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ENRICHED MODEL CATEGORIES AND AN APPLICATION TO ADDITIVE ENDOMORPHISM SPECTRA

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ABSTRACT. We define the notion of an additive model category and prove that any stable, additive, combinatorial model category $\mathcal{M}$ has a model enrichment over $Sp^{\Sigma}(s\mathcal{A}b)$ (symmetric spectra based on simplicial abelian groups). So to any object $X \in \mathcal{M}$ one can attach an endomorphism ring object, denoted $\mathrm{hEnd}_{\mathrm{ad}}(X)$, in the category $Sp^{\Sigma}(s\mathcal{A}b)$. We establish some useful properties of these endomorphism rings.

We also develop a new notion in enriched category theory which we call 'adjoint modules'. This is used to compare enrichments over one symmetric monoidal model category with enrichments over a Quillen equivalent one. In particular, it is used here to compare enrichments over $Sp^{\Sigma}(s\mathcal{A}b)$ and chain complexes.

1. Introduction

A model category is called **additive** if two conditions are satisfied. First, its hom-sets must have natural structures of abelian groups with respect to which composition is biadditive. Secondly, the abelian group structures on these hom-sets must interact well with the notion of 'higher homotopies'. We give a precise definition in Section 6. Examples of additive model categories include chain complexes over a ring and differential graded modules over a differential graded algebra, as one should expect.

Recall that a category is **locally presentable** if it is cocomplete and all objects are small with respect to large enough filtered colimits; see [AR]. A model category is called **combinatorial** if it is cofibrantly-generated and its underlying category is locally presentable. Finally, a model category is **stable** if it is pointed and the suspension functor is an auto-equivalence of the homotopy category.

In [D4] it was shown that any stable, combinatorial model category can be naturally enriched over the category Sp^{Σ} of symmetric spectra. This enrichment is invariant under Quillen equivalences in a certain sense. In the present paper we extend the results of [D4] to the additive case. We show that any stable, combinatorial, additive model category has a natural enrichment over $Sp^{\Sigma}(s\mathcal{A}b)$—the category of symmetric spectra based on simplicial abelian groups. This enrichment is *not* an invariant of Quillen equivalence, however: if $\mathcal{M}$ and $\mathcal{N}$ are stable, combinatorial, additive model categories which are

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connected by a zig-zag of Quillen equivalences (with the intermediate model categories not necessarily additive) then the additive enrichments for $\mathcal{M}$ and $\mathcal{N}$ need not be related. We can prove that the enrichments are equivalent, however, if all the intermediate steps in the zig-zag are additive.

1.1. Remark. The tools developed in this paper are applied in [DS2]. Two additive model categories $\mathcal{M}$ and $\mathcal{N}$ are called **additively Quillen equivalent** if there is a zig-zag of Quillen equivalences between $\mathcal{M}$ and $\mathcal{N}$ in which every intermediate model category is additive. It is a strange fact, established in [DS2], that additive model categories can be Quillen equivalent but not additively Quillen equivalent. The demonstration of this fact uses the model enrichments developed in the present paper.

We should explain up front that there are really three separate things going on in this paper. One is the development of the theory of additive model categories, taken up in Sections 6 and 7. The second is the construction of the model enrichment over $Sp^{\Sigma}(s\mathcal{A}b)$, which is begun in Section 8. Most of the details of the model enrichment exactly follow the pattern in [D4]. There is one extra result we wish to consider, though, which involves comparing model enrichments over $Sp^{\Sigma}(s\mathcal{A}b)$ to model enrichments over the Quillen equivalent category Ch of chain complexes of abelian groups. For this last issue we need to develop quite a bit more about enriched model categories than is available in the literature. Since this foundational material is important in its own right, we include it at the very beginning as Sections 2 through 5.

1.2. A closer look at the results. To describe the results in more detail we need to recall some enriched model category theory; specifically, we need the notions of **model enrichment** and **quasi-equivalence** from [D4]. Let $\mathcal{M}$ be a model category and $\mathcal{V}$ be a symmetric monoidal model category. Briefly, a model enrichment is a bifunctor $\tau\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{V}$ together with composition maps $\tau(Y,Z) \otimes \tau(X,Y) \to \tau(X,Z)$ which are associative and unital. The bifunctor must interact well with the model category structure—see [D4] for an explicit list of the necessary axioms, or Section 2.3 for a summary.

There is a notion of when two model enrichments of $\mathcal{M}$ by $\mathcal{V}$ are 'quasi-equivalent', which implies that they carry the same homotopical information. This takes longer to describe, but the reader can again find it in Section 2.3. We let $ME_0(\mathcal{M},\mathcal{V})$ denote the quasi-equivalence classes of model enrichments.

If $L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$ is a Quillen pair, there are induced functors $L_*\colon ME_0(\mathcal{M},\mathcal{V}) \to ME_0(\mathcal{N},\mathcal{V})$ and $L^*\colon ME_0(\mathcal{N},\mathcal{V}) \to ME_0(\mathcal{M},\mathcal{V})$. When (L,R) is a Quillen equivalence these are inverse bijections.

Using the above language, we can state the basic results. These are proved in Sections 8 and 9.

1.3. Theorem. *If $\mathcal{M}$ is a stable, additive, combinatorial model category, then there is a canonical element $\sigma_{\mathcal{M}} \in ME_0(\mathcal{M}, Sp^{\Sigma}(s\mathcal{A}b))$. If $L\colon \mathcal{M} \to \mathcal{N}$ is a Quillen equivalence then $L_*(\sigma_{\mathcal{M}}) = \sigma_{\mathcal{N}}$ and $L^*(\sigma_{\mathcal{N}}) = \sigma_{\mathcal{M}}$.*

If $X \in \mathcal{M}$, choose a cofibrant-fibrant object $\hat{X}$ which is weakly equivalent to X. Also, choose a specific model enrichment $\sigma_{\mathcal{M}}$ representing the equivalence class in the above theorem. Then $\sigma_{\mathcal{M}}(\hat{X},\hat{X})$ gives a ring object in $Sp^{\Sigma}(s\mathcal{A}b)$. The resulting isomorphism class in the homotopy category $\mathrm{Ho}\,(\mathcal{R}ing[Sp^{\Sigma}(s\mathcal{A}b)])$ only depends on the homotopy type of X, not on the choice of $\hat{X}$ or the representative $\sigma_{\mathcal{M}}$. We write $\mathrm{hEnd}_{\mathrm{ad}}(X)$ for any ring object in this isomorphism class, and call it the **additive homotopy endomorphism spectrum of** X.

1.4. Proposition. *Let $\mathcal{M}$ and $\mathcal{N}$ be additive, stable, combinatorial model categories. Suppose $\mathcal{M}$ and $\mathcal{N}$ are Quillen equivalent through a zig-zag of additive (but not necessarily combinatorial) model categories. Let $X \in \mathcal{M}$, and let $Y \in \mathcal{N}$ correspond to X under the derived equivalence of homotopy categories. Then $\mathrm{hEnd}_{\mathrm{ad}}(X)$ and $\mathrm{hEnd}_{\mathrm{ad}}(Y)$ are weakly equivalent in $\mathcal{R}ing[Sp^{\Sigma}(s\mathcal{A}b)]$.*

Any ring object R in $Sp^{\Sigma}(s\mathcal{A}b)$ gives rise to a ring object in Sp^{Σ} by forgetting the abelian group structure—this is called the Eilenberg-Mac Lane spectrum associated to R. Recall that in [D4] it was shown how to attach to any X in a stable, combinatorial model category an isomorphism class in $\mathrm{Ho}\,(\mathcal{R}ing[Sp^{\Sigma}])$. This was called the **homotopy endomorphism spectrum of** X, and denoted $\mathrm{hEnd}(X)$. We have the following:

1.5. Proposition. *Given $X \in \mathcal{M}$ as above, the homotopy endomorphism spectrum $\mathrm{hEnd}(X)$ is the Eilenberg-Mac Lane spectrum associated to $\mathrm{hEnd}_{\mathrm{ad}}(X)$.*

Finally, we have two results explaining how to compute $\mathrm{hEnd}_{\mathrm{ad}}(X)$ when the model category $\mathcal{M}$ has some extra structure. Recall that if $\mathcal{C}$ is a symmetric monoidal model category then a **$\mathcal{C}$-model category** is a model category equipped with compatible tensors, cotensors, and enriched hom-objects over $\mathcal{C}$ satisfying the analogue of $SM7$. See Section 2 for more detailed information. For $X, Y \in \mathcal{M}$ we denote the enriched hom-object by $\underline{\mathcal{M}}_{\mathcal{C}}(X,Y)$.

Note that a $Sp^{\Sigma}(s\mathcal{A}b)$-model category is automatically additive and stable. This follows from Corollary 6.9 below, and the appropriate analogue of [SS2, 3.5.2] or [GS, 3.2].

1.6. Proposition. *Let $\mathcal{M}$ be a combinatorial $Sp^{\Sigma}(s\mathcal{A}b)$-model category. Let $X \in \mathcal{M}$ be cofibrant-fibrant. Then $\mathrm{hEnd}_{\mathrm{ad}}(X)$ is weakly equivalent to the enriched hom-object $\underline{\mathcal{M}}_{Sp^{\Sigma}(s\mathcal{A}b)}(X,X)$.*

In [S] it is shown that the model categories of rings in $Sp^{\Sigma}(s\mathcal{A}b)$ and in Ch are Quillen equivalent. This is recalled in Section 9. Note that the rings in the category Ch are just differential graded algebras (dgas). The associated derived functors will be denoted $H'\colon \mathcal{DGA} \rightleftarrows \mathcal{R}ingSp^{\Sigma}(s\mathcal{A}b)\colon \Theta'$. (The reason for the 'primes' is that in [S] the functors H and Θ are functors between $\mathcal{DGA}$ and $H\mathbb{Z}$-algebras with $\mathcal{R}ingSp^{\Sigma}(s\mathcal{A}b)$ an intermediate category.) We then define the **homotopy endomorphism dga of X** to be $\Theta'[\mathrm{hEnd}_{\mathrm{ad}}(X)]$ and write $\mathbf{hEnd_{dga}(X)}$. Obviously, this carries exactly the same information as $\mathrm{hEnd}_{\mathrm{ad}}(X)$. In fact, $H'[\mathrm{hEnd}_{\mathrm{dga}}(X)]$ is weakly equivalent to $\mathrm{hEnd}_{\mathrm{ad}}(X)$ since H' and Θ' are inverse equivalences on the homotopy category level.

As above, we remark that a Ch-model category is automatically additive and stable, by Corollary 6.9 and the appropriate analogue of [SS2, 3.5.2].

1.7. Proposition. *Let $\mathcal{M}$ be a combinatorial* Ch*-model category. Assume $\mathcal{M}$ has a generating set of compact objects, as defined in (5.1) below. Let $X \in \mathcal{M}$ be cofibrant-fibrant. Then $\underline{\mathcal{M}}_{\mathrm{Ch}}(X, X)$ is weakly equivalent to* $\mathrm{hEnd}_{\mathrm{dga}}(X)$.

The assumption about the generating set in the above proposition is probably unnecessary, but we don't know how to remove it. It is satisfied in most cases of interest.

1.8. Remark. The study of **dg-categories** seems to be of current interest—see, for example, [Dr, T]. A dg-category is simply a category enriched over unbounded chain complexes Ch_k, where k is some commutative ground ring. We remark that the homotopy theory of dg-categories over $\mathbb{Z}$ is essentially the same as that of $Sp^\Sigma(s\mathcal{A}b)$-categories (this follows from results of [S] and [SS3]). So the present paper may be regarded as attaching to any stable, combinatorial, additive model category an associated dg-category.

The proof of Proposition 1.7 is not hard, but it requires a careful comparison of enrichments over Ch and $Sp^\Sigma(s\mathcal{A}b)$. This reduces to an abstract problem in enriched model category theory, but the necessary tools do not seem to be available in the literature. The first part of the paper is spent developing them.

We can briefly describe the main issue that arises. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal categories, and that there is an adjoint pair $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ in which the right adjoint G is lax monoidal. Also suppose that $\mathcal{M}$ is a $\mathcal{C}$-module category, $\mathcal{N}$ is a $\mathcal{D}$-module category, and that

$$L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$$

is another adjoint pair (all the terminology is defined later in the paper).

Then for any object $X \in \mathcal{N}$ we may consider the endomorphism monoid $\underline{\mathcal{N}}_{\mathcal{D}}(X, X)$. This is a monoid in $\mathcal{D}$, and applying G to it gives a monoid in $\mathcal{C}$. One can also consider the endomorphism monoid $\underline{\mathcal{M}}_{\mathcal{C}}(RX, RX)$, which is another monoid in $\mathcal{C}$. In order to compare these two monoids, one needs some compatibility conditions between the adjoint pairs (F, G) and (L, R). This leads to our definition of what we call an *adjoint module*. The basic theory of such things is developed in Sections 3–4. This notion has other applications, most notably in [GS].

1.9. Organization of the paper. Section 2 recalls the basics of enriched model category theory as used in [D4]. The new work begins in Sections 3 and 4 where we develop the notion of adjoint modules. This is used in Section 5 to prove a technical theorem about transporting enrichments over one symmetric monoidal model category to a Quillen equivalent one. Sections 6 through 9 contain the main results on additive model categories and $Sp^\Sigma(s\mathcal{A}b)$-enrichments. Appendix A reviews and expands on material from [SS3], which is needed in Section 4.9.

1.10. Notation and terminology. This paper is a sequel to [D4], and we will assume the reader is familiar with the machinery developed therein. In particular, we assume a familiarity with model enrichments and quasi-equivalences; see Section 2.3 for quick summaries, though. If $\mathcal{M}$ and $\mathcal{N}$ are model categories then by a **Quillen map** $L\colon \mathcal{M} \to \mathcal{N}$ we mean an adjoint pair of Quillen functors $L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$, where L is the left adjoint. Also, if $\mathcal{C}$ is a category then we'll write $\mathcal{C}(X,Y)$ for the set of maps from X to Y in $\mathcal{C}$. If $\mathcal{C}$ is a symmetric monoidal category, then the monoids in $\mathcal{C}$ are sometimes called "monoids" and sometimes called "rings"—we use these terms interchangeably.

2. Enriched model categories

In this section we review the notion of a model category $\mathcal{M}$ being enriched over a second model category $\mathcal{C}$. This situation comes in two varieties. If for every two objects $X, Y \in \mathcal{M}$ one has a 'mapping object' $\underline{\mathcal{M}}_{\mathcal{C}}(X,Y)$ in $\mathcal{C}$ together with composition maps (subject to certain axioms), then this is called a model enrichment. If for every $X \in \mathcal{M}$ and $c \in \mathcal{C}$ one also has objects $X \otimes c$ and $F(c,X)$ in $\mathcal{M}$, related by adjunctions to the mapping objects and also subject to certain axioms, then we say that $\mathcal{M}$ is a $\mathcal{C}$-model category. Thus, a $\mathcal{C}$-model category involves a model enrichment plus extra data.

There are two main examples to keep in mind. A *simplicial* model category is just another name for an $s\mathcal{S}et$-model category. And if $\mathcal{M}$ is any model category, then the hammock localization of Dwyer-Kan [DK] is an example of a model enrichment of $\mathcal{M}$ over $s\mathcal{S}et$.

2.1. Symmetric monoidal model categories. Let $\mathcal{C}$ be a closed symmetric monoidal category. This says that we are given a bifunctor $\otimes$, a unit object $\mathbf{1}_{\mathcal{C}}$, together with associativity, commutativity, and unital isomorphisms making certain diagrams commute (see [Ho1, Defs. 4.1.1, 4.1.4] for a nice summary). The 'closed' condition says that there is also a bifunctor $(a,b) \mapsto \underline{\mathcal{C}}(a,b) \in \mathcal{C}$ together with a natural isomorphism

$$\mathcal{C}(a, \underline{\mathcal{C}}(b,c)) \cong \mathcal{C}(a \otimes b, c).$$

Note that this gives isomorphisms $\mathcal{C}(\mathbf{1}_{\mathcal{C}}, \underline{\mathcal{C}}(a,b)) \cong \mathcal{C}(\mathbf{1}_{\mathcal{C}} \otimes a, b) \cong \mathcal{C}(a,b)$.

A **symmetric monoidal model category** consists of a closed symmetric monoidal category $\mathcal{C}$, together with a model structure on $\mathcal{C}$, satisfying two conditions:

(1) The analogue of SM7, as given in either [Ho1, 4.2.1] or [Ho1, 4.2.2(2)].

(2) A unit condition given in [Ho1, 4.2.6(2)].

2.2. $\mathcal{C}$-model categories. Let $\mathcal{C}$ be a symmetric monoidal category. One defines a **closed $\mathcal{C}$-module category** to be a category $\mathcal{M}$ equipped with natural constructions which assign to every $X, Z \in \mathcal{M}$ and $c \in \mathcal{C}$ objects

$$X \otimes c \in \mathcal{M}, \qquad F(c,Z) \in \mathcal{M}, \qquad \text{and} \qquad \underline{\mathcal{M}}_{\mathcal{C}}(X,Z) \in \mathcal{C}.$$

One requires, first, that there are natural isomorphisms $(X \otimes a) \otimes b \cong X \otimes (a \otimes b)$ and $X \otimes \mathbf{1}_{\mathcal{C}} \cong X$ making certain diagrams commute (see [Ho1, Def. 4.1.6]). One of these diagrams is a pentagon for four-fold associativity. We also require natural isomorphisms

$$\mathcal{M}(X \otimes a, Z) \cong \mathcal{M}(X, F(a, Z)) \cong \mathcal{C}(a, \underline{\mathcal{M}}_{\mathcal{C}}(X, Z)) \tag{2.2}$$

(see [Ho1, 4.1.12]).

Finally, suppose $\mathcal{C}$ is a symmetric monoidal model category. A **$\mathcal{C}$-model category** is a model category $\mathcal{M}$ which is also a closed $\mathcal{C}$-module category and where the two conditions from [Ho1, 4.2.18] hold: these are again the analogue of SM7 and a unit condition.

