Theory and Applications of Categories, Vol. 18, No. 13, 2007, pp. 348–371.

COHERENT UNIT ACTIONS ON REGULAR OPERADS AND HOPF ALGEBRAS

KURUSCH EBRAHIMI-FARD AND LI GUO

Abstract. J.-L. Loday introduced the concept of coherent unit actions on a regular operad and showed that such actions give Hopf algebra structures on the free algebras. Hopf algebras obtained this way include the Hopf algebras of shuffles, quasi-shuffles and planar rooted trees. We characterize coherent unit actions on binary quadratic regular operads in terms of linear equations of the generators of the operads. We then use these equations to classify operads with coherent unit actions. We further show that coherent unit actions are preserved under taking products and thus yield Hopf algebras on the free object of the product operads when the factor operads have coherent unit actions. On the other hand, coherent unit actions are never preserved under taking the dual in the operadic sense except for the operad of associative algebras.

1. Introduction

While the original motivation for the study of the dendriform dialgebra (also called the dendriform algebra) [20, 21] was to study the periodicity of algebraic K-groups, it soon became clear that dendriform dialgebras are an interesting subject on its own. This can be seen on one hand by its quite extensive study by several authors in areas related to operads [25], homology [10, 11], combinatorics [2, 3, 9, 28], arithmetic [24], quantum field theory [9] and especially Hopf algebras [2, 5, 15, 30, 36]. On the other hand it has several generalizations and extensions that share properties of the original dendriform dialgebra. These new structures include the dendriform trialgebra (also called the tridendriform algebra) [29], the dipterous algebra [30], the quadri-algebra [4], the 2-associative algebra [30, 31, 34], the magmatic algebra [12] and the ennea-algebra. In fact, they are special cases of a class of binary quadratic regular operads that will be made precise later in Section 2.1.

It is remarkable that many of these algebra structures have a Hopf algebra structure on the free algebras. For example, the free commutative dendriform dialgebras and trialgebras are the shuffle and quasi-shuffle Hopf algebras, and the free dendriform dialgebra and trialgebras are the Hopf algebra of binary planar rooted trees [23, 28] and planar rooted trees [29]. These findings were put in a general framework recently by Loday [25] who showed that the existence of a coherent unit action on a binary quadratic regular operad with a splitting of associativity endows the free objects with a Hopf algebra struc-

Received by the editors 2007-06-02 and, in revised form, 2007-08-08.
Transmitted by Jean-Louis Loday. Published on 2007-08-09.
2000 Mathematics Subject Classification: 18D50, 17A30, 16W30.
Key words and phrases: dendriform algebras, coherent unit actions, regular operads, Hopf algebras.

ture (Theorem 2.4). Since then, this method has been applied to obtain Hopf algebra structures on several other operads [17, 18, 19, 25].

Thus it is desirable to obtain a good understanding of such operads with a coherent unit action and therefore with a Hopf algebra structure on the free objects. This is our goal to achieve in this paper, by working with the generators and relations of these regular operads [7, 26]. As a result, we explicitly describe a large class of operads that give rise to Hopf algebras.

After briefly recalling related concepts and results, we first give in Section 2 a simple criterion for a unit action to be coherent. This criterion reduces the checking of the coherence condition to the verification of a system of linear equations, called coherence equations. Then in Section 3 we use the coherence equations to obtain a classification of binary, quadratic, regular operads that allows a coherent unit action. Special cases are studied and are related to examples in the current literature.

The compatibility of coherent unit actions on operads with taking operad products and duals is studied in Section 4. We show that the coherence condition is preserved by taking products. Thus the Hopf algebra structure on the product operad follows automatically from those on the factor operads, as long as the factor operads have coherent unit actions. In contrast to products, we show that the coherence condition is never preserved by taking the dual in the operadic sense, except for the trivial case when the operad is the one for associative algebras.

We give a similar study of the related notion of compatible unit actions [25]. It is related to, but weaker than the notion of coherent unit actions.

ACKNOWLEDGEMENTS: The first named author would like to thank the Ev. Studienwerk Villigst and the theory department of the Physikalisches Institut, at Bonn University for generous support. The second named author is supported in part by NSF grant DMS 0505643 and a grant from the Research Council of the Rutgers University.

2. Compatible and coherent unit actions

2.1. ABQR OPERADS. We will consider binary quadratic regular operads. We recall their standard definition of algebraic operads before rephrasing them in the more explicit form in terms of generators and relations. Since we will not need the general definition in the rest of the paper, we refer the interested reader to find further details in the standard references, such as [13, 22, 32, 33].

Let $\mathbf{k}$ be a field of characteristic zero and let **Vect** be the category of $\mathbf{k}$-vector spaces. An algebraic operad over $\mathbf{k}$ is an analytic functor $\mathcal{P}: \mathbf{Vect} \to \mathbf{Vect}$ such that $\mathcal{P}(0) = 0$, and is equipped with a natural transformation of functors $\gamma : \mathcal{P}\circ\mathcal{P} \to \mathcal{P}$ which is associative and has a unit $1 : \mathrm{id} \to \mathcal{P}$.

By considering free $\mathcal{P}$-algebras, an operad gives a sequence $\{\mathcal{P}(n)\}$ of finitely generated $\mathbf{k}[S_n]$-modules that satisfy certain composition axioms. An operad is called **binary** if $\mathcal{P}(1) = \mathbf{k}$ and $\mathcal{P}(2)$ generates $\mathcal{P}(n), n \geq 3$ by composition; is called **quadratic** if all relations among the binary operations $\mathcal{P}(2)$ are derived from relations in $\mathcal{P}(3)$; is called

regular if, moreover, the binary operations have no symmetries and the variables x, y and z appear in the same order in the relations (such as $(x \cdot y) \cdot z = x \cdot (y \cdot z)$, not $(x \cdot y) \cdot z = x \cdot (z \cdot y)$).

By regularity, the space $\mathcal{P}(n)$ is of the form $\mathcal{P}_n \otimes \mathbf{k}[S_n]$ where $\mathcal{P}_n$ is a vector space. So the operad $\{\mathcal{P}(n)\}$ is determined by $\{\mathcal{P}_n\}$. Then a binary, quadratic, regular operad is determined by a pair (Ω, Λ) where $\Omega = \mathcal{P}_2$, called the **space of generators**, and Λ is a subspace of $\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$, called the **space of relations**. So we write $\mathcal{P} = (\Omega, \Lambda)$ for the operad. Since (Ω, Λ) is determined by (ω, λ) where ω is a basis of Ω and λ is a basis of Λ, we also use (ω, λ) to denote a binary, quadratic, regular operad, as is usually the case in the literature.

For such a $\mathcal{P} = (\Omega, \Lambda)$, a $\mathbf{k}$-vector space A is called a $\mathcal{P}$-algebra if it has binary operations Ω and if, for

$$\Big(\sum_{i=1}^{k} \odot_i^{(1)} \otimes \odot_i^{(2)}, \sum_{j=1}^{m} \odot_j^{(3)} \otimes \odot_j^{(4)}\Big) \in \Lambda \subseteq \Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$$

with $\odot_i^{(1)}, \odot_i^{(2)}, \odot_j^{(3)}, \odot_j^{(4)} \in \Omega$, $1 \leq i \leq k$, $1 \leq j \leq m$, we have

$$\sum_{i=1}^{k} (x \odot_i^{(1)} y) \odot_i^{(2)} z = \sum_{j=1}^{m} x \odot_j^{(3)} (y \odot_j^{(4)} z), \forall\, x, y, z \in A. \tag{1}$$

We say that a binary, quadratic, regular operad (Ω, Λ) **has a splitting associativity** if there is a choice of $\star$ in Ω such that $(\star \otimes \star, \star \otimes \star)$ is in Λ [25]. As an abbreviation, we call such an operad an **associative BQR operad**, or simply an **ABQR operad**. Equivalently [7, Lemma 2.1], a binary, quadratic, regular operad is ABQR if and only if there is a basis $\omega = \{\omega_i\}$ of Ω such that $\star = \sum_i \omega_i$ and there is a basis $\lambda = \{\lambda_j\}_j$ of Λ such that the associativity of $\star$ is given by the sum of λ_j, giving a splitting of the associativity of $\star$: $(\star \otimes \star, \star \otimes \star) = \sum_j \lambda_j$. Note that a binary quadratic regular operad (Ω, Λ) might have different choices of associative operations. For example, if $\star$ is associative, then so is $c\star$ for any nonzero $c \in \mathbf{k}$. As we will see later, some operads even have linear independent associative operations.

To be precise, we let $(\Omega, \Lambda, \star)$, or $(\omega, \lambda, \star)$, denote an ABQR operad with $\star$ as the chosen associative operation. Let $(\Omega, \Lambda, \star)$ and $(\Omega', \Lambda', \star')$ be ABQR operads with associative operations $\star$ and $\star'$ respectively. A **morphism** $f : (\Omega, \Lambda, \star) \to (\Omega', \Lambda', \star)$ is a linear map from Ω to Ω' sending $\star$ to $\star'$ and inducing a linear map from Λ to Λ'. An invertible morphism is called an **isomorphism**.

The following examples of ABQR operads will be used later in the paper.

2.2. Example.

1. An associative $\mathbf{k}$-algebra is a $\mathbf{k}$-vector space A with an associative product $\cdot$. The corresponding operad is $\mathcal{P}_A = (\omega_A, \lambda_A, \cdot)$ with $\omega_A = \{\cdot\}$ and $\lambda_A = \{(\cdot \otimes \cdot, \cdot \otimes \cdot)\}$.

2. The **dendriform dialgebra** of Loday [23] corresponds to the operad $\mathcal{P}_D = (\omega_D, \lambda_D, \star_D)$ with $\omega_D = \{\prec, \succ\}$, $\star_D = \prec + \succ$ and

$$\lambda_D = \{(\prec \otimes \prec, \prec \otimes(\prec + \succ)), (\succ \otimes \prec, \succ \otimes \prec), ((\prec + \succ)\otimes \succ, \succ \otimes \succ)\}. \qquad (2)$$

3. The **dendriform trialgebra** of Loday and Ronco [29] corresponds to the operad $\mathcal{P}_T = (\omega_T, \lambda_T, \star_T)$ with $\omega_T = \{\prec, \succ, \circ\}$, $\star_T = \prec + \succ + \circ$ and

$$\begin{aligned}\lambda_T = \ & \{(\prec \otimes \prec, \prec \otimes \star), (\succ \otimes \prec, \succ \otimes \prec), (\star \otimes \succ, \succ \otimes \succ), (\succ \otimes \circ, \succ \otimes \circ), \\ & (\prec \otimes \circ, \circ \otimes \succ), (\circ \otimes \prec, \circ \otimes \prec), (\circ \otimes \circ, \circ \otimes \circ)\}. \end{aligned} \qquad (3)$$

4. Leroux's **NS-algebra** [18] corresponds to the operad $\mathcal{P}_N = (\omega_N, \lambda_N, \star_N)$ with $\omega_N = \{\prec, \succ, \bullet\}$, $\star_N = \prec + \succ + \bullet$ and

$$\begin{aligned}\lambda_N = \ & \{(\prec \otimes \prec, \prec \otimes \star), (\succ \otimes \prec, \succ \otimes \prec), (\star \otimes \succ, \succ \otimes \succ), \\ & (\star \otimes \bullet + \bullet \otimes \prec, \succ \otimes \bullet + \bullet \otimes \star)\}. \end{aligned} \qquad (4)$$

2.3. Compatible and coherent unit actions. We now review the concepts of coherent and compatible actions on regular operads, and the theorem of Loday showing that the existence of coherent unit actions yields Hopf algebras.

