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TOWARDS AN N-CATEGORY OF COBORDISMS

EUGENIA CHENG AND NICK GURSKI

ABSTRACT.

We discuss an approach to constructing a weak n-category of cobordisms. First we present a generalisation of Trimble's definition of n-category which seems most appropriate for this construction; in this definition composition is parametrised by a contractible operad. Then we show how to use this definition to define the n-category **nCob**, whose k-cells are k-cobordisms, possibly with corners. We follow Baez and Langford in using "manifolds embedded in cubes" rather than general manifolds. We make the construction for 1-manifolds embedded in 2- and 3-cubes. For general dimensions k and n we indicate what the construction should be.

Introduction

There are many definitions of weak higher-dimensional category but so far not many actual examples of such structures have been given precisely. In low dimensions there are examples – we have bicategories of modules, profunctors, monads, spans, and categories themselves, and many examples of monoidal categories, the degenerate version of bicategories. We also have examples of higher-dimensional structures in which all cells above a certain dimension are weakly invertible (such as quasi-categories, $(\infty, 1)$-categories, and categories enriched in topological spaces or simplicial sets) but the known examples of these are also "essentially" low-dimensional in that the only non-invertible cells occur in low dimensions.

One of the main examples of a "genuinely" higher-dimensional structure that is yet to be well-understood is the totality of weak n-categories themselves, which should be a weak $(n+1)$-category. Simpson has made such a construction for the Tamsamani theory of n-categories [16] but most existing theories have yet to achieve this important goal.

Another motivating example of a higher-dimensional structure is that of cobordisms with corners. Cobordisms without corners simply form a monoidal category – for each dimension $n \geq 1$ there is a monoidal category **nCob** whose objects are $(n-1)$-manifolds and whose morphisms are n-cobordisms between them. (Note that **nCob** is one of the few categories customarily named after its morphisms rather than its objects.) Below is an example of the famous "pair of pants" morphism in **2Cob**.

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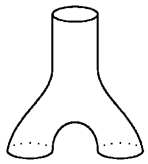
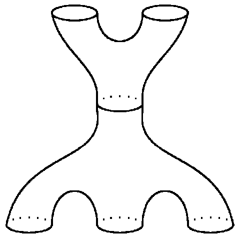
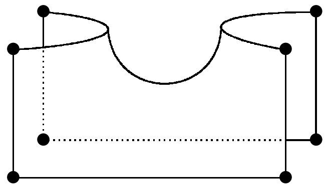


Since the $(n-1)$-manifolds have no boundary we have no need to refer to any lower-dimensions in this structure; composition is by glueing along the boundary as shown in the example below.

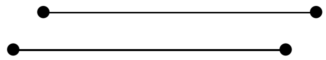


However, for n-cobordisms with corners we need to include information about all the lower-dimensional manifolds involved. The diagram below shows an example of a 2-manifold with corners (with the corners emphasised).



Reading from top to bottom, the "source" of this morphism is the following 1-manifold with boundary:

and the "target" is the following 1-manifold with boundary:



So we expect this structure to have the following cells:

0-cells	0-manifolds
1-cells	1-manifolds with appropriate boundary 0-manifolds
2-cells	diffeomorphism classes of 2-manifolds with corners.

In general for n-manifolds with corners we expect the following cells:

0-cells	0-manifolds
1-cells	1-manifolds with appropriate boundary
$\vdots$	
k-cells	k-manifolds with corners,
$\vdots$	
n-cells	diffeomorphism classes of n-manifolds with corners.

One of the benefits of organising cobordisms into an abstract structure in this way is that it gives a convenient definition of a "topological quantum field theory", or TQFT. TQFTs were originally defined by Atiyah [2] by means of explicit data and axioms; it is now understood that an n-TQFT can be equivalently defined as a representation of **nCob**, that is, a symmetric monoidal functor

$$\mathbf{nCob} \longrightarrow \mathbf{Vect}$$

where **Vect** is the symmetric monoidal category of vector spaces and linear maps between them. Evidently for some purposes this abstract definition is more useful unpacked and made explicit, but the abstract formulation has benefits besides sheer concision: it helps to guide the way for generalisation to related structures where data and axioms are not easy to write down directly, either because there are simply too many of them, or because the structures in question are not yet well enough understood. This is the case for "extended TQFT", the version of TQFT which should be based on cobordisms with corners rather than ordinary cobordisms. The idea is that an extended TQFT should be some sort of representation of the n-category $\mathbf{nCob}_*$ of n-cobordisms with corners; however the notion of "representation" needs to be interpreted in a suitably higher-dimensional way. In particular in place of **Vect** we should have an n-category of "n-vector spaces" and this structure is also not well understood in general, though some progress has been made on **2-Vect** [3, 10].

The "Stabilisation Hypothesis", the "Tangle Hypothesis" and the "Extended TQFT Hypothesis" of Baez and Dolan [4] combine to suggest a general framework in which the structure of $\mathbf{nCob}_*$ may be understood. The Extended TQFT Hypothesis proposes an algebraic formulation of $\mathbf{nCob}_*$ as a certain free n-category; the Tangle Hypothesis proposes a concrete formulation of $\mathbf{nCob}_*$ as a certain n-category of tangles, that is, manifolds with boundary embedded in $\mathbb{R}^m$ for some appropriate dimension m. It is this latter formulation that we aim towards in this paper.

Some work has been done on a low-dimensional case – in [5] Baez and Langford construct a braided monoidal 2-category of "2-tangles in 4-space" modelled by 2-tangles embedded in the unit 4-cube I^4. Although some generality is lost in using 4-cubes, there is a great computational advantage as the cubes are easily "stacked" in various directions to perform composition of tangles. Once they are stacked it is simply a matter of reparametrising the resulting "4-cuboid" to make it a unit cube again. We follow Baez and Langford in studying tangles as "k-manifolds with boundary embedded in n-cubes".

In order to express the totality of tangles as an n-category, we need to decide which definition of n-category we are going to use. However, none of the existing definitions seem to fit this purpose particularly naturally. One definition that almost fits is that of Trimble [17]. In this definition, composition is parametrised by maps $[0,1] \longrightarrow [0,k]$. Essentially, given a string of k composable cells, instead of having one unique composite (which would yield a strict n-category structure) we have one composite for each continuous, endpoint-preserving map $[0,1] \longrightarrow [0,k]$. These maps form a space $E(k)$ for each k, and Trimble's definition makes highly efficacious use of the good properties of these spaces – the fact that not only are they spaces, but moreover contractible spaces, and furthermore, together they form a topological operad. An operad can be thought of as a means of encoding operations with multiple arguments and one output; composition of cells in an n-category is one such operation, taking a number of composable cells and producing a composite.

Trimble's use of maps $[0,1] \longrightarrow [0,k]$ is highly relevant to our situation of stacking and reparametrising unit cubes, since once we have stacked k such cubes we can use just such a map to reparametrise the result and produce a unit cube again, as required. However we cannot use Trimble's definition directly since in our case we cannot use *all* the maps in $E(k)$: not all of them will preserve the smoothness of our embedded manifold in the reparametrisation process. At the very least we will want to restrict to smooth maps, and we will perhaps need to make further restrictions as well.

This leads us to ask: is there a more general form of Trimble's definition using a suboperad of E to parametrise composition? Indeed, is there a general definition of n-category in which any operad E in any category $\mathcal{B}$ may be used in this role, as long as E and $\mathcal{B}$ satisfy some good properties (to be determined)?

From an abstract point of view this latter question is a valid and interesting question in its own right, and a proposal by May [15] seeks to provide an answer. In this work, May proposes a framework of "good properties" for E and $\mathcal{B}$ and suggests a project for defining n-categories using any such operads. However, this work omits many details and in fact the project is much harder than envisaged. At the n-categories workshop at the IMA in 2004, Batanin pointed out that the proposal cannot work exactly as stated. The problem is essentially this: the definition is inductive, but the "good properties" demanded of n-categories in order to be able to define $(n+1)$-categories cannot then be exhibited for the $(n+1)$-categories themselves, thus the induction cannot proceed.

