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DIALECTICA AND CHU CONSTRUCTIONS: COUSINS?

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ABSTRACT. This note investigates two generic constructions used to produce categorical models of linear logic, the Chu construction and the Dialectica construction, in parallel. The constructions have the same objects, but are rather different in other ways. We discuss similarities and differences and prove that the Dialectica construction can be done over a symmetric monoidal closed basis. We also point out interesting open problems concerning the Dialectica construction.

1. Introduction

Linear Logic [G87] has been much investigated using categorical methods. In particular two generic constructions, the Chu construction [Barr79, LS90] and the Dialectica construction [dP89a, dP89b] were used to provide general ways of building classes of (categorical) models for Linear Logic. The constructions themselves are similar in many ways, but different in significant others, so it is difficult to compare them abstractly. While the Dialectica construction has been mainly explored by the author ([dP89a, dP89b, dP91c]), the Chu construction has a much bigger following, with more than ten authors and dozens of papers written about it (see, for example, `http://chu.stanford.edu/guide.html`).

The goal of this paper is to bring out the similarities between Chu and Dialectica constructions, to encourage work on the (underprivileged) Dialectica construction. We start small: we show that the Dialectica construction can be done over a *symmetric monoidal closed* category. This is important, as it shows that the two constructions can be done in the same general setting. The Chu construction was originally given over a symmetric monoidal closed category. Many applications of the Chu construction use, instead of a symmetric monoidal closed category, the special case of a *cartesian closed* category, or even the rather special cartesian closed category **Sets**, but the original version, proposed by Barr and Chu in [Barr79] was the general one. Meanwhile all the *published* versions of Dialectica constructions, so far, are over cartesian closed categories.

It has been known for a while ([HdP91]) that the Dialectica construction can be done over a symmetric monoidal closed category, but there is no published account of it[1]. Given the main use of both Chu and Dialectica constructions as means of produc-

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[1]The way the generalization must be done can be deduced from work of Hyland and Schalk[S02] on categories of games, but reconstructing it would require a reader with considerable expertise on the

ing models of linear logic, the non-existence in print of the Dialectica construction over symmetric monoidal closed categories might give the reader the wrong impression that the construction requires cartesian closed structure of the basis. Moreover, this might give the impression that some untoward use of cartesian closedness happens within the Dialectica construction itself. The reader may have the (mistaken!) impression that so much structure was put into the basis of the construction, that it is obvious that one should get the desired symmetric monoidal closed structure at the end. This is not true, as we hope to demonstrate in this note.

The rest of this note is organized as follows. We first briefly recap what is now traditional material on categorical modelling of Linear Logic. Then we recall the easy cases of the Chu and Dialectica constructions over **Sets**, bringing out their similarities. Thirdly we introduce the new, mildly generalized, Dialectica construction over a symmetric monoidal closed category. Fourthly we describe two examples of application of the generalized construction and draw some conclusions.

2. Categorical Semantics of Linear Logic

It is always good to repeat that categorical semantics (as considered in this paper) models derivations (i.e. proofs) and not simply whether theorems are true or not. Thus instead of sending formulae (proposed theorems) to some truth-value, when doing categorical semantics we need to have a function that maps full Natural Deduction proofs (coded as terms of a suitable lambda-calculus) to morphisms in an appropriate category. Formulae of the calculus are mapped to objects of that category, and theorems, which are formulae provable from no assumptions, are a special case and get mapped to special morphisms.

This style of semantics of proofs, prototypically defined in [LS85] is usually associated with intuitionistic logics, formalized as Natural Deduction systems. This is the celebrated "extended Curry-Howard isomorphism". Linear Logic makes an interesting case for the extended Curry-Howard isomorphism. On the one hand, it was the first non-intuitionistic logic for which semantics of proofs was considered [See89, Bie95]. The point here is that Linear Logic has an involutive negation, which unlike the involutive negation of classical logic, does not collapse the category into a poset. On the other hand, a Natural Deduction formulation for Intuitionistic Linear Logic was harder to obtain than originally thought [BBHdP93, Bie95], as the first versions of natural deduction for intuitionistic linear logic were not closed under substitution. Moreover, for Classical Linear Logic, other formalisms, such as proof-nets, needed to be devised, in lieu of Natural Deduction.

Several other formulations, both of Natural deduction and of categorical models for intuitionistic linear logic have been discussed in the literature. The main issue here is how to best deal with what Girard calls the *exponentials*, which logically behave like S4 -modalities. Recent surveys are [MMPR01, Mel03, S04].

In a nutshell the situation, as far as categorical models of Linear Logic are concerned, is as follows:

subject.

1. To model the multiplicatives (except linear negation), we need a symmetric monoidal closed category (sometimes called an autonomous category or an smcc). This much was uncontroversial from the beginning.

2. To model multiplicatives and additives (except negation) we need a symmetric monoidal closed category plus categorical products and coproducts. (Here one might dispute the need for the uniqueness of products and coproducts, i.e. one might might prefer *weak* categorical products and coproducts.)

3. To model linear negation, we need a categorical involution. A symmetric monoidal closed category with an appropriate involution is called an $*$-autonomous category ([Barr79]). Categorical products and coproducts can be added at will.

4. To model modalities/exponentials we need a *linear exponential comonad* and a corresponding linear exponential monad. The term linear exponential comonad was coined by Hyland ([HS99]) and is a neat short-hand for all that is involved.

To be precise (cf. [MMPR01]) one could say that a "linear exponential comonad" is a monoidal comonad, whose category of Eilenberg-Moore coalgebras is cartesian. But then one would have to explain what is a "monoidal comonad", what is the "Eilenberg-Moore category of coalgebras" for a comonad and what this category being cartesian entails. That is, the conditions defining a linear exponential comonad are long to state and convoluted to explain. Thus we refer the interested reader to the recent surveys mentioned and try to indicate here simply the intuitions behind the definitions. To begin with, we want to model a unary logical operator such as the modality ! as a functor. More than simply a functor, given the special form of the rules that ! satisfies (for each object A we have maps $!A \to A$ and $!A \to !!A$), we need it to be a *comonad*. This comonad must respect the monoidal structure of contexts, represented by the the tensor product. Thus we say that the comonad required is *monoidal*. Objects to which the monoidal comonad ! is applied, i.e objects of the form $!A$ are special, in that they satisfy the logical rules of contraction and weakening, which are not valid for other formulae/objects of the system. So these objects must have uniform collections of morphisms of the form $\mathsf{er}\colon !A \to I$ (for erasing) and $\mathsf{dupl}\colon !A \to !A \otimes !A$ (for duplication). This means that these objects are *(commutative) comonoids*. Finally we have to make sure that the commutative comonoid structure of these $!A$ objects (which are co-free coalgebras, thanks to some traditional category theoretical results) interacts nicely with their co-free coalgebra structure. We actually pack a lot of information into this nice behaviour: every map of free coalgebras is also a map of comonoids. All these conditions give rise to neat commutative diagrams, that need to be checked. Other formulations ([Ben95, Bar96]) look simpler, but 'unpack' to similar diagrams.

Summing up, to model intuitionistic linear logic we need a symmetric monoidal closed category, with finite products and coproducts, equipped with a linear exponential comonad. To model classical linear logic we need a $*$-autonomous category, with finite products and a linear exponential comonad. The involution (already part of the *-autonomous structure) automatically provides the coproducts and the linear exponential monad, in the presence of the products and linear exponential comonad. But this is simply what is *required* for a model of Linear Logic. More interesting is to discuss "real-life" or "mathematical-life" examples of such models. This is where both the Chu and Dialectica constructions come into play.

3. The Original Constructions

The first point of similarity between Chu and the Dialectica constructions is that both produce models of Linear Logic, given an underlying category $\mathbf{C}$ and a given object Ω of $\mathbf{C}$ (In some of the literature this given, dualizing object is called $\perp$, but since it does not have to behave like a notion of falsity, we changed the notation to Ω, hoping no one will get it confused with a subobject classifier.) The second point of similarity is that both constructions can be seen as producing categories with the *same objects*, but whose morphisms are different, in an interesting way. Since morphisms are different, the categorical structures (of the categories obtained) and the properties of the constructions are different, which makes more surprising the fact that so many applications can be made "parallel", as it were.