Let $\mathcal{P} = (\Omega, \Lambda, \star)$ be an ABQR operad. A **unit action** on $\mathcal{P}$ is a choice of two linear maps

$$\alpha,\ \beta : \Omega \to \mathbf{k}$$

such that $\alpha(\star) = \beta(\star) = 1$, the unit of $\mathbf{k}$.

Let A be a $\mathcal{P}$-algebra. A unit action (α, β) allows us to extend a binary operation $\odot \in \Omega$ on A to a restricted binary operation on $A_+ := \mathbf{k}.1 \oplus A$ by defining

$$\tilde{\odot} : A_+ \otimes A_+ \to A_+, \quad a\tilde{\odot}b := \begin{cases} a \odot b, & a, b \in A, \\ \alpha(\odot)\, a, & a \in A, b = 1, \\ \beta(\odot)\, b, & a = 1, b \in A, \\ 1, & a = b = 1, \odot = \star, \\ \text{undefined}, & a = b = 1, \odot \neq \star. \end{cases} \qquad (5)$$

Thus the extended binary operation $\tilde{\odot}$ is defined on the subspace $(\mathbf{k}.1 \otimes A) \oplus (A \otimes \mathbf{k}.1) \oplus (A \otimes A)$ of $A_+ \otimes A_+$, and on the full space $A_+ \otimes A_+$ when $\odot = \star$. The unit action is called **compatible** if the relations of $\mathcal{P}$ are still valid on A_+ for each $\mathcal{P}$-algebra A whenever the terms are defined. More precisely, if $(\sum_i \odot_i^{(1)} \otimes \odot_i^{(2)}, \sum_j \odot_j^{(3)} \otimes \odot_j^{(4)})$ is a relation of $\mathcal{P}$, then

$$\sum_i (x\tilde{\odot}_i^{(1)}y)\tilde{\odot}_i^{(2)}z = \sum_j x\tilde{\odot}_j^{(3)}(y\tilde{\odot}_j^{(4)}z),\ \forall\, x, y, z \in A_+,$$

whenever the two sides make sense.

Next let A, B be two $\mathcal{P}$-algebras where $\mathcal{P}$ has a compatible unit action (α, β). Consider the subspace $A \boxtimes B := (A \otimes \mathbf{k}.1) \oplus (\mathbf{k}.1 \otimes B) \oplus (A \otimes B)$. For $\odot \in \Omega$, define a binary operation

$$\boxdot : (A \boxtimes B) \otimes (A \boxtimes B) \to A \boxtimes B,$$

$$(a \otimes b) \boxdot (a' \otimes b') := \begin{cases} (a \tilde{\star} a') \otimes (b \tilde{\odot} b'), & \text{if } b \otimes b' \neq 1 \otimes 1, \\ (a \tilde{\odot} a') \otimes 1, & \text{otherwise.} \end{cases} \tag{6}$$

See [35] for the case of dendriform dialgebras. We say that the unit action (α, β) is **coherent** if, for any $\mathcal{P}$-algebras A and B, the subspace $A \boxtimes B$, equipped with these operations is still a $\mathcal{P}$-algebra. We note that $A_+ \otimes B_+ = \mathbf{k}.1 \oplus (A \boxtimes B) = (A \boxtimes B)_+$. Thus the associative operation $\star$ gives an associative operation $\star$ on $A \boxtimes B$ which, as in (5), extends to an associative operation $\tilde{\star}$ on $A_+ \otimes B_+$.

By definition, a coherent unit action on $\mathcal{P}$ is also compatible. The converse is not true. See Example 2.8.

The significance of the coherence property can be seen in the following theorem of Loday [25]. We refer the reader to the original paper for further details.

2.4. Theorem. *Let $\mathcal{P}$ be an ABQR operad. Let $\mathcal{P}(V)_+$ be the augmented free $\mathcal{P}$-algebra on a $\mathbf{k}$-vector space V. Any coherent unit action on $\mathcal{P}$ equips $\mathcal{P}(V)_+$ with a connected Hopf algebra structure.*

The Hopf algebra structure is in fact a $\mathcal{P}$-Hopf algebra structure in the sense that the coproduct is a morphism of augmented $\mathcal{P}$-algebras.

For example, it was shown in [25] that, for the dendriform trialgebra $\mathcal{P}_T = (\omega_T, \lambda_T, \star_T)$ with $\omega_T = \{\prec, \succ, \circ\}$, the unit action

$$\alpha(\prec) = \beta(\succ) = 1, \alpha(\succ) = \alpha(\circ) = \beta(\prec) = \beta(\circ) = 0$$

is coherent with the relations of $\mathcal{P}_T$, and thus the free $\mathcal{P}_T$-algebra on a $\mathbf{k}$-vector space has a Hopf algebra structure. This is the Hopf algebra of planar rooted trees [28]. The same method also recovers the Hopf algebra structure on the free dendriform dialgebra as planar binary rooted trees [23], and applies to obtain Hopf algebra structures on several other algebras [17, 18, 19, 25] (see Corollary 4.5). Furthermore, a variation of Theorem 2.4 applies to Zinbiel algebras to recover the shuffle Hopf algebra [25] and applies to commutative trialgebras to recover the quasi-shuffle Hopf algebra of Hoffman [14]. See [27].

The understandings gained in this paper on coherent unit actions on ABQR operads, especially the classification (Theorem 3.2), give us the precise information on the kind of operads that we should expect a Hopf algebra structure on the free objects.

Coherent unit actions are also defined by Loday [25] for binary, quadratic operads without the regularity condition and are recently extended to general algebraic operads by Holtkamp [16]. In these cases, the free objects of the operad $\mathcal{P}$ have the structure of a $\mathcal{P}$-Hopf algebra, a more general concept than Hopf algebra. It would be interesting to extend results in this paper to the general case.

2.5. Coherence equations. We provide an equivalent condition of the compatibility and coherence of unit actions in terms of linear relations among the binary operations in an operad. This criterion will be applied in Theorem 3.2 to classify ABQR operads with coherent or compatible unit actions.

2.6. Theorem. *Let $\mathcal{P} = (\Omega, \Lambda, \star)$ be an ABQR operad.*

1. *A unit action (α, β) on $\mathcal{P}$ is coherent if and only if, for every*
$$(\sum_i \odot_i^{(1)} \otimes \odot_i^{(2)}, \sum_j \odot_j^{(3)} \otimes \odot_j^{(4)}) \in \Lambda,$$
the following **coherence equations** *hold.*
   - **(C1)** $\sum_i \beta(\odot_i^{(1)})\odot_i^{(2)} = \sum_j \beta(\odot_j^{(3)})\odot_j^{(4)}$,
   - **(C2)** $\sum_i \alpha(\odot_i^{(1)})\odot_i^{(2)} = \sum_j \beta(\odot_j^{(4)})\odot_j^{(3)}$,
   - **(C3)** $\sum_i \alpha(\odot_i^{(2)})\odot_i^{(1)} = \sum_j \alpha(\odot_j^{(4)})\odot_j^{(3)}$,
   - **(C4)** $\sum_i \beta(\odot_i^{(2)})\odot_i^{(1)} = \sum_j \beta(\odot_j^{(3)})\beta(\odot_j^{(4)})\star$,
   - **(C5)** $\sum_i \alpha(\odot_i^{(1)})\alpha(\odot_i^{(2)})\star = \sum_j \alpha(\odot_j^{(3)}) \odot_j^{(4)}$.
2. *A unit action (α, β) on $\mathcal{P}$ is compatible if and only if, for every*
$$(\sum_i \odot_i^{(1)} \otimes \odot_i^{(2)}, \sum_j \odot_j^{(3)} \otimes \odot_j^{(4)}) \in \Lambda,$$
equations (C1), (C2) and (C3) above hold.

Before proving Theorem 2.6, we give examples to show how it can be used to determine compatible and coherent unit actions.

2.7. Example. Consider the dendriform dialgebra $\mathcal{P}_D = (\omega_D, \lambda_D)$ in Eq.(2). So $\omega_D = \{\prec, \succ\}$ and
$$\lambda_D = \{(\prec \otimes \prec, \prec \otimes(\prec + \succ)), (\succ \otimes \prec, \succ \otimes \prec), ((\prec + \succ)\otimes \succ, \succ \otimes \succ)\}.$$
Suppose (α, β) is a coherent unit action of $\mathcal{P}_D$. Then the three equations in λ_D satisfy (C1) – (C5). Applying (C1) to the first relation in λ_D, we obtain $\beta(\prec) \prec = \beta(\prec)(\prec + \succ)$. So $\beta(\prec) \succ = 0$. Therefore $\beta(\prec) = 0$. Applying (C2) to the first relation, we have $\alpha(\prec) \prec = \prec$ since $\beta(\prec + \succ) = \beta(\star) = 1$. Thus $\alpha(\prec) = 1$. Similarly, applying (C2) to the second equation, we have $\alpha(\succ) \prec = \beta(\prec) \succ$. So $\alpha(\succ) = 0$. Applying (C1) to the third equation gives $\beta(\prec + \succ) \succ = \succ = \beta(\succ) \succ$. Thus $\beta(\succ) = 1$. Therefore, the only coherent unit action on $\mathcal{P}_D$ is the one given in [23]:
$$\alpha(\prec) = \beta(\succ) = 1, \alpha(\succ) = \beta(\prec)) = 0.$$
Note that we have only used (C1) – (C3). So the above is also the only compatible unit action of $\mathcal{P}_D$.

We will comment on the trialgebra case in Corollary 3.6.

2.8. Example. We consider the 2-associative algebra in [30] and [34]. It is given by generators $\omega = \{*, \cdot\}$ and relations $\lambda = \{(* \otimes *, * \otimes *), (\cdot \otimes \cdot, \cdot \otimes \cdot)\}$. Consider the unit action (α, β) in [25] given by $\alpha(*) = \alpha(\cdot) = \beta(*) = \beta(\cdot) = 1$. We show that the action is not coherent regardless of the choice of the associative operation $\star$. Suppose (α, β) is coherent. Then applying (C4) to $(* \otimes *, * \otimes *)$, we have $\beta(*)* = \beta(*)\beta(*)\star$. So $* = \star$. Applying (C4) to $(\cdot \otimes \cdot, \cdot, \otimes \cdot)$, we have $\beta(\cdot)\cdot = \beta(\cdot)\beta(\cdot)\star$. So $\cdot = \star$. This is impossible. So Theorem 2.4 cannot be applied to give a Hopf algebra structure on free 2-associative algebras.

However, by verifying (C1) - (C3), we see that the unit action is compatible when $\star$ is taken to be $*$. Loday and Ronco [30, 31] have equipped a free 2-associative algebra with a Hopf algebra structure with $*$ as the product. This suggests a possible connection between compatibility and Hopf algebras.

