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FUNCTIONAL ANALYSIS ON NORMED SPACES: THE BANACH SPACE COMPARISON

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ABSTRACT. It is the aim of this paper to compute the category of Eilenberg-Moore algebras for the monad arising from the dual unit-ball functor on the category of (semi)normed spaces. We show that this gives rise to a stronger algebraic structure than the totally convex one obtained from the closed unit ball functor on the category of Banach spaces.

1. Introduction and basic notations

It was shown in [5] that contravariant Hom-functors, which we will call dualization functors for the rest of the paper, often give rise to meaningful categorical dualities such as the celebrated Gelfand-Naimark duality. A beautiful way to infer the Gelfand-Naimark duality in a purely categorical way is given in [4]. Throughout functional analysis, dual spaces and dualization functors into the scalar field occur everywhere. Another famous example is provided by the Riesz representation theorem which describes the dual of the Banach space of all (necessarily) bounded continuous functions on a given compact Hausdorff space in measure-theoretic terms. In all what follows, all modules or vectorspaces will be considered over $\mathbb{R}$ and we will write $\mathbf{sNorm}_1$ (resp. $\mathbf{Ban}_1$) for the category of seminormed (resp. Banach) spaces and non-expansive linear maps.

Moreover, it is well-known that the topological dual of a seminorned space comes equipped with a canonical dual norm, making it into a Banach space. This allows for the formation of (countably infinite) totally convex combinations on the closed dual unit ball, giving it the algebraic structure of a totally convex module in the sense of [7, 8]. Let us recall that in [7, 8] the authors showed that the category $\mathbf{TC}$ of totally convex modules and totally affine maps is the category of Eilenberg-Moore algebras for the monad arising from an adjunction having the closed unit ball-functor from $\mathbf{Ban}_1$ to $\mathbf{Set}$ as a right adjoint. Loosely speaking, this expresses the fact that the algebraic structure (i.e. describable in terms of generators and relations) which is intrinsically present on the closed unit ball of a Banach space is exactly the one of a totally convex module. Answering the analogous question for the closed unit ball-functor on the category $\mathbf{sNorm}_1$ instead of $\mathbf{Ban}_1$, the authors obtained in [7, 8] the (larger) category $\mathbf{AC}$ of absolutely convex modules and absolutely affine maps as the category of Eilenberg-Moore algebras.

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As shown in [12], the category **AC** arises also in so called "Quantified Functional Analysis". We use this term for the theory of (locally convex) approach spaces as developed e.g. in [11, 12], where the key idea is to work with canonical numerical structures overlying (locally convex) vector topologies, instead of only considering the topological level. For more information we refer to [11, 12]. In [12] we proved that **AC** is also the category of Eilenberg-Moore algebras for the monad arising from a dual adjunction having the dualization functor $\mathrm{Hom}_{\mathbf{lcApVec}}(-,\mathbb{R})$ on $\mathbf{lcApVec}^{\mathrm{op}}$ as a right adjoint. Here **lcApVec** denotes the category of locally convex approach spaces and contractive linear maps, and $\mathbb{R}$ is equipped with the absolute value as norm.

If we define $C : \mathbf{sNorm}_1^{\mathrm{op}} \to \mathbf{Set}$ to be the restriction of $\mathrm{Hom}_{\mathbf{lcApVec}}(-,\mathbb{R})$ to $\mathbf{sNorm}_1^{\mathrm{op}}$, then C is in fact the dualization functor $\mathrm{Hom}_{\mathbf{sNorm}_1}(-,\mathbb{R})$ which is nothing but the closed *dual* unit ball functor on $\mathbf{sNorm}_1^{\mathrm{op}}$. Again C is the right adjoint of a (dual) adjunction, giving rise to a monad. The question we address in this paper is finding a description of the category of Eilenberg-Moore algebras for this monad. Speaking more loosely as we did above this answers the question of which canonical algebraic structure is present on the closed *dual* unit ball of a seninormed space.