To make matters concrete, in this background section, let us fix on the category **Sets** and on a particular set Ω, say $\mathbf{2}$, the two-element partially ordered set, where $0 \leq 1$. Since our motivation is logic, we think of 0 as meaning *false* and of 1 as meaning *true*.

On this fairly concrete case, both constructions, Dialectica (written as $\mathsf{Dial}_2(\mathbf{Sets})$) and Chu (written as $\mathsf{Chu}_2(\mathbf{Sets})$) give us categories which have as objects relations, that is, functions of the form[2] $\alpha\colon U \times X \to 2$. Since we want to distinguish strongly between the first and the second components of the relation, we write the function α as a triple $(U \overset{\alpha}{\longleftrightarrow} X)$, where U, X are sets and $\alpha\colon U \times X \to 2$ is a set-theoretic relation. But we talk about elements of the relation, so we say "when $u\alpha x$ is true", i.e. when $\alpha(u,x) = 1$ and "u does not alpha-relate to x", i.e. $\alpha(u,x) = 0$. As mentioned the main difference between the two constructions is the notion of morphism in each category, explained in the next two definitions.

3.1. Definition. [Chu construction] *The category* $\mathsf{Chu}_2(\mathbf{Sets})$ *has as objects triples* A *written as* $(U \overset{\alpha}{\longleftrightarrow} X)$*, where* U *and* X *are sets and* $\alpha\colon U \times X \to 2$ *is a set-theoretic relation. Given two objects* A *and* B*, where* B *is the triple* $(V \overset{\beta}{\longleftrightarrow} Y)$*, with* $\beta\colon V \times Y \to 2$*, a morphism in* $\mathsf{Chu}_2(\mathbf{Sets})$ *from* A *to* B *consists of a pair of functions* (f, F) *where* $f\colon U \to V$ *and* $F\colon Y \to X$ *(note the contravariance of the second coordinate) such that*

$$u\alpha F(y) \text{ iff } f(u)\beta y$$

[2]Note that in **Sets** we can think of relations as either subsets of the cartesian product, $\alpha \subseteq U \times X$ or as maps into $\mathbf{2}$, using the characteristic function of the relation.

Graphically we have that the diagram below commutes

$$\begin{tikzcd}
U \times Y \arrow[r, "U \times F"] \arrow[d, "f \times Y"'] & U \times X \arrow[d, "\alpha"] \\
V \times Y \arrow[r, "\beta"'] & 2
\end{tikzcd}$$

Pratt calls an object of $\mathsf{Chu}_2(\mathbf{Sets})$, a **(dyadic) Chu space**, a morphism of Chu spaces, a **Chu transform** and the condition that morphisms must satisfy the **adjointness condition**.

Clearly work has to be done to prove that this is really a category: one needs to define composition of morphisms and identities for each object and one needs to check that they behave as expected. But we refer the reader to the literature[Barr79] and concentrate on explaining the logical connection.

First, we hope that the infix notation in $u\alpha F(y)$ iff $f(u)\beta y$ does not cause problems: α is a relation between u's and x's and since $F(y)$ is an x, $u\alpha F(y)$ typechecks and it is either true (1) or false (0). Similarly for the β relation applied to u and y. In $f(u)\beta y$, $f(u)$ is an element of V, some given v, β is a relation between v's and y's. Thus for a given pair of functions (f, F) either for every pair of elements (u, y) whenever $u\alpha F(y)$ is 0, so is $f(u)\beta y$ and whenever $u\alpha F(y)$ is 1 so is $f(u)\beta y$, and we have a morphism, or there exists at least one pair (u, y) where they disagree and (f, F) is not a morphism.

Secondly and more importantly, logical bi-implication here can be seen as equality: $u\alpha F(y) = f(u)\beta y$ means that $u\alpha F(y)$ is less or equal $f(u)\beta y$ and $u\alpha F(y)$ is greater or equal $f(u)\beta y$. If we read the less or equal sign $\leq$ as a logical implication, $u\alpha F(y) \leq f(u)\beta y$ means "if $u\alpha F(y)$ then $f(u)\beta y$", where implication can be classical or intuitionistic, and the equality is simply logical bi-implication or "if and only if" as in the definition.

This is the bridge to the Dialectica categories. If we read the "less or equal" sign as intuitionistic implication and insist that morphisms consists only of logical implication and not logical bi-implication, we have (the main variant) of the Dialectica construction. This was introduced in [dP89b]. The formal definition is as follows.

3.2. Definition. [Dialectica construction] *The category* $\mathsf{Dial}_2(\mathbf{Sets})$ *has as objects A triples such as $(U \overset{\alpha}{\longleftrightarrow} X)$, where U and X are sets and $\alpha\colon U \times X \to 2$ is a relation. Given two objects A and B of* $\mathsf{Dial}_2(\mathbf{Sets})$*, where B is the triple $(V \overset{\beta}{\longleftrightarrow} Y)$, with $\beta\colon V \times Y \to 2$, a morphism in* $\mathsf{Dial}_2(\mathbf{Sets})$ *from A to B consists of a pair of functions (f, F) where $f\colon U \to V$ and $F\colon Y \to X$ (note the contravariance of the second coordinate) such that*

$$\textit{If } u\alpha F(y) \textit{ then } f(u)\beta y$$

Or, in other words, for all u in U, for all y in Y, $u\alpha F(y) \leq f(u)\beta y$. Graphically we have that, instead of commuting, the diagram has a two-cell.

$$\begin{tikzcd}
U \times Y \arrow[r, "U \times F"] \arrow[d, "f \times Y"'] \arrow[dr, phantom, "\geq"] & U \times X \arrow[d, "\alpha"] \\
V \arrow[r, "\beta"'] & 2
\end{tikzcd}$$

By analogy to Pratt's notation we call the inequality defining Dialectica morphisms the **semi-adjointness condition**. Of course, just as for Chu spaces, we also have to prove that the composition of these morphisms is well-defined, (it is simply composition in both coordinates, but we need to check that if (f, F) and (g, G) both satisfy these semi-adjointness condition, so does their composition $((f; g), (G; F))$; that identities exist (they are simply identities in each coordinate) and that composition and identities interact as expected. These are all easy calculations.

When reading presentations of this material [dP91a, dP89b], one might not realise that the original dialectica[3] category **GC** has exactly the same objets as $\mathsf{Chu}_2(\mathbf{Sets})$. This is because the original definition of the dialectica category **GC** talks about objects being monics $A \hookrightarrow U \times X$, instead of functions $U \times X \to 2$. Using monics has one advantage (the fact that you can talk about (un)decidable predicates), but using monics makes the mathematics more complicated and looses the connection with Chu spaces, which we are trying to emphasize here.

Before comparing the constructions, we pause to explain the name "dialectica". Gödel's Dialectica Interpretation [G58] is the origin of the name of the dialectica categories. The Dialectica (or functional) interpretation was Gödel's way of proving consistency of Heyting's Arithmetic, published in the journal "Dialectica" on a special issue dedicated to Bernay's 70th birthday. As Troelstra explains[G90], Gödel presented his result as a contribution to a liberalized version of Hilbert's program. His aim was to justify classical systems, in particular arithmetic, in terms of notions as intuitively clear as possible – in this case in terms of functionals of finite type. The connection between Gödel's interpretation and a different, but similar categorical construction, was first presented in [dP89a]. There a category D**Sets**, which is an internalized version of Gödel's dialectica interpretation is presented and shown to be a model of intuitionistic linear logic (ILL). For the sake of completeness we repeat the definition here.

3.3. Definition. *The category D**Sets** has as objects triples A of the form $(U \overset{\alpha}{\leftarrow\!\!\!+} X)$, where $\alpha\colon U \times X \to 2$. Given two objects A and B, where B is the triple $(V \overset{\beta}{\leftarrow\!\!\!+} Y)$, with $\beta\colon V \times Y \to 2$, a morphism in D**Sets** from A to B consists of a pair of functions*

[3]The name **GC** cames from 'Girard category' over **C**, in this case **Sets**.