Similar arguments show that the associative dialgebra (see Example 4.7) and the operads $\mathcal{X}_\pm$ in [26] have no coherent unit actions.

Proof. (1) Let (α, β) be a unit action on $\mathcal{P}$. We say that the unit action (α, β) is coherent with a relation $(\sum_i \odot_i^{(1)} \otimes \odot_i^{(2)}, \sum_j \odot_j^{(3)} \otimes \odot_j^{(4)})$ in Λ if, for any $\mathcal{P}$-algebras A and B, the operations $\odot \in \Omega$, when extended to $A \boxtimes B$ by Eq. (6), still satisfy the same relation. Then to prove the theorem, we only need to prove that (α, β) is coherent with a given relation $(\sum_i \odot_i^{(1)} \otimes \odot_i^{(2)}, \sum_j \odot_j^{(3)} \otimes \odot_j^{(4)}) \in \Lambda$ if and only if (C1) – (C5) hold for this relation.

Further, by definition, (α, β) is coherent with $(\sum_i \odot_i^{(1)} \otimes \odot_i^{(2)}, \sum_j \odot_j^{(3)} \otimes \odot_j^{(4)})$ means that, for any $\mathcal{P}$-algebra A and B and for any $a, a', a'' \in A_+$ and $b, b', b'' \in B_+$ such that at least one of b, b', b'' is not 1, we have the equation

$$\sum_i \big((a \otimes b) \odot_i^{(1)} (a' \otimes b')\big) \odot_i^{(2)} (a'' \otimes b'') = \sum_j (a \otimes b) \odot_j^{(3)} \big((a' \otimes b') \odot_j^{(4)} (a'' \otimes b'')\big). \quad (7)$$

Thus there are 7 mutually disjoint cases for the choice of such b, b', b'': the case when none of b, b', b'' is 1, the three cases when exactly one of b, b', b'' is 1, and the three cases when exactly two of b, b', b'' are 1. Note that when none of b, b', b'' is 1, Eq. (7) just means that $(\sum_i \odot_i^{(1)} \otimes \odot_i^{(2)}, \sum_j \odot_j^{(3)} \otimes \odot_j^{(4)})$ is a relation for $\mathcal{P}$, so is automatic true. Thus to prove the theorem we only need to prove

Case 1. Eq. (7) holds for $b = 1$, $b' \neq 1 \neq b''$ if and only if (C1) is true;

Case 2. Eq. (7) holds for $b' = 1$, $b \neq 1 \neq b''$ if and only if (C2) is true;

Case 3. Eq. (7) holds for $b'' = 1$, $b \neq 1 \neq b'$ if and only if (C3) is true;

Case 4. Eq. (7) holds for $b = b' = 1$, $b'' \neq 1$ if and only if (C4) is true;

Case 5. Eq. (7) holds for $b' = b'' = 1$, $b \neq 1$ if and only if (C5) is true;

Case 6. Eq. (7) holds for $b = b'' = 1$, $b' \neq 1$ if (C1) is true.

We first consider the three cases when exactly one of b, b', b'' is 1. Then by the definition of the operation $\odot$ on $A \boxtimes B$ in Eq. (6), we can rewrite Eq. (7) as

$$\sum_i \Big(\big((a \tilde{\star} a')\tilde{\star} a''\big) \otimes \big((b \tilde{\odot}_i^{(1)} b')\tilde{\odot}_i^{(2)} b''\big)\Big) = \sum_j \Big(\big(a \tilde{\star}(a' \tilde{\star} a'')\big) \otimes \big(b \tilde{\odot}_j^{(3)} (b' \tilde{\odot}_j^{(4)} b'')\big)\Big), \qquad (8)$$

for all $1 \otimes 1 \otimes 1 \neq b \otimes b' \otimes b'' \in (B_+)^{\otimes 3}$. Since $\tilde{\star}$ is associative, by the arbitrariness of A and $a, a', a'' \in A_+$ (say by taking $a = a' = a'' = 1$), we see that Eq. (8), and hence Eq. (7), is equivalent to

$$\sum_i \big((b \tilde{\odot}_i^{(1)} b')\tilde{\odot}_i^{(2)} b''\big) = \sum_j \big(b \tilde{\odot}_j^{(3)} (b' \tilde{\odot}_j^{(4)} b'')\big). \qquad (9)$$

Case 1. Assume $b = 1$ and $b', b'' \neq 1$. Then Eq. (9) is

$$\sum_i (1 \tilde{\odot}_i^{(1)} b')\tilde{\odot}_i^{(2)} b'' = \sum_j 1 \tilde{\odot}_j^{(3)} (b' \tilde{\odot}_j^{(4)} b'')$$

and, by Eq. (5), this means

$$\sum_i \beta(\odot_i^{(1)}) b' \odot_i^{(2)} b'' = \sum_j \beta(\odot_j^{(3)})(b' \odot_j^{(4)} b'').$$

That is,

$$b' \big(\sum_i \beta(\odot_i^{(1)}) \odot_i^{(2)} \big) b'' = b' \big(\sum_j \beta(\odot_j^{(3)}) \odot_j^{(4)} \big) b'',$$

for every $\mathcal{P}$-algebra B and $b', b'' \in B$. By the following Lemma 2.9, this is true if and only if

$$\sum_i \beta(\odot_i^{(1)}) \odot_i^{(2)} = \sum_j \beta(\odot_j^{(3)}) \odot_j^{(4)},$$

giving (C1).

2.9. Lemma. *For $\odot_1, \odot_2 \in \Omega$, we have $\odot_1 = \odot_2$ if and only if $a \odot_1 a' = a \odot_2 a'$ for all $\mathcal{P}$-algebras A and $a, a' \in A$.*

Proof. The only if part is clear. Now suppose $a \odot_1 a' = a \odot_2 a'$ for all $\mathcal{P}$-algebras A and $a, a' \in A$. Let A be the free $\mathcal{P}$-algebra on one generator x. Then

$$A = \mathbf{k}x \oplus \mathcal{P}_2 \oplus \cdots .$$

Here $\mathcal{P}_2 = \oplus_i \mathbf{k}\omega_i$ in which $\{\omega_i\}$ is a basis of Ω. Also, a binary operation $\odot \in \Omega$ acts on $A \otimes A$ by $x \odot x = \odot \in \mathcal{P}_2$. Thus we have $\odot_1 = x \odot_1 x = x \odot_2 x = \odot_2$. This proves the if part. ∎

Case 2. Assume $b' = 1$ and $b, b'' \neq 1$. As in Case 1, we have

$$\begin{aligned}
\text{Eq.(9)} \quad & \Leftrightarrow \sum_i (b \tilde{\odot}_i^{(1)} 1) \tilde{\odot}_i^{(2)} b'' = \sum_j b \tilde{\odot}_j^{(3)} (1 \tilde{\odot}_j^{(4)} b''), \ \forall\, b, b'' \in B, \\
& \overset{(5)}{\Leftrightarrow} \sum_i b\alpha(\odot_i^{(1)}) \odot_i^{(2)} b'' = \sum_j b\beta(\odot_j^{(4)}) \odot_j^{(3)} b'', \ \forall\, b, b'' \in B, \\
& \Leftrightarrow \sum_i \alpha(\odot_i^{(1)}) \odot_i^{(2)} = \sum_j \beta(\odot_j^{(4)}) \odot_j^{(3)} \\
& \Leftrightarrow \text{(C2)}.
\end{aligned}$$

Case 3. Assume $b, b' \neq 1$ and $b'' = 1$. As in Case 1, we have

$$\text{Eq.(9)} \quad \overset{(5)}{\Leftrightarrow} \sum_i (b \odot_i^{(1)} b') \alpha(\odot^{(2)}) = \sum_j b \odot_j^{(3)} b' \alpha(\odot_j^{(4)}), \ \forall\, b, b' \in B \ \Leftrightarrow \text{(C3)}.$$

We next consider the three cases when exactly two of b, b', b'' are the identity.

Case 4. Assume $b = b' = 1$ and $b'' \neq 1$. Now Eq. (8) does not apply. Directly from Eq. (6), we see that Eq. (7) means

$$\sum_i \big((a \tilde{\odot}_i^{(1)} a') \tilde{\star} a''\big) \otimes (1 \tilde{\odot}_i^{(2)} b'') = \sum_j \big(a \tilde{\star} (a' \tilde{\star} a'')\big) \otimes \big(1 \tilde{\odot}_j^{(3)} (1 \tilde{\odot}_j^{(4)} b'')\big).$$

Then by Eq. (5), we equivalently have

$$\sum_i \big((a \tilde{\odot}_i^{(1)} a') \tilde{\star} a''\big) \otimes (\beta(\odot_i^{(2)}) b'') = \sum_j \big(a \tilde{\star} (a' \tilde{\star} a'')\big) \otimes (\beta(\odot_j^{(3)}) \beta(\odot_j^{(4)}) b'').$$

This means, by moving the scalars $\beta(\odot_i^{(j)})$ across the tensor product,

$$\sum_i \beta(\odot_i^{(2)}) \big((a \tilde{\odot}_i^{(1)} a') \tilde{\star} a''\big) \otimes b'' = \sum_j \beta(\odot_j^{(3)}) \beta(\odot_j^{(4)}) \big(a \tilde{\star} (a' \tilde{\star} a'')\big) \otimes b''.$$

Since this is true for all B and $b'' \in B$, we equivalently have

$$\sum_i \beta(\odot_i^{(2)}) \big((a \tilde{\odot}_i^{(1)} a') \tilde{\star} a''\big) = \sum_j \beta(\odot_j^{(3)}) \beta(\odot_j^{(4)}) \big(a \tilde{\star} (a' \tilde{\star} a'')\big). \tag{10}$$

Taking $a'' = 1$, we have

$$\sum_i \beta(\odot_i^{(2)}) (a \tilde{\odot}_i^{(1)} a') = \sum_j \beta(\odot_j^{(3)}) \beta(\odot_j^{(4)}) (a \tilde{\star} a'). \tag{11}$$

Conversely, right multiplying a'' (by $\tilde{\star}$) to this equation and using the associativity of $\tilde{\star}$, we obtain Eq. (10). So Eq. (10) and Eq. (11) are equivalent.

When a and a' are in A, Eq. (11) becomes

$$\sum_i \beta(\odot_i^{(2)})(a \odot_i^{(1)} a') = \sum_j \beta(\odot_j^{(3)})\beta(\odot_j^{(4)})(a \star a'),$$

or, equivalently,

$$a\big(\sum_i \beta(\odot_i^{(2)}) \odot_i^{(1)}\big)a' = a\big(\sum_j \beta(\odot_j^{(3)})\beta(\odot_j^{(4)}) \star\big)a'.$$

By Lemma 2.9, we have

$$\sum_i \beta(\odot_i^{(2)})\odot_i^{(1)} = \sum_j \beta(\odot_j^{(3)})\beta(\odot_j^{(4)}) \star . \tag{12}$$

This is (C4). To go backwards, assuming (C4), then by the linearity of the map $\odot \mapsto \tilde{\odot}$ in Eq. (5), we get Eq. (11) for all augmented $\mathcal{P}$-algebras A_+. So we are done with Case 4.