From the previous discussion one might be tempted to guess that this category of Eilenberg-Moore algebras is concretely isomorphic with **TC**. Quite surprisingly this is not the case as we shall prove. The resulting category **SC** of Eilenberg-Moore algebras has as objects sets which allow for an abstract integration with respect to certain finitely additive measures, also called charges, of total variation at most one in the sense of [9]. The crucial ingredient to obtain this result is a Riesz-type representation theorem for charges, to be found in [9] which we state in precise terms later on.

2. Basic definitions

Let S be a set and let $\mathcal{F}$ be a field of subsets of S, i.e. a collection $\mathcal{F} \subset 2^S$ satisfying the following axioms:

1. $\emptyset, S \in \mathcal{F}$,
2. $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$,
3. $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$.

A bounded charge on $(S,\mathcal{F})$ is an additive map $\alpha : \mathcal{F} \to \mathbb{R}$, i.e. a map satisfying $\alpha(A \cup B) = \alpha(A) + \alpha(B)$ for all $A, B \in \mathcal{F}$ with $A \cap B = \emptyset$ (note that then $\alpha(\emptyset) = 0$ follows), such that the total variation

$$||\alpha|| := \sup\{\sum_{i=1}^{n} |\alpha(A_i)| \mid \{A_1, \dots, A_n\} \subset \mathcal{F} \text{ finite partition of } S\}$$

is finite. We write $\mathrm{ba}(S,\mathcal{F})$ for the space of all bounded charges on $(S,\mathcal{F})$, equipped with the total variation norm defined above. To ease the notation we put $\mathrm{ba}(S)$ instead of $\mathrm{ba}(S, 2^S)$, A simple function on $(S,\mathcal{F})$ is a map $f:S\to\mathbb{R}$ for which there exist a finite partition $\{A_1,\dots,A_n\}\subset\mathcal{F}$ of S and scalars $a_1,\dots,a_n\in\mathbb{R}$, such that

$$f=\sum_{i=1}^{n} a_i 1_{A_i}.$$

We put $B(S,\mathcal{F})$ for the closure, with respect to the supremum–norm $\|f\|_\infty := \sup_{s\in S}|f(s)|$, of the set of simple functions on $(S,\mathcal{F})$ in the space of all bounded functions from S to $\mathbb{R}$. The space $(B(S,\mathcal{F}))$ is equipped with the norm $||-||_\infty)$. If $\mathcal{F}$ is the collection of all subsets of S, we write BS instead of $B(S,\mathcal{F})$, which in this case equals the space of all bounded real valued functions on S.

For $\alpha\in \mathrm{ba}(S,\mathcal{F})$ and $a\in B(S,\mathcal{F})$ the integral $\int_{s\in S} a(s)\mathrm{d}\alpha(s)$ can be introduced as the limit of the integrals of a sequence of simple functions converging uniformly to a [9], where the integral of a simple function is defined in the obvious way:

$$\int_{s\in S}\sum_{i=1}^{n} a_i 1_{A_i}(s)\mathrm{d}\alpha(s) := \sum_{i=1}^{n} a_i\alpha(A_i).$$

We will use the notational convention to write OX for the closed unit ball of a semi-normed space X and CX for the closed dual unit ball, i.e.

$$CX := \{\varphi : X \longrightarrow \mathbb{R}\ |\varphi \text{ is linear and for all } x\in X\text{: } |\varphi(x)|\le \|x\|\}.$$

In the sequel $L_C X$ denotes the topological dual of X, equipped with the dual norm

$$\|\varphi\|_C := \inf\{k>0 \mid \frac{1}{k}\varphi\in CX\}.$$

The following representation theorem identifies the dual space of $(B(S,\mathcal{F}),||-||_\infty)$.