2.3. Model enrichments. Let $\mathcal{M}$ be a model category and let $\mathcal{C}$ be a symmetric monoidal model category. Recall from [D4, 3.1] that a **model enrichment** of $\mathcal{M}$ by $\mathcal{C}$ is a bifunctor $\sigma\colon \mathcal{M}^{\mathrm{op}} \times \mathcal{M} \to \mathcal{C}$ which is equipped with composition pairings $\sigma(Y, Z) \otimes \sigma(X, Y) \to \sigma(X, Z)$ and unit maps $\mathbf{1}_{\mathcal{C}} \to \sigma(X, X)$ satisfying associativity and unital conditions. There is also a compatibility condition between the functor structure and the unit maps. Finally, one assumes that if $X \to X'$ is a weak equivalence between cofibrant objects and $Y \to Y'$ is a weak equivalence between fibrant objects then the maps $\sigma(X, Y) \to \sigma(X, Y')$ and $\sigma(X', Y) \to \sigma(X, Y)$ are weak equivalences. See [D4, Section 3.1].

There is a notion of quasi-equivalence encoding when two model enrichments are 'the same'. This is also given in [D4, Section 3.1]. To define this we need two preliminary notions.

Let σ and τ be two model enrichments of $\mathcal{M}$ by $\mathcal{C}$. By a $\sigma - \tau$ **bimodule** we mean a collection of objects $M(a, b) \in \mathcal{C}$ for every $a, b \in \mathcal{C}$, together with multiplication maps

$$\sigma(b, c) \otimes M(a, b) \to M(a, c) \qquad \text{and} \qquad M(b, c) \otimes \tau(a, b) \to M(a, c)$$

which are natural in a and c. Associativity and unital conditions are again assumed, although we will not write these down. One also requires that for any $a, b, c, d \in \mathcal{C}$ the two obvious maps

$$\sigma(c, d) \otimes M(b, c) \otimes \tau(a, b) \rightrightarrows M(a, d)$$

are equal.

It is perhaps not quite obvious, but M becomes a bifunctor via the multiplication maps from σ and τ and the fact that σ and τ are bifunctors. See [D4, Section 2.2].

A **pointed $\boldsymbol{\sigma - \tau}$ bimodule** is a bimodule M together with a collection of maps $\mathbf{1}_{\mathcal{C}} \to M(c, c)$ for every $c \in \mathcal{C}$, such that for any map $a \to b$ the square

$$\begin{tikzcd}
\mathbf{1}_{\mathcal{C}} \arrow[r] \arrow[d] & M(a, a) \arrow[d] \\
M(b, b) \arrow[r] & M(a, b)
\end{tikzcd}$$

commutes.

A **quasi-equivalence** between two model enrichments σ and τ consists of a pointed $\sigma - \tau$ bimodule M such that the compositions

$$\sigma(a,b) \otimes \mathbf{1}_{\mathcal{C}} \to \sigma(a,b) \otimes M(a,a) \to M(a,b) \qquad \text{and}$$
$$\mathbf{1}_{\mathcal{C}} \otimes \tau(a,b) \to M(b,b) \otimes \tau(a,b) \to M(a,b)$$

are weak equivalences whenever a is cofibrant and b is fibrant.

The notion of quasi-equivalence generates an equivalence relation on the class of model enrichments of $\mathcal{M}$ by $\mathcal{C}$. We write $ME_0(\mathcal{M},\mathcal{C})$ for the collection of equivalence classes of model enrichments. When we say that two enrichments σ and τ are 'quasi-equivalent' we mean that they are in the same equivalence class; note that this means there is a chain of model enrichments $\sigma = \sigma_1, \sigma_2, \ldots, \sigma_n = \tau$ and pointed $\sigma_i - \sigma_{i+1}$ bimodules M_i giving quasi-equivalences between each step in the chain.

If $L\colon \mathcal{M} \to \mathcal{N}$ is a Quillen map then by [D4, Prop. 3.14] there are induced maps $L_*\colon ME_0(\mathcal{M},\mathcal{C}) \to ME_0(\mathcal{N},\mathcal{C})$ and $L^*\colon ME_0(\mathcal{N},\mathcal{C}) \to ME_0(\mathcal{M},\mathcal{C})$. When L is a Quillen equivalence these are inverse bijections.

2.4. Monoidal functors. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal model categories, and that $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ is a Quillen pair.

First of all, recall that G is called **lax monoidal** if there is a natural transformation

$$G(X) \otimes G(Y) \to G(X \otimes Y)$$

and a map $\mathbf{1}_{\mathcal{C}} \to G(\mathbf{1}_{\mathcal{D}})$ which are compatible with the associativity and unital isomorphisms in $\mathcal{C}$ and $\mathcal{D}$. A lax monoidal functor takes monoids in $\mathcal{D}$ to monoids in $\mathcal{C}$.

A lax monoidal functor is called **strong monoidal** if the above maps are actually isomorphisms.

If G is lax monoidal then the adjunction gives rise to induced maps $F(\mathbf{1}_{\mathcal{C}}) \to \mathbf{1}_{\mathcal{D}}$ and $F(A \otimes B) \to F(A) \otimes F(B)$. Following [SS3, Section 3], we say that (F,G) is a **weak monoidal Quillen equivalence** if G is lax monoidal and two extra conditions hold. First, for some cofibrant replacement $A \to \mathbf{1}_{\mathcal{C}}$, the induced map $F(A) \to F(\mathbf{1}_{\mathcal{C}}) \to \mathbf{1}_{\mathcal{D}}$ is a weak equivalence. Second, for any two cofibrant objects $A, B \in \mathcal{C}$ the map $F(A \otimes B) \to F(A) \otimes F(B)$ is a weak equivalence.

3. Adjoint modules

In this section and the next we deal with the general situation of one Quillen pair enriched over another Quillen pair. Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal model categories, let $\mathcal{M}$ be a $\mathcal{C}$-model category, and let $\mathcal{N}$ be a $\mathcal{D}$-model category. Let

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G \qquad \text{and} \qquad L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$$

be two Quillen pairs, where we assume that G is lax monoidal (see Section 2.4). As usual, we'll write $\underline{\mathcal{M}}_{\mathcal{C}}(X,Y)$ and $\underline{\mathcal{N}}_{\mathcal{D}}(X,Y)$ for the enriched morphism objects over $\mathcal{C}$ and $\mathcal{D}$, respectively.

Finally, let Y be a cofibrant-fibrant object in $\mathcal{N}$. Then $\underline{\mathcal{N}}_{\mathcal{D}}(Y,Y)$ is a monoid in $\mathcal{D}$, and so $G(\underline{\mathcal{N}}_{\mathcal{D}}(Y,Y))$ is a monoid in $\mathcal{C}$. Alternatively, we may choose a cofibrant-replacement $\mathcal{Q}RY \xrightarrow{\sim} RY$ and consider the $\mathcal{C}$-monoid $\underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}RY, \mathcal{Q}RY)$. How can we compare these two monoids, and under what conditions will they be weakly equivalent?

This question can be answered by requiring certain compatibility conditions between (L,R) and (F,G). The goal of the present section is to write down these conditions; this culminates in Definition 3.8, where we define what it means for (L,R) to be an **adjoint module** over (F,G). The next section uses this to tackle the problem of comparing enrichments.

3.1. Compatibility structure. Before we can develop the definition of an adjoint module we need the following statement. For the moment we only assume that (F,G) and (L,R) are adjunctions. That is, we temporarily drop the assumptions that they are Quillen pairs and that G is lax monoidal.

3.2. Proposition. *There is a canonical bijection between natural transformations of the following four types:*

(i) $G\underline{\mathcal{N}}_{\mathcal{D}}(LX,Y) \to \underline{\mathcal{M}}_{\mathcal{C}}(X,RY)$

(ii) $L(X \otimes c) \to LX \otimes Fc$

(iii) $RY \otimes Gd \to R(Y \otimes d)$

(iv) $G\underline{\mathcal{N}}_{\mathcal{D}}(X,Y) \to \underline{\mathcal{M}}_{\mathcal{C}}(RX,RY)$.

Proof. This is a routine exercise in adjunctions. We will only do some pieces of the argument and leave the rest to the reader.

Suppose given a natural transformation $G\underline{\mathcal{N}}_{\mathcal{D}}(LX,Y) \to \underline{\mathcal{M}}_{\mathcal{C}}(X,RY)$. For any $c \in \mathcal{C}$ one therefore has the composition

$$\begin{tikzcd}[column sep=small] & \mathcal{C}(c, G\underline{\mathcal{N}}_{\mathcal{D}}(LX,Y)) \arrow[r] \arrow[d, "\cong"] & \mathcal{C}(c, \underline{\mathcal{M}}_{\mathcal{C}}(X,RY)) \arrow[d, "\cong"] & \\ \mathcal{N}(LX \otimes Fc, Y) \arrow[r, "\cong"] & \mathcal{D}(Fc, \underline{\mathcal{N}}_{\mathcal{D}}(LX,Y)) & \mathcal{M}(X \otimes c, RY) \arrow[r, "\cong"] & \mathcal{N}(L(X \otimes c), Y). \end{tikzcd} \tag{3.3}$$

By the Yoneda Lemma this gives a map $L(X \otimes c) \to LX \otimes Fc$, and this is natural in both X and c.

Likewise, suppose given a natural transformation $L(X \otimes c) \to LX \otimes Fc$. Then for $Y \in \mathcal{N}$ and $d \in \mathcal{D}$ we obtain

$$L(RY \otimes Gd) \to LRY \otimes FGd \to Y \otimes d$$

where the second map uses the units of the adjunctions. Taking the adjoint of the composition gives $RY \otimes Gd \to R(Y \otimes d)$, as desired.

Finally, suppose again that we have a natural transformation $G\underline{\mathcal{N}}_{\mathcal{D}}(LX, Y) \to \underline{\mathcal{M}}_{\mathcal{C}}(X, RY)$. For $X, Y \in \mathcal{N}$ consider the composite

$$G\underline{\mathcal{N}}_{\mathcal{D}}(X, Y) \to G\underline{\mathcal{N}}_{\mathcal{D}}(LRX, Y) \to \underline{\mathcal{M}}_{\mathcal{C}}(RX, RY)$$

where the first map is obtained by applying G to $\underline{\mathcal{N}}_{\mathcal{D}}(X, Y) \to \underline{\mathcal{N}}_{\mathcal{D}}(LRX, Y)$ induced by the unit $LRX \to X$. The above composite is our natural transformation of type (iv).

We have constructed maps $(i) \to (ii)$, $(ii) \to (iii)$, and $(i) \to (iv)$. We leave it to the reader to construct maps in the other directions and verify that one obtains inverse bijections. ∎

3.4. Remark. Suppose we are given a natural transformation $\gamma\colon G\underline{\mathcal{N}}_{\mathcal{D}}(LX, Y) \to \underline{\mathcal{M}}_{\mathcal{C}}(X, RY)$. Using the bijections from the above result, we obtain natural transformations of types (ii), (iii), and (iv). We will also call each of these γ, by abuse.

The next proposition lists the key homotopical properties required for (L, R) to be a Quillen adjoint module over (F, G).

3.5. Proposition. *Assume that (F, G) and (L, R) are Quillen pairs and that we have a natural transformation $\gamma\colon G\underline{\mathcal{N}}_{\mathcal{D}}(LX, Y) \to \underline{\mathcal{M}}_{\mathcal{C}}(X, RY)$.*

(a) The following two conditions are equivalent:

- *The map $\gamma\colon G\underline{\mathcal{N}}_{\mathcal{D}}(LX, Y) \to \underline{\mathcal{M}}_{\mathcal{C}}(X, RY)$ is a weak equivalence whenever X is cofibrant and Y is fibrant.*
- *The map $\gamma\colon L(X \otimes c) \to LX \otimes Fc$ is a weak equivalence whenever X and c are both cofibrant.*

(b) If (L, R) is a Quillen equivalence, the conditions in (a) are also equivalent to:

- *For any cofibrant replacement $\mathcal{Q}RX \to RX$, the composite map*

$$G\underline{\mathcal{N}}_{\mathcal{D}}(X, Y) \xrightarrow{\gamma} \underline{\mathcal{M}}_{\mathcal{C}}(RX, RY) \to \underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}RX, RY)$$

is a weak equivalence whenever X is cofibrant-fibrant and Y is fibrant.

(c) Assume that both (L, R) and (F, G) are Quillen equivalences. Then the conditions in (a) and (b) are also equivalent to:

- *For any cofibrant replacements $\mathcal{Q}RY \to RY$ and $\mathcal{Q}'Gd \to Gd$ and any fibrant replacement $Y \otimes d \to \mathcal{F}(Y \otimes d)$, the composite*

$$\mathcal{Q}RY \otimes \mathcal{Q}'Gd \to RY \otimes Gd \xrightarrow{\gamma} R(Y \otimes d) \to R\mathcal{F}(Y \otimes d)$$

is a weak equivalence whenever Y and d are cofibrant and fibrant.

PROOF. This is routine and basically follows from the adjunctions in Proposition 3.2 with the following two additions. For the equivalence in part (a), consider the maps from 3.3 in the respective homotopy categories. For the equivalence with (b), note that the composite in (b) agrees with the composite

$$G\underline{\mathcal{N}}_{\mathcal{D}}(X,Y) \to G\underline{\mathcal{N}}_{\mathcal{D}}(L\mathcal{Q}RX,Y) \xrightarrow{\gamma} \underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}RX,RY).$$

■

The above homotopical properties need to be supplemented by categorical associativity and unital properties which are listed in the next two propositions. Then, after stating these categorical properties, we finally state the definition of a Quillen adjoint module.

3.6. PROPOSITION. *Assume G is lax monoidal. Note that this gives a lax comonoidal structure on F, by adjointness. Let γ again denote a set of four corresponding natural transformations of types (i)–(iv). Then the conditions in (a) and (b) below are equivalent:*

(a) The diagrams

$$\begin{tikzcd} L((X\otimes c)\otimes c') \arrow[r, "\gamma"] \arrow[d, "\cong"] & L(X\otimes c)\otimes Fc' \arrow[r, "\gamma\otimes 1"] & (LX\otimes Fc)\otimes Fc' \arrow[d, "\cong"] \\ L(X\otimes(c\otimes c')) \arrow[r, "\gamma"] & LX\otimes F(c\otimes c') \arrow[r] & LX\otimes(Fc\otimes Fc') \end{tikzcd}$$

all commute, for any X, c, c'.

(b) The diagrams

$$\begin{tikzcd} RY\otimes(Gd\otimes Gd') \arrow[r] \arrow[d, "\cong"] & RY\otimes G(d\otimes d') \arrow[r, "\gamma"] & R(Y\otimes(d\otimes d')) \arrow[d, "\cong"] \\ (RY\otimes Gd)\otimes Gd' \arrow[r, "\gamma\otimes 1"] & R(Y\otimes d)\otimes Gd' \arrow[r, "\gamma"] & R((Y\otimes d)\otimes d') \end{tikzcd}$$

all commute, for any Y, d, d'.

If G is lax symmetric monoidal, then the above (equivalent) conditions imply the following one:

(c) The diagrams

$$\begin{tikzcd} G\underline{\mathcal{N}}_{\mathcal{D}}(Y,Z)\otimes G\underline{\mathcal{N}}_{\mathcal{D}}(X,Y) \arrow[r] \arrow[d, "\gamma\otimes\gamma"'] & G\Big(\underline{\mathcal{N}}_{\mathcal{D}}(Y,Z)\otimes\underline{\mathcal{N}}_{\mathcal{D}}(X,Y)\Big) \arrow[r] & G\underline{\mathcal{N}}_{\mathcal{D}}(X,Z) \arrow[d, "\gamma"] \\ \underline{\mathcal{M}}_{\mathcal{C}}(RY,RZ)\otimes\underline{\mathcal{M}}_{\mathcal{C}}(RX,RY) \arrow[rr] & & \underline{\mathcal{M}}_{\mathcal{C}}(RX,RZ) \end{tikzcd}$$

commute for any X, Y, and Z.

Proof. The equivalence of (a) and (b) is extremely tedious but routine; we leave it to the reader. For (c), note that by using the adjunction $\mathcal{C}(c, \underline{\mathcal{M}}_{\mathcal{C}}(RX, RZ)) \cong \mathcal{M}_{\mathcal{C}}(RX \otimes c, RZ)$ the two ways of going around the diagram correspond to two maps

$$RX \otimes [G\underline{\mathcal{N}}_{\mathcal{D}}(Y,Z) \otimes G\underline{\mathcal{N}}_{\mathcal{D}}(X,Y)] \longrightarrow RZ.$$

One of these is the composite

$$\begin{tikzcd}
 & RX \otimes [G\underline{\mathcal{N}}_{\mathcal{D}}(Y,Z) \otimes G\underline{\mathcal{N}}_{\mathcal{D}}(X,Y)] \arrow[r] & RX \otimes G[\underline{\mathcal{N}}_{\mathcal{D}}(Y,Z) \otimes \underline{\mathcal{N}}_{\mathcal{D}}(X,Y)] \arrow[d] \\
RZ & R(X \otimes \underline{\mathcal{N}}_{\mathcal{D}}(X,Z)) \arrow[l] & RX \otimes G[\underline{\mathcal{N}}_{\mathcal{D}}(X,Z)] \arrow[l, "\gamma"']
\end{tikzcd}$$

The other is the composite

$$\begin{tikzcd}
RX \otimes [G\underline{\mathcal{N}}_{\mathcal{D}}(Y,Z) \otimes G\underline{\mathcal{N}}_{\mathcal{D}}(X,Y)] \arrow[r, "\cong"] & RX \otimes [G\underline{\mathcal{N}}_{\mathcal{D}}(X,Y) \otimes G\underline{\mathcal{N}}_{\mathcal{D}}(Y,Z)] \arrow[d] \\
{[R(X \otimes \underline{\mathcal{N}}_{\mathcal{D}}(X,Y))] \otimes G\underline{\mathcal{N}}_{\mathcal{D}}(Y,Z)} \arrow[d] & {[RX \otimes G\underline{\mathcal{N}}_{\mathcal{D}}(X,Y)] \otimes G\underline{\mathcal{N}}_{\mathcal{D}}(Y,Z)} \arrow[l, "\gamma\otimes 1"'] \\
RY \otimes G\underline{\mathcal{N}}_{\mathcal{D}}(Y,Z) \arrow[r, "\gamma"] & R(Y \otimes \underline{\mathcal{N}}_{\mathcal{D}}(Y,Z)) \arrow[r] & RZ.
\end{tikzcd}$$

The commutativity isomorphism comes into the first stage of this composite because of how the composition map $\underline{\mathcal{M}}_{\mathcal{C}}(RY, RZ) \otimes \underline{\mathcal{M}}_{\mathcal{C}}(RX, RY) \to \underline{\mathcal{M}}_{\mathcal{C}}(RX, RZ)$ relates to the evaluation maps under adjunction—see [D4, Prop. A.3], for instance.