Case 5. Assume $b \neq 1$ and $b' = b'' = 1$. The proof is the same as for Case 4.

Case 6. Assume $b = b'' = 1$ and $b' \neq 1$. Then Eq. (8) still applies and we get

$$\sum_i (1\tilde{\odot}_i^{(1)}b')\tilde{\odot}_i^{(2)}1 = \sum_j 1\tilde{\odot}_j^{(3)}(b'\tilde{\odot}_j^{(4)}1)$$

and, by Eq. (5), we get

$$\sum_i \big(\beta(\odot_i^{(1)})\alpha(\odot_i^{(2)})\big)b' = \sum_j \big(\beta(\odot_j^{(3)})\alpha(\odot_j^{(4)})\big)b'.$$

Thus

$$\sum_i \beta(\odot_i^{(1)})\alpha(\odot_i^{(2)}) = \sum_j \beta(\odot_j^{(3)})\alpha(\odot_j^{(4)}).$$

We note that this follows from applying α to (C1). So we do not get a new relation.

(2) We note that the precise meaning of compatibility of the unit action (α, β) with a relation $(\sum_i \odot_i^{(1)} \otimes \odot_i^{(2)}, \sum_j \odot_j^{(3)} \otimes \odot_j^{(4)})$ in the space of relations Λ of $\mathcal{P}$ is the requirement that Eq. (9) holds for any $\mathcal{P}$-algebra B in the above proof. Thus the verification of the compatibility condition is equivalent to the verification of (C1), (C2) and (C3). This proves (2) of Theorem 2.6. ∎

3. Classification of operads with coherent unit actions

We now apply Theorem 2.6 to classify ABQR operads with coherent unit actions and compatible unit actions. We then discuss some special cases.

3.1. The classifications. We display the relations of ABQR operads $(\Omega, \Lambda, \star)$ that admit a coherent or compatible unit action.

3.2. Theorem. *Let $\mathcal{P} = (\Omega, \Lambda, \star)$ be an ABQR operad of dimension n (that is,* $\dim \Omega = n$*).*

1. *There is a coherent unit action (α, β) on $\mathcal{P}$ with $\alpha \neq \beta$ if and only if there is a basis $\{\odot_i\}$ of Ω with $\star = \sum_i \odot_i$ such that Λ is contained in the subspace $\Lambda'_{n,\mathrm{coh}}$ of $\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$ with the basis*

$$\lambda'_{n,\mathrm{coh}} := \left\{ \begin{array}{l} (\star \otimes \odot_2, \odot_2 \otimes \odot_2), \\ (\odot_1 \otimes \odot_1, \odot_1 \otimes \star), \\ (\odot_i \otimes \odot_1, \odot_i \otimes \odot_1),\ 2 \leq i \leq n, \\ (\odot_2 \otimes \odot_j, \odot_2 \otimes \odot_j),\ 3 \leq j \leq n, \\ (\odot_1 \otimes \odot_i, \odot_i \otimes \odot_2),\ 3 \leq i \leq n, \\ (\odot_i \otimes \odot_j, 0),\ (0, \odot_i \otimes \odot_j),\ 3 \leq i, j \leq n. \end{array} \right\} \tag{13}$$

2. *There is a coherent unit action (α, β) on $\mathcal{P}$ with $\alpha = \beta$ if and only if there is a basis $\{\odot_i\}$ of Ω with $\star = \sum_i \odot_i$ such that Λ is contained in the subspace $\Lambda''_{n,\mathrm{coh}}$ of $\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$ with the basis*

$$\lambda''_{n,\mathrm{coh}} := \left\{ \begin{array}{l} (\odot_1 \otimes \star, \odot_1 \otimes \star) + (\star \otimes \odot_1, \star \otimes \odot_1) - (\odot_1 \otimes \odot_1, \odot_1 \otimes \odot_1), \\ (\odot_i \otimes \odot_j, 0),\ (0, \odot_i \otimes \odot_j),\ 2 \leq i, j \leq n. \end{array} \right\}. \tag{14}$$

3. *There is a compatible unit action (α, β) on $\mathcal{P}$ with $\alpha \neq \beta$ if and only if there is a basis $\{\odot_i\}$ of Ω with $\star = \sum_i \odot_i$ such that Λ is contained in the subspace $\Lambda'_{n,\mathrm{comp}}$ of $\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$ with the basis*

$$\lambda'_{n,\mathrm{comp}} := \left\{ \begin{array}{l} ((\odot_1 + \odot_2) \otimes \odot_2, \odot_2 \otimes \odot_2), \\ (\odot_1 \otimes \odot_1, \odot_1 \otimes (\odot_1 + \odot_2)), \\ (\odot_i \otimes \odot_1, \odot_i \otimes \odot_1),\ 2 \leq i \leq n, \\ (\odot_2 \otimes \odot_j, \odot_2 \otimes \odot_j),\ 3 \leq j \leq n, \\ (\odot_1 \otimes \odot_i, \odot_i \otimes \odot_2),\ 3 \leq i \leq n, \\ (\odot_i \otimes \odot_2, 0), (0, \odot_1 \otimes \odot_i),\ 3 \leq i \leq n, \\ (\odot_i \otimes \odot_j, 0),\ (0, \odot_i \otimes \odot_j),\ 3 \leq i, j \leq n. \end{array} \right\} \tag{15}$$

4. *There is a compatible unit action (α, β) on $\mathcal{P}$ with $\alpha = \beta$ if and only if there is a basis $\{\odot_i\}$ of Ω with $\star = \sum_i \odot_i$ such that Λ is contained in the subspace $\Lambda''_{n,\mathrm{comp}}$ of $\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$ with the basis*

$$\lambda''_{n,\mathrm{comp}} := \left\{ \begin{array}{l} (\odot_1 \otimes \odot_1, \odot_1 \otimes \odot_1), \\ (\odot_1 \otimes \odot_i, \odot_1 \otimes \odot_i) + (\odot_i \otimes \odot_1, \odot_i \otimes \odot_1),\ 2 \leq i \leq n, \\ (\odot_i \otimes \odot_j, 0),\ (0, \odot_i \otimes \odot_j),\ 2 \leq i, j \leq n. \end{array} \right\}. \tag{16}$$

PROOF. (1) Let $\mathcal{P} = (\Omega, \Lambda, \star)$ be an ABQR operad. Suppose there is a basis $\{\odot_i\}$ with $\star = \sum_i \odot_i$ such that Λ is contained in the subspace of $\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$ generated by the relations (13). Define linear maps $\alpha, \beta : \Omega \to \mathbf{k}$ by

$$\alpha(\odot_i) = \delta_{1,i}1,\ \beta(\odot_i) = \delta_{2,i}1,\ 1 \leq i \leq n,$$

where $\delta_{i,j}$ is the Kronecker delta. Then $\alpha(\star) = 1 = \beta(\star)$ and $\alpha \neq \beta$. It is straightforward to check that each element in (13) satisfies the equations (C1)-(C5) in Theorem 2.6. For example, we check the first element $(\star \otimes \odot_2, \odot_2 \otimes \odot_2)$ against (C1)-(C5) and see that

- (C1) means $\beta(\star)\odot_2 = \beta(\odot_2)\odot_2$ which holds since $\beta(\star) = 1 = \beta(\odot_2)$;
- (C2) means $\alpha(\star)\odot_2 = \beta(\odot_2)\odot_2$ which holds since $\alpha(\star) = 1 = \beta(\odot_2)$;
- (C3) means $\alpha(\odot_2)\star = \alpha(\odot_2)\odot_2$ which holds since $\alpha(\odot_2) = 0$;
- (C4) means $\beta(\odot_2)\star = \beta(\odot_2)\beta(\odot_2)\star$ which holds since $\beta(\odot_2) = 1$;
- (C5) means $\alpha(\star)\alpha(\odot_2)\alpha = \alpha(\odot_2)\odot_2$ which holds since $\alpha(\odot_2) = 0$.

Therefore, each element in Λ satisfies the equations (C1)-(C5). Thus the unit action (α, β) is coherent with $\mathcal{P}$. This proves the "if" part.

To prove the "only if" part, we assume that there is a unit action (α, β) on $\mathcal{P}$ that is coherent and $\alpha \neq \beta$. In particular, $\alpha(\star) = \beta(\star) = 1$. Then there are direct sum decompositions

$$\Omega = \mathbf{k} \star \oplus \ker \alpha = \mathbf{k} \star \oplus \ker \beta.$$

This, together with $\alpha(\star) = \beta(\star) = 1$, implies that $\alpha = \beta$ if and only if $\ker \alpha = \ker \beta$. So we have $\ker \alpha \neq \ker \beta$. In fact, $\ker \alpha \subsetneq \ker \beta$ and $\ker \beta \subsetneq \ker \alpha$ since $\dim \ker \alpha = \dim \ker \beta = n - 1$. So there are elements $\odot_1 \in \ker \beta$ such that $\alpha(\odot_1) \neq 0$ and $\odot_2 \in \ker \alpha$ such that $\beta(\odot_2) \neq 0$. By rescaling, we can assume that

$$\alpha(\odot_1) = 1 = \beta(\odot_2),\ \alpha(\odot_2) = 0 = \beta(\odot_1).$$

Note that $\ker(\alpha)$ and $\ker(\beta)$ are the solution spaces of the linear equations

$$\odot_1\alpha(\odot_1)x_1 + \cdots + \alpha(\odot_n)x_n (= \alpha(x_1 \odot_1 + \cdots + x_n\odot_n)) = 0$$

and

$$\beta(\odot_1)x_1 + \cdots + \beta(\odot_n)x_n (= \beta(x_1 \odot_1 + \cdots x_n\odot_n)) = 0.$$

So $\ker \alpha \cap \ker \beta$ is the solution space of the linear system

$$\begin{cases} \alpha(\odot_1)x_1 + \cdots + \alpha(\odot_n)x_n = 0, \\ \beta(\odot_1)x_1 + \cdots + \beta(\odot_n)x_n = 0. \end{cases}$$

Since $\alpha \neq \beta$, we see that $\ker \alpha \cap \ker \beta$ has dimension $n - 2$. The intersection also contains $\star - \odot_1 - \odot_2$. So there is a basis $\odot_3, \cdots, \odot_n$ of $\ker \alpha \cap \ker \beta$ such that $\star - \odot_1 - \odot_2 = \odot_3 + \cdots + \odot_n$. Therefore $\star = \odot_1 + \cdots + \odot_n$.