2.1. Theorem. *[9, 1] The assignment*

$$\begin{array}{rcl}\gamma_{(S,\mathcal{F})} : \mathrm{ba}(S,\mathcal{F}) & \longrightarrow & L_C(B(S,\mathcal{F}),||-||_\infty)\\ \alpha & \longmapsto & \int_{s\in S} ev(-,s)\,d\alpha(s),\end{array}\tag{1}$$

with $\big(\int_{s\in S} ev(-,s)\,d\alpha(s)\big)(a) := \int_{s\in S} a(s)\,d\alpha(s)$ *is a linear isometry.*

Another notational convention is to put TS for the closed unit ball of $\mathrm{ba}(S)$. From Theorem 2.1 we deduce that the map

$$\begin{array}{rrcl}\gamma_S : & TS = O\mathrm{ba}(S,2^S) & \longrightarrow & CBS\\ & \alpha & \longmapsto & \int_{s\in S}\mathrm{ev}(-,s)\mathrm{d}\alpha(s)\end{array}\tag{2}$$

is a one-one correspondence.

It is easy to verify that we have an endofunctor

$$\begin{array}{rl}T:\mathbf{Set}\longrightarrow\mathbf{Set} : (S_1\xrightarrow{f} S_2)\longmapsto & (TS_1\xrightarrow{Tf} TS_2)\\ & \alpha\longmapsto\alpha_f,\end{array}\tag{3}$$

with $\alpha_f(A) := \alpha(f^{-1}(A))$ $(A \in \mathcal{F}_2)$.

We put

$$\eta_S : S \longrightarrow TS : s \longmapsto \delta_s, \tag{4}$$

with $\delta_s(A) := 1$ if $s \in A$ and $\delta_s(A) := 0$ if $s \notin A$.

Recall that for $\alpha \in \mathrm{ba}(S, \mathcal{F})$ and $a \in B(S, \mathcal{F})$, we have

$$\left| \int_{s \in S} a(s) \mathrm{d}\alpha(s) \right| \leq \int_{s \in S} |a(s)| \mathrm{d}|\alpha|(s),$$

where $|\alpha|$ is the total variation of α, given by $|\alpha| := \alpha^+ + \alpha^-$, with $\alpha^+(A) := \sup\{\alpha(B) \mid B \in \mathcal{F},\ B \subset A\}$ and $\alpha^- := -\inf\{\alpha(B) \mid B \in \mathcal{F},\ B \subset A\}$ for each $A \in \mathcal{F}$. Also recall that we have the identity $|\alpha|(S) = ||\alpha||$. Let $\beta : 2^{O\mathrm{ba}(S,\mathcal{F})} \to \mathbb{R}$ be in $TO\mathrm{ba}(S, \mathcal{F})$. Then the map $O\mathrm{ba}(S, \mathcal{F}) \longrightarrow \mathbb{R} : \alpha \longmapsto \alpha(A)$ is bounded, because for each $\alpha \in O\mathrm{ba}(S, \mathcal{F})$ we have $|\alpha(A)| \leq |\alpha(A)| + |\alpha(A^c)| \leq ||\alpha|| \leq 1$. This implies that this map is integrable with respect to β, i.e. the integral

$$\int_{\alpha \in O\mathrm{ba}(S,\mathcal{F})} \alpha(A) \mathrm{d}\beta(\alpha)$$

exists and is finite. One moreover shows, in the same way as in any standard measure theory course, that the assignment

$$\int_{\alpha \in O\mathrm{ba}(S,\mathcal{F})} \alpha(-) \mathrm{d}\beta(\alpha) : \mathcal{F} \longrightarrow \mathbb{R}$$

$$A \longmapsto \int_{\alpha \in O\mathrm{ba}(S,\mathcal{F})} \alpha(A) \mathrm{d}\beta(\alpha).$$

is actually a bounded charge on $O\mathrm{ba}(S, \mathcal{F})$. We thus obtain a map

$$\int_{O\mathrm{ba}(S,\mathcal{F})} : \quad \begin{array}{l} TO\mathrm{ba}(S, \mathcal{F}) \longrightarrow O\mathrm{ba}(S, \mathcal{F}) \\ \beta \longmapsto \int_{\alpha \in O\mathrm{ba}(S,\mathcal{F})} \alpha(-) \mathrm{d}\beta(\alpha). \end{array} \tag{5}$$

which in case $\mathcal{F} = 2^S$ is denote by

$$\mu_S : T^2 S \longrightarrow TS. \tag{6}$$

2.2. Definition. *A* **space of charges** *is a pair (M, I_M) consisting of a set M and a structure map*

$$I_M : TM \longrightarrow M : \alpha \longmapsto I_M(\alpha)$$

which satisfies

(SC1) for all $x \in M$: $I_M(\delta_x) = x$,

(SC2) for all $\beta \in T^2 M$: $I_M(\int_{\alpha \in TM} \alpha(-) d\beta(\alpha)) = (I_M \circ T I_M)(\beta)$.