(f, F) where $f\colon U \to V$ and $F\colon U \times Y \to X$ (not only this is contravariant on the second coordinate, but also it 'requests information' from the covariant side U) such that

$$\text{If } u\alpha F(u, y) \text{ then } f(u)\beta y$$

Or for all u and y, $u\alpha F(u, y) \leq f(u)\beta y$. Graphically we have that that the diagram has a two-cell.

$$\begin{tikzcd}
U \times Y \arrow[r, "\langle \pi_1, F \rangle"] \arrow[d, "\langle f, \pi_2 \rangle"'] \arrow[dr, phantom, "\geq"] & U \times X \arrow[d, "\alpha"] \\
V \arrow[r, "\beta"'] & 2
\end{tikzcd}$$

The dialectica morphisms in D**Sets** correspond precisely to the functionals implementing the interpretation of logical implication in Gödel's Dialectica. The fact that the category obtained also corresponds to a model of a version of linear logic was a pleasant surprise in 1987. This work is also discussed and improved in [H02]. Recently Shirahata proposed a dialectica interpretation of linear logic (instead of one for intuitionistic logic) based on this work, see Shirahata's contribution in this special volume. Also connections to other kinds of functional interpretations have been uncovered by Oliva [Oli07]. More work on functional interpretations and categorical models would be welcome though.

3.4. COMPARING $\mathsf{Chu}_2(\mathbf{Sets})$ AND $\mathsf{Dial}_2(\mathbf{Sets})$. The first main similarity between the two constructions is that they share the same objects. One first, big difference between the constructions has to do with the structure required to obtain the categories $\mathsf{Chu}_2(\mathbf{Sets})$ and $\mathsf{Dial}_2(\mathbf{Sets})$. As Pratt remarks on his lectures notes on Chu spaces[Pra99], as far as the Chu construction is concerned, the order on 2 and the consequent logical reading of equality as logical bi-implication is in the "eye of the beholder". For the Chu construction, whatever structure one has on Ω is invisible, while for the Dialectica construction this logical structure on Ω is essential. The logic associated to the structure of Ω is part of the logic one obtains on the end-product category; as the notation indicates, the logic of $\mathsf{Dial}_\Omega(\mathsf{C})$ is parametrized both by the logic of $\mathbf{C}$ and by the 'logic' of Ω, while the logic of $\mathsf{Chu}_\Omega(\mathsf{C})$ depends only on the logic of $\mathbf{C}$ and on the existence (of equality) on Ω. This dual dependency makes for flexibility of modelling using $\mathsf{Dial}_\Omega(\mathsf{C})$: getting a non-commutative version of the Dialectica construction ([dP91c]) was much easier than getting a non-commutative version of the Chu construction ([Barr96, Barr95]). But requiring a logical structure in Ω also means fewer examples.

Since we settled on the same representation for dialectica objects as for Chu spaces, one can define notions of *separable*, *extensional* and *biextensional* dialectica spaces just as defined for Chu spaces. We can also consider them as matrices and the operation of *transposition* of a dialectica space is a well-defined functor. But I know of no work that uses these restricted classes of objects of dialectica categories.

The first thing to note about comparing morphisms in the two constructions is that every map between objects A and B of $\mathsf{Chu}_2(\mathbf{Sets})$ is also a map between A and B of $\mathsf{Dial}_2(\mathbf{Sets})$, but $\mathsf{Dial}_2(\mathbf{Sets})$ has many more maps than $\mathsf{Chu}_2(\mathbf{Sets})$. This gives us a hint on how functions spaces in the two categories $\mathsf{Chu}_2(\mathbf{Sets})$ and $\mathsf{Dial}_2(\mathbf{Sets})$ will be related. But before working out similarities and differences between classes of morphisms let us compare the traditional categorical structure of the two constructions.

3.5. Additives in $\mathsf{Chu}_2(\mathbf{Sets})$ and $\mathsf{Dial}_2(\mathbf{Sets})$. Products and coproducts, a staple of category theory, are called the additive structure in the linear logic literature. Since the additive structure is a very easy one for both constructions, we start by comparing them.

Let us first note that the initial object $(0 = (0 \overset{\bullet}{\longleftrightarrow} 1))$ and the terminal object $(1 = (1 \overset{\bullet}{\longleftrightarrow} 0))$ exist in both Chu and Dialectica categories. Notice that the relations $\bullet$ play no role in these definitions, since $\bullet$ is supposed to be the unique map from the empty product (either 0×1 or 1×0) to 2. But while in the category **Sets** there is a single morphism from the empty set to 2, as objects of the categories $\mathsf{Chu}_2(\mathbf{Sets})$ and $\mathsf{Dial}_2(\mathbf{Sets})$ 0 and 1 are different objects, as objects are triples where the order counts.

Binary categorical products and coproducts also exist and coincide in $\mathsf{Chu}_2(\mathbf{Sets})$ and $\mathsf{Dial}_2(\mathbf{Sets})$, given, respectively by

$$\begin{aligned} A\&B &= (U \times V \xleftarrow{\alpha.\beta} X + Y) \\ A \oplus B &= (U + V \xleftarrow{\alpha.\beta} X \times Y) \end{aligned}$$

Again we do not need a combination $\alpha \cdot \beta$ of the relations α and β, as only one of the relations will be used once an element of $X+Y$ or $U+V$ is chosen. Given that the relations play almost no part on the definitions of products/coproducts and nullary versions, we have an easy proposition.

3.6. Proposition. *Both categories* $\mathsf{Chu}_2(\mathbf{Sets})$ *and* $\mathsf{Dial}_2(\mathbf{Sets})$ *have the same binary products and coproducts, including initial and terminal objects. Their additive structure is the same.*

3.7. Multiplicatives in $\mathsf{Chu}_2(\mathbf{Sets})$ and $\mathsf{Dial}_2(\mathbf{Sets})$. While in the case of Chu spaces, transposition corresponds directly to linear negation, in Dialectica spaces, to obtain linear negation we must do transposition *and* complementation of relations.

So while in $\mathsf{Chu}_2(\mathbf{Sets})$ given an object $A = (U \overset{\alpha}{\longleftrightarrow} X)$ its linear negation $A^{\perp}_{\mathsf{chu}}$ is $(X \overset{\alpha^t}{\longleftrightarrow} U)$, with the relation α simply transposed, in $\mathsf{Dial}_2(\mathbf{Sets})$ the negation of A, considered as a relation $u\alpha x$ is not simply $x\alpha u$, but $\neg(x\alpha u)$. In other words, $A^{\perp}_{\mathsf{dial}} = (X \overset{\alpha^*}{\longleftrightarrow} U)$ where $x\alpha^* u$ iff $\neg u\alpha x$, which seems a reasonable notion of negation. This also corresponds to considering linear negation as linear implication into (multiplicative) falsity, $A^{\perp} = A \multimap \perp$ where $\perp$ is unit for *par* ($\parr$) the (multiplicative) disjunction, as traditional in constructive logic.

Comparing the monoidal closed structure of the categories $\mathsf{Dial}_2(\mathbf{Sets})$ and $\mathsf{Chu}_2(\mathbf{Sets})$, the simplicity of the Dialectica construction shows up. The function space between objects A and B of $\mathsf{Dial}_2(\mathbf{Sets})$ is easily seen as the internalization of dialectica morphisms. Let us calculate it: we need to represent pairs of maps $f\colon U \to V$ and $F\colon Y \to X$, so that a special condition (the semi-adjointness condition) holds when these functions are applied to pairs of elements (u, y) of $U \times Y$. So we take the full set of maps from U to V, V^U, pair it with the full set of maps from Y to X, X^Y, and with the full set of pairs (u, y) in $U \times Y$. This gives us the domain of our function-space relation and the actual relation, which we call $\alpha{\multimap}\beta$ (as this is supposed to be a linear function space) does all the work. Thus $\alpha{\multimap}\beta\colon (V^U \times X^Y \times U \times Y) \to 2$ is the set of all functions f (of the form $f\colon U \to V$) and all functions F (of the form $F\colon Y \to X$) such that for all u in U and y in Y, $u\alpha F(y) \leq f(u)\beta y$.

