connections are thin, and composites of thin elements are thin.*

Proof. Let $C(n)$ be the sub-ω-category of C consisting of the elements which are identities for $\#_n$ (see Proposition 3.1), and let $X(n)$ be the nerve of $C(n)$; then $X(n)$ is a sub-precubical set of X closed under degeneracies, connections and compositions. We have $X_n = X(n)_n$, and the thin elements of X_n are precisely the members of $X(n-1)_n$. If an n-cube x is a degeneracy or connection of an $(n-1)$-cube y, then $y \in X_{n-1} = X(n-1)_{n-1}$, so $x \in X(n-1)_n$ and x is therefore thin. If an n-cube x is a composite of thin n-cubes, then x is thin because $X(n-1)$ is closed under composition. This completes the proof. ∎

5.2. Proposition. *Let X be the cubical nerve of an ω-category. Then there are operations*

$$\psi_1, \ldots, \psi_{n-1} \colon X_n \to X_n$$

such that

$$\begin{aligned}
&\psi_k x = \Gamma_k^+ \partial_{k+1}^- x \circ_{k+1} x \circ_{k+1} \Gamma_k^- \partial_{k+1}^+ x,\\
&\partial_{k+1}^\alpha \psi_k x = \epsilon_k \partial_k^\alpha \partial_{k+1}^\alpha x,\\
&\partial_i^\alpha \psi_k x = \psi_k \partial_i^\alpha x \textit{ for } i > k+1,\\
&\psi_k \epsilon_{k+1} x = \epsilon_k x,\\
&x = (\epsilon_k \partial_k^- x \circ_{k+1} \Gamma_k^+ \partial_{k+1}^+ x) \circ_k \psi_k x \circ_k (\Gamma_k^- \partial_{k+1}^- x \circ_{k+1} \epsilon_k \partial_k^+ x).
\end{aligned}$$

Proof. We know from [1] that X is a precubical set with operations satisfying the conditions of Definitions 4.1–4.3, Propositions 4.7–4.8 and Propositions 4.10–4.11. Using these conditions, it is easy to check that the composite on the right side of the first equality exists, and one can therefore use this equality to define $\psi_k x$. It is then straightforward to verify the next three equalities. As to the last equality, we have

$$\begin{aligned}
x &= x \circ_{k+1} \epsilon_{k+1} \partial_{k+1}^+ x\\
&= (\epsilon_k \partial_k^- x \circ_k x) \circ_{k+1} (\Gamma_k^+ \partial_{k+1}^+ x \circ_k \Gamma_k^- \partial_{k+1}^+ x)\\
&= (\epsilon_k \partial_k^- x \circ_{k+1} \Gamma_k^+ \partial_{k+1}^+ x) \circ_k (x \circ_{k+1} \Gamma_k^- \partial_{k+1}^+ x)
\end{aligned}$$

and

$$\begin{aligned} x \circ_{k+1} \Gamma_k^- \partial_{k+1}^+ x &= \epsilon_{k+1} \partial_{k+1}^- x \circ_{k+1} (x \circ_{k+1} \Gamma_k^- \partial_{k+1}^+ x) \\ &= (\Gamma_k^+ \partial_{k+1}^- x \circ_k \Gamma_k^- \partial_{k+1}^- x) \circ_{k+1} [(x \circ_{k+1} \Gamma_k^- \partial_{k+1}^+ x) \circ_k \epsilon_k \partial_k^+ x] \\ &= [\Gamma_k^+ \partial_{k+1}^- x \circ_{k+1} (x \circ_{k+1} \Gamma_k^- \partial_{k+1}^+ x)] \circ_k (\Gamma_k^- \partial_{k+1}^- x \circ_{k+1} \epsilon_k \partial_k^+ x) \\ &= \psi_k x \circ_k (\Gamma_k^- \partial_{k+1}^- x \circ_{k+1} \epsilon_k \partial_k^+ x), \end{aligned}$$

from which the result follows. ■

5.3. Proposition. *Let x be an n-cube in the cubical nerve of an ω-category with $n > 0$. Then there is an n-cube Ψx such that Ψx can be obtained by composing x with connections, such that x can be obtained by composing Ψx with degeneracies and connections, and such that*

$$\partial_i^\alpha \Psi x = \epsilon_1 \partial_{i-1}^\alpha \partial_1^+ \Psi x \text{ for } i > 1.$$