Thus the aim of the present work is to address the following two questions:

1. Is there a general version of Trimble's definition of weak n-category which avoids the problems with May's proposal?

2. Can we use such a definition to define a weak n-category of n-cobordisms with corners?

In seeking to answer these two questions simultaneously, we seek to build a much-needed bridge between the abstract theory of n-categories and the "concrete" examples. Our more general version of Trimble's definition undoubtedly loses some of the remarkable compactness and satisfying concision of Trimble's original, and it is likely to be but another stepping stone towards a theory of n-categories which is both abstractly appealing and concretely applicable.

We begin in Section 1 by describing some of the principal features of Trimble's original definition, and the way in which May's proposal and the present work treats each of those features. In Section 2 we give Trimble's original definition. In Section 3 we briefly describe May's proposed definition and the way in which it fails to work. In Section 4 we give our generalised version of Trimble's definition and in Section 5 we give two examples of operads that can be used in this new framework. The first example is the original operad E used by Trimble, showing that our definition allows Trimble's as an example; note that May's proposal, if it had worked, would not have allowed Trimble's original definition as an example. Our second example is the operad E_s that we then use in Section 6. In this last section, we show how to use our generalised form of Trimble's definition and the operad E_s to construct some n-categories of tangles in very low dimensions: 1-manifolds embedded in 2-cubes and 3-cubes. We also give a proposed construction for the general case of k-manifolds embedded in n-cubes, but a proof in this case will be much harder and requires further work.

Acknowledgements. We would like to thank Peter May for many useful discussions, and Aaron Lauda for his cobordism diagrams.

1. Overview of the definitions

We begin by listing some principal features of Trimble's original definition of n-category. These help give the general idea of the definition and also serve to guide us in our process of generalising it; we also provide the comparison, describing the way in which each feature is treated by May's proposal and the present work.

1.1. Overview of Trimble's original definition.

1. An n-category is defined as a category enriched in $(n-1)$-categories. In particular, the definition is inductive so does not *a priori* provide a definition of ω-category.

2. k-fold composites are specified for all $k \geq 0$, not just $k = 0, 2$.

3. There are many composites for any given composable string of k cells, but exactly how many is specified by the space $E(k)$ (see below). This is different from a "non-algebraic"' approach, in which there *may* be many composites but exactly how many is not prescribed.

4. $E(k)$ is the space of continuous, endpoint preserving maps from $[0,1]$ to $[0,k]$. The $E(k)$ form a topological operad, and this structure allows us to use the $E(k)$ to define composition in a meaningful way, that is, we can make sense of composites of composites. For more on the subject of operads and their use in parametrising composition, see [14].

5. Composition in an n-category should be an $(n-1)$-functor of hom-$(n-1)$-categories. In Trimble's definition, k-fold composition is parametrised by the "fundamental $(n-1)$-category" of $E(k)$. The fundamental n-category functor will be defined inductively along with the notion of n-category.

6. Each $E(k)$ is a contractible space (and so the "fundamental n-category" of each $E(k)$ should in fact be its fundamental n-groupoid), which gives us a sense in which the composition is "coherent". Intuitively, we have that any two composites of the same k-fold string of cells are equivalent thanks to the existence of a homotopy in between their parametrising elements of $E(k)$.

7. For each n we will define a category **nCat** of n-categories and strict n-functors between them, and a fundamental n-category functor

$$\Pi_n : \mathbf{Top} \longrightarrow \mathbf{nCat}.$$

We also prove that **nCat** and Π_n have enough good properties to make the induction go through.

8. We start the induction off by defining $\mathbf{0Cat} = \mathbf{Set}$.

9. The use of *strict* n-functors for composition means that this definition seems a little stricter than would be desired in generality, as it means that interchange (given by the functoriality of the composition functors) may be too strict. However, the work of Kock [12] suggests that weak units may be enough to yield a structure weak enough for some important purposes, for example modelling homotopy types and proving coherence for fully weak n-categories.

1.2. Overview of May's proposal.

1. As before, an n-category is defined as a category enriched in $(n-1)$-categories. k-fold composites are specified for all $k \geq 0$, but are now parametrised by $P(k)$, the kth term of a more general operad. The topological operad E is replaced by an operad P in a category $\mathcal{B}$; much of the difficulty of the definition is in determining which such categories and operads should be allowed.

2. As before, composition in an n-category should be an $(n-1)$-functor of hom-$(n-1)$-categories. However now instead of parametrising this composition by the "fundamental $(n-1)$-category" of our operad P, we define an action of $\mathcal{B}$ on the category **(n-1)Cat** satisfying some good properties. That is, instead of a functor

$$\Pi_n : \mathcal{B} \longrightarrow \mathbf{nCat}$$

for each n, we will have a functor

$$\mathcal{B} \times \mathbf{nCat} \longrightarrow \mathbf{nCat}$$

Observe that the functor Π_n gives an action via

$$(X, A) \mapsto \Pi_n(X) \times A$$

and thus Trimble's use of Π_n can be thought of as a particular action. However, May demands more than this from the "action" and we will show that Π_n does not satisfy these demands, and that in fact nor does May's inductively defined action.

3. Previously we used the fact that each $E(k)$ is a contractible space; in order to have a notion of "contractibility" for our general operad P, May asks for a Quillen model category structure on $\mathcal{B}$ and then stipulates that each $P(k)$ should be weakly equivalent to the unit object in $\mathcal{B}$. ($\mathcal{B}$ must be at least braided monoidal in order for us to have any operads in $\mathcal{B}$ at all.) However May does not use any other aspects of the model category structure for the actual definition of n-category so we will not go into this feature of the definition here. (The aim was evidently to induce a model category structure on **nCat**). For the rest of this paper it is sufficient to know that a model category has a notion of weak equivalence; we will not use the property in any way.

4. For each n May proposes a category **nCat** of n-categories and strict n-functors between them, an action of $\mathcal{B}$ on it, satisfying properties we will discuss later.

5. May is looking for an enriched version of n-categories so starts the induction off by defining $\mathbf{0Cat} = \mathcal{B}$ instead of **Set**. (In fact there does not seem to be a reason for this to be the same category $\mathcal{B}$ in which we defined the operad P.)

1.3. Overview of our generalised version of Trimble's definition. Finding the exact properties that make Trimble's induction work turns out to be quite a delicate matter. The work of May helps us get a clearer idea of which properties it is difficult to do without.

1. As before, an n-category is defined as a category enriched in $(n-1)$-categories. k-fold composites are specified for all $k \geq 0$, and are parametrised by the kth term of an operad. Here the topological operad E is replaced by an operad E in the category **GSet** of globular sets. (This might seem rather specific, but in fact the rest of the structure used gives us such an operad automatically, whether we demand it *a priori* or not.)

2. As before, composition in an n-category should be an $(n-1)$-functor of hom-$(n-1)$-categories, parametrised by an action of $E(k)$. We need to demand enough structure from our operad E to enable us to construct the required action at each stage of the induction; this raises the question – what is the "required action"? In Trimble's definition we had an action arising via a "fundamental n-category" functor

$$\mathbf{Top} \longrightarrow \mathbf{nCat}$$

but now we do not expect such a functor

$$\mathbf{GSet} \longrightarrow \mathbf{nCat}$$

as it does not make sense to try to take a "fundamental n-category" of an arbitrary globular set as globular sets do not have enough structure for this. However, we observe that in Trimble's definition we only ever need to take fundamental n-categories of a very restricted collection of spaces – essentially just the $E(k)$ and all their path spaces. In turn, this construction is enabled by the crucial action of E on path spaces. So we now ask for E to act on some subset $\mathcal{G}$ of **GSet** that at least contains all the $E(k)$ and is closed under taking path spaces. This may seem a little contrived but it does at least work, providing us with the required action at each dimension.

3. As before, we demand that each $E(k)$ is contractible; we do have a notion of contractible globular set.

4. We define, for each n a category **nCat** of n-categories and strict n-functors between them, and now a "fundamental n-category" functor

$$\mathcal{G} \longrightarrow \mathbf{nCat}$$

satisfying enough properties to make the induction work.