By contrast in the category $\mathsf{Chu}_2(\mathbf{Sets})$ we need to take a pullback $\mathcal{P}_1$ in the first coordinate, which cuts down the set of functions (f, F) to the 'right size'. This pullback is defined by the following diagram:

$$\begin{tikzcd}
\mathcal{P}_1 \arrow[r] \arrow[d] & V^U \arrow[d, "\beta^U"] \\
X^Y \arrow[r, "\alpha^Y"'] & 2^{U\times Y}
\end{tikzcd}$$

this intuitively says that the pullback $\mathcal{P}_1$ consists of pairs of maps (h_1, h_2) of the form

$$\{(h_1, h_2) | h_1\colon U \to V, h_2\colon Y \to X \text{ such that } \alpha(u, h_2 y) = \beta(h_1 u, u)\}$$

Similarly for the tensor product. In the dialectica category $\mathsf{Dial}_2(\mathbf{Sets})$ all the work is done by the relation, while in the Chu construction a different pullback $\mathcal{P}_2$ needs to be taken. Thus

$$\begin{aligned}
A{\multimap}_{\mathsf{Chu}}B &= (\mathcal{P}_1 \xleftarrow{\alpha\multimap\beta} U \times Y) \\
A \otimes_{\mathsf{Chu}} B &= (U \times V \xleftarrow{\alpha\otimes\beta} \mathcal{P}_2)
\end{aligned}$$

for specific pullbacks $\mathcal{P}_1$ and $\mathcal{P}_2$, while for Dialectica

$$\begin{aligned}
A{\multimap}_{\mathsf{Dial}}B &= (V^U \times X^Y \xleftarrow{\alpha\multimap\beta} U \times Y) \\
A \otimes_{\mathsf{Dial}} B &= (U \times V \xleftarrow{\alpha\otimes\beta} Y^U \times X^V)
\end{aligned}$$

Comparing the tensor products $A \otimes_{\mathsf{Dial}} B$ and $A \otimes_{\mathsf{Chu}} B$ we see that there is a generic morphism, call is $m_{A,B} \colon A \otimes_{\mathsf{Dial}} B \to A \otimes_{\mathsf{Chu}} B$, as in the diagram

$$\begin{tikzcd}
U \times V \arrow[d, "\mathrm{id}_{U\times V}"'] & X^V \times Y^U \arrow[l, "\alpha \otimes \beta"', "|" marking] \\
U \times V & \mathcal{P}_2 \arrow[l, "\alpha \otimes \beta", "|" marking] \arrow[u, "\mathrm{inc}"']
\end{tikzcd}$$

the pullback $\mathcal{P}_2$ is contained in $X^V \times Y^U$. The units for tensor I_{Dial} and I_{Chu} also satisfy $I_{\mathsf{Dial}} \vdash I_{\mathsf{Chu}}$, as I_{Dial} is the object $(1 \overset{\mathrm{id}}{\leftarrow\!\!+\!\!-} 1)$ with the identity relation, while I_{Chu} is the object $(1 \leftarrow\!\!+\!\!- 2)$ with the relation the identity on 2, hence we obtain a morphism $m_I \colon I_{\mathsf{Dial}} \to I_{\mathsf{Chu}}$:

$$\begin{tikzcd}
1 \arrow[d, "\mathrm{id}_1"'] & 1 \arrow[l, "\mathrm{id}"', "|" marking] \\
1 & 2 \arrow[l, "\mathrm{id}", "|" marking] \arrow[u, "!_2"']
\end{tikzcd}$$

Similar to these two constructions the multiplicative disjunction, *par* can be given by the involution in $\mathsf{Chu}_2(\mathbf{Sets})$, as $A \parr_{\mathsf{Chu}} B = (A^{\perp} \otimes B^{\perp})^{\perp}$, or can be seen as a third pullback, when compared to $\mathsf{Dial}_2(\mathbf{Sets})$.

$$\begin{aligned}
A \parr_{\mathsf{Dial}} B &= (V^X \times U^Y \xleftarrow{\;\alpha \parr \beta\;} X \times Y) \\
A \parr_{\mathsf{Chu}} B &= (\mathcal{P}_3 \xleftarrow{\quad\alpha \parr \beta\quad} X \times Y)
\end{aligned}$$

So far, so good: both categories $\mathsf{Chu}_2(\mathbf{Sets})$ and $\mathsf{Dial}_2(\mathbf{Sets})$ have symmetric monoidal structures, modeling the multiplicatives (tensor $\otimes$, linear implication $\multimap$ and par $\parr$) and these structures are comparable. We have:

3.8. PROPOSITION. *The category* $\mathsf{Dial}_2(\mathbf{Sets})$ *is a symmetric monoidal closed category with an extra monoidal bifunctor which models the linear disjunction 'par' and its unit. Considering linear negation as implication into multiplicative falsum,* $\mathsf{Dial}_2(\mathbf{Sets})$ *models the multiplicative fragment of classical linear logic.*

The structure of $\mathsf{Dial}_2(\mathbf{Sets})$ is summarized as follows:

$$
\begin{array}{lcl}
A \multimap_{\mathsf{Dial}} B & = & (V^U \times Y^Y \xleftarrow{\alpha \multimap \beta} U \times Y) \\
A \otimes_{\mathsf{Dial}} B & = & (U \times V \xleftarrow{\alpha \otimes \beta} Y^U \times X^V) \\
I_{\mathsf{Dial}} & = & (1 \xleftarrow{\mathrm{id}} 1) \\
A \parr_{\mathsf{Dial}} B & = & (U^Y \times V^X \xleftarrow{\alpha \parr \beta} X \times Y) \\
\bot_{\mathsf{Dial}} & = & (1 \xleftarrow{\emptyset} 1)
\end{array}
$$

Both $\mathsf{Dial}_2(\mathbf{Sets})$ and $\mathsf{Chu}_2(\mathbf{Sets})$ are symmetric monoidal closed categories. They have the same objects and each morphism of $\mathsf{Chu}_2(\mathbf{Sets})$ is also a morphism of $\mathsf{Dial}_2(\mathbf{Sets})$. Moreover we have:

$$A \otimes_{\mathsf{Dial}} B \vdash A \otimes_{\mathsf{Chu}} B$$

$$I_{\mathsf{Dial}} \vdash I_{\mathsf{Chu}}$$

$$A \parr_{\mathsf{Chu}} B \vdash A \parr_{\mathsf{Dial}} B$$

$$\bot_{\mathsf{Chu}} \vdash \bot_{\mathsf{Dial}}$$

$$A \multimap_{\mathsf{Chu}} B \vdash A \multimap_{\mathsf{Dial}} B$$

This flip in the direction of entailments may seem odd, until we realize that this is exactly what is required to make the inclusion of Chu spaces into Dialectica a symmetric monoidal functor.

3.9. Proposition. *The inclusion of* $\mathsf{Chu}_2(\mathbf{Sets})$ *into* $\mathsf{Dial}_2(\mathbf{Sets})$ *is a symmetric monoidal functor* inc: $\mathsf{Chu}_2(\mathbf{Sets}) \rightarrow \mathsf{Dial}_2(\mathbf{Sets})$ *preserving the additive structure.*

Since the inclusion functor is the identity on both objects and morphisms of $\mathsf{Chu}_2(\mathbf{Sets})$, the morphisms $m_{A,B}\colon A \otimes_{\mathsf{Dial}} B \rightarrow A \otimes_{\mathsf{Chu}} B$ and $m_I\colon I_{\mathsf{Dial}} \rightarrow I_{\mathsf{Chu}}$ above provide us with the required morphisms $m_{A,B}$ and m_I to show the monoidicity[4] of the inclusion functor.