Proof. Let

$$\Psi x = \psi_1 \psi_2 \ldots \psi_{n-1} x,$$

where the ψ_k are as in Proposition 5.2. From the first equality in Proposition 5.2, Ψx is a composite of x with connections; from the last equality in Proposition 5.2, x is a composite of Ψx with degeneracies and connections. For $i > 1$, the middle equalities in Proposition 5.2 give

$$\begin{aligned} \partial_i^\alpha \Psi x &= \partial_i^\alpha (\psi_1 \ldots \psi_{i-2}) \psi_{i-1} (\psi_i \ldots \psi_{n-1}) x \\ &= (\psi_1 \ldots \psi_{i-2}) \partial_i^\alpha \psi_{i-1} (\psi_i \ldots \psi_{n-1}) x \\ &= (\psi_1 \ldots \psi_{i-2}) \epsilon_{i-1} \partial_{i-1}^\alpha \partial_i^\alpha (\psi_i \ldots \psi_{n-1}) x \\ &= \epsilon_1 \partial_{i-1}^\alpha \partial_i^\alpha (\psi_i \ldots \psi_{n-1}) x \\ &= \epsilon_1 y, \end{aligned}$$

say, we then get

$$y = \partial_1^+ \epsilon_1 y = \partial_1^+ \partial_i^\alpha \Psi x = \partial_{i-1}^\alpha \partial_1^+ \Psi x,$$

and we deduce that $\partial_i^\alpha \Psi x = \epsilon_1 y = \epsilon_1 \partial_{i-1}^\alpha \partial_1^+ \Psi x$. This completes the proof. ■

We now give the two results showing that the functors are mutually inverse.

5.4. Theorem. *Let X be a complete stratified precubical set. Then the stratification on X obtained from its structure as the cubical nerve of an ω-category is the same as the original stratification.*

Proof. We use the method of Higgins [6]. Let x be an n-cube in X with $n > 0$, and let Ψx be as in Proposition 5.3. From Propositions 5.1 and 5.3 we see that in the ω-category stratification x is thin if and only if Ψx is thin. But Proposition 5.1 is also true in the original stratification: degeneracies and connections are thin by construction, and composites of thin elements are thin by Proposition 4.4. Hence, in the original stratification, it also follows from Proposition 5.3 that x is thin if and only if Ψx is thin.

It therefore suffices to show that Ψx is thin in the ω-category stratification if and only if it is thin in the original stratification.

From Proposition 5.3 we see that $\partial_i^\alpha \Psi x = \partial_i^\alpha \epsilon_1 \partial_1^+ \Psi x$ for $i > 1$, and we also have $\partial_1^+ \Psi x = \partial_1^+ \epsilon_1 \partial_1^+ \Psi x$, so Ψx and $\epsilon_1 \partial_1^+ \Psi x$ are fillers for the same n-box b opposite ∂_1^-. In each of the stratifications degeneracies are thin, so b is admissible as in Example 2.9 (see the justification of Definition 4.1), and b therefore has a unique thin filler. Since degeneracies are thin in each stratification, the unique thin filler of b in each stratification is given by $\epsilon_1 \partial_1^+ \Psi x$. In each stratification, it follows that Ψx is thin if and only if $\Psi x = \epsilon_1 \partial_1^+ \Psi x$. Therefore Ψx is thin in the ω-category stratification if and only if it is thin in the original stratification. This completes the proof. ∎

5.5. Theorem. *Let X be the cubical nerve of an ω-category. Then the cubical nerve structure obtained from the induced stratification is the same as the original cubical nerve structure.*

Proof. We must show that the degeneracies, connections and compositions constructed from the stratification are the same as the original degeneracies, connections and compositions. Now the original degeneracies are thin by Proposition 5.1, so they satisfy the conditions of Definition 4.1, and it follows that they are the same as the degeneracies constructed from the stratification. The same argument applies to connections. As to compositions, let x and y be n-cubes such that $\partial_k^+ x = \partial_k^- y$ for some k. In the original structure one can check that there is a composite $\Gamma_k^- x \circ_{k+1} \epsilon_k y$, and this composite is thin by Proposition 5.1. By an inductive argument one finds that this composite satisfies the conditions for $G_k(x, y)$ in Definition 4.3; see Figure 10 for the case $n = k = 1$. One therefore gets

$$G_k(x, y) = \Gamma_k^- x \circ_{k+1} \epsilon_k y,$$

where the left side is the composer constructed from the stratification and the right side is the composite in the original structure. Applying ∂_k^- to both sides now shows that the composite $x \circ_k y$ constructed form the stratification is the same as the original composite $\partial_k^- \Gamma_k^- x \circ_k \partial_k^- \epsilon_k y = x \circ_k y$. This completes the proof. ∎