A morphism between the spaces of charges (M_1, I_{M_1}) and (M_2, I_{M_2}) is a map $f : M_1 \longrightarrow M_2$ such that $I_{M_2} \circ Tf = f \circ I_{M_1}$ and the category of spaces of charges is denoted **SC***.*

These axioms may seem to be far fetched at first sight. However, notice that similar laws already where obtained in [10], where it was shown that the category of compact convex sets is monadic over the category of compacta.

3. On the algebraic structure of linear contractions

On the opposite category of $\mathbf{sNorm_1}$, the category of seminormed spaces with linear non–expansive maps, we consider the following dualisation functor:

$$C : \mathbf{sNorm}_1^{\mathrm{op}} \longrightarrow \mathbf{Set} : (Y \overset{f}{\leftarrow} X) \longmapsto (CY \overset{Cf}{\to} CX),$$

with $Cf(\varphi) := \varphi \circ f$ $(\varphi \in CY)$. It immediately follows from the fact that the product $\prod_{s\in S}(\mathbb{R}, |\ |)$ in the category $\mathbf{sNorm_1}$ is equal to BS, that the map

$$\eta'_S : S \longrightarrow CBS : s \longmapsto \mathrm{ev}(-, s) \tag{7}$$

is universal for C. We therefore obtain a functor

$$B : \mathbf{Set} \longrightarrow \mathbf{sNorm}_1^{\mathrm{op}} : (S_1 \overset{f}{\to} S_2) \longmapsto (BS_2 \overset{Bf}{\to} BS_1 : a \longmapsto a \circ f)$$

that is left adjoint to C. Let $\mathbf{T}' := (T', \eta', \mu')$ be the monad induced by the adjunction $B \dashv C$. So $T' := CB$, η' is defined above and if $\mu' := C\epsilon' B$ is applied on a set S we have

$$\mu'_S : T'^2 S \longrightarrow T'S : \Psi \longmapsto \mu'_S(\Psi), \tag{8}$$

with $\mu'_S(\Psi) : BS \longrightarrow \mathbb{R} : a \longmapsto \Psi(\mathrm{ev}(-, a))$.

For each point x in a seminormed space X, the map $\mathrm{ev}(-, x) : CX \longrightarrow \mathbb{R}$ is bounded and therefore integrable w.r.t. any $\alpha \in TCX$, i.e. the integral $\int_{\varphi\in CX} \varphi(x)\mathrm{d}\alpha(\varphi)$ exists and is finite. Moreover, we have the inequality

$$\left| \int_{\varphi\in CX} \varphi(x)\mathrm{d}\alpha(\varphi) \right| \leq ||x||||\alpha||.$$

and, since the assignment $\int_{\varphi\in CX} \varphi(-)\mathrm{d}\alpha(\varphi) : X \longrightarrow \mathbb{R}$ is linear, we therefore have an action

$$I_{CX} : TCX \longrightarrow CX : \alpha \longmapsto \int_{\varphi\in CX} \varphi(-)\mathrm{d}\alpha(\varphi).$$

We now come to our main theorem.

3.1. Theorem. *The pair $\widehat{C}X := (CX, I_{CX})$ is an object in* $\mathbf{SC}$. *Moreover,* $\mathbf{SC}$ *is a representation of the Eilenberg-Moore category of the adjunction $B \dashv C$ and the resp. comparison functor is given by*

$$\widehat{C} : \ \mathbf{sNorm}_1^{op} \longrightarrow \mathbf{SC} : (Y \overset{f}{\leftarrow} X) \longmapsto (\widehat{C}Y \overset{Cf}{\to} \widehat{C}X).$$

Proof. The proof is built upon the following series of facts.