To show that we only have to check that the morphisms $m_{A,B}$ and m_I defined earlier satisfy the commuting diagrams associated with a symmetric *monoidal* functor. In particular we have that $m_{A,B}\colon \mathrm{inc}(A) \otimes_{\mathsf{Dial}} \mathrm{inc}(B) \rightarrow \mathrm{inc}(A \otimes_{\mathsf{Chu}} B)$ and $m_I\colon I_{\mathsf{Dial}} \rightarrow \mathrm{inc}(I_{\mathsf{Chu}})$ have to satisfy the four following diagrams:

[4]This way of phrasing the relationship between the categories was kindly pointed out by a referee.

$$\begin{tikzcd}
\mathrm{inc}(I)\otimes\mathrm{inc}(A) \arrow[r, "m_{I,A}"] \arrow[d, "m_I\otimes \mathrm{id}"'] & \mathrm{inc}(I\otimes A) \arrow[d, "\mathrm{inc}(\mathsf{left})"] \\
I\otimes\mathrm{inc}(A) \arrow[r, "\mathsf{left}"'] & \mathrm{inc}(A)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{inc}(A)\otimes\mathrm{inc}(I) \arrow[r, "m_{A,I}"] \arrow[d, "\mathrm{id}\otimes m_I"'] & \mathrm{inc}(A\otimes I) \arrow[d, "\mathrm{inc}(\mathsf{right})"] \\
\mathrm{inc}(A)\otimes I \arrow[r, "\mathsf{right}"'] & \mathrm{inc}(A)
\end{tikzcd}$$

$$\begin{tikzcd}
(\mathrm{inc}(A)\otimes in(B))\otimes\mathrm{inc}(C) \arrow[r, "m\otimes\mathrm{id}"] \arrow[d, "\mathsf{assoc}"'] & \mathrm{inc}(A\otimes B)\otimes\mathrm{inc}(C) \arrow[r, "m"] & \mathrm{inc}((A\otimes B)\otimes C) \arrow[d, "\mathrm{inc}(\mathsf{assoc})"] \\
\mathrm{inc}(A)\otimes(in(B)\otimes\mathrm{inc}(C)) \arrow[r, "\mathrm{id}\otimes m"'] & \mathrm{inc}(A)\otimes\mathrm{inc}(B\otimes C) \arrow[r, "m"'] & \mathrm{inc}(A\otimes(B\otimes C))
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{inc}(A)\otimes\mathrm{inc}(B) \arrow[r, "m_{A,B}"] \arrow[d, "\mathrm{inc}(f)\otimes\mathrm{inc}(g)"'] & \mathrm{inc}(A\otimes B) \arrow[d, "\mathrm{inc}(f\otimes g)"] \\
\mathrm{inc}(C)\otimes\mathrm{inc}(D) \arrow[r, "m_{C,D}"'] & \mathrm{inc}(C\otimes D)
\end{tikzcd}$$

The calculations are lengthy, but reasonably straightforward.

One might be tempted to say that $\mathsf{Dial}_2(\mathbf{Sets})$ has too many maps and $\mathsf{Chu}_2(\mathbf{Sets})$ requires too many pullbacks. When dealing with Petri nets (both categories have been used for this application), if one desires morphisms which behave like *simulations* the dialectica construction [BGdP91] seems appropriate. On the other hand, since one usually tries to avoid huge collections of morphisms, the more restrained Chu morphisms may seem a better choice [Gup94]. But the comparison turns trickier when exponentials (or modalities) enter the picture.

3.10. Modalities in $\mathsf{Chu}_2(\mathbf{Sets})$ and $\mathsf{Dial}_2(\mathbf{Sets})$. The Chu construction[Barr79], as a general way of building $*$-autonomous categories, predates Linear Logic by some eight years. But this original construction had nothing to say about the modalities ! and ? (or about the additives, categorical products and coproducts) at least to begin with. By contrast the dialectica construction [dP89a, dP89b] has, from the beginning, discussed modalities and their modeling, since the cartesian closed structure they provide was the initial goal.

Lafont and Streicher [LS90] produced modalities for the Chu construction based on the dialectica modalities of [dP89b]. The next proposition recalls and compares the definitions, for details see [dP91b].

3.11. PROPOSITION. *The modality* ! *in* $\mathsf{Dial}_2(\mathbf{Sets})$ *is given by the comonad, whose functor on objects is given by* $!_{\mathsf{Dial}}A = (U \overset{!\alpha}{\longleftrightarrow} (X^*)^U)$, *where the relation* $!_{\mathsf{Dial}}\alpha$ *applies to* u *in* U *and to a map* $\phi\colon U \to X^*$ *iff each component of* $\phi(u) = \langle x_1, \ldots, x_k\rangle$ *alpha relates to* u, *for for all* u *in* U. *Thus we have that* $\alpha(u, x_1)$ *and* $\alpha(u, x_2)$ *and* $\ldots$ *and* $\alpha(u, x_k)$.

The modality Lafont and Streicher define for $\mathsf{Chu}_2(\mathbf{Sets})$ *is given by the comonad whose functor on objects is given by* $!_{\mathsf{Chu}}A = (U \overset{!\alpha}{\longleftrightarrow} \mathcal{L}_3)$, *where* $\mathcal{L}_3$ *denotes the subset of the functions* ϕ *from* U *to* X^* *such that if* $\phi(u) = \langle x_1, \ldots, x_k\rangle$ *then* $\alpha(u, x_1) = \alpha(u, x_2) = \ldots = \alpha(u, x_k)$.

Following the same pattern of the multiplicatives, $\mathsf{Dial}_2(\mathbf{Sets})$ considers all functions and the relation $!\alpha$ cuts them down to size, while $\mathsf{Chu}_2(\mathbf{Sets})$ considers only the subset of the functions for which equality holds. It is easy to see that we have in $\mathsf{Dial}_2(\mathbf{Sets})$ a map κ_A

$$\begin{tikzcd}
U \arrow[d, "\mathrm{id}"'] & (X^*)^U \arrow[l, "!_{\mathsf{Dial}}\alpha"', "+" marking] \\
U & \mathcal{L}_3(A) \arrow[l, "!_{\mathsf{Chu}}\alpha", "+" marking] \arrow[u]
\end{tikzcd}$$

The class of morphisms above $\kappa_A\colon !_{\mathsf{Dial}}in(A) \to in(!_{\mathsf{Chu}}(A)$ is the main ingredient (a distributive law in Beck's sense) for what Hyland and Schalk call a *linearly distributive functor* in [HS99]. Since this notion is relatively recent, we recap:

3.12. DEFINITION. [Hyland and Schalk] *Given two categorical models of linear logic,* $\mathbf{C}$ *and* $\mathbf{D}$, *the functor* $F\colon \mathbf{C} \to \mathbf{D}$ *is a* linearly distributive functor *if and only if* F *is monoidal (with structure* $n_{\mathbf{I}}$ *and* $n_{C,C'}$*) and is equipped with a distributive law* $\lambda\colon !F \to F!$, *respecting the comonad structure in the sense that the following diagram commutes:*

$$\begin{tikzcd}
I \arrow[d, "n_I"'] & !F(C) \arrow[l, "e_{F(C)}"'] \arrow[r, "d_{F(C)}"] \arrow[d, "\lambda_C"'] & !F(C)\otimes !F(C) \arrow[d, "n_{!C,!C} \circ (\lambda_C \otimes \lambda_C)"] \\
F(I) & F(!C) \arrow[l, "F(e_C)"] \arrow[r, "F(d_C)"'] & F(!C\otimes !C)
\end{tikzcd}$$

We have:

3.13. PROPOSITION. *The inclusion functor in*: $\mathsf{Chu}_2(\mathbf{Sets}) \rightarrow \mathsf{Dial}_2(\mathbf{Sets})$ *taking objects onto themselves and* **Chu** *spaces morphisms onto themselves is a* linearly distributive functor*, hence an appropriate morphism of models of linear logic.*