6. Proof of Theorem 3.14

Let ∂_k^γ be a face operation on n-cubes; we must construct a factorisation of $d_{n-1}^{(-)^{k-1}\gamma}\langle u_n \rangle$ in νI^n with certain properties. We will write

$$\sigma = \check{\partial}_k^\gamma u_{n-1} = u_{k-1} \otimes \partial^\gamma u_1 \otimes u_{n-k},$$

so that an n-box opposite ∂_k^γ is a morphism on $B(\sigma)$.

Consider the self-map of a geometric n-cube got by projection onto the face corresponding to σ. It is a cellular map cellularly homotopic to the identity, so it induces a chain endomorphism of I^n chain homotopic to the identity. To be explicit, we get the following two results.

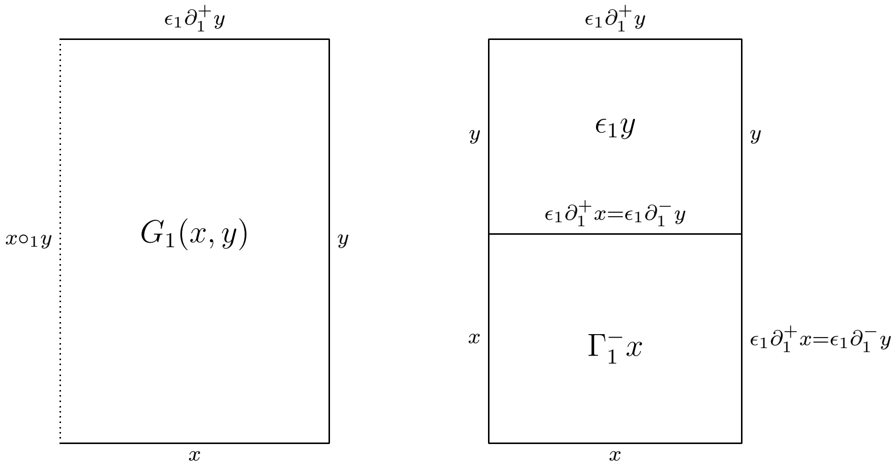

Figure 10: The equality $G_1(x,y) = \Gamma_1^- x \circ_2 \epsilon_1 y$

6.1. Proposition. *There is a chain map $f\colon I^n \to I^n$ given for x a chain in I^{k-1} and for y a chain in I^{n-k} by*

$$f(x \otimes u_1 \otimes y) = 0,$$
$$f(x \otimes \partial^\gamma u_1 \otimes y) = f(x \otimes \partial^{-\gamma} u_1 \otimes y) = x \otimes \partial^\gamma u_1 \otimes y,$$

such that

$$f\langle u_n\rangle_i^\alpha = \langle\sigma\rangle_i^\alpha.$$

Proof. It is straightforward to check that there is a chain map f as given, and it follows from Proposition 3.7 that $f\langle u_n\rangle_i^\alpha = \langle\sigma\rangle_i^\alpha$. ∎

6.2. Proposition. *There are abelian group homomorphisms $D\colon I^n \to I^n$ of degree 1, given for x an i-chain in I^{k-1} and for y a chain in I^{n-k} by*

$$D(x \otimes u_1 \otimes y) = 0,$$
$$D(x \otimes \partial^\gamma u_1 \otimes y) = 0,$$
$$D(x \otimes \partial^{-\gamma} u_1 \otimes y) = -(-)^i\gamma(x \otimes u_1 \otimes y),$$

such that

$$\partial D + D\partial = \mathrm{id} - f.$$

Proof. This is a straightforward computation. ∎

The chain homotopy D is related to the precubical operations complementary to ∂_k^γ as follows.

6.3. Proposition. *The chains $D\langle u_n\rangle_{q-1}^+$ and $-D\langle u_n\rangle_{q-1}^-$ are sums of basis elements*

$$\breve\partial_{i(n-q)}^{\alpha(n-q)} \ldots \breve\partial_{i(1)}^{\alpha(1)} u_q$$

corresponding to precubical operations $\partial_{i(1)}^{\alpha(1)} \ldots \partial_{i(n-q)}^{\alpha(n-q)}$ complementary to ∂_k^γ.