It is now a tedious but routine exercise to prove that the above two maps

$$RX \otimes [G\underline{\mathcal{N}}_{\mathcal{D}}(Y,Z) \otimes G\underline{\mathcal{N}}_{\mathcal{D}}(X,Y)] \rightrightarrows RZ$$

are indeed the same. One forms the adjoints and then writes down a huge commutative diagram. A very similar result (in fact, a special case of the present one) is proven in [D4, A.9]. ∎

Note that if G is lax monoidal then it comes with a prescribed map $\mathbf{1}_{\mathcal{C}} \to G(\mathbf{1}_{\mathcal{D}})$; adjointing gives $F(\mathbf{1}_{\mathcal{C}}) \to \mathbf{1}_{\mathcal{D}}$. The following result concerns compatibility between these maps and γ:

3.7. Proposition. *Assume again that G is lax monoidal, and let γ denote a set of four corresponding natural transformations of types (i)–(iv). The following three conditions are equivalent:*

(a) For any X, the following square commutes:

$$\begin{tikzcd}
LX \arrow[r, "\cong"] \arrow[d, "\cong"'] & LX \otimes \mathbf{1}_{\mathcal{D}} \\
L(X \otimes \mathbf{1}_{\mathcal{C}}) \arrow[r, "\gamma"] & LX \otimes F(\mathbf{1}_{\mathcal{C}}). \arrow[u]
\end{tikzcd}$$

(b) For any Y, the following square commutes:

$$\begin{tikzcd} RY \arrow[r, "\cong"] \arrow[d, "\cong"'] & RY \otimes \mathbf{1}_{\mathcal{C}} \arrow[d] \\ R(Y \otimes \mathbf{1}_{\mathcal{D}}) & RY \otimes G(\mathbf{1}_{\mathcal{D}}). \arrow[l, "\gamma"'] \end{tikzcd}$$

(c) For any Y, the following square commutes:

$$\begin{tikzcd} \mathbf{1}_{\mathcal{C}} \arrow[r] \arrow[d] & G(\mathbf{1}_{\mathcal{D}}) \arrow[d] \\ \underline{\mathcal{M}}_{\mathcal{C}}(RY, RY) & G\underline{\mathcal{N}}_{\mathcal{D}}(Y, Y) \arrow[l, "\gamma"'] \end{tikzcd}$$

Proof. Left to the reader. ∎

Finally we have the main definition:

3.8. Definition. *Assume given adjoint pairs (F, G) and (L, R) where G is lax monoidal. We will say that (L, R) is an **adjoint module** over (F, G) if there exists a natural transformation $\gamma\colon L(X \otimes c) \to LX \otimes Fc$ such that the conditions of Propositions 3.6(a) and 3.7(a) are both satisfied.*

*If in addition (F, G) and (L, R) are both Quillen pairs and the equivalent conditions of Proposition 3.5(a) are satisfied we will say that (L, R) is a **Quillen adjoint module** over (F, G).*

3.9. Basic properties. Below we give three properties satisfied by Quillen adjoint modules. Recall the notion of a $\mathcal{C}$-Quillen adjunction between $\mathcal{C}$-model categories, as in [D4, A.7]. This is a Quillen pair $L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$ where $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{C}$-model categories, together with natural isomorphisms $L(X \otimes c) \cong L(X) \otimes c$ which reduce to the canonical isomorphism for $c = \mathbf{1}_{\mathcal{C}}$ and which are compatible with the associativity isomorphisms in $\mathcal{M}$ and $\mathcal{N}$. See also [Ho1, Def. 4.1.7].

3.10. Proposition. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{C}$-model categories and $L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$ is a $\mathcal{C}$-Quillen adjunction. Then (L, R) is a Quillen adjoint module over the pair $(id_{\mathcal{C}}, id_{\mathcal{C}})$.*

Proof. Since (L, R) is a $\mathcal{C}$-adjunction, there are natural isomorphisms $LX \otimes c \to L(X \otimes c)$ which satisfy the associativity and unital properties listed in Propositions 3.6(a) and 3.7(a). This also fulfills the second condition listed in Proposition 3.5(a) ∎

3.11. Proposition. *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ be a Quillen pair between symmetric monoidal model categories, where G is lax monoidal. Let $F'\colon \mathcal{D} \rightleftarrows \mathcal{E}\colon G'$ be another such pair. Let $L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$ and $L'\colon \mathcal{N} \rightleftarrows \mathcal{P}\colon R'$ be Quillen pairs such that (L, R) is a Quillen adjoint module over (F, G) and (L', R') is a Quillen adjoint module over (F', G'). Then $(L'L, RR')$ is a Quillen adjoint module over $(F'F, GG')$.*

PROOF. For $X \in \mathcal{M}$ and $c \in \mathcal{C}$ we have natural maps

$$L'L(X \otimes c) \to L'(LX \otimes Fc) \to L'(LX) \otimes F'(Fc)$$

using the adjoint module structure on (L, R) over (F, G) first, and the module structure on (L', R') over (F', G') second. One just has to check the axioms to see that these maps make $(L'L, RR')$ a Quillen adjoint module over $(F'F, GG')$. This is a routine exercise in categorical diagramming which we will leave to the reader. ∎

3.12. COROLLARY. *Suppose (L, R) is a Quillen adjoint module over (F, G), and also suppose that $\mathcal{P}$ is a $\mathcal{C}$-model category and $J\colon \mathcal{P} \rightleftarrows \mathcal{M}\colon K$ is a $\mathcal{C}$-Quillen adjunction. Then (LJ, KR) is a Quillen adjoint module over (F, G).*

PROOF. This is an immediate consequence of the above two propositions. ∎

4. Applications of adjoint modules

Recall from the last section that $\mathcal{C}$ and $\mathcal{D}$ are symmetric monoidal model categories, $\mathcal{M}$ is a $\mathcal{C}$-model category, and $\mathcal{N}$ is a $\mathcal{D}$-model category. We have Quillen pairs

$$F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G \qquad \text{and} \qquad L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$$

in which G is lax monoidal, and we assume that (L, R) is a Quillen adjoint module over (F, G) as defined in Definition 3.8.

Recall the notion of model enrichment from Section 2.3. The assignment $X, Y \mapsto \underline{\mathcal{N}}_{\mathcal{D}}(X, Y)$ is a $\mathcal{D}$-model enrichment of $\mathcal{N}$, as in [D4, Example 3.2]. The induced assignment $X, Y \mapsto G\underline{\mathcal{N}}_{\mathcal{D}}(X, Y)$ is a $\mathcal{C}$-model enrichment of $\mathcal{N}$, by Proposition 4.6 below. Alternatively, if $\mathcal{Q}W \xrightarrow{\sim} W$ is a cofibrant-replacement functor for $\mathcal{M}$ and $W \overset{\sim}{\rightarrowtail} \mathcal{F}W$ is a fibrant-replacement functor for $\mathcal{N}$, then one obtains another $\mathcal{C}$-model enrichment of $\mathcal{N}$ via $X, Y \mapsto \underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}R\mathcal{F}X, \mathcal{Q}R\mathcal{F}Y)$. (This is precisely the enrichment $L_*[\underline{\mathcal{M}}_{\mathcal{C}}]$, as defined in [D4, Section 3.4].) If R preserves all weak equivalences, the simpler assignment $X, Y \mapsto \underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}RX, \mathcal{Q}RY)$ is also a $\mathcal{C}$-model enrichment.

4.1. THEOREM. *Assume the pair (L, R) is a Quillen adjoint module over (F, G). Also assume that G is lax symmetric monoidal and that (L, R) is a Quillen equivalence. Then the two $\mathcal{C}$-model enrichments on $\mathcal{N}$ given by $X, Y \mapsto G\underline{\mathcal{N}}_{\mathcal{D}}(X, Y)$ and $X, Y \mapsto \underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}R\mathcal{F}X, \mathcal{Q}R\mathcal{F}Y)$ are quasi-equivalent. That is to say, $L_*\underline{\mathcal{M}}_{\mathcal{C}} \simeq G\underline{\mathcal{N}}_{\mathcal{D}}$.*

If R preserves all weak equivalences, then the above enrichments are also quasi-equivalent to $X, Y \mapsto \underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}RX, \mathcal{Q}RY)$.

We also have the following corollary:

4.2. Corollary. *Under the assumptions of the theorem, the two $\mathcal{C}$-model enrichments on $\mathcal{M}$ given by $X, Y \mapsto G\underline{\mathcal{N}}_{\mathcal{D}}(\mathcal{F}L\mathcal{Q}X, \mathcal{F}L\mathcal{Q}Y)$ and $X, Y \mapsto \underline{\mathcal{M}}_{\mathcal{C}}(X, Y)$ are quasi-equivalent. That is, $L^*[G\underline{\mathcal{N}}_{\mathcal{D}}]$ is quasi-equivalent to $\underline{\mathcal{M}}_{\mathcal{C}}$. Here $\mathcal{Q}A \xrightarrow{\sim} A$ and $X \overset{\sim}{\rightarrowtail} \mathcal{F}X$ are now cofibrant- and fibrant- replacement functors in $\mathcal{M}$ and $\mathcal{N}$, respectively.*

The quasi-equivalences in Theorem 4.1 and Corollary 4.2 are used in a key argument in [GS] to translate a construction in $H\mathbb{Q}$-algebras into rational dgas. The following immediate corollary of the above theorem is what we will mainly need in the present paper.

4.3. Corollary. *Assume that $\mathcal{C}$ is combinatorial, satisfies the monoid axiom, and that $\mathbf{1}_{\mathcal{C}}$ is cofibrant. Under the assumptions of the theorem, let $X \in \mathcal{N}$ be a cofibrant-fibrant object. Let $A \in \mathcal{M}$ be any cofibrant-fibrant object which is weakly equivalent to RX. Then the $\mathcal{C}$-monoids $G\underline{\mathcal{N}}_{\mathcal{D}}(X, X)$ and $\underline{\mathcal{M}}_{\mathcal{C}}(A, A)$ are weakly equivalent.*

The extra assumptions on $\mathcal{C}$ are necessary in order to apply a certain proposition from [D4], saying that quasi-equivalent enrichments give weakly equivalent endomorphism monoids.

The above theorem and corollaries compare enrichments which have been transferred over the right adjoint G. It is more difficult to transfer enrichments over the left adjoint F. The assignment $A, B \mapsto F\underline{\mathcal{M}}_{\mathcal{C}}(A, B)$ is an enrichment only if F is assumed to be strong monoidal; and since $\underline{\mathcal{M}}_{\mathcal{C}}(A, B)$ will not usually be cofibrant, it will only be a model enrichment under very strong assumptions such as that F preserves all weak equivalences. Finally, the construction $F\underline{\mathcal{M}}_{\mathcal{C}}(A, B)$ effectively amounts to mixing a left and right adjoint (the mapping object is a right adjoint), and so is not so well-behaved categorically.

The following result is given only for completeness; we have no applications for it at present.

4.4. Proposition. *Assume that (L, R) is a Quillen adjoint module over (F, G). Suppose that F and G are both strong monoidal, and that F preserves all weak equivalences. Also assume that G is symmetric monoidal, and that both (F, G) and (L, R) are Quillen equivalences. Then the model enrichments $A, B \mapsto F\underline{\mathcal{M}}_{\mathcal{C}}(A, B)$ and $A, B \mapsto \underline{\mathcal{N}}_{\mathcal{D}}(\mathcal{F}L\mathcal{Q}A, \mathcal{F}L\mathcal{Q}B)$ are quasi-equivalent.*

4.5. Proofs of the above results.

4.6. Proposition. *The assignment $n, n' \mapsto G\underline{\mathcal{N}}_{\mathcal{D}}(n, n')$ is a $\mathcal{C}$-model enrichment on $\mathcal{N}$.*

Proof. One uses the monoidal structure on G to produce the associative and unital composition maps. Since G preserves equivalences between all fibrant objects and $\underline{\mathcal{N}}_{\mathcal{D}}(n, n')$ is fibrant if n is cofibrant and n' is fibrant, we see that $G\underline{\mathcal{N}}_{\mathcal{D}}(a', x) \to G\underline{\mathcal{N}}_{\mathcal{D}}(a, x)$ and $G\underline{\mathcal{N}}_{\mathcal{D}}(a, x) \to G\underline{\mathcal{N}}_{\mathcal{D}}(a, x')$ are weak equivalences whenever $a \to a'$ is a weak equivalence between cofibrant objects and $x \to x'$ is a weak equivalence between fibrant objects. ∎

Proof of Theorem 4.1. For $X, Y \in \mathcal{N}$ define $\sigma(X,Y) = G\underline{\mathcal{N}}_{\mathcal{D}}(\mathcal{F}X, \mathcal{F}Y)$ and $\tau(X,Y) = \underline{\mathcal{M}}_{\mathcal{C}}(QR\mathcal{F}X, QR\mathcal{F}Y)$. These are both $\mathcal{C}$-model enrichments on $\mathcal{N}$, and the former is quasi-equivalent to $X, Y \mapsto G\underline{\mathcal{N}}_{\mathcal{D}}(X,Y)$ by [D4, Prop. 3.9].

Define $W(X,Y) = \underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}R\mathcal{F}X, R\mathcal{F}Y)$. This is a $\sigma - \tau$ bimodule via the maps

$$\begin{tikzcd} G\underline{\mathcal{N}}_{\mathcal{D}}(\mathcal{F}Y,\mathcal{F}Z) \otimes \underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}R\mathcal{F}X, R\mathcal{F}Y) \arrow[r, "\gamma\otimes 1"] & \underline{\mathcal{M}}_{\mathcal{C}}(R\mathcal{F}Y, R\mathcal{F}Z) \otimes \underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}R\mathcal{F}X, R\mathcal{F}Y) \arrow[d] \\ & \underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}R\mathcal{F}X, R\mathcal{F}Z) \end{tikzcd}$$

and

$$\underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}R\mathcal{F}Y, R\mathcal{F}Z) \otimes \underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}R\mathcal{F}X, \mathcal{Q}R\mathcal{F}Y) \longrightarrow \underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}R\mathcal{F}X, R\mathcal{F}Z).$$

Some routine but tedious checking is required to see that this indeed satisfies the bimodule axioms of [D4, Section 2.2]. This uses the conditions from Proposition 3.6(c) and Proposition 3.7(a).

The canonical maps $\mathcal{Q}R\mathcal{F}X \to R\mathcal{F}X$ give maps $\mathbf{1}_{\mathcal{C}} \to W(X,X)$ making W into a pointed bimodule, and one checks using the condition from Proposition 3.5(b) that this is a quasi-equivalence. This last step uses our assumption that (L,R) is a Quillen equivalence.

If R preserves all weak equivalences, then the above proof works even if every appearance of the functor $\mathcal{F}$ is removed. ∎

Proof of Corollary 4.2. The result [D4, 3.14(d)] shows that since L is a Quillen equivalence the maps L^* and L_* are inverse bijections. Since we have already proven $L_*\underline{\mathcal{M}}_{\mathcal{C}} \simeq G\underline{\mathcal{N}}_{\mathcal{D}}$, we must have $L^*[G\underline{\mathcal{N}}_{\mathcal{D}}] \simeq \underline{\mathcal{M}}_{\mathcal{C}}$. ∎

Proof of Corollary 4.3. Using the above theorem together with [D4, Cor. 3.6] (which requires our assumptions on $\mathcal{C}$) we find that if $X \in \mathcal{N}$ is cofibrant-fibrant then the $\mathcal{C}$-monoids $G\underline{\mathcal{N}}_{\mathcal{D}}(X,X)$ and $\underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}R\mathcal{F}X, \mathcal{Q}R\mathcal{F}X)$ are weakly equivalent. However, note that one has a weak equivalence $A \xrightarrow{\sim} \mathcal{Q}R\mathcal{F}X$. By applying [D4, Cor. 3.7] (in the case where $\mathcal{I}$ is the category with one object and an identity map) one finds that the $\mathcal{C}$-monoids $\underline{\mathcal{M}}_{\mathcal{C}}(\mathcal{Q}R\mathcal{F}X, \mathcal{Q}R\mathcal{F}X)$ and $\underline{\mathcal{M}}_{\mathcal{C}}(A,A)$ are weakly equivalent. ∎

Proof of Proposition 4.4. We only sketch this, as the result is not needed in the paper. Consider the following two model enrichments on $\mathcal{M}$:

$$A, B \mapsto \underline{\mathcal{N}}_{\mathcal{D}}(\mathcal{F}L\mathcal{Q}A, \mathcal{F}L\mathcal{Q}B) \tag{4.7}$$

and

$$A, B \mapsto FG\underline{\mathcal{N}}_{\mathcal{D}}(\mathcal{F}L\mathcal{Q}A, \mathcal{F}L\mathcal{Q}B). \tag{4.8}$$

The strong monoidal assumptions on F and G show that for any map $P\otimes Q\to R$ in $\mathcal{D}$, the induced square

$$\begin{tikzcd} FG(P)\otimes FG(Q) \arrow[r] \arrow[d] & FG(R) \arrow[d] \\ P\otimes Q \arrow[r] & R \end{tikzcd}$$

is commutative. Here the vertical maps come from the counit of the adjunction (F,G), and the top horizontal map uses the isomorphism $FG(P\otimes Q)\cong FG(P)\otimes FG(Q)$ coming from the strong monoidal structures on F and G. This commutative square shows that we have a map of enrichments $FG\underline{\mathcal{N}}_{\mathcal{D}}(X,Y)\to\underline{\mathcal{N}}_{\mathcal{D}}(X,Y)$ on $\mathcal{N}$. Using this, we get a map of model enrichments from (4.8) to (4.7), which allows us to think of $A,B\mapsto\underline{\mathcal{N}}_{\mathcal{D}}(\mathcal{F}L\mathcal{Q}A,\mathcal{F}L\mathcal{Q}B)$ as a bimodule over (4.7) and (4.8) (in either order). Using that (F,G) is a Quillen equivalence one readily checks that this is a quasi-equivalence.