5. Like Trimble, we begin with **0Cat** = **Set**.

6. As before, note the use of *strict* n-functors for composition.

2. Trimble's definition

Our starting point for the rest of this work is Trimble's definition of weak n-category [17], so we begin by recalling this definition. Since none of our n-categories are strict we will omit the word "weak" throughout.

Trimble's definition was given in a talk at Cambridge University in 1999, and first appeared in print in Leinster's survey of definitions of n-category [13]. Here we give the definition essentially exactly as in [13]; for further explanations we also refer the reader to [7].

2.1. Basic data. First recall that an operad E in a symmetrical monoidal category $\mathcal{B}$ is a sequence $(E(k))_{k\geq 0}$ of objects of $\mathcal{B}$ together with an "identity" morphism $U \longrightarrow E(1)$ (where U is the unit object of $\mathcal{B}$), and for each $k, r_1, \dots, r_k \geq 0$ a "composition" morphism

$$E(k) \otimes E(r_1) \otimes \cdots \otimes E(r_k) \longrightarrow E(r_1 + \cdots + r_k)$$

obeying unit and associativity laws.

Let **Top** denote a good category of topological spaces as usual; this is a symmetric monoidal closed category with tensor given by cartesian product. There is an operad E in **Top** in which $E(k)$ is the space of continuous endpoint-preserving maps $[0,1] \longrightarrow [0,k]$. The identity element of $E(1)$ is the identity map and composition in the operad is by substitution.

Crucially, E "acts on" path spaces in the following sense. Fix a space X. For any $k \geq 0$ and $x_0, \ldots, x_k \in X$, there is a canonical map

$$E(k) \times X(x_{k-1}, x_k) \times \cdots \times X(x_0, x_1) \longrightarrow X(x_0, x_k).$$

These maps are compatible with the composition and identity of the operad E, and the construction is functorial in X.

We will define inductively for each $n \geq 0$ a category **nCat** with finite products and a functor $\Pi_n : \mathbf{Top} \to \mathbf{nCat}$ preserving finite products.

2.2. The case $n = 0$. For the case $n = 0$ we put $\mathbf{nCat} = \mathbf{Set}$ and define Π_0 to be the functor sending each space to its set of path-components.

N.B.. All the spaces to which we will need to apply Π_0 are in fact contractible. Π_0 clearly preserves products.

2.3. Objects of **(n+1)Cat**. Inductively, an $(n+1)$-category A is given by

- a set A_0 of 0-cells
- $\forall a, a' \in A_0$, a hom-n-category $A(a, a') \in \mathbf{nCat}$
- $\forall k \geq 0$ and $a_0, \ldots, a_k$, a composition functor (i.e., a morphism of **nCat**)

$$\gamma = \gamma^A_{a_0,\ldots,a_k} : \Pi_n\big(E(k)\big) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \to A(a_0, a_k)$$

Note that for the case $k = 0$ this should be interpreted as a map

$$\gamma_{a_o} : \Pi_n\big(E(0)\big) \longrightarrow A(a_0, a_0).$$

The above data are required to satisfy axioms ensuring compatibility with the composition and identity of the operad E. (This makes sense because Π_n preserves finite products and **nCat** has them.)

2.4. Maps in **(n+1)Cat**. A map $F : A \to B$ in **(n+1)Cat** is called an $(n+1)$-functor and is given by

- a function $F = F_0 : A_0 \to B_0$
- $\forall a, a' \in A_0$, an n-functor

$$F = F_{a,a'} : A(a, a') \to B(Fa, Fa')$$

such that $\forall k \geq 0$ and $\forall a_0, \ldots, a_k$, the following diagram commutes.

$$\begin{tikzcd}
\Pi_n(E(k)) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \arrow[r, "\gamma^A"] \arrow[d, "1 \times F \times \cdots \times F"] & A(a_0, a_k) \arrow[d, "F"] \\
\Pi_n(E(k)) \times B(Fa_{k-1}, Fa_k) \times \cdots \times B(Fa_0, Fa_1) \arrow[r, "\gamma^B"'] & B(Fa_0, Fa_k)
\end{tikzcd}$$

Composition of and identities for $(n+1)$-functors are obvious.

2.5. The functor Π_{n+1}. For X in **Top**, we define $\Pi_{n+1}(X) = A$, where A is given by

- A_0 is the underlying set of X
- $A(x, x') = \Pi_n\big(X(x, x')\big)$
- for $x_0, \ldots, x_k$ the composition functor γ is given by

$$\begin{tikzcd}
\Pi_n(E(k)) \times \Pi_n\big(X(x_{k-1}, x_k)\big) \times \cdots \times \Pi_n\big(X(x_0, x_1)\big) \arrow[d, "\Pi_n \text{preserves products}"] \\
\Pi_n\Big(E(k) \times X(x_{k-1}, x_k) \times \cdots \times X(x_0, x_1)\Big) \arrow[d] \\
\Pi_n\Big(X(x_0, x_k)\Big)
\end{tikzcd}$$

  where the second map is Π_n applied to the action of E on the path spaces of X.

Π_{n+1} is defined on maps in the obvious way; it is not hard to show that **(n+1)Cat** has finite products and that Π_{n+1} preserves them, so the inductive definition goes through.

Remarks. The compactness of this definition owes much to the rich structure and good behaviour of the operad E in **Top**. A natural question to ask then is: what are the features of **Top** and E that make this work, and can we thus generalise the definition to other categories and operads? This is the subject of the next two sections. It turns out that the question is a bit tricky to answer; the necessary features of **Top** and the operad E are not easy to distil.

3. May's proposed definition

May sketched a proposal for a definition of n-category in [15]. Many details were not given, and in fact at the n-categories workshop at the IMA in 2004 Batanin pointed out that the definition could not make sense as proposed, as the induction step would not go through. Essentially the problem is as follows. For each n May demands a certain amount of structure in the category **(n-1)Cat** in order to be able to define n-categories. To make this induction work one would then have to exhibit all this structure in the category **nCat**, but Batanin showed that this would not be possible in the way May described – or indeed at all.

A natural question to ask next is: is it possible to demand less structure in **(n-1)Cat** and still be able to define n-categories? And if so, can we exhibit this reduced amount of structure in **nCat** to make the induction go through? This is essentially the approach of Section 4. We do at least have one example of such a structure – if we examine Trimble's definition we see that his categories **nCat** for each n do *not* have all the structure that May originally demanded.

In the present section we present May's proposal briefly and point out where it does not work; we leave the full account of this to Batanin whose result this is. We include this section for the record, but also because the proposal may be seen as a first step in the direction of our definition.

3.1. The basic data. Let $\mathcal{B}$ be a symmetric monoidal closed category with a Quillen model category structure that is cofibrantly generated and proper. (This is a particularly tractable form of model category; it would be a lengthy and irrelevant digression to give the definition here.) Let P be an operad in $\mathcal{B}$ such that each $P(k)$ is weakly equivalent to the unit object I. We will define, for each $n \geq 0$ a category **nCat** which is symmetric monoidal closed, enriched in and tensored over $\mathcal{B}$.

Recall that if a category $\mathcal{V}$ is enriched in $\mathcal{B}$ then instead of homsets it has "hom-objects" of $\mathcal{B}$, that is, for any objects $V, W \in \mathcal{V}$ we have a hom-object $\mathcal{V}(V,W) \in \mathcal{B}$ and these satisfy various sensible axioms. Recall also that a category $\mathcal{V}$ is called *tensored over* $\mathcal{B}$ if there is a functor

$$\odot : \mathcal{B} \times \mathcal{V} \longrightarrow \mathcal{V}$$

(we write $\odot(B,V) = B \odot V$) such that for all $V \in \mathcal{V}$ there is an adjunction

$$- \odot V \dashv \mathcal{V}(V,-) : \mathcal{V} \longrightarrow \mathcal{B}$$

i.e. for all $B \in \mathcal{V}$ and $W \in \mathcal{W}$ we have

$$\mathcal{V}(B \odot V, W) \cong \mathcal{B}(B, \mathcal{V}(V,W))$$

as objects of $\mathcal{B}$, naturally in B and W.