Proof. This follows from Proposition 3.7, according to which $\langle u_n\rangle_{q-1}^{\alpha}$ is the sum of the basis elements

$$\check{\partial}_{i(n-q+1)}^{\alpha(n-q+1)}\dots\check{\partial}_{i(1)}^{\alpha(1)}u_q$$

such that $i(1)<i(2)<\dots<i(n-q+1)$ and such that $(-)^{i(r)-r}\alpha(r)=\alpha$ for all r. Applying D picks out the terms involving $\check{\partial}_k^{-\gamma}$, omits the factor $\check{\partial}_k^{-\gamma}$, and multiplies by α. This makes $D\langle u_n\rangle_{q-1}^{+}$ and $-D\langle u_n\rangle_{q-1}^{-}$ into sums of basis elements corresponding to precubical operations complementary to ∂_k^{γ}, as required. ■

The factors A_q^{β} of $d_{n-1}^{(-)^{k-1}\gamma}\langle u_n\rangle$ are defined as follows.

6.4. Proposition. *There are elements A_q^- and A_q^+ of νI^n for $1\le q\le n-1$ given as double sequences by the formulae*

$$\begin{aligned}
&(A_q^{\beta})_i^{\alpha}=\langle u_n\rangle_i^{\alpha}\ \textit{for } i<q-1,\\
&(A_q^{\beta})_{q-1}^{\beta}=\langle u_n\rangle_{q-1}^{\beta},\\
&(A_q^{\beta})_{q-1}^{-\beta}=(\mathrm{id}-\partial D)\langle u_n\rangle_{q-1}^{\beta},\\
&(A_q^{\beta})_q^{\alpha}=\beta D\langle u_n\rangle_{q-1}^{\beta},\\
&(A_q^{\beta})_i^{\alpha}=0\ \textit{for } i>q.
\end{aligned}$$

Proof. We must verify the conditions of Definition 3.5. Note that $(A_q^{\beta})_{q-1}^{-\beta}$ can be written in the form

$$(f+D\partial)\langle u_n\rangle_{q-1}^{\beta}=\langle\sigma\rangle_{q-1}^{\beta}+D\langle u_n\rangle_{q-2}^{+}-D\langle u_n\rangle_{q-2}^{-}$$

(interpret $\langle u_n\rangle_{-1}^{\alpha}$ as zero in the case $q=1$). Using Proposition 6.3 where necessary, we see that $(A_q^{\beta})_i^{\alpha}$ is a sum of i-dimensional basis elements in I^n. From the expressions in the statement of the proposition, it is easy to check that $\epsilon(A_q^{\beta})_0^{\alpha}=1$ and $(A_q^{\beta})_i^{+}-(A_q^{\beta})_i^{-}=\partial(A_q^{\beta})_{i+1}^{\alpha}$. This completes the proof. ■

We must now show that the elements A_q^{β} have the properties stated in Theorem 3.14. We begin with the following observation.

6.5. Proposition. *The elements A_q^{β} are members of $\nu B(\sigma)$ such that $b(A_q^{\beta})$ is an identity for $\#_{q-1}$ whenever b is an admissible n-box opposite ∂_k^{γ}.*

Proof. The A_q^{β} are members of $\nu B(\sigma)$ because the basis element σ is never a term in $(A_q^{\beta})_{n-1}^{\alpha}$ and because u_n is never a term in $(A_q^{\beta})_n^{\alpha}$.

Now let b be an admissible n-box opposite ∂_k^{γ}. By Proposition 6.3 and the definition of admissible box, $b\langle\tau\rangle$ is an identity for $\#_{q-1}$ whenever τ is a term in $\beta D\langle u_n\rangle_{q-1}^{\beta}$. But $(A_q^{\beta})_i^{\alpha}=0$ for $i>q$ and

$$(A_q^{\beta})_q^{-}=(A_q^{\beta})_q^{+}=\beta D\langle u_n\rangle_{q-1}^{\beta};$$

hence, by Theorem 3.9, A_q^{β} is a composite of atoms $\langle\tau\rangle$ such that τ is a term in $\beta D\langle u_n\rangle_{q-1}^{\beta}$ or τ has dimension less than q. It follows that $b(A_q^{\beta})$ is a composite of identities for $\#_{q-1}$. By Proposition 3.1, $b(A_q^{\beta})$ itself is an identity for $\#_{q-1}$.

This completes the proof. ■

Next we show how to compose the elements A_q^β.