Fact 1. Let σ_S denote the inverse of the map γ_S given in (2). Then the collection $\sigma := (\sigma_S)_{S\in\mathbf{Set}}$ defines a natural transformation $\sigma : T' \longrightarrow T$.

In order to show this, take a map $f : S_1 \longrightarrow S_2$. We have to verify commutation of the diagram

$$\begin{tikzcd} T'S_1 \arrow[r, "T'f"] \arrow[d, "\sigma_{S_1}"'] & T'S_2 \arrow[d, "\sigma_{S_2}"] \\ TS_1 \arrow[r, "Tf"'] & TS_2, \end{tikzcd}$$

i.e. $\sigma_{S_2} \circ CBf = Tf \circ \sigma_{S_1}$. Since σ_{S_1} and σ_{S_2} are bijections, this follows since, for each $\alpha \in TS_1$ and $\varphi \in BS_2$, the following string of equalities hold:

$$\begin{aligned} (CBf \circ \gamma_{S_1})(\alpha)(\varphi) &= CBf(\sigma_{S_1}^{-1}(\alpha))(\varphi) \\ &= \sigma_{S_1}^{-1}(\alpha)(\varphi \circ f) \\ &= \int_{S_1} \varphi \circ f \mathrm{d}\alpha \\ &= \int_{S_2} \varphi \mathrm{d}\alpha_f \\ &= \gamma_{S_2}(\alpha_f)(\varphi) \\ &= (\gamma_{S_2} \circ Tf)(\alpha)(\varphi). \end{aligned}$$

Fact 2. The diagram

$$\begin{tikzcd} S \arrow[r, "\eta_S"] \arrow[dr, "\eta'_S"'] & TS. \\ & T'S \arrow[u, "\sigma_S"'] \end{tikzcd} \tag{9}$$

commutes. Indeed, for $s \in S$ and $\varphi \in T'S$ we have

$$\begin{aligned} (\gamma_S \circ \eta_S)(s)(\varphi) &= \gamma_S(\delta_s)(\varphi) \\ &= \int_{t\in S} \varphi(t)\mathrm{d}\delta_s(t) \\ &= \varphi(s) \\ &= \eta'_S(s)(\varphi), \end{aligned}$$

from which the desired identity follows since γ_S is a bijection.

Fact 3. The diagram

$$\begin{tikzcd} T'^2S \arrow[r, "\mu'_S"] \arrow[d, "\sigma_S^2"'] & T'S \arrow[d, "\sigma_S"] \\ T^2S \arrow[r, "\mu_S"'] & TS \end{tikzcd} \tag{10}$$

is commutative.

Bearing in mind the factorization $T'^2S \xrightarrow[T'\sigma_S]{} T'TS \xrightarrow[\sigma_{TS}]{} T^2S$ (with composite σ^2_S), we have to verify that $\sigma_S \circ \mu'_S = \mu_S \circ \sigma_{TS} \circ T'\sigma_S$. Take $\Phi : BCBS \longrightarrow \mathbb{R}$ in T'^2S and $A \subset S$ arbitrary. We have

$$\mu_S((\sigma_{TS} \circ T'\sigma_S)(\Phi)) = \int_{\alpha \in TS} \alpha(-)\mathrm{d}(\sigma_{TS} \circ T'\sigma_S)(\Phi)(\alpha),$$

so

$$\begin{aligned}
(\mu_S \circ \sigma_{TS} \circ T'\sigma_S)(\Phi)(A) &= \int_{\alpha \in TS} \alpha(A)\mathrm{d}(\sigma_{TS} \circ T'\sigma_S)(\Phi)(\alpha) \\
&= \int_{\alpha \in TS} \mathrm{ev}(-, A)(\alpha)\mathrm{d}\sigma_{TS}(T'\sigma_S(\Phi))(\alpha) \\
&= T'\sigma_S(\Phi)(\mathrm{ev}(-, A)) \\
&= \Phi(\mathrm{ev}(-, A) \circ \sigma_S).
\end{aligned}$$