3.2. The case $n = 0$. For the case $n = 0$ we put $\mathbf{nCat} = \mathcal{B}$ and define the functor $\odot$ to be the monoidal tensor $\otimes$ in $\mathcal{B}$.

3.3. Objects of **(n+1)Cat**. Inductively, an $(n+1)$-category A is given by

- a set A_0 of 0-cells

- $\forall a, a' \in A_0$, a hom-n-category $A(a, a') \in$ **nCat**
- $\forall k \geq 0$ and $a_0, \ldots, a_k$, a composition functor (i.e., a morphism of **nCat**)

$$\gamma = \gamma^A_{a_0,\ldots,a_k} : P(k) \odot \big(A(a_{k-1}, a_k) \otimes \cdots \otimes A(a_0, a_1)\big) \to A(a_0, a_k)$$

satisfying axioms ensuring compatibility with the composition and identity of the operad P. (This makes sense because $\odot$ is required to distribute over $\otimes$.)

3.4. Maps in **(n+1)Cat**. A map $F : A \to B$ in **(n+1)Cat** is called an $(n + 1)$-functor and is given by

- a function $F = F_0 : A_0 \to B_0$
- $\forall a, a' \in A_0$, an n-functor

$$F = F_{a,a'} : A(a, a') \to B(Fa, Fa')$$

such that $\forall k \geq 0$ and $\forall a_0, \ldots, a_k$, the following diagram commutes.

$$\begin{tikzcd}
P(k) \odot \big(A(a_{k-1}, a_k) \otimes \cdots \otimes A(a_0, a_1)\big) \arrow[r, "\gamma^A"] \arrow[d, "1\odot(F\otimes\cdots\otimes F)"] & A(a_0, a_k) \arrow[d, "F"] \\
P(k) \odot \big(B(Fa_{k-1}, Fa_k) \otimes \cdots \otimes B(Fa_0, Fa_1)\big) \arrow[r, "\gamma^B"'] & B(Fa_0, Fa_k)
\end{tikzcd}$$

Composition of and identities for $(n + 1)$-functors are obvious.

3.5. The proposed action of $\mathcal{B}$ on **(n+1)Cat**. The difficulty comes now in showing that **(n+1)Cat** is tensored over $\mathcal{B}$. May suggests that it is, but for some time it was not clear how to produce this structure; eventually Batanin produced a counterexample. Note that in Trimble's definition we have an action of **Top** on **nCat** via the functor Π_n. However, this is not given as a tensor; *a priori* we do not now that **nCat** is enriched in **Top** at all.

4. Generalised Trimble definition

In this section we present a generalisation of Trimble's definition which we will be using throughout the rest of the paper. The idea is, much as in May's proposal, to copy Trimble's definition but with a general operad E in some category $\mathcal{B}$, in place of the operad in **Top** used by Trimble. The delicate issue is to make the setting sufficiently general to include more examples of such operads, but to retain enough structure to enable us to make the induction step; this is the lesson learnt from May's proposal. The difficulty in the induction step is in knowing how to construct the action of $\mathcal{B}$ on **nCat**, which, unlike

May, we now choose to do via a "fundamental n-category" functor Π_n. This is essentially because the framework we find ourselves forced to use for the rest of the definition forces any suitable action to be of this form in any case.

Our approach starts from the observation that, for the definition of n-category and n-functor, we only need the "fundamental n-category" of *certain* spaces, not all of them. We also observe that although Trimble's operad E is an operad in **Top**, the induction step of the definition of n-category is enabled by the associated operad in globular sets. This arises from considering the "underlying" globular set associated with any space, with

0-cells	the points in the space
1-cells	paths between these points
2-cells	paths between paths i.e. maps $D^2 \longrightarrow X$
$\vdots$	
m-cells	maps $D^m \longrightarrow X$
$\vdots$	

So in fact we make the definition using an operad E defined directly in the category **GSet** of globular sets (we recall the definition below). Among other things, this also enables us to construct more variants of the original definition by restricting the operad at each dimension as desired, giving us finer control over the situation. This is different from the situation in **Top** where it is not possible to change the m-cells of the underlying globular set at will once the lower-dimensional cells have been fixed.

In using globular sets we frequently make use of the fact that the cells $x \longrightarrow y$ in a globular set X form a globular set themselves. That is, $\forall x, y \in X_k$ we have a globular set $X(x, y)$ whose 0-cells are $(k+1)$-cells $x \longrightarrow y$, whose 1-cells are $(k+2)$ cells between those, and so on. The globular set $X(x, y)$ can be thought of as analogous to a path space.

Now that we are using **GSet** instead of **Top** we might think we need to be able to take the "fundamental n-category" of an arbitrary globular set, which does not make any obvious sense. However, as we remarked above it turns out that we only ever need the fundamental n-category of a very restricted collection of objects of $\mathcal{B}$, and for these objects the notion of fundamental n-category does make sense.

4.1. Basic data. First we recall the definition of globular sets. The idea is that we have, for each natural number n, a set $X(n)$ of "n-cells", each of which has a source $(n-1)$-cell and a target $(n-1)$-cell. The source and target of a cell must themselves share the same source and target; this is the "globularity" condition, ensuring that the cells have "globular" shape rather than any other kind of shape.

Formally, the category of globular sets **GSet** has objects X consisting of sets X_n for each natural number n and maps $s_n, t_n : X_n \to X_{n-1}$ subject to the relations $ss = st, tt = ts$; morphisms $f : X \to Y$ in **GSet** consist of maps $f_n : X_n \to Y_n$ that strictly commute with the s_n and t_n.

We are now ready to describe the basic data needed for the definition of n-categories. We begin with:

- a contractible operad E in **GSet**
- a subcategory $\mathcal{G}$ of **GSet** satisfying
  i) $1 \in \mathcal{G}$
  ii) $\forall k \geq 0$, $E(k) \in \mathcal{G}$
  iii) $\forall X \in \mathcal{G}$ and $\forall x, y \in X_0$, $X(x, y) \in \mathcal{G}$
  iv) $\forall X, Y \in \mathcal{G}$, $X \times Y \in \mathcal{G}$
- $\forall X \in \mathcal{G}$, an action of E on X (in the sense of Section 2.1) such that every map $f : X \to Y$ in $\mathcal{G}$ is an E-action map. Note that in particular this means E must then act on products via the diagonal map.

We call an operad in **GSet** contractible if its underlying globular set is contractible, that is, given any parallel pair of m-cells f and g, there exists an $(m+1)$-cell $\alpha : f \longrightarrow g$. All 0-cells are parallel; for $m > 0$ a pair of m-cells is parallel if they have the same source and target.

We will define inductively for each $n \geq 0$ a category **nCat** with finite products and a functor $\Pi_n : \mathcal{G} \to$ **nCat** preserving the above finite products. Π_n can be thought of as a "fundamental n-category" functor, but in fact we will only ever apply it to contractible globular sets, giving a fundamental n-groupoid. Note that the action of E on the members of $\mathcal{G}$ is what enables us to make some sense of the notion of fundamental n-category for these globular sets despite the fact that *a priori* they do not come with a composition.

4.2. The case $n = 0$. For the case $n = 0$ we put **nCat** $=$ **Set** and define the functor Π_0 to send everything to 1.

N.B. Π_0 can be thought of as taking path components, but all objects of $\mathcal{G}$ in which we are interested are contractible. Π_0 clearly preserves products.

4.3. Objects of **(n+1)Cat**. Inductively, an $(n+1)$-category A is given by

- a set A_0 of 0-cells
- $\forall a, a' \in A_0$, a hom-n-category $A(a, a') \in$ **nCat**
- $\forall k \geq 0$ and $a_0, \ldots, a_k$, a composition functor (i.e., a morphism of **nCat**)

$$\gamma = \gamma^A_{a_0,\ldots,a_k} : \Pi_n\big(E(k)\big) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \to A(a_0, a_k)$$

satisfying axioms ensuring compatibility with the composition and identity of the operad E. (This makes sense because Π_n preserves finite products and **nCat** has them.)