By Corollary 4.2, the model enrichment $A,B\mapsto G\underline{\mathcal{N}}_{\mathcal{D}}(\mathcal{F}L\mathcal{Q}A,\mathcal{F}L\mathcal{Q}B)$ is quasi-equivalent to $A,B\mapsto\underline{\mathcal{M}}_{\mathcal{C}}(A,B)$. But the assumptions on F readily show that appyling F to everything preserves quasi-equivalence. This shows that (4.8) is quasi-equivalent to $F\underline{\mathcal{M}}_{\mathcal{C}}$, so (4.7) is also quasi-equivalent to $F\underline{\mathcal{M}}_{\mathcal{C}}$. This is what we wanted. ∎

4.9. Applications to module categories. We'll now apply the above results to the homotopy theory of $\mathcal{C}I$-categories. Readers may want to review Appendix A before proceeding further.

Let $\mathcal{C}$ and $\mathcal{D}$ be cofibrantly-generated symmetric monoidal model categories satisfying the monoid axiom, and assume that $\mathbf{1}_{\mathcal{C}}$ and $\mathbf{1}_{\mathcal{D}}$ are cofibrant. Let $F\colon\mathcal{C}\rightleftarrows\mathcal{D}\colon G$ be a Quillen pair where G is lax monoidal. Let I be a set and consider the notion of $\mathcal{C}I$-category (a category enriched over $\mathcal{C}$ with object set I) from Appendix A. Note that when I consists of one object then a $\mathcal{C}I$-category is just a monoid in $\mathcal{C}$.

Let $\mathcal{R}$ be a $\mathcal{D}I$-category, and consider the category Mod-$\mathcal{R}$ of right $\mathcal{R}$-modules. By [SS3, 6.1] the category Mod-$\mathcal{R}$ has a model structure in which the weak equivalences and fibrations are obtained by forgetting objectwise to $\mathcal{D}$. This is a $\mathcal{D}$-model category in a natural way. The SM7 (or pushout product) condition follows from $\mathcal{D}$ using [SS1, 3.5] since the $\mathcal{D}$ action is pointwise, and the unit condition follows from our assumption that $\mathbf{1}_{\mathcal{D}}$ is cofibrant (since this implies that the cofibrant $\mathcal{R}$-modules are objectwise cofibrant).

Since G is lax monoidal, $G\mathcal{R}$ is a $\mathcal{C}I$-category and we may consider the corresponding module category Mod-$G\mathcal{R}$. This is a $\mathcal{C}$-model category. If M is an $\mathcal{R}$-module then GM becomes a $G\mathcal{R}$-module in a natural way, and there is an adjoint pair $F^{\mathcal{R}}\colon$ Mod-$G\mathcal{R}\rightleftarrows$ Mod-$\mathcal{R}\colon G$ by Proposition A.6(a). The functors $(F^{\mathcal{R}},G)$ are a Quillen pair since G preserves the objectwise fibrations and trivial fibrations.

We have two Quillen pairs $F\colon\mathcal{C}\rightleftarrows\mathcal{D}\colon G$ and $F^{\mathcal{R}}\colon$ Mod-$G\mathcal{R}\rightleftarrows$ Mod-$\mathcal{R}\colon G$. The categories Mod-$G\mathcal{R}$ and Mod-$\mathcal{R}$ are $\mathcal{C}$- and $\mathcal{D}$-model categories, respectively.

4.10. Proposition. *Under the above assumptions on $\mathcal{C}$, $\mathcal{D}$, and G one has:*

(a) $(F^{\mathcal{R}},G)$ is an adjoint module over (F,G).

(b) If F is strong monoidal, then $(F^{\mathcal{R}}, G)$ is a Quillen adjoint module over (F, G).

(c) Assume that $\mathcal{C}$ is a stable model category whose homotopy category is generated by $\mathbf{1}_{\mathcal{C}}$. Assume as well that $F(\mathbf{1}_{\mathcal{C}}) \to \mathbf{1}_{\mathcal{D}}$ is a weak equivalence. Then $(F^{\mathcal{R}}, G)$ is a Quillen adjoint module over (F, G).

PROOF. In terms of the notation of Section 3 we have $L = F^{\mathcal{R}}$ and $R = G$. A natural transformation γ of the type in Proposition 3.2(iii) is therefore obtained using the lax monoidal structure on G. This automatically satisfies the axioms of Proposition 3.6(b) and Proposition 3.7(b), so that we have an adjoint module over (F, G). This proves (a).

To prove (b) we show that $L(X \otimes c) \to LX \otimes Fc$ is an isomorphism, and hence a weak equivalence. Here $L = F^{\mathcal{R}} = F(-) \otimes_{FG\mathcal{R}} \mathcal{R}$ since F is strong monoidal; see the discussion above [SS3, 3.11]. It is then easy to verify that $F^{\mathcal{R}}(X \otimes c) = F(X \otimes c) \otimes_{FG\mathcal{R}} \mathcal{R} \cong (FX \otimes Fc) \otimes_{FG\mathcal{R}} \mathcal{R} \cong F^{\mathcal{R}}(X) \otimes Fc$.

To prove (c), we will verify that $G\underline{\mathcal{N}}_{\mathcal{D}}(LX, Y) \xrightarrow{\gamma} \underline{\mathcal{M}}_{\mathcal{C}}(X, RY)$ is a weak equivalence whenever X is cofibrant and Y is fibrant. Using our assumption about $\mathbf{1}_{\mathcal{C}}$ generating $\mathrm{Ho}(\mathcal{C})$, it suffices to show that

$$[\mathbf{1}_{\mathcal{C}}, G\underline{\mathcal{N}}_{\mathcal{D}}(LX, Y)]_* \to [\mathbf{1}_{\mathcal{C}}, \underline{\mathcal{M}}_{\mathcal{C}}(X, RY)]_*$$

is an isomorphism of graded groups, where $[-,-]_*$ denotes the graded group of maps in a triangulated category.

By adjointness, the problem reduces to showing that the map

$$[LX \otimes F(\mathbf{1}_{\mathcal{C}}), Y]_* \to [L(X \otimes \mathbf{1}_{\mathcal{C}}), Y]_*$$

is an isomorphism—or in other words, that $LX \otimes F(\mathbf{1}_{\mathcal{C}}) \to L(X \otimes \mathbf{1}_{\mathcal{C}})$ is a weak equivalence. But this follows easily from our assumption that $F(\mathbf{1}_{\mathcal{C}}) \to \mathbf{1}_{\mathcal{D}}$ is a weak equivalence. ∎

Now assume that $\mathcal{O}$ is a cofibrant $\mathcal{C}I$-category. By Proposition A.3 there is an adjunction $F^{\mathcal{D}I} \colon \mathcal{C}I - \mathcal{C}at \rightleftarrows \mathcal{D}I - \mathcal{C}at \colon G$, so that we get a $\mathcal{D}I$-category $F^{\mathcal{D}I}\mathcal{O}$. By Proposition A.6(b) there is a Quillen pair

$$F_{\mathcal{O}} \colon \text{Mod-}\,\mathcal{O} \rightleftarrows \text{Mod-}\,F^{\mathcal{D}I}\mathcal{O} \colon G_{\mathcal{O}}.$$

4.11. PROPOSITION. *In the above setting one has:*

(a) $(F_{\mathcal{O}}, G_{\mathcal{O}})$ is an adjoint module over (F, G).

(b) If F is strong monoidal, then $(F_{\mathcal{O}}, G_{\mathcal{O}})$ is a Quillen adjoint module over (F, G).

(c) Assume that $\mathcal{C}$ is a stable model category whose homotopy category is generated by $\mathbf{1}_{\mathcal{C}}$, and that $F(\mathbf{1}_{\mathcal{C}}) \to \mathbf{1}_{\mathcal{D}}$ is a weak equivalence. Then $(F_{\mathcal{O}}, G_{\mathcal{O}})$ is a Quillen adjoint module over (F, G).

Proof. Write $\mathcal{R} = F^{\mathcal{D}I}\mathcal{O}$. The adjunction $(F_{\mathcal{O}}, G_{\mathcal{O}})$ is the composite of the two adjunctions

$$\text{Mod-}\,\mathcal{O} \underset{\beta^*}{\overset{\beta_*}{\rightleftarrows}} \text{Mod-}\,G\mathcal{R} \underset{G}{\overset{F^{\mathcal{R}}}{\rightleftarrows}} \text{Mod-}\,\mathcal{R}$$

where $\beta\colon \mathcal{O} \to G\mathcal{R} = GF^{\mathcal{D}I}\mathcal{O}$ is the unit of the adjunction $(F^{\mathcal{D}I}, G)$.

But (β_*, β^*) is a $\mathcal{C}$-Quillen adjunction and by Proposition 4.10, under either set of conditions, we know $(F^{\mathcal{R}}, G)$ is a Quillen adjoint module over (F, G). The result now follows immediately from Corollary 3.12. ∎

4.12. Corollary. *In addition to our previous assumptions, assume that G is lax symmetric monoidal and $\mathcal{O}$ is a cofibrant $\mathcal{C}I$-category. Suppose also that $(F_{\mathcal{O}}, G_{\mathcal{O}})$ is a Quillen equivalence and the hypotheses in either part (b) or (c) hold from Proposition 4.11. Let $X \in \text{Mod-}(F^{\mathcal{D}I}\mathcal{O})$ be a cofibrant-fibrant object and let $A \in \text{Mod-}\,\mathcal{O}$ be any module weakly equivalent to $G_{\mathcal{O}}X$. Then the $\mathcal{C}$-monoids*

$$\underline{\text{Mod-}\,\mathcal{O}}_{\mathcal{C}}(A, A) \qquad \textit{and} \qquad G\big[\underline{\text{Mod-}(F^{\mathcal{D}I}\mathcal{O})}_{\mathcal{D}}(X, X)\big]$$

are weakly equivalent.

Proof. This follows from the above proposition and Corollary 4.3. ∎

4.13. Example. The adjoint pair $L\colon Sp^{\Sigma}(\text{ch}_+) \rightleftarrows Sp^{\Sigma}(s\mathcal{A}b)\colon \nu$ from [S, 4.3] forms one example for (F, G). The result [S, 3.4] shows that $Sp^{\Sigma}(\text{ch}_+)$ and $Sp^{\Sigma}(s\mathcal{A}b)$ are cofibrantly generated symmetric monoidal model categories which satisfy the monoid axiom. The conditions in Proposition 4.10(c) or 4.11(c) are verified in the last paragraph of the proof of [S, 4.3]. Note, though, that L is not strong monoidal. This failure is due to the fact that the adjunction $N\colon s\mathcal{A}b \rightleftarrows \text{ch}_+\colon \Gamma$ is not monoidal [SS3, 2.14].

Corollary 4.12 holds for (L, ν) in place of (F, G) because N is lax symmetric monoidal, so its prolongation and ν are also lax symmetric monoidal. The fact that $(L_{\mathcal{O}}, (\nu)_{\mathcal{O}})$ is a Quillen equivalence follows from [S, 3.4, 4.3] and [SS3, 6.5(1)]. See also Proposition A.6(c).

5. Transporting enrichments

In this section we prove a technical result about transporting enrichments. This will be needed later, in the proof of Proposition 9.4. The basic idea is as follows. Suppose $\mathcal{M}$ is a $\mathcal{C}$-model category, where $\mathcal{C}$ is a certain symmetric monoidal model category. Assume also that $\mathcal{D}$ is another symmteric monoidal model category, and that one has a Quillen equivalence $\mathcal{C} \rightleftarrows \mathcal{D}$ which is compatible with the monoidal structure. Then one might hope to find a $\mathcal{D}$-model category $\mathcal{N}$ which is Quillen equivalent to $\mathcal{M}$, and where the Quillen equivalence aligns the $\mathcal{C}$- and $\mathcal{D}$-structures. In this section we prove one theorem along these lines, assuming several hypotheses on the given data.

We begin with the following two definitions:

5.1. DEFINITION. *Let $\mathcal{T}$ be a triangulated category with infinite coproducts.*

(a) An object $P \in \mathcal{T}$ is called ***compact*** *if $\oplus_\alpha \mathcal{T}(P, X_\alpha) \to \mathcal{T}(P, \oplus_\alpha X_\alpha)$ is an isomorphism for every set of objects $\{X_\alpha\}$;*

(b) A set of objects $S \subseteq \mathcal{T}$ is a ***generating set*** *if the only full, triangulated subcategory of $\mathcal{T}$ which contains S and is closed under arbitrary coproducts is $\mathcal{T}$ itself. If S is a singleton set $\{P\}$ we say that P is a* ***generator****.*

When $\mathcal{M}$ is a stable model category we will call an object compact if it is compact in $\mathcal{H}o(\mathcal{M})$, and similarly for the notion of generating set. Most stable model categories of interest have a generating set of compact objects. For example, Hovey shows in [Ho1, 7.4.4] that this is true for any finitely-generated, stable model category.

Let $\mathcal{C}$ and $\mathcal{D}$ be symmetric monoidal, stable model categories. Let $\mathcal{M}$ be a pointed $\mathcal{C}$-model category (so that $\mathcal{M}$ is also stable). We make the following assumptions:

(a) $\mathcal{C}$ and $\mathcal{D}$ are combinatorial model categories satisfying the monoid axiom, and their units are cofibrant.

(b) There is a weak monoidal Quillen equivalence $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$, where G is lax symmetric monoidal.

(c) $\mathcal{C}$ satisfies axioms (QI1-2) from Appendix A.

(d) $\mathcal{C}$ is a stable model category whose homotopy category is generated by $\mathbf{1}_\mathcal{C}$, and $F(\mathbf{1}_\mathcal{C}) \to \mathbf{1}_\mathcal{D}$ is a weak equivalence.

(e) $\mathcal{M}$ has a generating set of compact objects.

If $\mathcal{N}$ is a $\mathcal{D}$-model category, let $G\underline{\mathcal{N}}_\mathcal{D}$ denote the assignment $X, Y \mapsto G\underline{\mathcal{N}}_\mathcal{D}(X, Y)$. By Proposition 4.6 this is a $\mathcal{C}$-model enrichment of $\mathcal{N}$.

5.2. PROPOSITION. *Under the above conditions there exists a combinatorial $\mathcal{D}$-model category $\mathcal{N}$ and a zig-zag of Quillen equivalences*

$$\mathcal{M} \xleftarrow{L_1} \mathcal{M}_1 \xrightarrow{L_2} \mathcal{N}$$

such that the model enrichment $\underline{\mathcal{M}}_\mathcal{C}$ is quasi-equivalent to $(L_1)_*(L_2)^*[G\underline{\mathcal{N}}_\mathcal{D}]$.

If, in addition, $\mathcal{C}$ and $\mathcal{D}$ are additive model categories (see the following section for the definition) then $\mathcal{M}_1$ and $\mathcal{N}$ may also be chosen to be additive.

By [D4, 3.6] this yields the following immediate corollary:

5.3. COROLLARY. *If $Y \in \mathcal{N}$ and $X \in \mathcal{M}$ are cofibrant-fibrant objects and Y is the image of X under the derived functors of the above Quillen equivalence $\mathcal{M} \simeq \mathcal{N}$, then the $\mathcal{C}$-monoids $\underline{\mathcal{M}}_\mathcal{C}(X, X)$ and $G\underline{\mathcal{N}}_\mathcal{D}(Y, Y)$ are weakly equivalent.*

Proof of Proposition 5.2. Constructing the model category $\mathcal{N}$ will require several steps, and we will start by just giving a sketch—then we will come back and provide detailed justifications afterwards.

Let I denote a set of cofibrant-fibrant, compact objects which generate $\mathcal{M}$. Let $\mathcal{O}$ be the $\mathcal{C}I$-category [Bo, 6.2] defined by $\mathcal{O}(i,j) = \underline{\mathcal{M}}_{\mathcal{C}}(i,j)$. Then there is a $\mathcal{C}$-Quillen equivalence

$$T\colon \text{Mod-}\mathcal{O} \rightleftarrows \mathcal{M}\colon S \tag{5.4}$$

where Mod-$\mathcal{O}$ is the model category of right $\mathcal{O}$-modules (see Proposition A.2).

Let $g\colon \overline{\mathcal{O}} \to \mathcal{O}$ be a cofibrant-replacement for $\mathcal{O}$ in the model category of $\mathcal{C}I$-categories (Proposition A.3(a)). Then tensoring and restricting give the left and right adjoints of a $\mathcal{C}$-Quillen equivalence

$$g_*\colon \text{Mod-}\overline{\mathcal{O}} \rightleftarrows \text{Mod-}\mathcal{O}\colon g^* \tag{5.5}$$

(see Proposition A.2(c)).

Next we use the functor $L^{\mathcal{D}I}$ from Proposition A.3(b). This gives us a $\mathcal{D}I$-category $L^{\mathcal{D}I}\overline{\mathcal{O}}$ and a Quillen equivalence

$$L_{\overline{\mathcal{O}}}\colon \text{Mod-}\overline{\mathcal{O}} \rightleftarrows \text{Mod-}(L^{\mathcal{D}I}\overline{\mathcal{O}})\colon \nu. \tag{5.6}$$

Let $\mathcal{N} = \text{Mod-}(L^{\mathcal{D}I}\overline{\mathcal{O}})$. This is a $\mathcal{D}$-model category, and we have established a zig-zag of Quillen equivalences

$$\mathcal{M} \xleftarrow{\sim} \text{Mod-}\mathcal{O} \xleftarrow{\sim} \text{Mod-}\overline{\mathcal{O}} \xrightarrow{\sim} \text{Mod-}(L^{\mathcal{D}I}\overline{\mathcal{O}}) = \mathcal{N}.$$

We set $\mathcal{M}_1 = \text{Mod-}\overline{\mathcal{O}}$. Note that if $\mathcal{C}$ and $\mathcal{D}$ are additive model categories then by Corollary 6.9 so are $\mathcal{M}_1$ and $\mathcal{N}$ (since $\mathcal{M}_1$ is a $\mathcal{C}$-model category and $\mathcal{N}$ is a $\mathcal{D}$-model category).