On the other hand we have that

$$\begin{aligned}
(\sigma_S \circ \mu'_S)(\Phi)(A) &= \int_{s \in S} 1_A(s)\mathrm{d}\sigma_S(\mu'_S(\Phi))(s) \\
&= \mu'_S(\Phi)(1_A) \\
&= \Phi(\mathrm{ev}(-, 1_A)).
\end{aligned}$$

We are done if we show that

$$\mathrm{ev}(-, A) \circ \sigma_S = \mathrm{ev}(-, 1_A).$$

This is true since for each $\varphi : BS \longrightarrow \mathbb{R}$ in $T'S$,

$$\begin{aligned}
(\mathrm{ev}(-, A) \circ \sigma_S)(\varphi) &= \sigma_S(\varphi)(A) \\
&= \int_{s \in S} 1_A(s)\mathrm{d}\sigma_S(\varphi) \\
&= \varphi(1_A).
\end{aligned}$$

Fact 4. For the moment we only know that $\mathbf{T}' = (T', \eta', \mu')$ is a monad. However, from a lengthy yet straightforward categorical computation [13] it follows that this information, together with Facts 1—3 suffice for the triple $\mathbf{T} = (T, \eta, \mu)$ to be a monad too. Moreover, $\sigma : \mathbf{T}' \to \mathbf{T}$ is an isomorphism of monads.

Fact 5. It is another straightforward exercise in category theory to show that the assignment

$$I_\gamma : \mathbf{Set}^{\mathbf{T}} \longrightarrow \mathbf{Set}^{\mathbf{T}'} : \Big((X,h) \xrightarrow{f} (Y,k)\Big) \longmapsto \Big((X, h\circ\gamma_X) \xrightarrow{f} (Y, k\circ\gamma_Y)\Big) \quad (11)$$

is an isomorphism of categories.

Fact 6. The category $\mathbf{SC}$ equals $\mathbf{Alg}^{\mathbf{T}}$ by definition [3, 6].

Now we proceed as follows. Let $K : \mathbf{sNorm}_1^{\mathrm{op}} \to \mathbf{Alg}^{\mathbf{T}'}$ be the usual comparison functor. We should compose K with the concrete isomorphism I_γ given above (11), in order to get the comparison functor of the adjunction $B \dashv C$ in the $\mathbf{SC}$-representation. From the definition of the comparison functor [3] we see that $I_\gamma \circ K$ is given by

$$I_\gamma \circ K : \mathbf{sNorm}_1^{\mathrm{op}} \longrightarrow \mathbf{SC} : X \xrightarrow{f} Y \longmapsto (CY, C\epsilon'_Y \circ \gamma_{CY}) \xrightarrow{Cf} (CX, C\epsilon'_X \circ \gamma_{CX}),$$

so we are done if we show that, for every seminormed space X,

$$C\epsilon'_X \circ \gamma_{CX} = I_{CX}.$$

Hereto, fix $\alpha \in TCX$ and $x \in X$. Then we obtain that

$$\begin{aligned}(C\epsilon'_X \circ \gamma_{CX})(\alpha)(x) &= (C\epsilon'_X(\gamma_{CX}(\alpha)))(x)\\ &= (\gamma_{CX}(\alpha)\circ \epsilon'_X)(x)\\ &= \gamma_{CX}(\alpha)(\mathrm{ev}(-,x))\\ &= \int_{\varphi\in CX} \mathrm{ev}(-,x)(\varphi)\mathrm{d}\alpha(\varphi),\end{aligned}$$

which completes the proof. ∎

4. The Banach space connection

We write

$$O : \mathbf{Ban}_1 \to \mathbf{Set} : (X \xrightarrow{f} Y) \longmapsto (OX \stackrel{f|_{OX}}{\to} OY)$$

for the closed unit ball-functor. For a set S, put

$$l_1 S := \{a : S \to \mathbb{R} \mid \{a \neq 0\} \text{ is (at most) countable and } ||a||_1 < \infty\},$$