4.4. Maps in **(n+1)Cat**. A map $F : A \to B$ in **(n+1)Cat** is called an $(n+1)$-functor and is given by

- a function $F = F_0 : A_0 \to B_0$
- $\forall a, a' \in A_0$, an n-functor

$$F = F_{a,a'} : A(a,a') \to B(Fa, Fa')$$

such that $\forall k \geq 0$ and $\forall a_0, \ldots, a_k$, the following diagram commutes.

$$\begin{tikzcd}
\Pi_n(E(k)) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1) \arrow[r, "\gamma^A"] \arrow[d, "1 \times F \times \cdots \times F"] & A(a_0, a_k) \arrow[d, "F"] \\
\Pi_n(E(k)) \times B(Fa_{k-1}, Fa_k) \times \cdots \times B(Fa_0, Fa_1) \arrow[r, "\gamma^B"'] & B(Fa_0, Fa_k)
\end{tikzcd}$$

Composition of and identities for $(n+1)$-functors are obvious.

4.5. Π_{n+1} on objects. For $X \in \mathcal{G}$, we define $\Pi_{n+1}(X) = A$, where A is given by

- $A_0 = X_0$
- $A(x, x') = \Pi_n\big(X(x, x')\big)$ (defined since $X(x, x') \in \mathcal{G}$ by hypothesis)
- for $x_0, \ldots, x_k$ the composition functor γ is given by

$$\begin{tikzcd}
\Pi_n(E(k)) \times \Pi_n\big(X(x_{k-1}, x_k)\big) \times \cdots \times \Pi_n\big(X(x_0, x_1)\big) \arrow[d, "\Pi_n \text{preserves products laxly}"] \\
\Pi_n\Big(E(k) \times X(x_{k-1}, x_k) \times \cdots \times X(x_0, x_1)\Big) \arrow[d] \\
\Pi_n\Big(X(x_0, x_k)\Big)
\end{tikzcd}$$

where the second map is Π_n applied to the action of E on X.

4.6. Π_{n+1} on maps. Given $\varphi : X \to Y$ in $\mathcal{G}$, we define $F = \Pi_{n+1}\varphi : \Pi_{n+1}X \to \Pi_{n+1}Y$ as follows.

- On objects, $F_0 = \varphi_0 : X_0 \to Y_0$.

- The functor $F_{x,x'} : \Pi_{n+1}\big(X(x,x')\big) \to \Pi_{n+1}\big(Y(Fx,Fx')\big)$ is given by Π_n applied to the restriction of φ to $X(x,x') \to Y(\varphi x, \varphi x') = Y(Fx, Fx')$.

We check that the following diagram commutes.

$$\begin{tikzcd}
\Pi_n(E(k)) \times \Pi_n\big(X(x_{k-1},x_k)\big) \times \cdots \times \Pi_n\big(X(x_0,x_1)\big) \arrow[dd, "1\times\Pi_n\varphi\times\cdots\times\Pi_n\varphi"'] \arrow[dr] & \\
& \Pi_n\Big(E(k) \times X(x_{k-1},x_k) \times \cdots \times X(x_0,x_1)\Big) \arrow[d, "\Pi_n\alpha"] \\
\Pi_n(E(k)) \times \Pi_n\big(Y(\varphi x_{k-1},\varphi x_k)\big) \times \cdots \times \Pi_n\big(Y(\varphi x_0,\varphi x_1)\big) \arrow[d] & \Pi_n\Big(X(x_0,x_k)\Big) \arrow[d, "\Pi_n\varphi"] \\
\Pi_n\Big(E(k) \times Y(\varphi x_{k-1},\varphi x_k) \times \cdots \times Y(\varphi x_0,\varphi x_1)\Big) \arrow[r, "\Pi_n\alpha"'] & \Pi_n\Big(Y(\varphi x_0,\varphi x_k)\Big)
\end{tikzcd}$$

This can be seen by dividing the diagram into two squares via the map

$$\begin{tikzcd}
\Pi_n\Big(E(k) \times X(x_{k-1},x_k) \times \cdots \times X(x_0,x_1)\Big) \arrow[d, "\Pi_n(1\times\varphi\times\cdots\times\varphi)"] \\
\Pi_n\Big(E(k) \times Y(\varphi x_{k-1},\varphi x_k) \times \cdots \times Y(\varphi x_0,\varphi x_1)\Big)
\end{tikzcd}$$

(where here α is the action map of E on an object of $\mathcal{G}$).

4.7. Finite products. A product of $(n+1)$-categories A and B is given as follows:

- $(A \times B)_0 = A_0 \times B_0$
- $(A \times B)\big((a,b),(a',b')\big) = A(a,a') \times B(b,b')$

- composition is given by

$$\begin{array}{c}
\Pi_n(E(k)) \times (A\times B)\big((a_{k-1},b_{k-1}),(a_k,b_k)\big) \times \cdots \times (A\times B)\big((a_0,b_0),(a_1,b_1)\big) \\
\Big\| \\
\Pi_n(E(k)) \times A(a_{k-1},a_k) \times B(b_{k-1},b_k) \times \cdots \times A(a_0,a_1) \times B(b_0,b_1) \\
\big\downarrow{\scriptstyle \cong} \\
\Pi_n(E(k)) \times A(a_{k-1},a_k) \times \cdots \times A(a_0,a_1) \times B(b_{k-1},b_k) \times \cdots \times B(b_0,b_1) \\
\big\downarrow{\scriptstyle \Delta\times 1} \\
\Pi_n(E(k)) \times \Pi_n(E(k)) \times A(a_{k-1},a_k) \times \cdots \times A(a_0,a_1) \times B(b_{k-1},b_k) \times \cdots \times B(b_0,b_1) \\
\big\downarrow{\scriptstyle \cong} \\
\Pi_n(E(k)) \times A(a_{k-1},a_k) \times \cdots \times A(a_0,a_1) \times \Pi_n(E(k)) \times B(b_{k-1},b_k) \times \cdots \times B(b_0,b_1) \\
\big\downarrow{\scriptstyle \gamma^A\times\gamma^B} \\
A(a_0,a_k) \times B(b_0,b_k) \\
\Big\| \\
(A\times B)\big((a_0,b_0),(a_k,b_k)\big),
\end{array}$$

  where Δ is the diagonal map in **GSet**.

It follows from the definition of the action of E on products via the diagonal that Π_{n+1} preserves products laxly, so the inductive definition goes through.

5. Examples

In this section, we will give two examples of operads that fit into our framework. The first example is to show that our framework really is just a generalisation of Trimble's; the second example is the motivating one and might be thought of as the motivating reason for making the generalisation at all.

5.1. Trimble's definition in the new framework. In this section we discuss how Trimble's original definition arises in the generalised setting. The key point is the "underlying globular set" functor

$$D : \mathbf{Top} \to \mathbf{GSet}$$

which is a lax monoidal functor, so takes the original topological operad to an operad in globular sets; also, it "preserves path spaces", that is, for all $x, y \in X$

$$DX(x,y) = D\big(X(x,y)\big).$$

For the subcategory $\mathcal{G}$ we simply take any globular set which is the image (under D) of a path space; likewise we take any morphism which is the image under D of a continuous map between the path spaces in question.

5.2. A smooth version of the original operad. The idea is that we now wish to parametrise composition by only *smooth* maps $[1] \longrightarrow [k]$, smooth homotopies between them, smooth homotopies between those, and so on. The motivating example is that of manifolds; when we parametrise composition of manifolds we will need to ensure that the smoothness is preserved. Note that for convenience we will also write $[k]$ for the interval $[0, k]$.

Note that the most naive approach to this might be to ask for a suboperad E' of Trimble's operad in **Top**, and then copy the rest of Trimble's definition verbatim. However, this is not so straightforward – we can fix each $E'(k)$ to be the subspace of $E(k)$ consisting of the smooth maps, but then we cannot demand only smooth homotopies between them; the homotopies are determined by the topology of $E(k)$ and the points of $E'(k)$. Furthermore, we will need to make other restrictions on our maps and homotopies to ensure that their smoothness is preserved under operadic composition.