Now we fill in the details of the above sketch. The category of right $\mathcal{O}$-modules Mod-$\mathcal{O}$ is defined in [SS3, Section 6], and the model structure on Mod-$\mathcal{O}$ is provided in [SS3, 6.1(1)]. See Appendix A for a review. To justify the Quillen equivalence in (5.4), define $S\colon \mathcal{M} \to \text{Mod-}\mathcal{O}$ by letting $S(Z)$ be the functor $i \mapsto \text{Hom}_{\mathcal{C}}(i,Z)$. This obviously comes equipped with a structure of right $\mathcal{O}$-module. The construction of the left adjoint can be copied almost verbatim from [SS2, 3.9.3(i)], which handled the case where $\mathcal{C}$ was Sp^{Σ}. The right adjoint obviously preserves fibrations and trivial fibrations, so we have a Quillen pair. It is readily seen to be a $\mathcal{C}$-Quillen pair.

Finally, that this is a Quillen equivalence follows just as in [SS2, 3.9.3(ii)]; this uses that I was a generating set of compact objects. The proof can be summarized quickly as follows. First, the compactness of the objects in I shows that the derived functor of S preserves all coproducts; this is trivially true for the derived functor of T because it is a left adjoint. One has canonical generators $Fr_i \in \text{Mod-}\mathcal{O}$ for each $i \in I$, and adjointness shows that $T(Fr_i) \cong i$. Likewise, $S(i) \cong Fr_i$. Using that the derived functors of S and T preserve coproducts and triangles, one now deduces that the respective composites are naturally isomorphic to the identities. This completes step (5.4) above.

We now turn to (5.5). The map of $\mathcal{C}I$-categories $g\colon \overline{\mathcal{O}} \to \mathcal{O}$ gives a Quillen map Mod-$\overline{\mathcal{O}} \to$ Mod-$\mathcal{O}$ by Proposition A.2(b). We will know this is a Quillen equivalence by Proposition A.2(c) as long as we know that $\mathcal{C}$ satisfies the axioms (QI1-2) of Appendix A.

The Quillen equivalence of (5.6) is a direct application of Proposition A.6(c).

At this point we have constructed the zig-zag $\mathcal{M} \xleftarrow{L_1} \mathcal{M}_1 \xrightarrow{L_2} \mathcal{N}$. We must verify that $(L_1)_*(L_2)^*[G\underline{\mathcal{N}}_{\mathcal{D}}]$ is quasi-equivalent to $\underline{\mathcal{M}}_{\mathcal{C}}$.

It follows from Proposition 4.11 and Theorem 4.1 that $(L_2)^*[G\underline{\mathcal{N}}_{\mathcal{D}}]$ is quasi-equivalent to $(\underline{\mathcal{M}}_1)_{\mathcal{C}}$. This is where the theory of adjoint modules was needed. Since L_1 is a $\mathcal{C}$-Quillen equivalence, it follows from [D4, 3.14(e)] that $(L_1)_*[(\underline{\mathcal{M}}_1)_{\mathcal{C}}]$ is quasi-equivalent to $\underline{\mathcal{M}}_{\mathcal{C}}$. So these two statements give exactly what we want. ∎

6. Additive model categories

Now the second half of the paper begins. We change direction and start to pursue our main results on additive enrichments. In the present section we define the notion of an *additive* model category, and prove some basic results for recognizing them.

A category is **preadditive** if its hom-sets have natural structures of abelian groups for which the composition pairing is biadditive. A category is **additive** if it is preadditive and it has finite coproducts. This forces the existence of an initial object (the empty coproduct), which will necessarily be a zero object. See [ML, Section VIII.2]. A functor $F\colon \mathcal{C} \to \mathcal{D}$ between additive categories is an **additive functor** if $F(f+g) = F(f)+F(g)$ for any two maps $f, g\colon X \to Y$.

Now let $\mathcal{M}$ be a model category whose underlying category is additive. Write $\mathcal{M}_{cof}$ for the full subcategory of cofibrant objects, and $c\mathcal{M}$ for the category of cosimplicial objects in $\mathcal{M}$. Recall from [Hi, Section 15.3] that $c\mathcal{M}$ has a Reedy model category structure. Also recall that a **cosimplicial resolution** is a Reedy cofibrant object of $c\mathcal{M}$ in which every coface and codegeneracy map is a weak equivalence.

6.1. Definition. *Let $\mathcal{I}$ be a small, additive subcategory of $\mathcal{M}_{\mathrm{cof}}$. By an* ***additive cosimplicial resolution*** *on $\mathcal{I}$ we mean an additive functor $\Gamma\colon \mathcal{I} \to c\mathcal{M}$ whose image lies in the subcategory of cosimplicial resolutions, together with a natural weak equivalence $\Gamma(X)^0 \xrightarrow{\sim} X$.*

By [Hi, 16.1.9], any small subcategory $\mathcal{I} \subseteq \mathcal{M}_{\mathrm{cof}}$ has a cosimplicial resolution; however, the existence of an *additive* cosimplicial resolution is not at all clear.

If Γ and Γ' are two additive cosimplicial resolutions on $\mathcal{I}$, then define a map $\Gamma \to \Gamma'$ to be a natural transformation of functors which gives commutative triangles

$$\begin{tikzcd} \Gamma(X)^0 \arrow[r] \arrow[d] & X \\ \Gamma'(X)^0 \arrow[ur] & \end{tikzcd}$$

for all $X \in \mathcal{I}$. The map is called a weak equivalence if all the maps $\Gamma(X) \to \Gamma'(X)$ are weak equivalences.

6.2. DEFINITION. *A model category $\mathcal{M}$ is **additive** if its underlying category is additive and if for every small, full subcategory $\mathcal{I}$ of $\mathcal{M}_{\mathrm{cof}}$ the following two statements are satisfied:*

(a) $\mathcal{I}$ has an additive cosimplicial resolution;

(b) The category of additive cosimplicial resolutions on $\mathcal{I}$, where maps are natural weak equivalences, has a contractible nerve.

6.3. PROPOSITION. *Let $\mathcal{M}$ be a model category whose underlying category is additive. Suppose that there is a functor $F\colon \mathcal{M}_{\mathrm{cof}} \to c\mathcal{M}$ together with a natural isomorphism $F^0(X) \cong X$. Assume that each $F(X)$ is a cosimplicial resolution, that F preserves colimits, and that if $X \rightarrowtail Y$ is a cofibration then $F(X) \to F(Y)$ is a Reedy cofibration. Then $\mathcal{M}$ is an additive model category.*

Note that the functor F will automatically be additive; since it preserves colimits, it preserves direct sums.

PROOF. The existence of additive cosimplicial resolutions is provided by F. So we must only prove the contractibility of the category of all such resolutions.

If $\Gamma \in c\mathcal{M}$ is any cosimplicial object, applying F to Γ yields a bi-cosimplicial object $F\Gamma$ given by $[m], [n] \mapsto F^m\Gamma^n$. Let $\widetilde{\Gamma} \in c\mathcal{M}$ denote the diagonal of this bi-cosimplicial object, and note that there is a natural map $\widetilde{\Gamma} \to \Gamma$. We claim that if Γ is a cosimplicial resolution then so is $\widetilde{\Gamma}$.

Suppose that $\Gamma \in c\mathcal{M}$ is a cosimplicial resolution of some object X. Then every latching map $L^n\Gamma \to \Gamma^n$ is a cofibration (see [Hi, 15.3] for a discussion of latching maps). From the bi-cosimplicial object $F\Gamma$, we get a 'vertical' latching map in $c\mathcal{M}$ of the form $L^{*,n}[F\Gamma] \to F(\Gamma^n)$. Here the domain is the cosimplicial object which in level m is the nth latching object of $[F\Gamma]^{m,*}$. Since the latching spaces are formed as colimits, and F preserves colimits, one has $L^{*,n}[F\Gamma] \cong F(L^n\Gamma)$. So our vertical latching map is $F(L^n\Gamma) \to F(\Gamma^n)$. But this is the result of applying F to a cofibration in $\mathcal{M}$, so it is a Reedy cofibration.

So we are in the situation of Lemma 6.5 below, in which every vertical latching map of $F\Gamma$ is a Reedy cofibration. By the lemma, this implies that the diagonal $\widetilde{\Gamma}$ is Reedy cofibrant. Since clearly every map in $F\Gamma$ is a weak equivalence, it is therefore a cosimplicial resolution of X.

If Γ is a cosimplicial resolution of X, then the weak equivalence $\Gamma(X)^0 \xrightarrow{\sim} X$ gives a map of cosimplicial objects $\Gamma \to cX$ (here cX denotes the constant cosimplicial object with X in every dimension). Applying $\widetilde{(-)}$, we get a diagram of cosimplicial objects

$$\begin{tikzcd} \widetilde{\Gamma} \arrow[r] \arrow[d] & \Gamma \arrow[d] \\ \widetilde{(cX)} \arrow[r] & cX. \end{tikzcd} \tag{6.4}$$

Note that $\widetilde{(cX)}$ is just $F(X)$, which is a cosimplicial resolution by the assumptions of the proposition. All the maps in the above square can be checked to be levelwise weak equivalences.

Now suppose that $\mathcal{I}$ is a small, full subcategory of $\mathcal{M}_{\mathrm{cof}}$. Let $\mathrm{CR}(\mathcal{I})$ be the category of all additive cosimplicial resolutions on $\mathcal{I}$, where the maps are natural weak equivalences. The restriction of F to a functor $\mathcal{I} \to c\mathcal{M}$ is an object of $\mathrm{CR}(\mathcal{I})$, which we also denote F by abuse. Consider the overcategory $(\mathrm{CR}(\mathcal{I}) \downarrow F)$ together with the forgetful functor $U\colon (\mathrm{CR}(\mathcal{I}) \downarrow F) \to \mathrm{CR}(\mathcal{I})$ sending an object $[\Gamma \to F]$ to Γ.

We define a functor $\mathrm{CR}(\mathcal{I}) \to (\mathrm{CR}(\mathcal{I}) \downarrow F)$ by $\Gamma \mapsto \widetilde{\Gamma}$ (we are applying the construction $\widetilde{(-)}$ to every $\Gamma(X)$, for $X \in \mathcal{I}$). The cosimplicial resolution $\widetilde{\Gamma}$ is augmented over F by the left vertical arrow in (6.4).

Consider the two functors

$$\mathrm{CR}(\mathcal{I}) \longrightarrow (\mathrm{CR}(\mathcal{I}) \downarrow F) \xrightarrow{U} \mathrm{CR}(\mathcal{I}).$$

Then there is a natural transformation from the composite to the identity, via the natural maps $\widetilde{\Gamma(X)} \to \Gamma(X)$. This shows that on the level of nerves, the identity map for $\mathrm{CR}(\mathcal{I})$ is homotopic to a map which factors through a contractible space—the nerve of $(\mathrm{CR}(\mathcal{I}) \downarrow F)$ is contractible because this category has a final object. So we conclude that $\mathrm{CR}(\mathcal{I})$ has contractible nerve. ∎

We need some notation for the following lemma. Let $X^{*,*}$ be a bi-cosimplicial object in a model category $\mathcal{M}$. Considering this as an object of $c(c\mathcal{M})$, one obtains a 'vertical' latching map $L^{*,n}X \to X^{*,n}$ in $c\mathcal{M}$. Here $L^{*,n}X$ denotes the cosimplicial object sending $[m]$ to the nth latching object of $X^{m,*}$.

6.5. Lemma. *Let $\mathcal{M}$ be any model category. Suppose that $X^{*,*}$ is a bi-cosimplicial object of $\mathcal{M}$—that is, $X \in c(c\mathcal{M})$. Assume that every latching map $L^{*,n}X \to X^{*,n}$ is a Reedy cofibration in $c\mathcal{M}$. Then the diagonal cosimplicial object $[n] \mapsto X^{n,n}$ is Reedy cofibrant.*

The proof of the above lemma is a little technical. We defer it until the end of the section.

6.6. Corollary. *Let $\mathcal{C}$ and $\mathcal{M}$ be model categories, where the underlying category of $\mathcal{M}$ is additive. Suppose there is a bifunctor $\otimes\colon \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ satisfying the pushout-product axiom for cofibrations: if $i\colon A \rightarrowtail B$ is a cofibration in $\mathcal{M}$ and $j\colon X \rightarrowtail Y$ is a cofibration in $\mathcal{C}$, then $(A\otimes Y) \amalg_{(A\otimes X)} (B \otimes X) \to B \otimes Y$ is a cofibration which is a weak equivalence if either i of j is. Suppose also that*

(i) For any $X \in \mathcal{C}$ the functor $(-) \otimes X$ preserves colimits;

(ii) For any $A \in \mathcal{M}$ the functor $A \otimes (-)$ preserves colimits;

(iii) There is a cofibrant object $\mathbf{1} \in \mathcal{C}$ and natural isomorphisms $A \otimes \mathbf{1} \cong A$.

Then $\mathcal{M}$ is an additive model category.

PROOF. Let $\Gamma \in c\mathcal{C}$ be a cosimplicial resolution of $\mathbf{1}$ with $\Gamma^0 = \mathbf{1}$. For any cofibrant object $A \in \mathcal{M}$, let $F(A)$ be the cosimplicial object $[n] \mapsto A \otimes \Gamma^n$. The pushout-product axiom, together with assumption (ii), shows that $F(A)$ is a cosimplicial resolution of A. Assumption (i) implies that F preserves colimits, and assumption (iii) says there are natural isomorphisms $F(A)^0 \cong A$. Finally, it is an easy exercise to use assumption (ii) and the pushout-product axiom to show that if $A \to B$ is a cofibration then $F(A) \to F(B)$ is a Reedy cofibration. The result now follows by applying Proposition 6.3. ∎

The above corollary lets one identify many examples of additive model categories. We only take note of the few obvious ones:

6.7. COROLLARY. *If R is a ring, consider the model category $s(R-\mathcal{M}od)$ where fibrations and weak equivalences are determined by the forgetful functor to $s\mathcal{S}et$. This is an additive model category. So is the model category* $\mathrm{Ch}(R)$ *of unbounded chain complexes, where weak equivalences are quasi-isomorphisms and fibrations are sujections.*

PROOF. This results from two applications of the previous corollary. For the first statement we take $\mathcal{M} = s(R-\mathcal{M}od)$, $\mathcal{C} = s(\mathbb{Z}-\mathcal{M}od)$, and $\otimes$ to be the levelwise tensor product over $\mathbb{Z}$. Here we are using that if M is an R-module and A is a $\mathbb{Z}$-module then $M \otimes_{\mathbb{Z}} A$ has a natural R-module structure from the left.

For the second statement we can take $\mathcal{M} = \mathrm{Ch}(R)$, $\mathcal{C} = \mathrm{Ch}_{\geq 0}(\mathbb{Z})$, and $\otimes$ the usual tensor product of chain complexes over $\mathbb{Z}$. (One could also take $\mathcal{C} = \mathrm{Ch}(\mathbb{Z})$, but verifying the pushout-product axiom is a little easier for bounded below complexes). ∎

If R is a dga, then $R - \mathcal{M}od$ has a model category structure where weak equivalences are quasi-isomorphisms and fibrations are surjections.

6.8. COROLLARY. *If R is a dga, then the model category $R - \mathcal{M}od$ is additive.*

PROOF. We again apply Corollary 6.6, this time with $\mathcal{M} = R - \mathcal{M}od$ and $\mathcal{C} = \mathrm{Ch}_{\geq 0}(\mathbb{Z})$. The $\otimes$ functor is the tensor product $M, C \mapsto M \otimes_{\mathbb{Z}} C$ with the induced left R-module structure. ∎

We also note the following result:

6.9. COROLLARY. *Let $\mathcal{C}$ be a symmetric monoidal model category in which the unit is cofibrant, and where the underlying category is additive. Then $\mathcal{C}$ is an additive model category. Any $\mathcal{C}$-model category is also additive.*

PROOF. The first statement follows immediately from Corollary 6.6, as the bifunctor $X, Z \mapsto X \otimes Z$ preserves colimits in both variables.

The second statement is also a direct application of Corollary 6.6, as soon as one notes that if $\mathcal{M}$ is a $\mathcal{C}$-model category then the underlying category of $\mathcal{M}$ is additive. This follows using the adjunctions $\mathcal{M}(X, Y) \cong \mathcal{M}(X \otimes \mathbf{1}_{\mathcal{C}}, Y) \cong \mathcal{C}(\mathbf{1}_{\mathcal{C}}, \underline{\mathcal{M}}_{\mathcal{C}}(X, Y))$, as there is a natural abelian group structure on the latter set. One checks that composition is biadditive with respect to this structure. ∎

6.10. Bisimplicial machinery. The last thing we must do in this section is prove Lemma 6.5. This requires some machinery which we briefly recall.

If $K \in s\mathcal{S}et$ and $A \in c\mathcal{M}$, one may form the coend $A \otimes K \in \mathcal{M}$. This is the coequalizer of the two arrows

$$\coprod_{[n]\mapsto[m]} A^n \otimes K_m \rightrightarrows \coprod_{[n]} A^n \otimes K_n \tag{6.11}$$

where $A^n \otimes K_m$ is shorthand for a coproduct of copies of A^n indexed by the set K_m. There are adjunctions

$$\mathcal{M}(A \otimes K, X) \cong c\mathcal{M}(A, X^K) \cong s\mathcal{S}et(K, \mathcal{M}(A, X)) \tag{6.12}$$

for $A \in c\mathcal{M}$, $K \in s\mathcal{S}et$, and $X \in \mathcal{M}$. Here X^K is the cosimplicial object $[n] \mapsto X^{K_n}$, where X^{K_n} denotes a product of copies of X indexed by the set K_n. One checks—using the above adjunctions or otherwise—that $A \otimes \Delta^n \cong A^n$, and $A \otimes \partial\Delta^n$ is isomorphic to the nth latching object of A [Hi, Def. 15.2.5]. See [D1, Section 4] for the dual situation with $s\mathcal{M}$ instead of $c\mathcal{M}$.