Now any element $\lambda \in \Lambda$ is of the form

$$\lambda = (\sum_{i,j=1}^{n} a_{ij} \odot_i \otimes \odot_j, \sum_{i,j=1}^{n} b_{ij} \odot_i \otimes \odot_j). \tag{17}$$

Since the unit action (α, β) is coherent on $\mathcal{P}$, λ satisfies each of the five coherence equations in Theorem 2.6. For (C1), the equation is $\sum_{ij} a_{ij}\beta(\odot_i)\odot_j = \sum_{ij} b_{ij}\beta(\odot_i)\odot_j$. By our choice of the basis $\{\odot_i\}$, this means $\sum_j a_{2,j}\odot_j = \sum_j b_{2,j} \odot_j$. Thus we have

$$a_{2,j} = b_{2,j},\ 1 \le j \le n. \tag{18}$$

Similarly, from (C3), we obtain

$$a_{i,1} = b_{i,1},\ 1 \le i \le n. \tag{19}$$

Applying (C2), we obtain $\sum_{ij} a_{ij}\alpha(\odot_i)\odot_j = \sum_{i,j} b_{ij}\beta(\odot_j) \odot_i$. This gives

$$a_{1,i} = b_{i,2},\ 1 \le i \le n. \tag{20}$$

Applying (C4), we have $\sum_{ij} a_{ij}\beta(\odot_j)\odot_i = \sum_{ij} b_{ij}\beta(\odot_i)\beta(\odot_j)\star$ which means

$$\sum_i a_{i,2}\odot_i = b_{2,2} \star .$$

If $b_{2,2} = 0$, then since $\{\odot_i\}$ is a basis, we have $a_{i,2} = 0,\ 1 \le i \le n$. If $b_{2,2} \ne 0$, then again since $\{\odot_i\}$ is a basis and $\star = \sum_i \odot_i$ by construction, we must have $a_{i,2} = b_{2,2}$. Thus we always have

$$a_{i,2} = b_{2,2},\ 1 \le i \le n. \tag{21}$$

As with (C4), applying (C5) gives

$$a_{1,1} = b_{1,j},\ 1 \le j \le n. \tag{22}$$

Some of the relations above are duplicated. For examples, $a_{1,1} = b_{1,1}$ is in both Eq. (19) and Eq. (22). To avoid this we list (21) and (22) first and list the rest only when needed. Note also that the above relations only involve coefficients with at least one of the subscripts in $\{1, 2\}$. This means that there are no restrictions among the relations

$$\{(\odot_i \otimes \odot_j, 0),\ (0, \odot_i \otimes \odot_j) |\, 3 \le i, j \le n\}.$$

Then we see that λ in Eq. (17) is of the following linear combination of linearly independent elements in $\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$.

$$\lambda \ = \ b_{2,2}\big(\sum_{i=1}^{n} \odot_i \otimes \odot_2, \odot_2 \otimes \odot_2\big) + a_{1,1}\big(\odot_1 \otimes \odot_1, \odot_1 \otimes \sum_{j=1}^{n} \odot_j\big)$$

$$+\sum_{i=2}^{n} a_{i,1}\big(\odot_i \otimes \odot_1, \odot_i \otimes \odot_1\big) + \sum_{j=3}^{n} a_{2,j}\big(\odot_2 \otimes \odot_j, \odot_2 \otimes \odot_j\big)$$

$$+\sum_{i=3}^{n} a_{1,i}\big(\odot_1 \otimes \odot_i, \odot_i \otimes \odot_2\big) + \sum_{i,j=3}^{n} a_{i,j}\big(\odot_i \otimes \odot_j, 0\big) + \sum_{i,j=3}^{n} b_{i,j}\big(0, \odot_i \otimes \odot_j\big).$$

Recall that $\star = \sum_i \odot_i$. We see that λ is in $\Lambda'_{n,\,\mathrm{coh}}$ defined by Eq. (13). So $\Lambda \subseteq \Lambda'_{n,\,\mathrm{coh}}$.

(2) The proof is similar to the last part. To prove the "if" direction, suppose there is a basis $\{\odot_i\}$ with $\star = \sum_i \odot_i$ such that Λ is contained in the subspace of $\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$ generated by the relations (14). Define linear maps $\alpha = \beta : \Omega \to \mathbf{k}$ by

$$\alpha(\odot_i) = \delta_{1,i}1,\ \ 1 \leq i \leq n.$$

Then $\alpha(\star) = 1 = \beta(\star)$. It is straightforward to check that elements in (14) satisfy the equations (C1)-(C5) in Theorem 2.6. Therefore, each element in Λ satisfies the equations (C1)-(C5). Thus the unit action (α, β) is coherent.

Now we consider the "only if" direction. Let (α, β) be a coherent unit action on $\mathcal{P}$ with $\alpha = \beta$. Then we have $\Omega = \mathbf{k}\star \oplus \ker\alpha$. Let $\{\odot_2, \cdots, \odot_n\}$ be a basis of $\ker\alpha = \ker\beta$ and define $\odot_1 = \star - (\odot_2 + \cdots + \odot_n)$. We have $\star = \sum_i \odot_i$, $\alpha(\odot_1) = 1$ and $\alpha(\odot_i) = 0, 2 \leq i \leq n$. Let

$$\lambda = (\sum_{i,j=1}^{n} a_{ij} \odot_i \otimes \odot_j, \sum_{i,j=1}^{n} b_{ij} \odot_i \otimes \odot_j) \tag{23}$$

be in Λ. Since (α, β) is coherent on $\mathcal{P}$, applying Theorem 2.6, we have

- by (C1), $\sum_j a_{1,j}\odot_j = \sum_j b_{1,j}\odot_j$, so $a_{1,j} = b_{1,j}$;
- by (C2), $\sum_j a_{1,j}\odot_j = \sum_i b_{i,1}\odot_i$, so $a_{1,j} = b_{j,1}$;
- by (C3), $\sum_i a_{i,1}\odot_i = \sum_i b_{i,1}\odot_i$, so $a_{i,1} = b_{i,1}$;
- by (C4), $\sum_i a_{i,1}\odot_i = b_{1,1}\star$, so $a_{i,1} = b_{1,1}$;
- by (C5), $a_{1,1}\star = \sum_j b_{1,j}\odot_j$, so $a_{1,1} = b_{1,j}$.

Therefore,

$$a_{i,1} = a_{1,j} = b_{i,1} = b_{1,j},\ \ 1 \leq i, j \leq n.$$

Thus

$$\begin{aligned}
& (\sum_{i=1 \text{ or } j=1} a_{ij} \odot_i \otimes \odot_j, \sum_{i=1 \text{ or } j=1} b_{ij} \odot_i \otimes \odot_j) \\
= \;& a_{1,1}(\sum_{i=1 \text{ or } j=1} \odot_i \otimes \odot_j, \sum_{i=1 \text{ or } j=1} \odot_i \otimes \odot_j) \\
= \;& a_{1,1}\Big((\odot_1 \otimes \star, \odot_1 \otimes \star) + ((\star - \odot_1) \otimes \odot_1, (\star - \odot_1) \otimes \odot_1)\Big)
\end{aligned}$$

$$= \quad a_{1,1}\Big((\odot_1\otimes\star,\odot_1\otimes\star)+(\star\otimes\odot_1,\star\otimes\odot_1)-(\odot_1\otimes\odot_1,\odot_1\otimes\odot_1)\Big).$$

On the other hand, the coherence equations impose no restriction on other elements in $\Omega^{\otimes 2}\oplus\Omega^{\otimes 2}$. Therefore λ in Eq. (23) is a linear combination of the elements in Eq. (14). So Λ is a subspace of the subspace $\Lambda''_{n,\,\mathrm{coh}}$.

(3) The proofs of part (3) of Theorem 3.2 follow from a similar analysis as for part (1). But we only need to consider (C1)-(C3) which give relations (18), (19) and (20).

(4) Likewise, for the proof of part (4), we only consider (C1)-(C3) in the proof of part (2). Grouping the resulting relations, we obtain Eq. (16). ∎

3.3. Special cases. We now consider the cases where Ω is of dimension 2 and 3. Suppose $\alpha\neq\beta$. Then from Theorem 3.2 we easily check that

$$\lambda'_{2,\,\mathrm{coh}}=\lambda'_{2,\,\mathrm{comp}}=\{(\star\otimes\odot_2,\odot_2\otimes\odot_2),(\odot_1\otimes\odot_1,\odot_1\otimes\star),(\odot_2\otimes\odot_1,\odot_2\otimes\odot_1)\}.$$

Replacing $\odot_1$ by $\prec$ and replacing $\odot_2$ by $\succ$, we obtain the following improvement of Proposition 1.2 in [25].

3.4. Corollary. *Let $\mathcal{P}=(\Omega,\Lambda,\star)$ be an ABQR operad with* $\dim\Omega=2$. *The following statements are equivalent.*

1. *There is a coherent unit action (α,β) on $\mathcal{P}$ with $\alpha\neq\beta$;*
2. *There is a compatible unit action (α,β) on $\mathcal{P}$ with $\alpha\neq\beta$;*
3. *There is a basis $(\prec,\succ)$ of Ω with $\star=\prec+\succ$ such that Λ is contained in the subspace of $\Omega^{\otimes 2}\oplus\Omega^{\otimes 2}$ with the basis*

$$\{(\star\otimes\succ,\succ\otimes\succ),(\prec\otimes\prec,\prec\otimes\star),(\succ\otimes\prec,\succ\otimes\prec)\}.$$

Next suppose $\alpha=\beta$.

3.5. Corollary. *Let $\mathcal{P}=(\Omega,\Lambda,\star)$ be an ABQR operad with* $\dim\Omega=2$.

1. *There is a coherent unit action (α,β) on $\mathcal{P}$ with $\alpha=\beta$ if and only there is a basis $(\odot_1,\odot_2)$ of Ω with $\star=\odot_1+\odot_2$ such that Λ is contained in the subspace of $\Omega^{\otimes 2}\oplus\Omega^{\otimes 2}$ with the basis*

$$\{(\odot_1\otimes\star,\odot_1\otimes\star)+(\odot_2\otimes\odot_1,\odot_2\otimes\odot_1),(\odot_2\otimes\odot_2,0),(0,\odot_2\otimes\odot_2)\}.$$

2. *There is a compatible unit action (α,β) on $\mathcal{P}$ with $\alpha=\beta$ if and only if there is a basis $(\odot_1,\odot_2)$ of Ω with $\star=\odot_1+\odot_2$ such that Λ is contained in the subspace of $\Omega^{\otimes 2}\oplus\Omega^{\otimes 2}$ with the basis*

$$\{(\odot_1\otimes\odot_1,\odot_1\otimes\odot_1),(\odot_1\otimes\odot_2,\odot_1\otimes\odot_2)+(\odot_2\otimes\odot_1,\odot_2\otimes\odot_1),(\odot_2\otimes\odot_2,0),(0,\odot_2\otimes\odot_2)\}.$$

Now we consider the dimension three case. Replacing $\{\odot_1,\odot_2,\odot_3\}$ with $\{\prec,\succ,\circ\}$ in Theorem 3.2, we get

3.6. Corollary. *Let* $\mathcal{P} = (\Omega, \Lambda, \star)$ *be an ABQR operad with* $\dim \Omega = 3$. *There is a coherent unit action* (α, β) *on* $\mathcal{P}$ *with* $\alpha \neq \beta$ *if and only if there is a basis* $(\prec, \succ, \circ)$ *of* Ω *with* $\star = \prec + \succ + \circ$ *such that* Λ *is contained in the subspace with the basis*

$$\begin{aligned}\lambda'_{3,\,\mathrm{coh}} = \quad & \{(\prec \otimes \prec, \prec \otimes \star), (\succ \otimes \prec, \succ \otimes \prec), (\star \otimes \succ, \succ \otimes \succ), (\succ \otimes \circ, \succ \otimes \circ), \\ & (\prec \otimes \circ, \circ \otimes \succ), (\circ \otimes \prec, \circ \otimes \prec), (\circ \otimes \circ, 0), (0, \circ \otimes \circ)\}.\end{aligned}$$

Clearly the relations in Eq. (3) and (4) of the dendriform trialgebra and NS-algebra, respectively, are contained in $\lambda'_{3,\,\mathrm{coh}}$, so have coherent unit actions, as were shown in [18, 25]. We also note that when $n \geq 3$, compatibility does not imply coherency for unit actions.