Using an operad in globular sets we can build each dimension individually, specifying exactly which paths and homotopies we wish to include. Our operad E_s does not come from a suboperad of Trimble's E in **Top**, but it is a suboperad of $D(E)$ the underlying globular set operad of E.

E_s is defined as follows. For each $k \geq 0$, $E_s(k)$ has

- 0-cells which are endpoint-preserving diffeomorphisms $f : [1] \to [k]$ such that there exists $\delta > 0$ for which $\forall x \in [0, \delta)$ $f(x) = x$ and $\forall x \in (1 - \delta, 1]$ $f(x) = x + k - 1$
- m-cells, $m > 0$, from α to β are smooth homotopies

$$\Theta : I^m \times [1] \to [k]$$

from α to β such that

i) each $\Theta(t, -, -, \ldots, -) : I^{m-1} \times [1] \to [k]$ satisfies the conditions for being an $(m-1)$-cell and

ii) there exists a $\delta > 0$ such that $\forall t \in [0, \delta)$

$$\Theta(t, -, -, \ldots, -) = \alpha$$

and $\forall t \in (1 - \delta, 1]$

$$\Theta(t, -, -, \ldots, -) = \beta.$$

Note that in effect we have demanded that “the 0-cells have derivative 1 on a neighbourhood of the boundary” to ensure that smoothness is preserved by operadic composition; E_s inherits its unit and composition from E so we only have to check closure to see that E_s is indeed an operad. The other conditions are to ensure the desired behavior of the n-category of manifolds when we eventually construct it. Furthermore we can check that each $E_s(k)$ is a contractible globular set.

Next we need to construct a suitable category $\mathcal{G}$ and show that $E_s(k)$ has the required action on it. We specify the objects of $\mathcal{G}$ by taking the smallest collection of objects satisfying the necessary conditions; essentially this amounts to taking each $E_s(k)$, all path spaces (of all dimensions), and all products of them. To specify the morphisms we take not *all* globular set maps between objects of $\mathcal{G}$, but only those that are restrictions of maps of the form DPf where $P : \mathbf{Top} \longrightarrow \mathbf{Top}$ is the path space functor. To show that E_s then acts as necessary, we check that the action of the original operad E preserves the properties used to define cells of $E(k)$ above.

N.B. The above characterisation is evidently only a sketch. Our precise characterisation is as yet unilluminating and it remains to be seen if a more illuminating characterisation can be found.

6. n-categories of manifolds in cubes

In this section we discuss how to construct operadic n-categories of “manifolds in cubes”. The aim is to define an n-category **nCob** of cobordisms with corners as below.

0-cells	0-manifolds
1-cells	1-manifolds with appropriate boundary
$\vdots$	
k-cells	k-manifolds with corners,
$\vdots$	
n-cells	diffeomorphism classes of n-manifolds with corners.

(In this paper, all manifolds are smooth and compact.) We follow Baez and Langford [5] and consider manifolds embedded in cubes. We aim to construct, for each $n \geq 1$ and $0 \leq k < n$ an n-category of “k-manifolds in n-cubes”. The n-cube is the space I^n, where $I = [0, 1]$, and we consider k-manifolds that are subsets of I^n, equipped with the smooth structure inherited from the standard smooth structure on I^n.

Ultimately, these should be degenerate n-categories with extra structure, since the m-cells are given as follows.

0-cells	trivial
1-cells	trivial
$\vdots$	$\vdots$
$(n-k)$-cells	0-manifolds in $(n-k)$-cubes
$(n-k+1)$-cells	1-manifolds in $(n-k+1)$-cubes
$\vdots$	$\vdots$
n-cells	diffeomorphism classes of k-manifolds in n-cubes

So this is an $(n-k)$-degenerate n-category or an $(n-k)$-tuply monoidal k-category. Note that for $m < n-k$, the unique m-cell can be thought of as being the empty set as a subset of I^m; note also that for all valid n and k the 0-cells are trivial.

The Tangle Hypothesis of Baez and Dolan [4] says "framed oriented n-tangles in $(n+k)$-dimensions are the n-cells of the free weak k-tuply monoidal n-category with duals on one object". Note that by the following reindexing the dimensions involved do match up.

Current paper		Tangle Hypothesis
$n-k$	$\longrightarrow$	k
k	$\longrightarrow$	n

The Stabilisation Hypothesis [4] then says that the notion of k-tuply monoidal n-category should stabilise when $k \geq n+2$. With our indexing that means

$$n-k \geq k+2$$

i.e.

$$n \geq 2k+2.$$

So if we fix k, the dimension of our manifolds, and embed them in cubes of higher and higher dimension n, the situation should stabilise. Later in this paper we will sketch this process for $k=1$, where we expect 1-manifolds in 4-cubes to give "the same structure" as 1-manifolds in n-cubes $\forall n \geq 4$. This corresponds to the fact that under- and over-crossings are the same in 4-space (and all higher dimensions), or the fact that all knots can be untied in space of dimension 4 and above.

Finally note that our construction differs from that of Baez and Langford as we are constructing a *weak* n-category where theirs is strict. Essentially this means that when we compose k-cells by "sticking cubes together" we need to keep track of how we reparametrise the resulting cuboid to become a unit cube again. By contrast, in the strict case the strictifying equivalence relation ensures that the different reparametrisations will not be detected by the n-category. Our use of the operad E_s gives us precisely the machinery we need to keep track of and make use of this reparameterising.

6.1. 1-MANIFOLDS IN 2-CUBES. We now construct a (degenerate) 2-category of 1-manifolds with boundary embedded in the 2-cube I^2.

0-cells. There is only one 0-cell, which we consider to be $\varnothing \subset I^0$.

1-cells. These are 0-manifolds in $(0,1)$.

2-CELLS. Given 1-cells x, y, a 2-cell $\theta : x \Rightarrow y$ is a 1-manifold with boundary, in I^2, satisfying the following conditions, subject to the equivalence relation below.

i) θ has empty intersection with $I \times \{0,1\}$.

ii) The boundary of θ is contained in $\{0,1\} \times I$ with $\partial\theta|_{\{0\}\times I} = x$ and $\partial\theta|_{\{1\}\times I} = y$.

iii) θ has a product structure near the top ($\{0\} \times I$) and the bottom ($\{1\} \times I$); we refer to this as a "collar" region.

The equivalence relation on these is defined as follows. Let θ_1, θ_2 be two 1-manifolds with the same boundary. We say $\theta_1 \sim \theta_2$ if there is a self-diffeomorphism of $I \times I$ taking θ_1 to θ_2 and keeping the boundary pointwise fixed.

Note that we can express a subset of I^m using a characteristic function $I^m \to \mathbf{2} = \{0,1\}$. In this formalism a 1-cell x is expressed as a function $x : I \to \mathbf{2}$, and a 2-cell $\alpha : x \Rightarrow y$ is expressed as a function $I \times I \to \mathbf{2}$ where

$$\begin{aligned} \alpha(0,-) &= x, \text{and} \\ \alpha(1,-) &= y. \end{aligned}$$

The function $\alpha(t,-) : I \to \mathbf{2}$ gives the cross-section of the manifold at height t.