Write $s^2\mathcal{S}et$ for the category of bisimplicial sets and $c^2\mathcal{M}$ for the category of bi-cosimplicial objects in $\mathcal{M}$. When drawing a bisimplicial set P we will draw each $P_{m,*}$ horizontally, and each $P_{*,n}$ vertically. If $K \in s\mathcal{S}et$ and $P \in s^2\mathcal{S}et$, let $\mathrm{vMap}(K, P)$ denote the simplicial set $[n] \mapsto s\mathcal{S}et(K, P_{*,n})$. We are mapping K into the *vertical* simplicial sets of P.

If $K, L \in s\mathcal{S}et$ write $K \boxtimes L$ for the bisimplicial set $[m], [n] \mapsto K_m \times L_n$. Observe that there is an adjunction formula

$$s^2\mathcal{S}et(K \boxtimes L, P) \cong s\mathcal{S}et(L, \mathrm{vMap}(K, P)). \tag{6.13}$$

Note in particular that $s^2\mathcal{S}et(\Delta^m \boxtimes \Delta^n, P) \cong P_{m,n}$.

If $P \in s^2\mathcal{S}et$ and $A \in c^2\mathcal{M}$, one can form a coend $A \otimes P \in \mathcal{M}$ similarly to what was done in (6.11). There are adjunction formulas analogous to (6.12). One checks that $A \otimes (\Delta^m \boxtimes \Delta^n) \cong A^{m,n}$, and more generally $A \otimes (\Delta^m \boxtimes L) \cong A^{m,*} \otimes L$ (use (6.13) for both). So, for instance, $A \otimes (\Delta^m \boxtimes \partial\Delta^n)$ is the nth latching object for the cosimplicial object $A^{m,*}$.

Finally, recall from [BF, p. 125] that the diagonal functor $\mathrm{diag}\colon s^2\mathcal{S}et \to s\mathcal{S}et$ has a left adjoint which we will call $d\colon s\mathcal{S}et \to s^2\mathcal{S}et$. It follows immediately from adjointness that $d\Delta^n \cong \Delta^n \boxtimes \Delta^n$. Since d preserves colimits and every simplicial set is a colimit of Δ^n's, this tells us what d does to any simpicial set.

By chasing through adjunctions one finds that if $X \in c^2\mathcal{M}$ and $K \in s\mathcal{S}et$ then $\mathrm{diag}(X) \otimes K \cong X \otimes dK$.

Proof of Lemma 6.5. Consider the object $X \in c^2\mathcal{M}$ given in the statement of the lemma. Our task is to show that $\mathrm{diag}(X) \otimes \partial\Delta^n \to \mathrm{diag}(X) \otimes \Delta^n$ is a cofibration, for each n. This is the condition for $\mathrm{diag}(X)$ to be Reedy cofibrant. Using the isomorphisms $\mathrm{diag}(X) \otimes K \cong X \otimes dK$, this is equivalent to showing that the map $X \otimes d(\partial\Delta^n) \to X \otimes d\Delta^n \cong X \otimes (\Delta^n \boxtimes \Delta^n)$ is a cofibration.

Let S denote the set of all maps $P \to Q$ of bisimplicial sets such that $X \otimes P \to X \otimes Q$ is a cofibration in $\mathcal{M}$. This set is closed under composition and cobase change. Our assumption about the latching maps of X amounts to saying that the maps

$$(\partial\Delta^k \boxtimes \Delta^n) \amalg_{(\partial\Delta^k \boxtimes \partial\Delta^n)} (\Delta^k \boxtimes \partial\Delta^n) \longrightarrow \Delta^k \boxtimes \Delta^n \tag{6.14}$$

belong to S, for all n and k. Our goal is to show that this forces $d(\partial\Delta^n) \to \Delta^n \boxtimes \Delta^n$ to also belong to S.

We have now reduced things to a problem in combinatorial homotopy theory. Namely, we must show that $d(\partial\Delta^n) \to \Delta^n \boxtimes \Delta^n$ can be obtained from the maps in (6.14) by iterated cobase changes and compositions. But a little thought shows that *every* monomorphism of bisimplicial sets can be obtained in this way from the maps in (6.14) (the point is that every monomorphism of simplicial sets can be obtained from the maps $\partial\Delta^n \to \Delta^n$ in the same way). So we are done. ∎

7. Universal additive model categories

Suppose $\mathcal{C}$ is a small category. The paper [D2] introduced the idea of a universal model category built from $\mathcal{C}$, there denoted $U\mathcal{C}$. This is just the category of functors $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, s\mathcal{S}et)$ with a well-known model structure. If $\mathcal{C}$ is also an additive category then one can ask for a universal *additive* model category built from $\mathcal{C}$. This section develops this concept.

7.1. Presheaves and additive presheaves. Let $\mathcal{C}$ be a small, additive category. Let $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$ denote the category of all functors. Note that for every $X \in \mathcal{C}$, the representable functor $rX\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$ defined by $U \mapsto \mathcal{C}(U, X)$ is additive.

The Yoneda Lemma does not hold in $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$: that is, if $F \in \mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$ one need not have $\mathrm{Hom}(rX, F) \cong F(X)$ for all $X \in \mathcal{C}$. But it is easy to check that this *does* hold when F is an additive functor.

Let $\mathrm{Func}_{\mathrm{ad}}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$ denote the full subcategory of additive functors. The following lemma records several basic facts about this category.

7.2. Lemma. *Let $\mathcal{C}$ be a small, additive category.*

(a) Colimits and limits in $\mathrm{Func}_{\mathrm{ad}}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$ *are the same as those in* $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$.

(b) Every additive functor $F \in \mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$ *is isomorphic to its canonical colimit with respect to the embedding* $r\colon \mathcal{C} \hookrightarrow \mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$. *That is, the natural map* $\left[\operatorname*{colim}_{rX \to F}(rX)\right] \to F$ *is an isomorphism.*

(c) The additive functors in $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$ *are precisely those functors which are colimits of representables.*

(d) The inclusion $i\colon \mathrm{Func}_{\mathrm{ad}}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b) \hookrightarrow \mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$ *has a left adjoint* Ad *(for 'additivization'), and the composite* $\mathrm{Ad} \circ i$ *is naturally isomorphic to the identity.*

(e) *Suppose given a co-complete, additive category $\mathcal{A}$ and an additive functor $\gamma\colon \mathcal{C} \to \mathcal{A}$. Define* Sing: $\mathcal{A} \to \mathrm{Func}_{\mathrm{ad}}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$ *by letting* $\mathrm{Sing}(a)$ *be the functor* $c \mapsto \mathcal{A}(\gamma c, a)$. *Then* Sing *has a left adjoint* Re, *and there are natural isomorphisms* $\mathrm{Re}(rX) \cong \gamma(X)$.

Proof. We mostly leave this to the reader. We note, however, that the fact that $\mathcal{C}$ has finite coproducts (which is part of the definition of an additive category) is needed in (b). This ensures that the categories indexing the canonical colimits are pseudo-filtered, in the sense that for any objects i and j there is a third object k and maps $i \to k$, $j \to k$. The proof uses the fact that if $D\colon I \to \mathcal{A}b$, where I is pseudo-filtered, then for every element $x \in \mathrm{colim}_I\, D$ there exists an $i \in I$ such that x is in the image of $D_i \to \mathrm{colim}_I\, D$.

Also, we define the additivization functor from (d). If F is any functor, then $(\mathrm{Ad}\, F)(X)$ is the quotient of $F(X)$ by the subgroup generated by all $(f+g)^*(s) - f^*(s) - g^*(s)$ for all objects Y, all functions $f, g\colon X \to Y$, and all $s \in F(Y)$.

For the proof of (e), note the following. If $X \in \mathcal{A}$ and B is an abelian group, one can define $X \otimes B$ as the coequalizer of two maps

$$\coprod_{B\times B} X \rightrightarrows \coprod_{B} X.$$

Here the objects are coproducts of copies of X, indexed by the sets $B \times B$ and B, respectively. To describe the two maps, we have to say what they do to each summand corresponding to a pair (b_1, b_2). The first map is just the inclusion into the summand indexed by $b_1 + b_2$. The second map is the sum of the two inclusion maps corresponding to the summands b_1 and b_2. One checks that with this definition there is a natural adjunction isomorphism $\mathcal{A}(X \otimes B, Y) \cong \mathcal{A}b(B, \mathcal{A}(X, Y))$.

Recall that we are given an additive functor $\gamma\colon \mathcal{C} \to \mathcal{A}$. Given a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{A}b$, we consider the coend

$$\gamma \otimes F = \mathrm{coeq}\left[\coprod_{c\to d} \gamma(c) \otimes F(d) \rightrightarrows \coprod_{c} \gamma(c) \otimes F(c)\right].$$

When F is an additive functor one defines $\mathrm{Re}(F) = \gamma \otimes F$. It is routine to check that this is a left adjoint to Sing. ∎

By [Hi, Th. 11.6.1] the category $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, s\mathcal{A}b)$ has a cofibrantly-generated model structure in which the weak equivalences and fibrations are defined objectwise. We will need the analogous result for the category of additive functors:

7.3. Lemma. *Let $\mathcal{C}$ be a small, preadditive category. Then* $\mathrm{Func}_{\mathrm{ad}}(\mathcal{C}^{\mathrm{op}}, s\mathcal{A}b)$ *has a cofibrantly-generated model structure in which the weak equivalences and fibrations are defined objectwise. This model structure is simplicial, left proper, combinatorial, and cellular.*

Proof. The proof uses the adjoint pair (Ad, i) to create the model structure, as in [Hi, Th. 11.3.2]. Recall that the model category $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, s\mathcal{A}b)$ has generating trivial cofibrations $J = \{rX \times \mathbb{Z}[\Lambda^{n,k}] \to rX \times \mathbb{Z}[\Delta^n] \mid X \in \mathcal{C}\}$. Our notation is that if $K \in s\mathcal{S}et$ then $\mathbb{Z}[K] \in s\mathcal{A}b$ is the levelwise free abelian group on K; and if $A \in s\mathcal{A}b$ then $rX \times A$ denotes the presheaf $U \mapsto \mathcal{C}(U, X) \times A$ (with the product performed levelwise). Note that we think of rX as a $\mathcal{S}et$-valued functor here, so $\mathcal{C}(U, X) \times A$ denotes a direct sum of copies of A indexed by the set $\mathcal{C}(U, X)$—this is *not* the same as as the direct product of the abelian groups $\mathcal{C}(U, X)$ and A.

To apply [Hi, 11.3.2] we must verify that the functor i takes relative $\mathrm{Ad}(J)$-cell complexes to weak equivalences. However, note that if A is an abelian group then $\mathrm{Ad}(rX \times A) \cong rX \otimes A$, where the latter refers to the presheaf $U \mapsto \mathcal{C}(U, X) \otimes A$. So $\mathrm{Ad}(J)$ is the set of maps $rX \otimes \mathbb{Z}[\Lambda^{n,k}] \to rX \otimes \mathbb{Z}[\Delta^n]$. Objectwise, these maps are monomorphisms and weak equivalences of simplicial abelian groups.

Now, the model category $s\mathcal{A}b$ has the special property that a pushout of a map which is both a monomorphism and a weak equivalence is still a monomorphism and weak equivalence. The fact that forming pushouts in $\mathrm{Func}_{\mathrm{ad}}(\mathcal{C}^{\mathrm{op}}, s\mathcal{A}b)$ and $\mathrm{Func}(\mathcal{C}^{op}, s\mathcal{A}b)$ give the same answers (by Lemma 7.2(a)) and are done objectwise therefore shows that the $\mathrm{Ad}(J)$-cell complexes are objectwise monomorphisms and objectwise weak equivalences. In particular, they are weak equivalences in $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, s\mathcal{A}b)$.

Finally, it is routine to check that the resulting model structure is simplicial, left proper, combinatorial, and cellular. ∎

From now on we will write $U_{\mathrm{ad}}\mathcal{C}$ for the category $\mathrm{Func}_{\mathrm{ad}}(\mathcal{C}^{\mathrm{op}}, s\mathcal{A}b)$ with the model structure provided by the above lemma. The reason for the notation is provided by the next result.

Recall that if $L_1, L_2 \colon \mathcal{M} \to \mathcal{N}$ are two Quillen maps then a **Quillen homotopy** from L_1 to L_2 is a natural transformation $L_1 \to L_2$ which is a weak equivalence on the cofibrant objects.

If $\mathcal{M}$ is a model category and S is a set of maps in $\mathcal{M}$, then we use $\mathcal{M}/S$ to denote the left Bousfield localization of $\mathcal{M}$ at S, if it exists. See [Hi, Chapters 3–4] and [D2] for a discussion. The localizations always exist when $\mathcal{M} = U_{\mathrm{ad}}\mathcal{C}$, since this model category is left proper and cellular.

7.4. Theorem. *Let $\mathcal{M}$ be an additive model category.*

(a) Let $\mathcal{C}$ be a small, additive category and $\gamma \colon \mathcal{C} \to \mathcal{M}$ an additive functor taking values in the cofibrant objects. Then there is a Quillen pair $\mathrm{Re} \colon U_{\mathrm{ad}}\mathcal{C} \rightleftarrows \mathcal{M} \colon \mathrm{Sing}$ *together with a natural weak equivalence* $\eta \colon \mathrm{Re} \circ r \xrightarrow{\sim} \gamma$. *The category whose objects are such triples* $(\mathrm{Re}, \mathrm{Sing}, \eta)$ *and whose maps are Quillen homotopies compatible with η—defined just as in [D2, paragraph preceding Proposition 1.1]—is contractible.*

(b) If $\mathcal{M}$ is combinatorial then there is a Quillen equivalence $U_{\mathrm{ad}}\mathcal{C}/S \xrightarrow{\sim} \mathcal{M}$ for some small, additive category $\mathcal{C}$ and some set of maps S in $U_{\mathrm{ad}}\mathcal{C}$.

(c) *Suppose* $\mathcal{M} \xleftarrow{\sim} \mathcal{M}_1 \xrightarrow{\sim} \cdots \xleftarrow{\sim} \mathcal{M}_n \xrightarrow{\sim} \mathcal{N}$ *is a zig-zag of Quillen equivalences in which all the model categories are additive. If* $\mathcal{M}$ *is combinatorial, there is a simple zig-zag of equivalences*

$$\mathcal{M} \xleftarrow{\sim} U_{\mathrm{ad}}\mathcal{C}/S \xrightarrow{\sim} \mathcal{N}$$

such that the derived equivalence $Ho(\mathcal{M}) \simeq Ho(\mathcal{N})$ *is isomorphic to the derived equivalence given by the original zig-zag.*

Proof. For (a), one shows that giving a Quillen pair $\mathrm{Re}\colon U_{\mathrm{ad}}\mathcal{C} \rightleftarrows \mathcal{M}\colon \mathrm{Sing}$ together with a natural weak equivalence $\mathrm{Re}(rX) \xrightarrow{\sim} \gamma(X)$ is precisely the same as giving an additive cosimplicial resolution on γ. The proof of this is exactly the same as [D2, Prop. 3.4]. Giving a Quillen homotopy between two such Quillen pairs exactly amounts to giving a natural weak equivalence between the corresponding cosimplicial resolutions. This proves (a), once one recalls our definition of additive model categories.

The proof for (c) now exactly follow the case for $U\mathcal{C}$ given in [D2, Cor. 6.5]. One uses along the way that adjoint functors between additive categories are necessarily additive functors.

The proof of (b) is slightly more complicated; we will return to it at the end of this section, after some discussion. ∎

7.5. Remark. Part of the result in (a) states that the category whose objects are triples $(\mathrm{Re}, \mathrm{Sing}, \eta)$ is contractible. This is what distinguishes $U_{\mathrm{ad}}\mathcal{C}$ as a 'homotopically universal' construction. In this paper we will not need the full power of contractibility, but we *will* need the connectedness of this category. This says that any two choices of such $(\mathrm{Re}, \mathrm{Sing}, \eta)$ can be connected by a zig-zag of Quillen homotopies.

7.6. Remark. The result in (c) is false if one does not assume that all the $\mathcal{M}_i$'s are additive. For an example, let R be the dga $\mathbb{Z}[e; de = 2]/(e^4)$ and let T be the dga $\mathbb{Z}/2[x; dx = 0]/(x^2)$, where e has degree 1 and x has degree 2. Let $\mathcal{M}$ and $\mathcal{N}$ be the categories of R- and T-modules, respectively. These turn out to be Quillen equivalent, but they cannot be linked by a zig-zag of Quillen equivalences between *additive* model categories. A verification of these claims can be found in [DS2, Section 8].

7.7. Additive presentations. We turn to the proof of Theorem 7.4(b). This will be deduced from the work of [D3] plus some purely formal considerations.

Let $\mathcal{M}$ be a combinatorial model category. By [D3, Prop. 3.3], there is a small category $\mathcal{C}$ and a functor $\mathcal{C} \to \mathcal{M}$ such that the induced map $L\colon U\mathcal{C} \to \mathcal{M}$ is *homotopically surjective* (see [D3, Def. 3.1] for the definition). Then [D3, Prop. 3.2] shows that this fact implies there is a set of maps S in $U\mathcal{C}$ which the derived functor of L takes to weak equivalences, and such that the resulting map $U\mathcal{C}/S \to \mathcal{M}$ is a Quillen equivalence.

Now suppose that $\mathcal{M}$ was also an additive model category. By examining the proof of [D3, Prop. 3.3] one sees that the $\mathcal{C}$ constructed there is actually an additive category and the functor $\gamma\colon \mathcal{C} \to \mathcal{M}$ an additive functor taking values in the cofibrant objects (the category $\mathcal{C}$ is a certain full subcategory of the cosimplicial objects over $\mathcal{M}$). By

Theorem 7.4(a) there is an induced map $F\colon U_{\mathrm{ad}}\mathcal{C} \to \mathcal{M}$. Again using [D3, Prop. 3.2], it will be enough to prove that this map is homotopically surjective.

Consider now the following sequence of adjoint pairs:

$$\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, s\mathcal{S}et) \underset{U}{\overset{\mathbb{Z}}{\rightleftarrows}} \mathrm{Func}(\mathcal{C}^{\mathrm{op}}, s\mathcal{A}b) \underset{i}{\overset{\mathrm{Ad}}{\rightleftarrows}} \mathrm{Func}_{\mathrm{ad}}(\mathcal{C}^{\mathrm{op}}, s\mathcal{A}b) \underset{\mathrm{Sing}}{\overset{F}{\rightleftarrows}} \mathcal{M}$$

The composite of the right adjoints is clearly the right adjoint of L, so the composite of the left adjoints is L. We have constructed things so that this composite is homotopically surjective, and we are trying to show that F is also homotopically surjective.