6.6. Proposition. *There are elements A_q in νI^n for $0 \le q \le n-1$ given inductively by $A_0 = \langle\sigma\rangle$ and by*

$$A_q = A_q^- \#_{q-1} A_{q-1} \# A_q^+ \text{ for } 1 \le q \le n-1$$

such that

$$\begin{aligned}(A_q)_i^\alpha &= \langle u_n\rangle_i^\alpha \text{ for } i < q,\\ (A_q)_q^\alpha &= (\mathrm{id} - \partial D)\langle u_n\rangle_q^\alpha,\\ (A_q)_i^\alpha &= \langle\sigma\rangle_i^\alpha \text{ for } i > q.\end{aligned}$$

Proof. The proof is by induction. To begin, let $A_0 = \langle\sigma\rangle$; then A_0 is certainly a member of νI^n such that the $(A_0)_i^\alpha$ are as described, since

$$(\mathrm{id} - \partial D)\langle u_n\rangle_0^\alpha = (f + D\partial)\langle u_n\rangle_0^\alpha = f\langle u_n\rangle_0^\alpha = \langle\sigma\rangle_0^\alpha.$$

For the inductive step, suppose that there is a member A_{q-1} of νI^n as described. Then one finds that $d_{q-1}^+ A_q^- = d_{q-1}^- A_{q-1}$ and $d_{q-1}^+ A_{q-1} = d_{q-1}^- A_q^+$, so one can define A_q in νI^n as the composite

$$A_q = A_q^- \#_{q-1} A_{q-1} \# A_q^+ = A_q^- - d_{q-1}^- A_{q-1} + A_{q-1} - d_{q-1}^+ A_{q-1} + A_q^+.$$

It now follows from the inductive hypothesis that the $(A_q)_i^\alpha$ are as required; for the case $i = q$ note that

$$-D\langle u_n\rangle_{q-1}^- + \langle\sigma\rangle_q^\alpha + D\langle u_n\rangle_{q-1}^+ = (f + D\partial)\langle u_n\rangle_q^\alpha = (\mathrm{id} - \partial D)\langle u_n\rangle_q^\alpha.$$

This completes the proof. ∎

Finally, we complete the proof of Theorem 3.14 by proving the following result.

6.7. Proposition. *There is an equality*

$$A_{n-1} = d_{n-1}^{(-)^{k-1}\gamma}\langle u_n\rangle.$$

Proof. We have $(A_{n-1})_i^\alpha = \langle u_n\rangle_i^\alpha$ for $i < n-1$ and $(A_{n-1})_i^\alpha = \langle\sigma\rangle_i^\alpha = 0$ for $i > n-1$, from which it follows that

$$(A_{n-1})_{n-1}^+ - (A_{n-1})_{n-1}^- = \partial(A_{n-1})_n^- = 0.$$

It therefore suffices to show that $(A_{n-1})_{n-1}^{(-)^{k-1}\gamma} = \langle u_n\rangle_{n-1}^{(-)^{k-1}\gamma}$. But

$$(A_{n-1})_{n-1}^{(-)^{k-1}\gamma} = (\mathrm{id} - \partial D)\langle u_n\rangle_{n-1}^{(-)^{k-1}\gamma} = \langle u_n\rangle_{n-1}^{(-)^{k-1}\gamma}$$

because $u_{k-1} \otimes \partial^{-\gamma} u_1 \otimes u_{n-k}$ is not a term in $\langle u_n\rangle_{n-1}^{(-)^{k-1}\gamma}$. This completes the proof. ∎

7. A thin filler with a single non-thin face

In this section we exhibit a thin 3-cube x in the nerve of an ω-category such that x has exactly one non-thin 2-face; moreover, x is the thin filler of an admissible box. It complies with the final condition of Definition 2.13, because the non-thin face is $\partial_1^- x$ and the box is opposite a different operation ∂_2^-, but it shows that this condition in Definition 2.13 cannot be weakened.

We need an ω-category containing elements with certain properties.