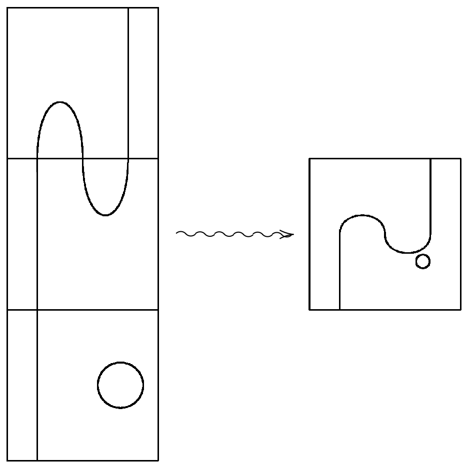
COMPOSITION ALONG BOUNDING 1-CELLS (VERTICAL COMPOSITION). For all $k \geq 0$ we need a function

$$\Pi_0(E_s(k)) \times X(x_{k-1}, x_k) \times \cdots \times X(x_0, x_1) \longrightarrow X(x_0, x_k).$$

However $\Pi_0(E_s(k)) = 1$. Given composable 2-cells $f_1, \ldots, f_k$ we can "stack" the copies of I^2, that is, take a colimit giving

$$f_1 + \cdots + f_k : [k] \longrightarrow \mathbf{2};$$

we can now pre-compose with any reparametrising map $[1] \longrightarrow [k]$ to give a subset of the unit square again. The choice of reparametrisation does not matter as any two will give equivalent manifolds under our equivalence relation for 2-cells. Note that the collar regions ensure that such vertical composites are smooth. The following diagram illustrates this composition.



COMPOSITION ALONG BOUNDING 0-CELLS (HORIZONTAL COMPOSITION). For all $k \geq 0$, we define a functor

$$\Pi_1(E_s(k)) \times X^k \to X$$

where X is the (unique) vertical hom-category.

On objects (i.e., 1-cells of our 2-category), we have on the left a tuple

$$(f, x_1, x_2, \ldots, x_k)$$

where f is a map $[1] \to [k]$ and each x_i is a subset of I, which we write as a function $x_i : I \to \mathbf{2}$. We can take a colimit to get

$$x_1 + x_2 + \cdots + x_k : [k] \to \mathbf{2}.$$

We now compose this with f to get

$$[1] \xrightarrow{f} [k] \xrightarrow{x_1+\cdots+x_k} \mathbf{2},$$

i.e., a subset of I given by $(x_1 + \cdots + x_k) \circ f$; we check that the resulting subset is a manifold in $(0, 1)$. This is the required composite on 1-cells.

On morphisms (i.e. 2-cells of our 2-category) we must find a composite for

$$\begin{pmatrix} f, & x_1, & \ldots, & x_k \\ \alpha \Downarrow, & \Downarrow \theta_1, & \ldots, & \Downarrow \theta_k \\ g, & y_1, & \ldots, & y_k \end{pmatrix}$$

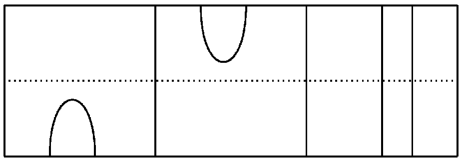
whose source must be $(x_1 + \cdots + x_k) \circ f$ and target must be $(y_1 + \cdots + y_k) \circ g$. We take the following composite

$$I \times [1] \xrightarrow{\Delta \times 1} I \times I \times [1] \xrightarrow{1 \times \alpha} I \times [k] \xrightarrow{\theta_1 + \cdots + \theta_k} \mathbf{2}$$

giving a subset of I^2 defined explicitly on elements by

$$(z_1, z_2) \mapsto (\theta_1 + \cdots + \theta_k)(z_1, \alpha(z_1, z_2))$$

and we then take the required equivalence class. The idea is simply that at each "height" z_1 we reparametrise the manifold by $\alpha(z_1, -)$. This is illustrated by the following diagram.



$\rightarrow$ reparam. by $\alpha(0, -)$

$\rightarrow$ reparam. by $\alpha(z_1, -)$

$\rightarrow$ reparam. by $\alpha(1, -)$

It is straightforward to check that this composite has the correct source and target, and that it satisfies the conditions for being a 2-cell. This follows from the collar regions and the various "endpoint neighbourhood" conditions we demanded when defining the operad $E_s(k)$.

Finally we check the remaining axioms for a 2-category, i.e., that this composition is functorial and interacts well with operad composition.

6.2. 1-manifolds in 3-cubes. We now define a 3-category of 1-manifolds in 3-cubes. The idea is that the 0- and 1-cells are now trivial, and the 2- and 3-cells should be 0-manifolds and 1-manifolds with boundary respectively, now embedded in cubes one dimension higher than before. Composition of cubes can now occur in three ways as required – by stacking cubes in each of the three possible directions.

Note that this is a doubly degenerate 3-category, so should in fact be a braided monoidal category of some sort if we regard the 2-cells as objects and the 3-cells as morphisms. (See also [6].)

0-cells. There is only one 0-cell, which we consider to be $\varnothing \subset I^0$.

1-cells. There is only one 1-cell, which we consider to be $\varnothing \subset I$.

2-cells. A 2-cell is a 0-manifold in $(0, 1) \times (0, 1)$.

3-cells. Given 1-cells x, y, a 3-cell $\theta : x \longrightarrow y$ is a 1-manifold-with-boundary in I^3, satisfying the following conditions, subject to the equivalence relation below.

i) The boundary of θ is contained in $\{0, 1\} \times I^2$ with $\partial\theta|_{\{0\}\times I^2} = x$ and $\partial\theta|_{\{1\}\times I^2} = y$.

ii) As before θ has a product structure near the top ($\{0\} \times I^2$) and the bottom ($\{1\} \times I^2$).

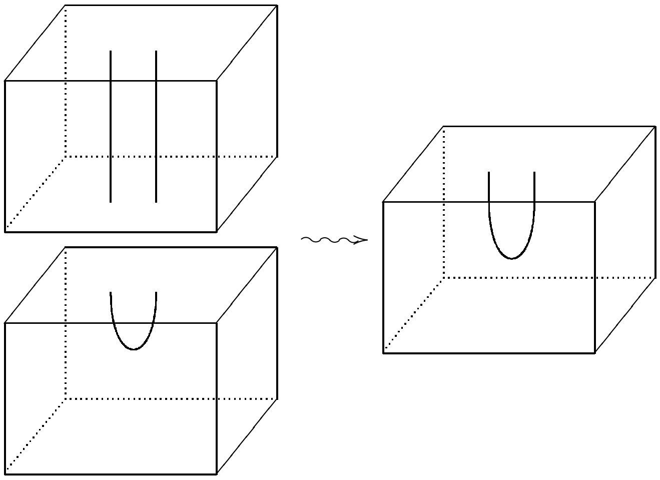
The equivalence relation on these is defined as follows. Let θ_1, θ_2 be two 1-manifolds with the same boundary. We say $\theta_1 \sim \theta_2$ if there is a self-diffeomorphism of I^3 taking θ_1 to θ_2 and keeping the boundary pointwise fixed.

We must now define three kinds of composition.

COMPOSITION ALONG A BOUNDING 2-CELL. $\forall k \geq 0$ we need a function

$$\Pi_0(E_s(k)) \times X(x_{k-1}, x_k) \times \cdots \times X(x_0, x_1) \longrightarrow X(x_0, x_k).$$

As with vertical composition in the previous example, this composition is achieved by stacking boxes in the direction that identifies the boundaries of the 1-manifolds, as illustrated in the following diagram.



As before, we then reparametrise to a unit cube using any reparametrisation, and take equivalences classes, and observe that

- a different choice of reparametrisation gives the same equivalence class, and
- collar regions ensure that the composite is smooth.

COMPOSITION ALONG A BOUNDING 1-CELL. Recall that there is only one 1-cell, so there is only one hom-category. We write this as X; then for all $k \geq 0$ we need a 1-functor

$$\Pi_1(E_s(k)) \times X^k \longrightarrow X.$$

On objects (i.e., 2-cells of our 3-category), we have on the left a tuple

$$(f, x_1, x_2, \ldots, x_k)$$

where f is a map $[1] \to [k]$ and each x_i is a subset of $I \times I$, which we write as a function $x_i : I \times I \to \mathbf{2}$. We make the following composite

$$I \times [1] \xrightarrow{1\times f} I \times [k] \xrightarrow{x_1+\cdots+x_k} \mathbf{2},$$

giving a subset of I^2 defined explicitly on elements by

$$(z_1, z_2) \mapsto (\theta_1 + \cdots + \theta_k)(z_1, f(z_2))$$

and we check that the resulting subset is a manifold in $(0,1)$.