4. Coherent unit actions on products and duals of operads

We briefly recall the concept of the black square product of ABQR operads [7, 26], and show that coherent and compatible unit actions are preserved by the black square product. We then recall the concept of the dual operad and show that, other than a trivial case, coherence and compatibility are not preserved by taking the duals.

4.1. Products of operads. For ABQR operads $(\Omega_1, \Lambda_1, \star_1)$ and $(\Omega_2, \Lambda_2, \star_2)$, and for $\odot^{(i)} \in \Omega_i,\ i = 1, 2$, we use a column vector $\begin{bmatrix} \odot_1 \\ \odot_2 \end{bmatrix}$ to denote the tensor product $\odot_1 \otimes \odot_2 \in \Omega_1 \otimes \Omega_2$. For $f_i = (\odot_i^{(1)} \otimes \odot_i^{(2)}, \odot_j^{(3)} \otimes \odot_j^{(4)}) \in \Omega_i^{\otimes 2} \oplus \Omega_i^{\otimes 2},\ i = 1, 2$, define

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \left(\begin{bmatrix} \odot_1^{(1)} \\ \odot_2^{(1)} \end{bmatrix} \otimes \begin{bmatrix} \odot_1^{(2)} \\ \odot_2^{(2)} \end{bmatrix}, \begin{bmatrix} \odot_1^{(3)} \\ \odot_2^{(3)} \end{bmatrix} \otimes \begin{bmatrix} \odot_1^{(4)} \\ \odot_2^{(4)} \end{bmatrix} \right) \in (\Omega_1 \otimes \Omega_2)^{\otimes 2} \oplus (\Omega_1 \otimes \Omega_2)^{\otimes 2}.$$

This extends by bilinearity to all $f_i \in \Omega_i^{\otimes 2} \oplus \Omega_i^{\otimes 2},\ i = 1, 2$. We define a subspace of $(\Omega_1 \otimes \Omega_2)^{\otimes 2} \oplus (\Omega_1 \otimes \Omega_2)^{\otimes 2}$ by

$$\Lambda_1 \blacksquare \Lambda_2 = \left\{ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \middle| f_i \in \Lambda_i,\ i = 1, 2 \right\}.$$

So $\Lambda_1 \blacksquare \Lambda_2$ is a space of relations for the operator space $\Omega_1 \otimes \Omega_2$.

It is easy to see [7] that the operad $\mathcal{P} := (\Omega_1 \otimes \Omega_2, \Lambda_1 \blacksquare \Lambda_2)$ with the operation $\begin{bmatrix} \star_1 \\ \star_2 \end{bmatrix}$ is also an ABQR operad [7], called the **black square** product of $\mathcal{P}_1$ and $\mathcal{P}_2$ and denoted $\mathcal{P}_1 \blacksquare \mathcal{P}_2$. See [37] for further studies of $\blacksquare$.

We recall the following results from [7] for later reference.

4.2. Proposition.

1. *The quadri-algebra of Aguiar and Loday [4], defined by four binary operations* $\nearrow, \nwarrow, \searrow, \swarrow$ *and 9 relations, is isomorphic to the black square product* $\mathcal{P}_D \blacksquare \mathcal{P}_D$ *where* $\mathcal{P}_D = (\omega_D, \lambda_D, \star_D)$ *is the dendriform dialgebra in Eq. (2).*

2. *The ennea-algebra of Leroux [17], defined by 9 binary operations*

$$\nwarrow, \uparrow, \nearrow, \prec, \circ, \succ, \swarrow, \downarrow, \searrow$$

and 49 relations, is isomorphic to the black square product $\mathcal{P}_T \blacksquare \mathcal{P}_T$ *where* $\mathcal{P}_T = (\omega_T, \lambda_T, \star_T)$ *is the dendriform trialgebra in Eq. (3).*

3. *The dendriform-Nijenhuis algebra [18], equipped with 9 binary operations*

$$\nearrow, \ \searrow, \ \swarrow, \ \nwarrow, \ \uparrow, \ \downarrow, \ \tilde{\prec}, \ \tilde{\succ}, \ \tilde{\bullet}$$

and 28 relations, is isomorphic to the black square product $\mathcal{P}_T \blacksquare \mathcal{P}_N$ *where* $\mathcal{P}_T = (\omega_T, \lambda_T, \star_T)$ *and* $\mathcal{P}_N = (\omega_N, \lambda_N, \star_N)$ *are the dendriform trialgebra and the NS-algebra in Eq. (4), respectively.*

4. *The octo-algebra [19], defined using 8 operations*

$$\nearrow_i, \nwarrow_i, \swarrow_i, \searrow_i, \ i = 1, 2$$

and 27 relations, is isomorphic to the third power $\mathcal{P}_D \blacksquare \mathcal{P}_D \blacksquare \mathcal{P}_D$ *of the dendriform dialgebra* $\mathcal{P}_D = (\omega_D, \lambda_D, \star_D)$*.*

4.3. Unit actions on products. We now use Theorem 2.6 to show that coherent unit actions on ABQR operads are preserved by the black square product and thus give rise to Hopf algebra structures on the free objects of the product operad.

For $i = 1, 2$, let $\mathcal{P}_i := (\Omega_i, \Lambda_i, \star_i)$ be an ABQR operad and let (α_i, β_i) be a unit action on $\mathcal{P}_i$. Let $\mathcal{P} = \mathcal{P}_1 \blacksquare \mathcal{P}_2$ and define

$$\alpha := \alpha_1 \otimes \alpha_2 \text{ (resp. } \beta := \beta_1 \otimes \beta_2) : \Omega_1 \otimes \Omega_2 \to \mathbf{k}$$

by

$$\alpha\left(\begin{bmatrix} \odot_1 \\ \odot_2 \end{bmatrix}\right) = \alpha_1(\odot_1)\alpha_2(\odot_2) \text{ (resp. } \beta\left(\begin{bmatrix} \odot_1 \\ \odot_2 \end{bmatrix}\right) = \beta_1(\odot_1)\beta_2(\odot_2)).$$

Then (α, β) defines a unit action on $\mathcal{P}$.

4.4. Theorem. *Let* $\mathcal{P}_i := (\Omega_i, \Lambda_i, \star_i)$, $i = 1, 2$, *be ABQR operads with coherent unit actions* (α_i, β_i)*. Then the unit action* $(\alpha_1 \otimes \alpha_2, \beta_1 \otimes \beta_2)$ *on the ABQR operad* $\mathcal{P} := \mathcal{P}_1 \blacksquare \mathcal{P}_2 := (\Omega_1 \otimes \Omega_2, \Lambda_1 \blacksquare \Lambda_2, \begin{bmatrix} \star_1 \\ \star_2 \end{bmatrix})$ *is also coherent. Therefore, The augmented free* $\mathcal{P}$-*algebra* $\mathcal{P}(V)_+$ *on a* $\mathbf{k}$-*vector space* V *is a connected Hopf algebra.*

It will be clear from the proof that, if $\mathcal{P}_1$ and $\mathcal{P}_2$ have compatible unit actions, then so does their product.

Proof. By Theorem 2.6, we just need to verify that the relations in $\Lambda_1 \blacksquare \Lambda_2$ satisfy (C1)–(C5) with the unit action (α, β) on $\mathcal{P}$. We recall that a relation in $\Lambda_1 \blacksquare \Lambda_2$ is of the form

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} := \left(\sum_{i_1, i_2} \begin{bmatrix} \odot^{(1)}_{i_1} \\ \odot^{(1)}_{i_2} \end{bmatrix} \otimes \begin{bmatrix} \odot^{(2)}_{i_1} \\ \odot^{(2)}_{i_2} \end{bmatrix}, \sum_{j_1, j_2} \begin{bmatrix} \odot^{(3)}_{j_1} \\ \odot^{(3)}_{j_2} \end{bmatrix} \otimes \begin{bmatrix} \odot^{(4)}_{j_1} \\ \odot^{(4)}_{j_2} \end{bmatrix} \right)$$

for $(\sum_{i_1} \odot^{(1)}_{i_1} \otimes \odot^{(2)}_{i_1}, \sum_{j_1} \odot^{(3)}_{j_1} \otimes \odot^{(4)}_{j_1}) \in \Lambda_1$ and $(\sum_{i_2} \odot^{(1)}_{i_2} \otimes \odot^{(2)}_{i_2}, \sum_{j_2} \odot^{(3)}_{j_2} \otimes \odot^{(4)}_{j_2}) \in \Lambda_2$.

Then by bilinearity of the product $\begin{bmatrix} \odot_1 \\ \odot_2 \end{bmatrix}$ and the equation (C1) for $\mathcal{P}_1$ and $\mathcal{P}_2$, we have

$$\begin{aligned}
\sum_{i_1, i_2} \beta\left(\begin{bmatrix} \odot^{(1)}_{i_1} \\ \odot^{(1)}_{i_2} \end{bmatrix} \right) \begin{bmatrix} \odot^{(2)}_{i_1} \\ \odot^{(2)}_{i_2} \end{bmatrix} &= \sum_{i_1, i_2} \beta_1(\odot^{(1)}_{i_1}) \beta_2(\odot^{(1)}_{i_2}) \begin{bmatrix} \odot^{(2)}_{i_1} \\ \odot^{(2)}_{i_2} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i_1} \beta_1(\odot^{(1)}_{i_1}) \odot^{(2)}_{i_1} \\ \sum_{i_2} \beta_2(\odot^{(1)}_{i_2}) \odot^{(2)}_{i_2} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j_1} \beta_1(\odot^{(3)}_{j_1}) \odot^{(4)}_{j_1} \\ \sum_{j_2} \beta_2(\odot^{(3)}_{j_2}) \odot^{(4)}_{j_2} \end{bmatrix} \\
&= \sum_{j_1, j_2} \beta_1(\odot^{(3)}_{j_1}) \beta_2(\odot^{(3)}_{j_2}) \begin{bmatrix} \odot^{(4)}_{j_1} \\ \odot^{(4)}_{j_2} \end{bmatrix} \\
&= \sum_{j_1, j_2} \beta\left(\begin{bmatrix} \odot^{(3)}_{j_1} \\ \odot^{(3)}_{j_2} \end{bmatrix} \right) \begin{bmatrix} \odot^{(4)}_{j_1} \\ \odot^{(4)}_{j_2} \end{bmatrix}.
\end{aligned}$$

This gives (C1) for the product operad $\mathcal{P}$. Conditions (C2) and (C3) can be verified in the same way.