To see that, it remains to check that the following diagrams commute:

$$\begin{tikzcd}
I_{\mathsf{Dial}} \arrow[d, "m_I"'] & !_{\mathsf{Dial}}(A) \arrow[l] \arrow[r] \arrow[d, "\kappa_A"'] & !_{\mathsf{Dial}}(A) \otimes !_{\mathsf{Dial}}(A) \arrow[d] \\
I_{\mathsf{Chu}} & !_{\mathsf{Chu}}(A) \arrow[l] \arrow[r] & !_{\mathsf{Chu}}(A) \otimes !_{\mathsf{Chu}}(A)
\end{tikzcd}$$

This means that we have to prove that the diagrams below commute:

$$\begin{tikzcd}
(1 \overset{\mathrm{id}}{\leftarrow\!\!\!+} 1) \arrow[d, "m_I"'] & (U \overset{!\alpha}{\leftarrow\!\!\!+} X^{*U}) \arrow[l] \arrow[r] \arrow[d, "\kappa_A"'] & (U \times U \overset{!\alpha \otimes !\alpha}{\leftarrow\!\!\!+} X^{*U \times U} \times X^{*U \times U}) \arrow[d] \\
(1 \overset{\mathrm{id}}{\leftarrow\!\!\!+} 2) & (U \overset{!\alpha}{\leftarrow\!\!\!+} \mathcal{L}_3(A)) \arrow[l] \arrow[r] & (U \times U \overset{!\alpha \otimes !\alpha}{\leftarrow\!\!\!+} \mathcal{L}_2(A \otimes A))
\end{tikzcd}$$

Again this is lengthy, but straightforward. This proposition is strong evidence for the notion of morphism of linear logic models proposed by Hyland and Schalk.

Barr has done extensive work([Barr91, Barr90, Barr96, Barr98]) on Chu categories from the perspective of models of Linear Logic. In [Barr90], for instance he proved that ! exists for $\mathsf{Chu}_{\Omega}(\mathsf{C})$ if $\mathbf{C}$ is symmetric monoidal closed and locally presentable and Ω is an internal cogenerator in $\mathbf{C}$ (thm 4.8, page 12). But in its generality, the paper gives us no direct construction for the operator !. In [Barr91] he proved the existence of ! for the subcategory of *separated objects* of $\mathbf{Chu}_{\Omega}\mathbf{C}$, when $\mathbf{C}$ is a cocomplete and complete cartesian closed category and Ω is an internal cogenerator (thm 9.2, page 19). In particular this holds when $\mathbf{C}$ is $\mathbf{Sets}$ and Ω is 2. But note the operative word *subcategory* here, the result applies only to the *separated* objects, not to the full category $\mathsf{Chu}_2(\mathbf{Sets})$.

In any case modalities, if they exist, are not unique in a category: so the fact that modalities can be made similar in $\mathsf{Chu}_2(\mathbf{Sets})$ and $\mathsf{Dial}_2(\mathbf{Sets})$ is interesting, but not that surprising. Also the fact that several constructions of ! are possible in Chu spaces only gives us plenty more of open problems to try the dialectica construction on.

4. A Generalized Dialectica Category

In this section we present the generalization of the category $\mathsf{Dial}_2(\mathbf{Sets})$ to $\mathsf{Dial}_{\Omega}(\mathcal{V})$, which is the goal of this paper. This generalization has, historically, proceeded by steps. In [dP91b] we described two constructions $\mathsf{Dial}_{\mathsf{N}}(\mathbf{Sets})$ (where the object 2 was generalised to $\mathbf{N}$ the set of natural numbers) and $\mathsf{Dial}_{\Omega}(\mathbf{C})$, where $\mathbf{C}$ is still a cartesian closed category[5], but instead of the object 2, we use a *lineale* Ω [dP02]. In this paper, we generalize from a category $\mathbf{C}$ cartesian closed to a category $\mathcal{V}$ symmetric monoidal closed, to give $\mathsf{Dial}_{\Omega}(\mathcal{V})$.

First of all our base category $\mathcal{V}$ has to be a symmetric monoidal closed category (Barr calls them autonomous categories) with finite categorical products. To settle notation we write the structure of $\mathcal{V}$ as $(\mathcal{V}, \otimes, \multimap, I, \times, 1)$. As mentioned before we need some structure on the chosen object Ω of $\mathbf{C}$, this needs to a *lineale* [dP02], which means that it has to be a monoidal closed poset. We write the lineale Ω as $(\Omega, \multimap, \cdot, \leq, e)$, where $\leq$ is the order of the poset, $\multimap$ is the internal hom of the poset, the dot $\cdot$ is its tensor product, and e the unit for the tensor product.

Now the objects of $\mathsf{Dial}_{\Omega}(\mathcal{V})$ are generalised relations between U and X or maps in $\mathcal{V}$ of the form $\alpha\colon U \otimes X \to \Omega$. The only (easy) modification being the use of the tensor product, instead of the product between components of the relation.

The maps of $\mathsf{Dial}_{\Omega}(\mathcal{V})$ are pairs of maps of $\mathcal{V}$, $f\colon U \to V$ and $F\colon Y \to X$ such that $u\alpha F(y) \leq f(u)\beta y$, where the $\leq$ sign should be thought of as a logical implication, inherited from Ω. Composition is as before, composition in each coordinate and associativity is inherited from the composition in $\mathcal{V}$ together with the fact that inequalities compose. Also identities are simply identities in each coordinate. Thus we have.

4.1. Proposition. *Given $\mathcal{V}$ a symmetric monoidal category with a lineale-like object Ω we can construct the category $\mathsf{Dial}_{\Omega}(\mathcal{V})$ whose objects are generalized relations $\alpha\colon U \otimes X \to \Omega$ and whose morphisms are generalized dialectica morphisms (f, F) where $f\colon U \to V$ and $F\colon Y \to X$ are maps in $\mathcal{V}$ satisfying the semi-adjointness condition, that is maps such that for all u in U and all y in Y, $u\alpha F(y) \leq_{\Omega} f(u)\beta y$.*

As before, proving that $\mathsf{Dial}_{\Omega}(\mathcal{V})$ really is a category requires checking quite a few diagrams, mostly found in [dP91b].

4.2. The logical structure of $\mathsf{Dial}_{\Omega}(\mathcal{V})$. Just as before let us start by checking categorical products and coproducts in $\mathsf{Dial}_{\Omega}(\mathcal{V})$. All works as expected as

$$A \times B = (U \times V \overset{\alpha \cdot \beta}{\longleftarrow\!\!\!+\!\!\!\longrightarrow} X + Y)$$

$$A + B = (U + V \overset{\alpha \cdot \beta}{\longleftarrow\!\!\!+\!\!\!\longrightarrow} X \times Y)$$

but to define the relations this time we have to choose between morphisms

$$p_1\colon (U \times V) \otimes X \overset{\pi_1 \otimes id_X}{\longrightarrow} U \otimes X \overset{\alpha}{\longrightarrow} \Omega$$

and

$$p_2\colon (U \times V) \otimes Y \overset{\pi_2 \otimes \mathrm{id}_Y}{\longrightarrow} V \otimes Y \overset{\beta}{\longrightarrow} \Omega$$

[5]Actually we tried to do the full generalization to a symmetric monoidal closed category, but could not get it to work for the exponentials/modalities.

Since

$$(U \times V) \otimes (X + Y) \simeq (U \times V) \otimes X + (U \times V) \otimes Y \xrightarrow{\binom{p_1}{p_2}} \Omega$$

we need to recall that tensor products distribute over coproducts. Similarly

$$q_1 \colon U \otimes (X \times Y) \xrightarrow{\mathrm{id}_U \otimes \pi_1} U \otimes X \xrightarrow{\alpha} \Omega$$

and

$$q_2 \colon V \otimes (X \times Y) \xrightarrow{\mathrm{id}_V \otimes \pi_2} V \otimes Y \xrightarrow{\beta} \Omega$$

Hence

$$(U + V) \otimes (X \times Y) \simeq U \otimes (X \times Y) + V \otimes (X \times Y) \xrightarrow{\binom{q_1}{q_2}} \Omega$$

Also unities (terminal and initial object) for these operations are a bit more complicated. We want them to look like:

$$I = (1 \longleftarrow\!\!\!\!\!\!\!+\!\!\!\!\!\!\!\longrightarrow 0)$$

$$0 = (0 \longleftarrow\!\!\!\!\!\!\!+\!\!\!\!\!\!\!\longrightarrow 1)$$

but what should the relations be? In the cartesian closed case, this was easy, as $0 \times 1 = 0$ and there is a single map from 0 into any object, be that 2 or Ω. Now if $1 = I$ in $\mathcal{V}$ then we can say that $0 \otimes 1 \simeq 0 \otimes I \simeq 0$ and use the unique map from 0 into Ω to provide relations for both 1 and 0 in $\mathsf{Dial}_{\Omega}(\mathcal{V})$.