7.1. Proposition. *There is an ω-category with elements A and b such that A is an identity for $\#_2$ but not for $\#_1$, such that b is an identity for $\#_1$, and such that $A \#_0 b$ exists and is an identity for $\#_1$.*

Proof. We take the ω-category to be the 2-category of small categories; thus the identities for $\#_0$ are the categories, the identities for $\#_1$ are the functors, the identities for $\#_2$ are the natural transformations, and every element is an identity for $\#_3$. Let A be a non-identity natural transformation and b be a functor such that $A \#_0 b$ exists and is an identity natural transformation. Then A and b have the required properties. ∎

7.2. Theorem. *There is an ω-category with a 3-cube x in its cubical nerve such that x is the thin filler of an admissible box opposite ∂_2^-, such that $\partial_1^- x$ is not thin, and such that $\partial_i^\alpha x$ is thin for $\partial_i^\alpha \neq \partial_1^-$.*

Proof. By Proposition 3.13, to construct the thin 3-cube x it suffices to construct a 3-shell s such that $s(d_2^-\langle u_3\rangle) = s(d_2^+\langle u_3\rangle)$. By Proposition 3.8, to construct the shell s, it suffices to assign ω-category elements to the atoms of dimension less than 3 in I^3 such that the restriction to each 2-face is as shown in Figure 1. Take an ω-category with elements A and b as in Proposition 7.1, let $a^- = d_1^- A$, and let $a^+ = d_1^+ A$. One can then check that there is a shell s as shown in Figure 11, where the equality signs denote identities for $\#_0$ and where the 2-faces are positioned as in Figure 4; that is,

$$\begin{gathered} x\langle \check{\partial}_1^- u_2\rangle = A,\ x\langle \check{\partial}_2^+ u_2\rangle = b,\ x\langle \check{\partial}_3^- u_2\rangle = a^+, \\ x\langle \check{\partial}_3^+ u_2\rangle = a^- \#_0 b,\ x\langle \check{\partial}_2^- u_2\rangle = A \#_0 b,\ x\langle \check{\partial}_1^+ u_2\rangle = a^+ \#_0 b. \end{gathered}$$

From the formulae for $d_2^-\langle u_3\rangle$ and $d_2^+\langle u_3\rangle$ in Example 3.10, we see that $s(d_2^-\langle u_3\rangle)$ and $s(d_2^+\langle u_3\rangle)$ are both equal to $A \#_0 b$, so s has a thin filler x. Since a^-, a^+, b and $A \#_0 b$ are identities for $\#_1$ and since A is not an identity for $\#_1$, it follows that $\partial_1^- x$ is the unique non-thin 2-face of x. In particular $\partial_1^+ x$ and $\partial_3^+ x$ are thin; so also is $\partial_1^+ \partial_3^+ x$, which is given by the common edge of $\partial_1^+ x$ and $\partial_3^+ x$. This means that x restricts to an admissible box opposite ∂_2^-, so x is the thin filler of an admissible box opposite ∂_2^-. This completes the proof. ∎

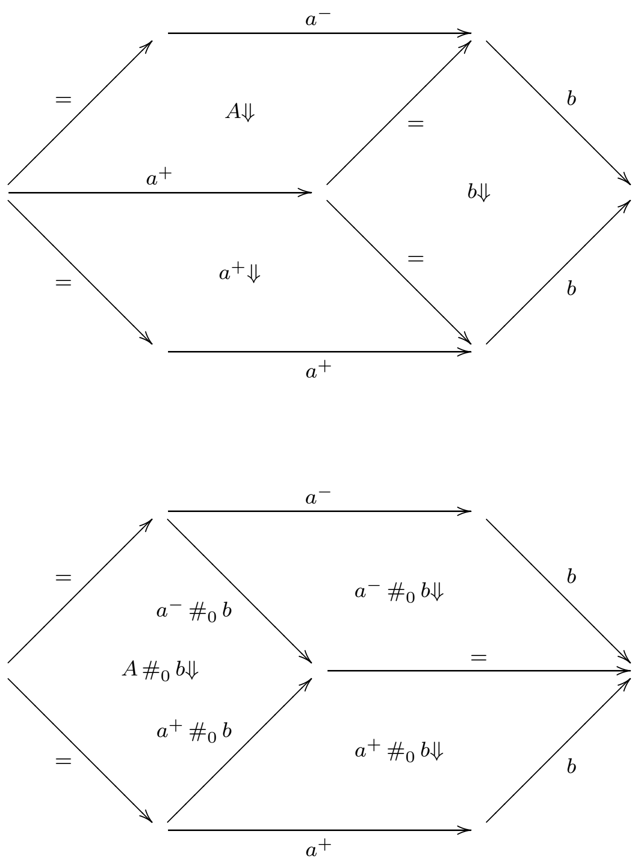

Figure 11: The shell s in the proof of Theorem 7.2

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