For (C4), we have

$$\begin{aligned}
\sum_{i_1, i_2} \beta\left(\begin{bmatrix} \odot^{(2)}_{i_1} \\ \odot^{(2)}_{i_2} \end{bmatrix} \right) \begin{bmatrix} \odot^{(1)}_{i_1} \\ \odot^{(1)}_{i_2} \end{bmatrix} &= \sum_{i_2, i_2} \beta_1(\odot^{(2)}_{i_1}) \beta_2(\odot^{(2)}_{i_2}) \begin{bmatrix} \odot^{(1)}_{i_1} \\ \odot^{(1)}_{i_2} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{i_1} \beta_1(\odot^{(2)}_{i_1}) \odot^{(1)}_{i_1} \\ \sum_{i_2} \beta_2(\odot^{(2)}_{i_2}) \odot^{(1)}_{i_2} \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j_1} \beta_1(\odot^{(3)}_{j_1}) \beta_1(\odot^{(4)}_{j_1}) \star_1 \\ \sum_{j_2} \beta_2(\odot^{(3)}_{j_2}) \beta_2(\odot^{(4)}_{j_2}) \star_2 \end{bmatrix} \\
&= \sum_{j_1, j_2} \beta_1(\odot^{(3)}_{j_1}) \beta_1(\odot^{(4)}_{j_1}) \beta_2(\odot^{(3)}_{j_2}) \beta_2(\odot^{(4)}_{j_2}) \begin{bmatrix} \star_1 \\ \star_2 \end{bmatrix} \\
&= \sum_{j_1, j_2} \beta\left(\begin{bmatrix} \odot^{(3)}_{j_1} \\ \odot^{(3)}_{j_2} \end{bmatrix} \right) \beta\left(\begin{bmatrix} \odot^{(4)}_{j_1} \\ \odot^{(4)}_{j_2} \end{bmatrix} \right) \begin{bmatrix} \star_1 \\ \star_2 \end{bmatrix}.
\end{aligned}$$

This gives (C4) for the product operad $\mathcal{P}$. The same works for (C5). ∎

By Proposition 4.2 and Theorem 4.4 we immediately obtain the following results on Hopf algebras.

4.5. Corollary.

1. **(Loday[25])** *Augmented free quadri-algebras are Hopf algebras;*
2. **(Leroux[17])** *Augmented free ennea-algebras are Hopf algebras;*
3. **(Leroux[18])** *Augmented free Nijenhuis-dendriform algebras are Hopf algebras;*
4. **(Leroux[19])** *Augmented free octo-algebras are Hopf algebras;*

4.6. Duality of ABQR operads. We now recall the definition of the dual [7, 23, 26] of an ABQR operad before we study the relation between coherent unit actions and taking duals.

For an ABQR operad $\mathcal{P} = (\Omega, \Lambda, \star)$, let $\check{\Omega} := \mathrm{Hom}(\Omega, \mathbf{k})$ be the dual space of Ω, giving the natural pairing

$$\langle\ ,\ \rangle_\Omega : \Omega \times \check{\Omega} \to \mathbf{k}.$$

Then $\check{\Omega}^{\otimes 2}$ is identified with the dual space of $\Omega^{\otimes 2}$, giving the natural pairing

$$\langle\ ,\ \rangle^{\otimes 2} : \Omega^{\otimes 2} \times \check{\Omega}^{\otimes 2} \to \mathbf{k}, \langle x \otimes y, a \otimes b\rangle^{\otimes 2} = \langle x, a\rangle_\Omega\ \langle y, b\rangle_\Omega.$$

We then define a pairing

$$\langle\ ,\ \rangle : (\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}) \times (\check{\Omega}^{\otimes 2} \oplus \check{\Omega}^{\otimes 2}) \to \mathbf{k} \tag{24}$$

by

$$\langle(\alpha, \beta), (\gamma, \delta)\rangle = \langle \alpha, \gamma\rangle^{\otimes 2} - \langle \beta, \delta\rangle^{\otimes 2}, \alpha, \beta \in \Omega^{\otimes 2},\ \gamma, \delta \in \check{\Omega}^{\otimes 2}.$$

We now define $\Lambda^\perp$ to be the annihilator of $\Lambda \subseteq \Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$ in $\check{\Omega}^{\otimes 2} \oplus \check{\Omega}^{\otimes 2}$ under the pairing $\langle\ ,\rangle$. We call $\mathcal{P}^! := (\check{\Omega}, \Lambda^\perp)$ the **dual operad** of $\mathcal{P} = (\Omega, \Lambda)$ which is the Koszul dual in our special case. It follows from the definition that $(\mathcal{P}^!)^! = \mathcal{P}$.

4.7. Example. (Associative dialgebra [23, Proposition 8.3]) Let (Ω_D, Λ_D) be the operad for the dendriform dialgebra. Let $\{\dashv, \vdash\} \in \check{\Omega}_D$ be the dual basis of $\{\prec, \succ\}$ (in this order). Then $\Lambda_D^\perp$ is generated by

$$\{(\dashv \otimes \dashv, \dashv \otimes \dashv), (\vdash \otimes \vdash, \vdash \otimes \vdash), (\dashv \otimes \dashv, \dashv \otimes \vdash), (\vdash \otimes \dashv, \vdash \otimes \dashv), (\dashv \otimes \vdash, \vdash \otimes \vdash)\}.$$

It is called the associative dialgebra and is denoted $\mathcal{P}_{AD} = (\Omega_{AD}, \Lambda_{AD})$. It has two associative operations $\dashv$ and $\vdash$, giving two ABQR operads $(\Omega_{AD}, \Lambda_{AD}, \dashv)$ and $(\Omega_{AD}, \Lambda_{AD}, \vdash)$. They are isomorphic under the correspondence $\dashv \leftrightarrow \vdash$.

Other examples of duals of ABQR operads can be found in [7, 29].

4.8. Unit actions on duals. A natural question to ask is whether the dual of an ABQR operad $\mathcal{P} = (\Omega, \Lambda, \star)$ with a coherent unit action still has a coherent unit action. When Ω has one generator, the answer is positive. This is because the generator must be the associative operation $\star$. Then $(\Omega, \Lambda, \star)$ is the operad for associative algebras, with the coherent unit action given by $\alpha(\star) = \beta(\star) = 1$. The dual operad is isomorphic to the operad itself, so again admits a coherent unit action. We now show that this is the only case that coherent unit action is preserved by taking the dual.

4.9. Theorem. *Suppose that an ABQR operad $\mathcal{P} = (\Omega, \Lambda, \star)$ with $n = \dim \Omega \geq 2$ has a compatible unit action. Let $\mathcal{P}^! = (\check{\Omega}, \Lambda^\perp)$ be the dual operad.*

1. *$\mathcal{P}^!$ has an associative operation, so is an ABQR operad. In fact it has n linearly independent associative operations.*
2. *$\mathcal{P}^!$ does not have a compatible unit action for any choice of associative operation.*

The same is true when compatible is replaced by coherent.

For example the associative dialgebra operad $\mathcal{P}_{AD}$ in Example 4.7 does not have a coherent unit action even though it has two linearly independent associative binary operations $\vdash$ and $\dashv$.

Proof. (1) First assume that the compatible unit action on the given ABQR operad (Ω, Λ) satisfies $\alpha \neq \beta$. Let $\{\odot_1, \cdots, \odot_n\}$ be the bases of Ω given in the proof of Theorem 3.2.(3). So we have

$$\sum_i \odot_i = \star,\ \alpha(\odot_i) = \delta_{1,i},\ \beta(\odot_i) = \delta_{2,i}.$$

Then by Theorem 3.2, Λ is a subspace of the subspace $\Lambda'_{n,\,\mathrm{comp}}$ of $\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$ with a basis

$$\lambda'_{n,\,\mathrm{comp}} := \left\{ \begin{array}{l} ((\odot_1 + \odot_2) \otimes \odot_2, \odot_2 \otimes \odot_2), \\ (\odot_1 \otimes \odot_1, \odot_1 \otimes (\odot_1 + \odot_2)), \\ (\odot_i \otimes \odot_1, \odot_i \otimes \odot_1),\ 2 \leq i \leq n, \\ (\odot_2 \otimes \odot_j, \odot_2 \otimes \odot_j),\ 3 \leq j \leq n, \\ (\odot_1 \otimes \odot_i, \odot_i \otimes \odot_2),\ 3 \leq i \leq n, \\ ((\odot_i \otimes \odot_2, 0), (0, \odot_1 \otimes \odot_i), 3 \leq i \leq n, \\ (\odot_i \otimes \odot_j, 0),\ (0, \odot_i \otimes \odot_j),\ 3 \leq i, j \leq n. \end{array} \right\} \tag{25}$$

Let $\{\check{\odot}_i\}$ be the dual basis of $\{\odot_i\}$. So $\check{\odot}_i(\odot_j) = \delta_{i,j},\ 1 \leq i, j \leq n$.

Then the pairing (24) between $\Omega^{\otimes 2} \oplus \Omega^{\otimes 2}$ and $\check{\Omega}^{\otimes 2} \oplus \check{\Omega}^{\otimes 2}$ is given by

$$\langle (\odot_i \otimes \odot_j, \odot_k \otimes \odot_\ell), (\check{\odot}_s \otimes \check{\odot}_t, \check{\odot}_u \otimes \check{\odot}_v) \rangle = \delta_{i,s}\delta_{j,t} - \delta_{k,u}\delta_{\ell,v}.$$

Consider $x = (\check{\odot}_1 \otimes \check{\odot}_1, \check{\odot}_1 \otimes \check{\odot}_1)$. Then the pairing between x and the second element in $\lambda'_{n,\,\mathrm{comp}}$ is $1 - 1 = 0$. The pairing between x and every other element in $\lambda'_{n,\,\mathrm{comp}}$ is also 0 since $\check{\odot}_1 \otimes \check{\odot}_1$ does not occur in the element. Thus $x = (\check{\odot}_1 \otimes \check{\odot}_1, \check{\odot}_1 \otimes \check{\odot}_1)$ is in

${\Lambda'_{n,\,\mathrm{comp}}}^{\perp}$ and hence in $\Lambda^{\perp}$, the relation space of $\mathcal{P}^!$. This shows that $\mathcal{P}^!$ has an associative operation. The same argument works for $(\check{\odot}_i \otimes \check{\odot}_i, \check{\odot}_i \otimes \check{\odot}_i)$, $2 \leq i \leq n$, giving n linearly independent associative operations.

Next assume that the compatible unit action on the given ABQR operad (Ω, Λ) satisfies $\alpha = \beta$. Let $\{\odot_1, \cdots, \odot_n\}$, $n = \dim \Omega$, be the basis of Ω given in the proof of Theorem 3.2.(4). Then as in the last case, we check that $(\check{\odot}_i \otimes \check{\odot}_i, \check{\odot}_i \otimes \check{\odot}_i)$, $1 \leq i \leq n$, are in ${\Lambda''_{n,\mathrm{comp}}}^{\perp} \subseteq \Lambda^{\perp}$, again showing that $\mathcal{P}^!$ is ABQR .