equipped with the sum-norm

$$||a||_1 := \sum_{s\in S} |a(s)|.$$

Then the functor

$$l_1 : \mathbf{Set} \to \mathbf{Ban}_1 : (S_1 \xrightarrow{g} S_2) \longmapsto (l_1 S_1 \xrightarrow{l_1 g} l_1 S_2),$$

with $l_1 g(a) := \sum_{s\in S_1} a(s)\delta^f_{g(s)}$, $a \in l_1 S_1$, is left adjoint to O [7]. The unit of the adjunction $l_1 \dashv O$ is given by

$$\eta''_S : S \hookrightarrow Ol_1 S : s \longmapsto \delta^f_s \ (S \in |\mathbf{Set}|),$$

where the Dirac function δ^f_s is given by

$$\delta^f_s(t) := \begin{cases} 1 \text{ if } s = t \\ 0 \text{ if } s \neq t \end{cases}.$$

The counit of this adjunction at a Banach space X is given by the assignment

$$\epsilon''_X : l_1 OX \to X : a \longmapsto \sum_{x\in OX} a(x)x.$$

Let $\mathbf{T}''$ be the monad induced by the adjunction $l_1 \dashv O$. We will proceed by providing an explicit description of E. By definition, an object in $\mathbf{Alg}^{\mathbf{T}''}$ is a pair (N, E_N) consisting of a set N and a structure map

$$E_N : Ol_1 N \longrightarrow N$$

that satisfies the following equations:

(TCF1) $E_N(\delta^f_x) = x$,

(TCF2) $E_N(\sum_{a\in Ol_1 N} B(a)a(-)) = E_N\left(\sum_{x\in N}\left(\sum_{\substack{a\in Ol_1 N\\ x=E_N(a)}} B(a)\right)\delta^f_x(-)\right)$.

4.1. Definition. [7] *A* **totally convex structure** *is a set M together with, for each $\alpha = (\alpha_i)_{i\in\mathbb{N}_0} \in Ol_1\mathbb{N}_0$, an operation $\widehat{\alpha} : M^{\mathbb{N}_0} \to M$, such that, with the notation*

$$\sum_{i=1}^{\infty} \alpha_i x_i := \widehat{\alpha}((x_i)_{i\in\mathbb{N}_0}),$$

the following identities are satisfied:

(TC1) $\sum_{i=1}^\infty \delta^f_i(k) x_i = x_k$,

(TC2) $\sum_{j=1}^\infty \beta_j(\sum_{i=1}^\infty \alpha_{ij}x_i) = \sum_{i=1}^\infty(\sum_{j=1}^\infty \beta_j\alpha_{ij})x_i$.

A totally affine map $f : M \to N$ between totally convex modules is a map between the underlying sets such that, for all $(\alpha_i)_i \in Ol_1\mathbb{N}_0$ and for all $(x_i)_i \in M^{\mathbb{N}_0}$,

$$f(\sum_{i=1}^{\infty} \alpha_i x_i) = \sum_{i=1}^{\infty} \alpha_i f(x_i).$$

The category of totally convex modules is denoted by $\mathbf{TC}$ *and we put* $U : \mathbf{TC} \longrightarrow \mathbf{Set}$ *for the forgetful functor.*

The closed unit ball of a Banach space X is in a pointwise way a totally convex module, which is denoted $\widehat{O}X$. Moreover, if $f : X \to Y$ is a linear non–expansive map between Banach spaces, then $f|_{OX} : \widehat{O}X \to \widehat{O}Y$ is a totally affine map between totally convex modules.

Let M be a totally convex module. We put

$$E_M(a) := \sum_{i=1}^{\infty} a(x_i)x_i, \ \ (a \in Ol_1 M) \tag{12}$$

where $(x_i)_{i\in\mathbb{N}_0}$ is a sequence in M such that $\{x_i \mid i \in \mathbb{N}_0\}$ contains the support of a.