On morphisms (i.e. 3-cells of our 3-category) we must find a composite for

$$\begin{pmatrix} f, & x_1, & \ldots, & x_k \\ \alpha \Downarrow, & \Downarrow \theta_1, & \ldots, & \Downarrow \theta_k \\ g, & y_1, & \ldots, & y_k \end{pmatrix}$$

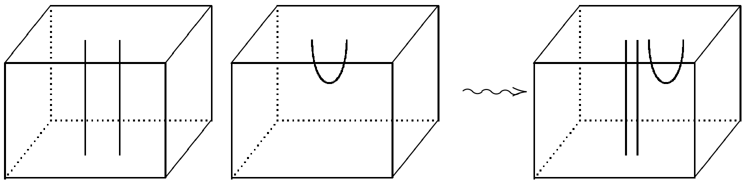
We take the equivalence class of the following composite.

$$I \times I \times [1] \overset{1\times\Delta\times 1}{\longrightarrow} I \times I \times I \times [1] \overset{1\times 1\times\alpha}{\longrightarrow} I \times I \times [k] \overset{\theta_1+\cdots+\theta_k}{\longrightarrow} \mathbf{2}$$

giving a subset of I^3 defined explicitly on elements by

$$(z_1, z_2, z_3) \mapsto (\theta_1 + \cdots + \theta_k)(z_1, z_2, \alpha(z_2, z_3))$$

and we then take the required equivalence class. This composition is illustrated by the following diagram.



As before, it is straightforward to check that this has the correct source and target, that it satisfies the conditions for being a 3-cell, and that the composition is a functor satisfying the required properties.

Composition along a bounding 0-cell. Recall that there is only one 0-cell, so there is only one hom-2-category. We write this as X; then for all $k \geq 0$ we need a 2-functor

$$\Pi_2(E_s(k)) \times X^k \longrightarrow X.$$

On 0-cells (i.e. 1-cells of our 3-category) this is trivial.
On 1-cells (i.e., 2-cells of our 3-category), we have on the left a tuple

$$(f, x_1, x_2, \ldots, x_k)$$

where f is a map $[1] \to [k]$ and each x_i is a subset of I^3, which we write as a function $x_i : I^3 \to \mathbf{2}$. We form the required composite by the following map:

$$I \times [1] \overset{\Delta\times 1}{\longrightarrow} I \times I \times [1] \overset{1\times\alpha}{\longrightarrow} I \times [k] \overset{\theta_1+\cdots+\theta_k}{\longrightarrow} \mathbf{2}$$

as for the 2-cells in the 2-category case, but now without the equivalence relation.

On 2-cells (i.e. 3-cells of our 3-category) we have on the left a tuple $(\phi, \gamma_1, \ldots, \gamma_k)$ where $\phi : \alpha \longrightarrow \alpha'$ is a map

$$I \times I \times [1] \longrightarrow [k]$$

and each $\gamma_i : \theta_i \longrightarrow \theta_i{}'$ is a map

$$I \times I \times I \longrightarrow \mathbf{2}.$$

We make the following composite

$$I_1 \times I_2 \times [1] \stackrel{\Delta\times 1}{\longrightarrow} I_1 \times I_2 \times I_1 \times I_2 \times [1] \stackrel{1\times 1\times\phi}{\longrightarrow} I_1 \times I_2 \times [k] \stackrel{\gamma_1+\cdots+\gamma_k}{\longrightarrow} \mathbf{2}$$

giving a subset of I^3 defined explicitly on elements by

$$(z_1, z_2, z_3) \mapsto (\gamma_1 + \cdots + \gamma_k)(z_1, z_2, \phi(z_1, z_2, z_3)).$$

(The subscripts on the I's here are simply to show which component is which.)

As before, we must check that this has the correct source and target, that it satisfies the conditions for being a 3-cell, and that the composition is a functor satisfying the required properties.

Note that we expect these constructions to generalise to 1-manifolds in n-cubes for any n, and indeed k-manifolds in n-cubes for any valid n and k. This is the subject of the next section.

6.3. k-MANIFOLDS IN n-CUBES. The above low-dimensional examples indicate how the construction should go for general k and n, albeit not how to prove that the construction "works", in the following sense. The constructions above can be thought of as having two components:

1. For all x we construct an n-category of "subsets of n-cubes" where the m-cells are given by *any* characteristic functions $I^m \longrightarrow \mathbf{2}$ and the n-cells also have an equivalence relation imposed – $\theta_1 \sim \theta_2$ if there is a self-diffeomorphism of I^m taking θ_1 to θ_2 and keeping the boundary pointwise fixed.

2. We take a sub-n-category of the above (rather crude) n-category, whose m-cells are those subsets of I^m which are in fact manifolds of the desired dimension with boundary. (The desired dimension works out to be $m - n + k$ in this case.)

The difficulty with part (1) is in describing the n-cells, that is, determining the equivalence classes of $\sim$. The difficulty with part (2) is in proving that this collection of cells is closed under composition and so does define a sub-n-category; this involves proving that some subsets of I^m are manifolds (with boundary), and in full generality this is beyond the scope of this paper.

However, making the construction for part (1) is not hard (it mostly involves counting dimensions in order to get subscripts correct) and this is the subject of the rest of this section.

We will use the same operad E_s as before, and the following m-cells.

- 0-cells are trivial, thought of as before as $\varnothing \in I^0$

- 1-cells are subsets of I, i.e. functions $f : I \longrightarrow \mathbf{2}$
- 2-cells $f \to g$ are subsets of I^2 i.e. functions $\alpha : I^2 \longrightarrow \mathbf{2}$ such that $\alpha(0,-) = f,\ \alpha(1,-) = g$
- m-cells $\theta \to \phi$ are subsets of I^m i.e. functions $\gamma : I^m \longrightarrow \mathbf{2}$ such that $\gamma(0,-,\ldots,-) = \theta,\ \gamma(1,-,\ldots,-) = \phi$
- n-cells are as for the general case above, subject to the equivalence relation defined earlier.

Now we define k-fold composition of m-cells along bounding j-cells. Given m-cells $\theta_1,\ldots,\theta_k$ with suitable source/target compatibility at the j-cell level, and α an $(m-j-1)$-cell of $\Pi_{n-j-1}(E_s(k))$, we need to specify a composite as a subset of I^m. We define this to be the characteristic function that takes $(z_1,\ldots,z_m) \in I^m$ to

$$(\theta_1 + \cdots + \theta_k)\big(z_1, z_2, \ldots, z_{m-1}, \alpha(z_{j+1},\ldots,z_m)\big).$$

For the case $m = n$ we use any reparametrisation α and then take the equivalence class of the result.

Note that the 0-cells of $\Pi_{n-j-1}(E_s(k))$ are certain maps $I \longrightarrow [k]$ and a quick dimension-check shows that the $(m-j-1)$-cells are thus certain maps $I^{m-j} \longrightarrow [k]$, so in the above formula α does have the correct number of inputs. Each θ_i is a function $I^m \longrightarrow \mathbf{2}$ so $\theta_1 + \cdots + \theta_k$ is a function $I^{m-1} \times [k] \longrightarrow \mathbf{2}$. Thus we have defined a function $I^m \longrightarrow \mathbf{2}$ as required.

Remarks. The condition forcing the 0-cells here to be trivial ensures that this putative n-category is globular (cells satisfying the globularity conditions $ss = st, ts = tt$) and not “cubical”. This might seem unnatural for an n-category of cubes, but in the case of manifolds in cubes it arises because our 0-manifolds can always be embedded in the open cube $(0,1)^m$ leaving the edges “empty”. However, for more complicated TQFTs such as open-closed TQFT it may be natural and/or necessary to drop this “globular” condition and build cubical n-categories. Cubical n-categories have a similar flavour to n-categories but raise some very different issues; as for n-categories they are currently only well understood in low dimensions or strict cases (see for example [1, 18, 8]).

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Department of Mathematics
University of Chicago
Email: eugenia@math.uchicago.edu, gurski@math.uchicago.edu

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