(2) Suppose that there is a choice of associative operation $\star$ in $\check{\Omega}$ and a unit action (α, β) that is compatible with $(\check{\Omega}, \Lambda^{\perp}, \star)$. First assume that $\alpha \neq \beta$. We easily verify that the three relations

$$(\check{\odot}_2 \otimes \check{\odot}_1, \check{\odot}_2 \otimes \check{\odot}_1), (\check{\odot}_1 \otimes \check{\odot}_1, \check{\odot}_1 \otimes \check{\odot}_1), (\check{\odot}_2 \otimes \check{\odot}_2, \check{\odot}_2 \otimes \check{\odot}_2) \tag{26}$$

are in ${\Lambda'_{n,\mathrm{comp}}}^{\perp}$ and hence in $\Lambda^{\perp}$. So they should satisfy the compatible equations (C1)-(C3) in Theorem 3.2. Applying (C2) to the first relation in Eq. (26), we obtain $\alpha(\check{\odot}_2)\check{\odot}_1 = \beta(\check{\odot}_1)\check{\odot}_2$. So

$$\alpha(\check{\odot}_2) = \beta(\check{\odot}_1) = 0. \tag{27}$$

Applying (C2) to the second relation in Eq. (26), we have $\alpha(\check{\odot}_1) = \beta(\check{\odot}_1)$, yielding $\alpha(\check{\odot}_1) = 0$ by Eq. (27). Applying (C2) to the third relation in Eq. (26), we have $\alpha(\check{\odot}_2) = \beta(\check{\odot}_2)$, giving $\beta(\check{\odot}_2) = 0$ by Eq. (27). For $i \geq 2$, we check that $(\check{\odot}_i \otimes \check{\odot}_2, 0)$ and $(0, \check{\odot}_1 \otimes \check{\odot}_i)$ are in ${\Lambda'_{n,\mathrm{comp}}}^{\perp}$ and hence in $\Lambda^{\perp}$, so satisfy (C1)-(C2). Applying (C1) to the first relation gives $\beta(\check{\odot}_i)\check{\odot}_2 = 0$. So $\beta(\check{\odot}_i) = 0$. Applying (C3) to the second equation gives $\alpha(\check{\odot}_i)\check{\odot}_1 = 0$. So $\alpha(\check{\odot}_i) = 0$. Therefore α and β are identically zero. But this is impossible, since $\alpha(\star)$ and $\beta(\star)$ should be 1.

Next assume that $\alpha = \beta$. For each $i \geq 2$ (there is such an i since $\dim \Omega \geq 2$), we check that $(\check{\odot}_1 \otimes \check{\odot}_i, \check{\odot}_1 \otimes \check{\odot}_i)$ and $(\check{\odot}_i \otimes \check{\odot}_1, \check{\odot}_i \otimes \check{\odot}_1)$ are in ${\Lambda''_{n,\mathrm{comp}}}^{\perp} \subseteq \Lambda^{\perp}$. Applying (C2) to them, we get

$$\alpha(\check{\odot}_1)\check{\odot}_i = \beta(\check{\odot}_i)\check{\odot}_1, \quad \alpha(\check{\odot}_i)\check{\odot}_1 = \beta(\check{\odot}_1)\check{\odot}_i.$$

So $\alpha(\check{\odot}_1) = \beta(\check{\odot}_i) = 0$ and $\alpha(\check{\odot}_i) = \beta(\check{\odot}_1) = 0, i \geq 2$. Thus α and β are identically zero, giving a contradiction.

Finally if $\mathcal{P}$ has a coherent unit action, then it automatically has a compatible unit action. So by part (1), $\mathcal{P}^!$ is an ABQR operad, but does not have a compatible unit action. It therefore does not have a coherent unit action. ∎

References

[1] M. Aguiar, Pre-Poison algebras, *Lett. Math. Phys.*, **54**, (2000), 263-277.

[2] M. Aguiar and F. Sottile, Structure of the Loday-Ronco Hopf algebra of trees, *J. Algebra* **295** (2006) 473-511.

[3] M. Aguiar and F. Sottile, Cocommutative Hopf algebras of permutations and trees, preprint, March 2004, arXiv:math.QA/0403101.

[4] M. Aguiar and J.-L. Loday, Quadri-algebras, *J. Pure Appl. Algebra* **191** (2004), 205-221.

[5] F. Chapoton, Un théorème de Cartier-Milnor-Moore-Quillen pour les bigèbres dendriformes et les algèbres braces, *J. Pure Appl. Alg.*, **168**, (2002), 1-18.

[6] K. Ebrahimi-Fard, Loday-type algebras and the Rota-Baxter relation, *Lett. Math. Phys.*, **61**, no. 2, (2002), 139-147.

[7] E. Ebrahimi-Fard and L. Guo, On products and duality of binary, quadratic, regular operads, *J. Pure Appl. Algebra*, **200** (2005), 293-317.

[8] E. Ebrahimi-Fard and L. Guo, Rota-Baxter algebras and dendriform algebras, to appear in *J. Pure Appl. Algebra*, arXiv: math.RA/0503647.

[9] L. Foissy, Les algèbres de Hopf des arbres enracinés décorés II, *Bull. Sci. Math.*, **126**, (2002), 249-288.

[10] A. Frabetti, Dialgebra homology of associative algebras, *C. R. Acad. Sci. Paris,* **325**, (1997), 135-140.

[11] A. Frabetti, Leibniz homology of dialgebras of matrices, *J. Pure Appl. Alg.*, **129**, (1998), 123-141.

[12] L. Gerritzen and R. Holtkamp, Hopf co-addition for free magma algebras and the non-associative Hausdorff series, *J. Algebra*, **265** (2003), 264-284.

[13] V. Ginzburg and M. Kapranov, Koszul duality for operads. *Duke Math. J.*, **76** (1994), 203-272.

[14] M. Hoffman, Quasi-shuffle products, *J. Algebraic Combin.*, **11**, no. 1, (2000), 49-68.

[15] R. Holtkamp, Comparison of Hopf algebras on trees, *Archiv der Mathematik*, Vol.**80**, (2003), 368-383.

[16] R. Holtkamp, On Hopf algebra structures over operads, *Adv. Math*, **207** (2006), 544-565.

[17] P. Leroux, Ennea-algebras, *J. Algebra*, **281** (2004), 287-302.

[18] P. Leroux, Construction of Nijenhuis operators and dendriform trialgebras, *Int. J. Math. Math. Sci.*, no. 49-52, (2004), 2595-2615.

[19] P. Leroux, On some remarkable operads constructed from Baxter operators, preprint, Nov. 2003, arXiv:math.QA/0311214.

[20] J.-L. Loday, Une version non commutative des algèbre de Lie: les algèbres de Leibniz, *Ens. Math.*, **39**, (1993), 269-293.

[21] J.-L. Loday, Algèbres ayant deux opérations associatives (digèbres), C. R. Acad. Sci. Paris 321 (1995), 141-146.

[22] J.-L. Loday, La renaissance des opérades, Séminaire Bourbaki, Vol. 1994/95. Astérisque No. 237 (1996), Exp. No. 792, 3, 47-74.

[23] J.-L. Loday, Dialgebras, in Dialgebras and related operads, *Lecture Notes in Math.*, 1763, (2002), 7-66.

[24] J.-L. Loday, Arithmetree, *J. Algebra*, **258** (2002), 275-309.

[25] J.-L. Loday, Scindement d'associativité et algèbres de Hopf, Actes des Journées Mathématiques à la Mémoire de Jean Leray, Sémin. Congr., 9, Soc. Math. France, Paris, (2004), 155-172.

[26] J.-L. Loday, Completing the operadic butterfly, *Georgian Math Journal* **13** (2006), no 4. 741–749.

[27] J.-L. Loday, On the algebra of quasi-shuffles, *Manuscripta Mathematica*, **123** (2007), 79–93.

[28] J.-L. Loday and M. O. Ronco, Hopf algebra of the planar binary trees, *Adv. Math.*, **139**, (1998), 293-309.

[29] J.-L. Loday and M. Ronco, Trialgebras and families of polytopes, in "Homotopy Theory: Relations with Algebraic Geometry, Group Cohomology, and Algebraic K-theory" Contemporary Mathematics 346 (2004).

[30] J.-L. Loday and M. Ronco, Algèbre de Hopf colibres, *C. R. Acad. Sci. Paris*, **337**, (2003), 153-158.

[31] J.-L. Loday and M. Ronco, On the structure of cofree Hopf algebras, *J. reine angew. Math.* **592** (2006) 123–155.

[32] J.-L. Loday, J. D. Stasheff and A. A. Voronov, Operads: Proceedings of Renaissance Conferences: Special Session and International Conference on Moduli Spaces, Operads, and Representation Theory (Contemporary Mathematics, Vol 202), AMS, (1997).

[33] M. Markl, S. Shnider and J. Stasheff, Operads in Algebra, Topology and Physics, AMS, 2002.

[34] T. Pirashvili, Sets with two associative operations, *C.E.J.M.* **2** (2003), 169-183.

[35] M. Ronco, Primitive elements in a free dendriform algebra. New trends in Hopf algebra theory (La Falda, 1999), Contemp. Math., 267, Amer. Math. Soc., Providence, RI, 2000, 245263.

[36] M. R. Ronco, Eulerian idempotents and Milnor-Moore theorem for certain non-commutative Hopf algebras, *J. Algebra*, **254**, (2002), 152-172.

[37] B. Vallette, Manin products, Koszul duality, Loday algebras and Deligne conjecture (2006), math.QA/0609002

I.H.É.S. Le Bois-Marie,
35, Route de Chartres,
F-91440 Bures-sur-Yvette,
France

Department of Mathematics and Computer Science,
Rutgers University,
Newark, NJ 07102, USA

Email: `kurusch@ihes.fr`
`liguo@newark.rutgers.edu`

This article may be accessed at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/18/13/18-13.{dvi,ps,pdf}`

THEORY AND APPLICATIONS OF CATEGORIES (ISSN 1201-561X) will disseminate articles that significantly advance the study of categorical algebra or methods, or that make significant new contributions to mathematical science using categorical methods. The scope of the journal includes: all areas of pure category theory, including higher dimensional categories; applications of category theory to algebra, geometry and topology and other areas of mathematics; applications of category theory to computer science, physics and other mathematical sciences; contributions to scientific knowledge that make use of categorical methods.
Articles appearing in the journal have been carefully and critically refereed under the responsibility of members of the Editorial Board. Only papers judged to be both significant and excellent are accepted for publication.
Full text of the journal is freely available in .dvi, Postscript and PDF from the journal's server at http://www.tac.mta.ca/tac/ and by ftp. It is archived electronically and in printed paper format.

SUBSCRIPTION INFORMATION. Individual subscribers receive abstracts of articles by e-mail as they are published. To subscribe, send e-mail to tac@mta.ca including a full name and postal address. For institutional subscription, send enquiries to the Managing Editor, Robert Rosebrugh, rrosebrugh@mta.ca.

INFORMATION FOR AUTHORS. The typesetting language of the journal is TeX, and LaTeX2e strongly encouraged. Articles should be submitted by e-mail directly to a Transmitting Editor. Please obtain detailed information on submission format and style files at http://www.tac.mta.ca/tac/.