As expected $\mathsf{Dial}_{\Omega}(\mathcal{V})$ will be a symmetric monoidal closed category, but it is interesting to see *where* the two different monoidal structures of the base category $\mathcal{V}$ (the tensor $\otimes$ and categorical product $\times$) are used.

The internal-hom (or function space) $[A, B]_{\mathsf{Dial}}$ is given by

$$([U, V]_{\mathcal{V}} \times [Y, X]_{\mathcal{V}} \xleftarrow{\alpha \multimap \beta} U \otimes Y)$$

We need to 'pair' the internal-homs of $\mathcal{V}$ using the categorical product in the first coordinate. The relation follows the pattern of $\mathsf{Dial}_2(\mathbf{Sets})$, but is more complicated.

Using heavily that tensor is associative and commutative we can sketch it as follows:

$$\begin{tikzcd}
([U,V]_{\mathcal{V}} \times [Y,X]_{\mathcal{V}}) \otimes (U \otimes Y) \arrow[r] & ([U,V]_{\mathcal{V}} \otimes (U \otimes Y)) \times ([Y,X]_{\mathcal{V}} \otimes (Y \otimes U)) \arrow[d, "(ev \otimes \mathrm{id}_Y) \times (ev \otimes \mathrm{id}_U)"] \\
& (V \otimes Y) \times (X \otimes U) \arrow[d, "\beta \times \alpha"] \\
& \Omega \times \Omega \arrow[d, "\cdot"] \\
& \Omega
\end{tikzcd}$$

The tensor product $A \otimes_{\mathsf{Dial}} B$ is similarly defined by

$$(U \otimes V \xleftarrow{\alpha \otimes \beta} [V,X]_{\mathcal{V}} \times [U,Y]_{\mathcal{V}})$$

Also the unit for the tensor product is given by

$$I = (I \longleftarrow 1)$$

where the relation is given by $I \otimes 1 \simeq 1 \to \Omega$ and the element function $1 \to \Omega$ simply "picks" the identity of the tensor, e, in Ω.

The category $\mathsf{Dial}_{\Omega}(\mathcal{V})$ also has an intuitionistic 'par' given by

$$A \parr B = ([X,V] \times [Y,U] \xleftarrow{\alpha \parr \beta} X \otimes Y)$$

whose construction is very similar to the tensor and the internal-hom. Its unit, when it exists, is given by

$$\perp = (1 \longleftarrow I)$$

where the relation $1 \otimes I \simeq 1 \to \Omega$ must be an element of Ω, dual to e. (This might not not exist, though.)

One observation about the structure of $\mathsf{Dial}_{\Omega}(\mathcal{V})$ is that this category does not necessarily satisfy the Mix rule, unlike $\mathsf{Dial}_2(\mathbf{Sets})$ where we have a morphism $\mathsf{mix}\colon \perp \to I$, given by

$$\begin{tikzcd}
1 \arrow[d, "\mathrm{id}"'] & 1 \arrow[l, "\emptyset"', "+" marking] \\
1 & 1 \arrow[l, "\mathrm{id}", "+" marking] \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

where I, the unit for tensor, corresponds to true or the identity relation on 1, $\mathrm{id}\colon 1 \times 1 \to 2$, while $\perp$ the unit for par ($\parr$) corresponds to false or the empty relation $\emptyset\colon 1 \times 1 \to 2$ on the same set.

4.3. The Modalities in $\mathsf{Dial}_\Omega(\mathcal{V})$. The modalities in $\mathsf{Dial}_\Omega(\mathcal{V})$ follow the same pattern of $\mathsf{Dial}_2(\mathbf{Sets})$, but these were complicated enough to begin with and now it gets a bit more so. The point is that, in any cartesian closed category, any object U has a given structure as a *comonoid*. This, given comonoid structure is provided by the diagonal map $\delta_U\colon U \to U \times U$ and the terminal map $!_U\colon U \to 1$ that necessarily exist, for any U. This is not necessarily the case in symmetric monoidal closed categories, so we have to ask for it to happen.

Already in $\mathsf{Dial}_2(\mathbf{Sets})$ we had to ask for free commutative monoids to exist in the base category, which they do, in **Sets**. We wrote X^* for the free commutative monoid generated by the set X. Then given an object A of $\mathsf{Dial}_2(\mathbf{Sets})$, say $(U \overset{\alpha}{\nleftarrow} X)$, we said that $!A$ was the object $(U \overset{!\alpha}{\nleftarrow} X^{*U})$ where the relation $!\alpha\colon U \times X^{*U} \to 2$ was given by $u(!\alpha)f$ if and only if $u\alpha x_1, \ldots u\alpha x_k$, where $f\colon U \to X^*$ and $f(u) = \langle x_1, \ldots, x_k\rangle$. Note that the element u of U has been duplicated as many times as necessary to fit the monoid X^*. Now that the base category is not cartesian closed anymore, we cannot duplicate the elements of U without due care. Thus we introduce the notation U_* to mean the free commutative *comonoid* structure on U. Thus the full definition becomes:

$$!A = (U_* \overset{!\alpha}{\nleftarrow} (X^*)^{U_*})$$

$$?A = ((U^*)^{X_*} \overset{?\alpha}{\nleftarrow} X_*)$$

5. Aplications of the Dialectica Construction

This section recapitulates some old results, putting them in the context of applications of one and the same construction. Basically we review the work on [BGdP91][6] and

[6]These results have not been formally published, as my co-authors (responsible for the concurrency side of the work) decided to leave academia, exactly after referee reports were collected, but before the required modifications were made.

the work in [dP91c]. It should be noted that the applications described here are not ideal applications of the construction in this paper, as both applications keep the base category as **Sets**, hence cartesian closed, instead of symmetric monoidal closed. From this viewpoint we ought to provide the calculations for $\mathcal{V}$ the category of, say, finite-dimensional vector spaces. But having applications over the category **Sets** makes the paper easier to understand and shows the versatility of the construction.

5.1. Petri Nets. The Dialectica construction has been applied to the modelling of Petri nets in two different, but related settings. First, Brown and Gurr[BG90] have developed a category of *safe* Petri nets, using the original dialectica construction **GC** from my thesis [dP91a] as a blueprint. The basic idea here is that a Petri net can be described (following Winskel's suggestions [Win88]) as a set of *events* E, a set of *conditions* B and two relations, the *precondition* and the *postcondition* relations, relating events and conditions. Given that the basic objects of the dialectica construction are relations, Brown and Gurr decided to consider a Petri net, as a double object, that is two relations $pre, post\colon E \times B \to 2$ over the same set $E \times B$. This is interesting, but given that relations are only either true or false, that is, they evaluate to 1 (true) or evaluate to 0 (false), the framework cannot cope well with the *multiplicity* intrinsic to Petri nets.

Thus in the second setting, having seen Brown and Gurr's original work, I joined them for a collaboration that resulted in [BGdP91] and whose mathematical foundations were described in [dP91b].

From the mathematical perspective of this paper, for modelling multiplicities in Petri nets, what we need is a dialectica construction where the base category is still **Sets** (after all we are still talking about sets of events and conditions) but where the lineale object that we map to is the set of natural numbers $\mathbf{N}$ (the obvious way of talking about multiplicities is using numbers such as $1, 2, 3, 4, \ldots$). The only complication is which logical structure should the set of natural numbers $\mathbf{N}$ have, so that we are allowed to define $\mathsf{Dial}_{\mathbf{N}}(\mathbf{Sets})$. It turns out that Lawvere had done work on a similar problem[7] before [Law] and the only thing we need to do is consider *truncated subtraction* as the notion of logical implication, once we take the opposite order in $\mathbf{N}$. Details can be found in [dP91b], a summary follows.