In the following lemma, note that the presheaf rX can be regarded as an object of either $\mathrm{Func}_{\mathrm{ad}}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b)$ or $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, \mathcal{S}et)$. It will usually be clear from context which one we intend.

7.8. Lemma. *If $X \in \mathcal{C}$ then $\mathrm{Ad}(\mathbb{Z}(rX)) \cong rX$. Or equivalently, $\mathrm{Ad}(\mathbb{Z}(Ui(rX))) \cong rX$.*

Proof. This follows from the fact that the two functors $\mathrm{Func}_{\mathrm{ad}}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}b) \to \mathcal{A}b$ given by $F \mapsto \mathrm{Func}_{\mathrm{ad}}(\mathrm{Ad}(\mathbb{Z}(rX)), F)$ and $F \mapsto \mathrm{Func}_{\mathrm{ad}}(rX, F)$ are both naturally isomorphic to $F \mapsto F(X)$. ∎

Let $G \in \mathrm{Func}_{\mathrm{ad}}(\mathcal{C}^{\mathrm{op}}, s\mathcal{A}b)$. Let QG be the simplicial presheaf whose nth level is

$$\coprod_{rX_n \to rX_{n-1} \to \cdots \to rX_0 \to G_n} (rX_n)$$

where the coproduct is in $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, s\mathcal{S}et)$. The simplicial presheaf QG is treated in detail in [D2, Sec. 2.6], as it is a cofibrant-replacement functor for $U\mathcal{C}$. Likewise, let $Q_{\mathrm{ad}}G$ be the simplicial presheaf whose nth level is

$$\bigoplus_{rX_n \to rX_{n-1} \to \cdots \to rX_0 \to G_n} (rX_n)$$

where the coproduct is now in $\mathrm{Func}(\mathcal{C}^{\mathrm{op}}, s\mathcal{A}b)$. The proof of [D3, Prop. 2.8] showing that Q is a cofibrant-replacement functor for $U\mathcal{C}$ adapts verbatim to show that Q_{ad} is a cofibrant-replacement functor for $U_{\mathrm{ad}}\mathcal{C}$. Note that by Lemma 7.8 we have $Q_{\mathrm{ad}}G \cong \mathrm{Ad}(\mathbb{Z}(Q(UiG)))$, since Ad and $\mathbb{Z}(-)$ are left adjoints and therefore preserve coproducts.

Finally we are in a position to conclude the

Proof of Theorem 7.4(b). We have reduced to showing that $F\colon U_{\mathrm{ad}}\mathcal{C} \to \mathcal{M}$ is homotopically surjective. Let Sing be the right adjoint of F. Then we must show that for every fibrant object $X \in \mathcal{M}$ the induced map $FQ_{\mathrm{ad}}(\mathrm{Sing}\, X) \to X$ is a weak equivalence.

However, we have seen above that

$$F[Q_{\mathrm{ad}}\,\mathrm{Sing}\, X] \cong F[\mathrm{Ad}\,\mathbb{Z}(QUi(\mathrm{Sing}\, X))] \cong L[QUi(\mathrm{Sing}\, X)].$$

Recall that $Ui\,\mathrm{Sing}$ is the right adjoint to L. Since $L\colon U\mathcal{C} \to \mathcal{M}$ is homotopically surjective we know $LQ(Ui\,\mathrm{Sing}\, X) \to X$ is a weak equivalence in $\mathcal{M}$, so we are done. ∎

8. Homotopy enrichments over $Sp^\Sigma(s\mathcal{A}b)$

In this section and the next we prove the main results stated in Section 1. Except for the work in the next section, the proofs are essentially the same as in [D4]—but they use Theorem 7.4 in place of [D4, Prop. 5.5].

8.1. Background on ring objects. If $\mathcal{M}$ is a monoidal model category which is combinatorial and satisfies the monoid axiom, then by [SS1, Th. 4.1(3)] the category of monoids in $\mathcal{M}$ has an induced model structure where the weak equivalences and fibrations are the same as those in $\mathcal{M}$. We'll write $\mathcal{R}ing[\mathcal{M}]$ for this model category. If $\mathcal{N}$ is another such monoidal model category and $L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$ is a Quillen pair which is weak monoidal in the sense of [SS3, Def. 3.6], then there is an induced Quillen map $\mathcal{R}ing[\mathcal{M}] \to \mathcal{R}ing[\mathcal{N}]$. This is a Quillen equivalence if $\mathcal{M} \to \mathcal{N}$ was a Quillen equivalence and the units in $\mathcal{M}$ and $\mathcal{N}$ are cofibrant [SS3, Th. 3.12].

The adjunction $\mathcal{S}et_* \rightleftarrows \mathcal{A}b$ is strong monoidal, and therefore induces strong monoidal Quillen functors $Sp^\Sigma(s\mathcal{S}et_*) \rightleftarrows Sp^\Sigma(s\mathcal{A}b)$. It then follows that one gets a Quillen pair $F\colon \mathcal{R}ing[Sp^\Sigma] \rightleftarrows \mathcal{R}ing[Sp^\Sigma(s\mathcal{A}b)]\colon U$. By the **Eilenberg-Mac Lane ring spectrum** associated to an $R \in \mathcal{R}ing[Sp^\Sigma(s\mathcal{A}b)]$ we simply mean the ring spectrum UR.

8.2. Additive enrichments. Let $\mathcal{M}$ be an additive, stable, combinatorial model category. By Theorem 7.4 there is a Quillen equivalence $U_{\mathrm{ad}}\mathcal{C}/S \to \mathcal{M}$ for some small, additive category $\mathcal{C}$ and some set of maps S in $U_{\mathrm{ad}}\mathcal{C}$. The category $U_{\mathrm{ad}}\mathcal{C}/S$ is simplicial, left proper, and cellular, so using [Ho2, Sections 8, 9] we may form $Sp^\Sigma(U_{\mathrm{ad}}\mathcal{C}/S)$. Since $U_{\mathrm{ad}}\mathcal{C}/S$ is stable (since $\mathcal{M}$ was), we obtain a zig-zag of Quillen equivalences

$$\mathcal{M} \xleftarrow{\sim} U_{\mathrm{ad}}\mathcal{C}/S \xrightarrow{\sim} Sp^\Sigma(U_{\mathrm{ad}}\mathcal{C}/S).$$

Applying $ME_0(-, Sp^\Sigma(s\mathcal{A}b))$ to this zig-zag gives a diagram of bijections by [D4, 3.14(d)].

The category $U_{\mathrm{ad}}\mathcal{C}$ is a $s\mathcal{A}b$-model category, and therefore $Sp^\Sigma(U_{\mathrm{ad}}\mathcal{C}/S)$ is a $Sp^\Sigma(s\mathcal{A}b)$-model category by [Ho2, 8.3]. So $Sp^\Sigma(U_{\mathrm{ad}}\mathcal{C}/S)$ comes with a natural model enrichment by $Sp^\Sigma(s\mathcal{A}b)$, as in [D4, Ex. 3.2]. We can transport this enrichment onto $\mathcal{M}$ via the Quillen equivalences, and therefore get an element $\sigma_{\mathcal{M}} \in ME_0(\mathcal{M}, Sp^\Sigma(s\mathcal{A}b))$. Just as in [D4, Prop. 6.1], one shows (using Theorem 7.4) that this quasi-equivalence class does not depend on the choice of $\mathcal{C}$, S, or the Quillen equivalence $U_{\mathrm{ad}}\mathcal{C}/S \xrightarrow{\sim} \mathcal{M}$.

We can now give the:

Proof of Theorem 1.3. We have just constructed the enrichment $\sigma_{\mathcal{M}}$. The proof that it is preserved by Quillen equivalences is exactly the same as in [D4, Prop. 6.2], but using Theorem 7.4. ∎

Let $X \in \mathcal{M}$, and let $\tilde{X}$ be a cofibrant-fibrant object weakly equivalent to X. We write $\mathbf{hEnd_{ad}}(\boldsymbol{X})$ for any object in $\mathcal{R}ing[Sp^\Sigma(s\mathcal{A}b)]$ having the homotopy type of $\sigma_{\mathcal{M}}(\tilde{X}, \tilde{X})$, and we'll call this the **additive homotopy endomorphism object of** $\boldsymbol{X}$. By [D4, Cors. 3.6, 3.7] this homotopy type depends only on the homotopy type of X and the quasi-equivalence class of $\sigma_{\mathcal{M}}$—and so it is a well-defined invariant of X and $\mathcal{M}$.

Proof of Proposition 1.4. This is entirely similar to the proof of [D4, Th. 1.4], but using Theorem 7.4(c). ∎

Proof of Proposition 1.6. Same as the proof of [D4, Prop. 1.5]. ∎

Proof of Proposition 1.5. We know that there exists a zig-zag of Quillen equivalences $\mathcal{M} \xleftarrow{\sim} U_{\mathrm{ad}}\mathcal{C}/S \xrightarrow{\sim} Sp^{\Sigma}(U_{\mathrm{ad}}\mathcal{C}/S)$. Therefore, using [D4, Thm. 1.4] and Proposition 1.4 we may as well assume $\mathcal{M} = Sp^{\Sigma}(U_{\mathrm{ad}}\mathcal{C}/S)$. This is an $Sp^{\Sigma}(s\mathcal{A}b)$-model category, and so for any object X we have a ring object $\underline{\mathcal{M}}(X,X)$ in $Sp^{\Sigma}(s\mathcal{A}b)$. The adjoint functors $\mathcal{S}et_* \rightleftarrows \mathcal{A}b$ induce a strong monoidal adjunction $F\colon Sp^{\Sigma}(s\mathcal{S}et_*) \rightleftarrows Sp^{\Sigma}(s\mathcal{A}b)\colon U$. The $Sp^{\Sigma}(s\mathcal{A}b)$-structure on $\mathcal{M}$ therefore yields an induced Sp^{Σ}-structure as well (see [D4, Lem. A.5]). In this structure, the endomorphism ring spectrum of X is precisely $U[\underline{\mathcal{M}}(X,X)]$. Using [D4, Prop. 1.5], we know that this has the homotopy type of the ring spectrum $\mathrm{hEnd}(X)$, at least when X is cofibrant-fibrant. And Proposition 1.6 says that $\underline{\mathcal{M}}(X,X)$ has the homotopy type of $\mathrm{hEnd}_{\mathrm{ad}}(X)$. This is all we needed to check. ∎

9. Chain enrichments

Proposition 1.6 says that if $\mathcal{M}$ is a $Sp^{\Sigma}(s\mathcal{A}b)$-model category then one can compute $\mathrm{hEnd}_{\mathrm{ad}}(X)$ using the $Sp^{\Sigma}(s\mathcal{A}b)$-structure. We would like to prove a similar result for Ch-model categories, where Ch denotes the model category of unbounded chain complexes of abelian groups. These are what arise most commonly in algebraic situations.

The monoidal model categories $Sp^{\Sigma}(s\mathcal{A}b)$ and Ch can be connected by a zig-zag of weak monoidal Quillen equivalences, as described in [S]. This zig-zag can be used to translate enrichment-type information between these two categories. However, this is not as straightforward as one might expect; there are complications arising from the monoidal properties of the Dold-Kan equivalence between $s\mathcal{A}b$ and ch_+, as analyzed in [SS3]. Our method for dealing with this requires some cumbersome machinery and gives a slightly weaker result than one would like. However, it is the best we can do at the moment.

9.1. Statement of the result. We give Ch the projective model structure, where weak equivalences are quasi-isomorphisms and fibrations are surjections. Recall again that a **Ch-model category** is a model category with compatible tensors, cotensors, and enrichments over Ch satisfying an analogue of SM7; see Section 2. For X, Y in $\mathcal{M}$, we denote the enriched hom-object in Ch by $\underline{\mathcal{M}}_{\mathrm{Ch}}(X,Y)$.

Note that a Ch-model category is automatically additive and stable. See Corollary 6.9 for the additivity, and [SS2, 3.5.2] or [GS, 3.2] for stability.

Recall from [S] that there are two Quillen equivalences

$$Sp^{\Sigma}(\mathrm{ch}_+) \underset{R}{\overset{D}{\rightleftarrows}} \mathrm{Ch} \qquad \text{and} \qquad Sp^{\Sigma}(\mathrm{ch}_+) \underset{\nu}{\overset{L}{\rightleftarrows}} Sp^{\Sigma}(s\mathcal{A}b)$$

in which (D,R) is strong monoidal and (L,ν) is weak monoidal. These induce Quillen equivalences between the corresponding model categories of rings:

$$\mathcal{R}ing(Sp^{\Sigma}(\mathrm{ch}_+)) \overset{\sim}{\rightleftarrows} \mathcal{DGA} \qquad \mathcal{R}ing(Sp^{\Sigma}(\mathrm{ch}_+)) \overset{\sim}{\rightleftarrows} \mathcal{R}ing(Sp^{\Sigma}(s\mathcal{A}b)). \tag{9.2}$$

In the first equivalence of (9.2) the left and right adjoints are just the restrictions of D and R, as these were strong monoidal. In the second, the right adjoint is just ν again, but the left adjoint is more complicated; see [SS3, 3.3].

Let $\underline{\nu}$ and $\underline{D}$ denote the derived functors of ν and D from (9.2), and write $\Theta' = \underline{D}\,\underline{\nu}$. So Θ' is a functor

$$\mathcal{H}o(\mathcal{R}ing(Sp^\Sigma(s\mathcal{A}b))) \to \mathcal{H}o(\mathcal{D}G\mathcal{A}).$$

Let $\mathcal{M}$ be a stable, combinatorial, additive model category and let $X \in \mathcal{M}$. We have shown how to associate to X an object $\mathrm{hEnd}_{\mathrm{ad}}(X) \in \mathcal{R}ing(Sp^\Sigma(s\mathcal{A}b))$. By applying Θ' we get the **homotopy endomorphism dga** of X. Denote this as $\mathrm{hEnd}_{\mathrm{dga}}(X) = \Theta'\big[\mathrm{hEnd}_{\mathrm{ad}}(X)\big]$.

The goal for this section is to prove Proposition 1.7. We restate the result here for the convenience of the reader.

9.3. Proposition. *Suppose that $\mathcal{M}$ is a combinatorial* Ch-*model category, and that $\mathcal{M}$ has a generating set of compact objects. Let $X \in \mathcal{M}$ be cofibrant and fibrant. Then the dga $\underline{\mathcal{M}}_{\mathrm{Ch}}(X,X)$ is quasi-isomorphic to* $\mathrm{hEnd}_{\mathrm{dga}}(X)$

Proposition 9.3 will be proven by reducing from a Ch-model category to a $Sp^\Sigma(s\mathcal{A}b)$-model category and then applying results of Section 8. The reduction from Ch to $Sp^\Sigma(\mathrm{ch}_+)$ will be simple because of the strong monoidal equivalence between these two categories. The following proposition provides the reduction from $Sp^\Sigma(\mathrm{ch}_+)$ to $Sp^\Sigma(s\mathcal{A}b)$. This is where all the enriched category theory from Sections 2 through 5 is needed. Recall that for a general $\mathcal{D}$-model category $\mathcal{N}$ we denote the morphism object in $\mathcal{D}$ by $\underline{\mathcal{N}}_{\mathcal{D}}(X,Y)$.

9.4. Proposition. *Let $\mathcal{M}$ be a combinatorial $Sp^\Sigma(\mathrm{ch}_+)$-model category with a generating set of compact objects. Let $X \in \mathcal{M}$ be a cofibrant-fibrant object. Then there exists*

(i) a combinatorial, $Sp^\Sigma(s\mathcal{A}b)$-model category $\mathcal{N}$,

(ii) a zig-zag of Quillen equivalences between $\mathcal{M}$ and $\mathcal{N}$, where the intermediate model categories are all additive, and

(iii) a cofibrant-fibrant object $Y \in \mathcal{N}$

such that Y is taken to X by the derived functors of the Quillen equivalences and $\underline{\nu}\big[\underline{\mathcal{N}}_{\mathrm{Sp}^\Sigma(\mathrm{s}\mathcal{A}\mathrm{b})}(Y,Y)\big]$ is weakly equivalent to $\underline{\mathcal{M}}_{Sp^\Sigma(\mathrm{ch}_+)}(X,X)$.

Proof. This is a special case of Proposition 5.2 and Corollary 5.3. We need to verify the properties for $\mathcal{C} = Sp^\Sigma(\mathrm{ch}_+)$ and $\mathcal{D} = Sp^\Sigma(s\mathcal{A}b)$ stated just prior to Proposition 5.2, with (F,G) replaced by (L,ν). Axioms (QI1-2) for $\mathcal{C}$ follow from [S, 3.2, 3.3]. The fact that (L,ν) is a weak monoidal Quillen equivalence is given in [S, 4.3]. All the other conditions are easy exercises, but see also Example 4.13 for more information. ∎

Using the above proposition, we can complete the following:

Proof of Proposition 9.3. Let $\mathcal{M}$ be a combinatorial Ch-model category with a generating set of compact objects. Let $\mathcal{C} = Sp^{\Sigma}(\mathrm{ch}_+)$. Using the strong monoidal adjunction (D, R), $\mathcal{M}$ becomes a $\mathcal{C}$-model category via the definitions $Z \otimes c = Z \otimes D(c)$, $Z^c = Z^{Dc}$, and $\underline{\mathcal{M}}_{\mathcal{C}}(W, Z) = R[\underline{\mathcal{M}}_{\mathrm{Ch}}(W, Z)]$ where $W, Z \in \mathcal{M}$ and $c \in \mathcal{C}$. See [D4, Lem. A.5].