5.2. Proposition. *The set of the natural numbers* $\mathbf{N}$ *taken with the opposite of its usual order, where addition is considered a product,* 0 *is the identity of the product and truncated subtraction is the internal-hom is a lineale.*

It's enough to check that, if $\dot{-}$ denotes truncated subtraction, we have the following adjunction equation:

$$\dot{-}(m+n)+p \geq 0 \text{ if and only if } \dot{-}m+(\dot{-}n+p \geq 0$$

5.3. Proposition. *The category* $\mathsf{Dial}_{\mathbf{N}}(\mathbf{Sets})$ *can be defined, as a special instance of the construction in the previous section. This category is symmetric monoidal closed, with*

[7]I only learnt about it, after writing the original version of this note, and getting it slight wrong. Many thanks to Pino Rosolini for setting me right.

products and coproducts. Hence this category is a model of the multiplicative fragment of Intuitionistic Linear Logic.

The structure of $\mathsf{Dial}_{\mathbf{N}}(\mathbf{Sets})$ is given by

$$\begin{array}{lcl}
A \multimap B & = & (V^U \times Y^Y \xleftarrow{\alpha \dot{-} \beta} U \times Y) \\
A \otimes B & = & (U \times V \xleftarrow{\alpha + \beta} Y^U \times X^V) \\
I & = & (1 \xleftarrow{\text{zero}} 1) \\
A \& B & = & (U \times V \xleftarrow{\alpha \cdot \beta} X + Y) \\
1 & = & (1 \xleftarrow{\text{empty}} 0) \\
A \oplus B & = & (U + V \xleftarrow{\alpha \cdot \beta} X \times Y) \\
0 & = & (0 \xleftarrow{\text{empty}} 1)
\end{array}$$

where one can see how the structure of $\mathbf{N}$ is used to define the relations for the multiplicative operators, tensor, its unit and linear implication.

5.4. Proposition. *A double copy of the category* $\mathsf{Dial}_{\mathbf{N}}(\mathbf{Sets})$ *can be used to define a general category of Petri nets* $\mathbf{GNet}$. *Objects of* $\mathbf{GNet}$ *are 4-tuples* $\langle E, B, pre, post \rangle$ *where* E, B *are sets and* $pre, post\colon E \times B \to N$ *are multirelations. Maps of* $\mathbf{GNet}$ *are pairs of functions* (f, F) *where* $f\colon E \to E'$ *and* $F\colon B' \to B$ *are such that for all* e *in* E *and all* b' *in* B', $pre'(f(e), b') \leq pre(e, F(b')$ *and* $post'(f(e), b') \geq post(e, F(b'))$.

In [BGdP91] it is shown that whenever we have a $\mathbf{GNet}$-morphism between nets N and N', then the net N' can simulate any evolution of N. In this sense, dialectica morphisms correspond to simulations. The paper also shows (proposition 6.16 in page 15) a relation between bisimulations of labelled transition systems and morphisms of $\mathbf{GNet}$.

5.5. Lambek Calculus. An unrelated application of the dialectica construction has been described in [dP91c]. In this case we use a modification of the dialectica construction to model the Lambek calculus[Lam58], a syntactic formalism devised by Lambek in the late fifties as an explanation of the methematics of sentence structure. The interested reader is referred to the paper. Here we briefly describe its main results and shortcomings.

Since [dP91c] was written when linear logic was still not much investigated, the paper takes its time explaining that the Lambek calculus is really equivalent to a non-commutative multiplicative intuitionistic linear logic and describes some systems related to it, in particular the commutative Lambek calculus, called by van Benthem **LP** for Lambek calculus with permutation and the system **LA**, which is the Lambek calculus with additives. The main point of the paper is to produce a categorical semantics for the

Lambek calculus, based on the Dialectica construction, but also other algebraic semantics for **L**, in terms of monoids and matrices are discussed.

The variant of the dialectica construction presented is very interesting, because we want to model a non-commutative system, the Lambek calculus **L**, but we use commutative products in our base category, **Sets**. The only place non-commutativity is introduced is in the lineale $\mathcal{N}$ and this is enough. The main definition (page 453, definition 4) is as follows:

5.6. Definition. *Given a biclosed poset* $\mathcal{N}$ *and the category of* **Sets** *(with usual functions), the category* $\mathsf{Dial}_{\mathcal{N}}(\mathbf{Sets})$ *has as objects triples* $U \overset{\alpha}{\nleftrightarrow} X$*, where* U, X *are sets and* $\alpha \colon U \times X \to \mathcal{N}$ *is a function into the special object* $\mathcal{N}$*. Morphisms of* $\mathsf{Dial}_{\mathcal{N}}(\mathbf{Sets})$ *are pairs of functions* (f, F)*, where* $f \colon U \to V$ *and* $F \colon Y \to X$ *satisfy, for all* u *in* U *and* y *in* Y*,* $\alpha(u, Fy) \leq \beta(fu, y)$ *where* $\leq$ *is the order in* $\mathcal{N}$*.*

This is clearly another instance of our general definition of $\mathsf{Dial}_{\Omega}(\mathbf{Sets})$ and most of the structure described in section applies here. In particular the tensor unit and the tensor product carry over. But it is important to note that, despite the fact that the "carrier" for the tensor products $A \otimes B$ and $B \otimes A$ is the same, the tensor products are not the same. One important difference is that instead of one internal-hom, we now have two such, $[A, B]_{\mathsf{left}}$ and $[A, B]_{\mathsf{right}}$.

5.7. Theorem. *The category* $\mathsf{Dial}_{\mathcal{N}}(\mathbf{Sets})$ *is a (non-symmetric) monoidal biclosed category.*

We also stated that a monoidal biclosed category is a categorical model of the Lambek calculus, hence we have that $\mathsf{Dial}_{\mathcal{N}}(\mathbf{Sets})$ is a sound categorical model of the Lambek calculus.

More importantly, the paper goes on to show that modalities could be produced for the Lambek calculus, following ideas of Yettter[Yet88]. Thus if one adds additives and restricted forms of permutation, the categorical model allows us to define a Girard-style modality. All this is done only categorically, i.e. semantically.

Thus this paper has two main shortcomings. First, it assumes that the Extended Curry-Howard Isomorphism has been established for the Lambek calculus. This, as far as I know, is folklore, but hasn't been written down with details. In particular, I know of no proof of completeness, fully worked out, but maybe this is to be found in recent literature. The second shortcoming is more substantial. Since all the work is done using categories, it is not clear to me that the proof theory that should accompany it, will work. I hope to come back to these issues at some other time.

6. Conclusions and Further Work

We have described a generic dialectica construction $\mathsf{Dial}_{\Omega}(\mathcal{V})$ and have discussed (at some length) some of its special cases, especially $\mathsf{Dial}_2(\mathbf{Sets})$ which we compared to $\mathsf{Chu}_2(\mathbf{Sets})$.

The main reason for this comparison was to draw attention to the dialectica version, which we feel is understudied when compared to the Chu construction. To incentivate work on the dialectica, I finish with some examples of research that I believe should be pursued:

- Iteration and recursion have not been considered for the dialectica construction at all. Some is known to work, but nothing has been published. Connections to traced monoidal categories need working out.
- Some of the connections to games are still to be worked out, it seems to me. In particular, it would be nice to know whether there is (or not) a modification of Devarajan et al's proof[DHPP99] that Chu spaces are fully complete that would work for dialectica.
- Generalisations of the dialectica construction to 2-categories or bicategories, in the style of Koslowski [Kos00] might be possible.
- Work on model theory of the chu construction, done by van Benthem [vBen00] and Feferman[F03] might be easy to adapt. If so, there would be relevant connections to Gödel's interpretation, I suspect.

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