Now we apply Proposition 9.4 to $\mathcal{M}$ with this $\mathcal{C}$-model structure to construct $\mathcal{N}$ and Y. By Proposition 1.4, the additive homotopy endomorphism spectra corresponding to X and Y are weakly equivalent. Let $\mathcal{D} = Sp^{\Sigma}(s\mathcal{A}b)$. Since $\mathcal{N}$ is a $\mathcal{D}$-model category, we have by Proposition 1.6 that $\mathrm{hEnd}_{\mathrm{ad}}(Y)$ is weakly equivalent to $\underline{\mathcal{N}}_{\mathcal{D}}(Y, Y)$. So we have

$$\mathrm{hEnd}_{\mathrm{dga}}(X) = \Theta'[\mathrm{hEnd}_{\mathrm{ad}}(X)] \simeq \Theta'[\mathrm{hEnd}_{\mathrm{ad}}(Y)] \simeq \underline{D}\,\underline{\nu}\big[\underline{\mathcal{N}}_{\mathcal{D}}(Y, Y)\big]$$

(recalling that $\Theta' = \underline{D}\,\underline{\nu}$).

But $\mathcal{N}$ and Y were chosen in such a way that we have $\underline{\nu}\big[\underline{\mathcal{N}}_{\mathcal{D}}(Y, Y)\big] \simeq \underline{\mathcal{M}}_{\mathcal{C}}(X, X)$. So in fact

$$\mathrm{hEnd}_{\mathrm{dga}}(X) \simeq \underline{D}\,\big[\underline{\mathcal{M}}_{\mathcal{C}}(X, X)\big] = \underline{D}\,\underline{R}\big[\underline{\mathcal{M}}_{\mathrm{Ch}}(X, X)\big] \simeq \underline{\mathcal{M}}_{\mathrm{Ch}}(X, X).$$

■

A. Homotopy theory of $\mathcal{C}I$-categories

The present section reviews and expands on results from [SS3]. In particular, [SS3] often states results in settings which are extremely general and therefore require somewhat awkward hypotheses. Here we will specialize, replacing those hypotheses with conditions more readily checked in practice.

We assume that $\mathcal{C}$ is a combinatorial, symmetric monoidal model category. Also, $\mathcal{C}$ is assumed to satisfy the *monoid axiom* of [SS1, 3.3]. We'll refer to those conditions as our *'standing assumptions'*. Finally, we will sometimes require the following two conditions as well:

(QI1) For any cofibrant object $A \in \mathcal{C}$ and any weak equivalence $X \to Y$, the map $A \otimes X \to A \otimes Y$ is also a weak equivalence.

(QI2) Suppose $A \rightarrowtail B$ is a cofibration, and X is any object. Then for any map $A \otimes X \to Z$, the map from the homotopy pushout of $B \otimes X \longleftarrow A \otimes X \to Z$ to the pushout is a weak equivalence.

The abbreviation (QI) is for 'Quillen invariance', as these conditions will be used to check what [SS3, 3.11] calls Quillen invariance for modules.

A.1. Example. The category ch_+ of non-negatively graded chain complexes with tensor product and its usual 'projective' model structure satisfies (QI1-2). It follows from [S, 3.2, 3.3] that $Sp^\Sigma(\mathrm{ch}_+)$ also satisfies (QI1-2). Typically, these axioms will follow from the existence of an 'injective' model structure for $\mathcal{M}$ in which all objects are cofibrant, provided such a model structure is a Quillen module over the corresponding projective version.

Let I be a set. We assume the reader is familiar with the notion of **$\mathcal{C}I$-category** (a category enriched over $\mathcal{C}$ with object set I) from [Bo, 6.2]. If $\mathcal{O}$ is a $\mathcal{C}I$-category, then the category of right $\mathcal{O}$-modules (contravariant $\mathcal{C}$-functors from $\mathcal{O}$ to $\mathcal{C}$) is defined in [Bo, 6.2]; see also [SS3, Section 6].

A.2. Proposition. *Let $\mathcal{O}$ be a $\mathcal{C}I$-category.*

(a) The category Mod-$\mathcal{O}$ *has a model category structure in which the weak equivalences and fibrations are defined objectwise.*

(b) Let $\mathcal{O} \to \mathcal{R}$ be a map of $\mathcal{C}I$-categories. Then there is a Quillen map Mod-$\mathcal{O} \to$ Mod-$\mathcal{R}$ *in which the right adjoint is restriction.*

(c) If $\mathcal{O} \to \mathcal{R}$ is a weak equivalence and $\mathcal{C}$ satisfies (QI1-2), then the above Quillen map is a Quillen equivalence.

Proof. Part (a) is [SS3, 6.1(1)]. For (b) we need only construct the left adjoint, as the restriction clearly preserves fibrations and trivial fibrations. This construction is given in the paragraph above [SS3, 6.1]. Denote this left adjoint by $X \mapsto X \otimes_{\mathcal{O}} \mathcal{R}$.

Part (c) requires a little work. First, for any $i \in I$ and $A \in \mathcal{C}$ let $A \otimes Fr_i(\mathcal{O})$ be the 'free $\mathcal{O}$-module generated by A at spot i'. This is defined by $j \mapsto A \otimes \mathcal{O}(j, i)$. It is easy to see that we have the adjunction Mod-$\mathcal{O}(A \otimes Fr_i(\mathcal{O}), X) \cong \mathcal{C}(A, X(i))$. From this it immediately follows that

$$[A \otimes Fr_i(\mathcal{O})] \otimes_{\mathcal{O}} \mathcal{R} \cong A \otimes Fr_i(\mathcal{R}).$$

As a another consequence of the adjunction, observe that Mod-$\mathcal{O}$ is cofibrantly-generated and the generating cofibrations are maps of the form $A \otimes Fr_i(\mathcal{O}) \to B \otimes Fr_i(\mathcal{O})$ where $A \rightarrowtail B$ is a generating cofibration of $\mathcal{C}$.

By [SS3, 6.1(2)], to prove (c) it suffices to check that for any cofibrant $\mathcal{O}$-module N the natural map $N \to U[N \otimes_{\mathcal{O}} \mathcal{R}]$ is a weak equivalence, where U is the restriction Mod-$\mathcal{R} \to$ Mod-$\mathcal{O}$. Let G denote the composite functor $X \mapsto U[X \otimes_{\mathcal{O}} \mathcal{R}]$, so that we are concerned with the natural transformation $Id \to G$. Note that when $X = A \otimes Fr_i(\mathcal{O})$ we have $G(X) = A \otimes Fr_i(\mathcal{R})$. If A is cofibrant, the map $X \to G(X)$ is an objectwise weak equivalence because $\mathcal{O} \to \mathcal{R}$ is (this uses (QI1)).

Apply the small object argument to factor $\emptyset \to N$ as a cofibration followed by a trivial fibration. This gives us a (possibly transfinite) sequence of cofibrations

$$\emptyset = W_0 \rightarrowtail W_1 \rightarrowtail W_2 \rightarrowtail \cdots$$

in which W_{i+1} is obtained from W_i by a pushout diagram

$$\begin{tikzcd} \coprod_j A_j \otimes Fr_j(\mathcal{O}) \arrow[r] \arrow[d] & W_i \arrow[d] \\ \coprod_j B_j \otimes Fr_j(\mathcal{O}) \arrow[r] & W_{i+1}, \end{tikzcd}$$

together with a trivial fibration $W_\infty = \operatorname{colim}_i W_i \to N$. Since N is cofibrant, N is a retract of W_∞. So it will suffice to show that $W_\infty \to G(W_\infty)$ is a weak equivalence, as $N \to GN$ is a retract of this map.

We first prove that if $W_{i-1} \to G(W_{i-1})$ is a weak equivalence then the same is true of $W_i \to G(W_i)$. To see this, note that we have the following diagram:

$$\begin{tikzcd} \coprod B_j \otimes Fr_j(\mathcal{O}) \arrow[d, "\sim"] & \coprod A_j \otimes Fr_j(\mathcal{O}) \arrow[l, tail] \arrow[r] \arrow[d, "\sim"] & W_{i-1} \arrow[d, "\sim"] \\ \coprod G(B_j \otimes Fr_j(\mathcal{O})) & \coprod G(A_j \otimes Fr_j(\mathcal{O})) \arrow[l] \arrow[r] & G(W_{i-1}). \end{tikzcd}$$

The pushout of the top row is W_i, and of the bottom row is $G(W_i)$ (the latter follows because G preserves colimits). Note that $G(A_j \otimes Fr_j(\mathcal{O})) \to G(B_j \otimes Fr_j(\mathcal{O}))$ is a cofibration, as it is just the map $A_j \otimes Fr_j(\mathcal{R}) \to B_j \otimes Fr_j(\mathcal{R})$. It follows that $G(W_{i-1}) \to G(W_i)$ is a cofibration.

Certainly the above diagram induces a weak equivalence of *homotopy* pushouts. We claim these homotopy pushouts are weakly equivalent to the corresponding pushouts. This is an objectwise question, since pushouts, homotopy pushouts, and weak equivalences in the module category are all determined objectwise. The claim for the top row then follows directly from (QI2). The claim for the bottom row is similar, but uses the identification $G(B_j \otimes Fr_j(\mathcal{O})) = B_j \otimes Fr_j(\mathcal{R})$, etc.

Thus, we have shown that $W_i \to G(W_i)$ is a weak equivalence whenever $W_{i-1} \to G(W_{i-1})$ is so. It is trivial that $W_0 \to G(W_0)$ is a weak equivalence. The result now follows by a transfinite induction, using [Hi, 17.9.1] to pass the weak equivalences to the limit ordinals. One again uses that G preserves colimits. ∎

A.3. Proposition. *Let I be a fixed set, and let $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ be a weak monoidal Quillen pair (see Section 2.4) where both $\mathcal{C}$ and $\mathcal{D}$ satisfy our standing assumptions.*

(a) The category $\mathcal{C}I - \mathcal{C}at$ (and likewise $\mathcal{D}I - \mathcal{C}at$) has a model category structure in which weak equivalences and fibrations are defined objectwise.

(b) There is a Quillen map $\mathcal{C}I - \mathcal{C}at \to \mathcal{D}I - \mathcal{C}at$ in which the right adjoint is 'apply R objectwise'. The left adjoint will be denoted $L^{\mathcal{D}I}$.

(c) Suppose $\mathbf{1}_\mathcal{C}$ and $\mathbf{1}_\mathcal{D}$ are cofibrant. If $\mathcal{O}$ is a cofibrant $\mathcal{C}I$-category then there are weak equivalences $L[\mathcal{O}(i,j)] \to (L^{\mathcal{D}I}\mathcal{O})(i,j)$ for every $i, j \in I$. These are adjoint to the maps provided by the adjunction unit $\mathcal{O} \to R(L^{\mathcal{D}I}\mathcal{O})$.

(d) *Suppose* (L, R) *is a Quillen equivalence and* $\mathbf{1}_{\mathcal{C}}$, $\mathbf{1}_{\mathcal{D}}$ *are cofibrant. Then the induced Quillen map* $\mathcal{C}I - \mathcal{C}at \to \mathcal{D}I - \mathcal{C}at$ *is also a Quillen equivalence.*

PROOF. Part (a) is [SS3, 6.3(1)]. For part (b) we argue as follows. Recall the category $\mathcal{C}I - \mathcal{G}raph$ from [SS3, 6.1], and that this category comes equipped with a monoidal product $\otimes$. A $\mathcal{C}I$-category is precisely a monoid with respect to this tensor product. The existence of the desired left adjoint follows from Lemma A.4 below. As the right adjoint obviously preserves fibrations and trivial fibrations, we have a Quillen pair.

For part (c), note that there is a Quillen map $\mathcal{C}I-\mathcal{G}raph \to \mathcal{C}I-\mathcal{C}at$ in which the right adjoint is the forgetful functor. The model structure on $\mathcal{C}I - \mathcal{C}at$ is 'created' by these adjoint functors from the cofibrantly-generated model structure on $\mathcal{C}I - \mathcal{G}raph$. [SS3, 6.4(1)] proves the desired claim in the case $\mathcal{O}$ is a cell complex, but since any cofibrant object is a retract of a cell complex one immediately obtains the more general statement.

Finally, we prove (d). Note that since the functor $\mathcal{D}I - \mathcal{C}at \to \mathcal{C}I - \mathcal{C}at$ is just 'apply R objectwise', a map of fibrant objects $X \to Y$ in $\mathcal{D}I - \mathcal{C}at$ is a weak equivalence if and only if $RX \to RY$ is a weak equivalence in $\mathcal{C}I - \mathcal{C}at$. So by Lemma A.5 below, we only need to show that if $\mathcal{O}$ is a cofibrant $\mathcal{C}I$-category and $L^{\mathcal{D}I}\mathcal{O} \xrightarrow{\sim} \mathcal{A}$ is a fibrant replacement in $\mathcal{D}I - \mathcal{C}at$, then $\mathcal{O} \to R\mathcal{A}$ is a weak equivalence.

Since weak equivalences are detected objectwise, we must check that $\mathcal{O}(i,j) \to R[\mathcal{A}(i,j)]$ is a weak equivalence for every $i, j \in I$. But $\mathcal{O}$ is cofibrant, so each $\mathcal{O}(i,j)$ is cofibrant in $\mathcal{C}$ (see [SS3, 6.3(2)] —this uses that $\mathbf{1}_{\mathcal{C}}$ is cofibrant). And since $\mathcal{A}$ is fibrant, each $\mathcal{A}(i,j)$ is fibrant. Using the Quillen equivalence (L, R), we are therefore reduced to checking that $L[\mathcal{O}(i,j)] \to \mathcal{A}(i,j)$ is a weak equivalence. But we are really looking at the composite

$$L[\mathcal{O}(i,j)] \to [L^{\mathcal{D}I}\mathcal{O}](i,j) \to \mathcal{A}(i,j).$$

The second map was assumed to be a weak equivalence, and the first map is a weak equivalence by part (c). So we are done. ∎

In this proof we used the following two lemmas.

A.4. LEMMA. *Let* $\mathcal{C}$ *be a monoidal category which is complete and co-complete. Assume that for any* $X \in \mathcal{C}$ *the functors* $X \otimes (-)$ *and* $(-) \otimes X$ *preserve filtered colimits and countable coproducts. Then*

(a) *The category of monoids in* $\mathcal{C}$ *is co-complete.*

(b) *If* $\mathcal{B}$ *is another monoidal category, and* $L\colon \mathcal{B} \rightleftarrows \mathcal{C}\colon R$ *is an adjunction where* R *is weak monoidal, then* R *induces a functor* $\mathcal{C} - \mathcal{M}onoid \to \mathcal{B} - \mathcal{M}onoid$ *and this functor has a left adjoint.*

PROOF. Let $T\colon \mathcal{C} \to \mathcal{C}$ be the 'free algebra' monad, where

$$T(X) = \mathbf{1} \amalg X \amalg (X \otimes X) \amalg \cdots .$$

(The fact that this is a monad uses the assumption that tensoring preserves countable coproducts). The monoids in $\mathcal{C}$ are precisely the T-algebras. Our assumptions imply that T preserves filtered colimits, so [Bo, 4.3.6] implies that $\mathcal{C} - \mathcal{M}onoid$ is co-complete.

Part (b) is an immediate consequence of (a) and [Bo, 4.5.6]. ∎

A.5. Lemma. *[Ho1, 1.3.16] Let* $L\colon \mathcal{M} \rightleftarrows \mathcal{N}\colon R$ *be a Quillen pair. Assume the following two conditions hold:*

(i) If X *and* Y *are fibrant objects in* $\mathcal{N}$*, a map* $X \to Y$ *is a weak equivalence whenever* $RX \to RY$ *is a weak equivalence.*

(ii) For every cofibrant object $A \in \mathcal{M}$ *and every fibrant replacement* $LA \xrightarrow{\sim} Z$ *in* $\mathcal{N}$*, the composite map* $A \to RLA \to RZ$ *is a weak equivalence.*

Then (L, R) *is a Quillen equivalence.*

Here is the final result we will need:

A.6. Proposition. *Again assume that* $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ *is a weak monoidal Quillen pair, where* $\mathcal{C}$ *and* $\mathcal{D}$ *satisfy our standing assumptions. Also assume that* $\mathbf{1}_{\mathcal{C}}$ *and* $\mathbf{1}_{\mathcal{D}}$ *are cofibrant.*

(a) If $\mathcal{A}$ *is a* $\mathcal{D}I$*-category then there is a Quillen map* Mod-$R\mathcal{A} \to$ Mod-$\mathcal{A}$ *in which the right adjoint is 'apply* R *objectwise'.*

(b) Let $\mathcal{O}$ *be a* $\mathcal{C}I$*-category. Then there is a Quillen map* Mod-$\mathcal{O} \to$ Mod-$(L^{\mathcal{D}I}\mathcal{O})$ *in which the right adjoint is the composition of 'applying* R *objectwise, then restricting across* $\mathcal{O} \to R(L^{\mathcal{D}I}\mathcal{O})$*'.*

(c) If $L\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon R$ *is a Quillen equivalence and* $\mathcal{O}$ *is a cofibrant* $\mathcal{C}I$*-category, then* Mod-$\mathcal{O} \to$ Mod-$(L^{\mathcal{D}I}\mathcal{O})$ *is also a Quillen equivalence.*

Proof. If X is an object in the functor category $\mathcal{D}^I$, let $T_{\mathcal{A}}X \in \mathcal{D}^I$ be the functor $j \mapsto \coprod_j X(j) \otimes \mathcal{A}(-, j)$. Note that this is a monad in an obvious way, and that the $T_{\mathcal{A}}$-algebras are precisely the $\mathcal{A}$-modules. We have the diagram of categories

$$\begin{tikzcd} \mathcal{C}^I & \mathcal{D}^I \arrow[l, "R"'] \\ \text{Mod-}(R\mathcal{A}) \arrow[u] & \text{Mod-}\mathcal{A} \arrow[l, "R"'] \arrow[u] \end{tikzcd}$$

where the vertical maps are forgetful functors. By [Bo, 4.5.6] the map Mod-$\mathcal{A} \to$ Mod-$(R\mathcal{A})$ has a left adjoint, since Mod-$\mathcal{A}$ is cocomplete. This clearly gives a Quillen pair.

For (b) we use the composite of the two Quillen maps

$$\text{Mod-}\mathcal{O} \to \text{Mod-}(RL^{\mathcal{D}I}\mathcal{O}) \to \text{Mod-}(L^{\mathcal{D}I}\mathcal{O}).$$

The first is provided by Proposition A.2(b), induced by the map $\mathcal{O} \to RL^{\mathcal{D}I}\mathcal{O}$. The second comes from (a) of the present result.

Finally, part (c) is just [SS3, 6.5(1)]. ∎

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