Conversely, let (N, E_N) be in $\mathbf{Alg}^{\mathbf{T}''}$. Then we define

$$\sum_{i=1}^{\infty} \alpha_i x_i := E_N(a), \tag{13}$$

where for every $(\alpha_i)_{i\in\mathbb{N}_0}$ such that $\sum_{i=1}^{\infty} |\alpha_i| \leq 1$ and for every $(x_i)_i \in N^{\mathbb{N}_0}$, we put

$$a(x) := \begin{cases} 0 & \text{if } x \notin \{x_n \mid n \in \mathbb{N}_0\} \\ \sum_{i:x_i=x} \alpha_i & \text{otherwise.} \end{cases}$$

We now have the following result.

4.2. Theorem. *[7] The correspondences $M \longmapsto (M, E_M)$ defined in (12) extends to a concrete isomorphism $\mathbf{TC} \simeq \mathbf{Alg}^{\mathbf{T}''}$, the inverse of which is described in (13). Moreover, $\widehat{O} : \mathbf{Ban}_1 \to \mathbf{TC}$ is the respective comparison functor.*

Another important issue in the theory of totally convex modules is the fact that $\widehat{O}$ has a left adjoint $S : \mathbf{TC} \to \mathbf{Ban}_1$, such that

$$S \circ \widehat{O} \simeq \mathrm{id}_{\mathbf{Ban}_1}. \tag{14}$$

For a more detailed account, we refer to [7].

Note that there is a dualisation functor into the category of Banach spaces:

$$L_C : \mathbf{sNorm}_1^{\mathrm{op}} \longrightarrow \mathbf{Ban}_1 : (X \xrightarrow{f} Y) \longmapsto (L_C Y \xrightarrow{L_C f} L_C X),$$

with $L_C f(\varphi) := \varphi \circ f \ (\varphi \in L_C Y)$. Thus, neglecting the dashed arrow, there is a commutative diagram

$$\begin{tikzcd} \mathbf{Ban}_1 \arrow[r, "\widehat{O}"] & \mathbf{TC} \arrow[dr, "U"] & \\ \mathbf{sNorm}_1^{\mathrm{op}} \arrow[u, "L_C"] \arrow[r, "\widehat{C}"'] & \mathbf{SC} \arrow[u, dotted, "E"'] \arrow[r, "V"'] & \mathbf{Set}, \end{tikzcd} \tag{15}$$

with $V : \mathbf{SC} \to \mathbf{Set}$ the canonical forgetful functor. Suppose that we could find a concrete functor $E : \mathbf{SC} \to \mathbf{TC}$ with the additional property that the square formed in diagram (15) commutes, then from (14) we see that

$$L_C \simeq S \circ E \circ \widehat{C}.$$

In other words, such E would yield a factorization of the dualisation L_C via the natural dual algebraic structure. The sequel of this section is devoted to the comparison of $\mathbf{SC}$, the algebraic theory of dual unit balls, with $\mathbf{TC}$, the algebraic component of Banach spaces.

Let (M, I_M) be a space of charges and fix $a \in Ol_1 M$. Then we put

$$\overline{a} := \sum_{x \in M} a(x)\delta_x \in TM.$$

As the closed unit ball of $\mathrm{ba}(M)$, TM carries a natural totally convex structure, so this assignment is well defined. Now we define

$$E^{I_M}(a) := I_M(\overline{a}),$$

so we have a map $E^{I_M} : Ol_1 M \longrightarrow M$.

4.3. Theorem. *The pair (M, E^{I_M}) is a totally convex module. Moreover, the assignment $(M, I_M) \longmapsto (M, E^{I_M})$ defines a concrete functor*

$$E : \mathbf{SC} \to \mathbf{TC}$$

such that the square formed in the diagram (15) commutes.

Proof. First we verify that (M, E^{I_M}) satisfies the axioms for the formal representation of a totally convex module.

(TCF1) follows trivially from (SC1).

In order to obtain (TCF2), fix $B \in (Ol_1)^2 M$ and define

$$\beta := \sum_{a \in Ol_1 M} B(a)\delta_{\overline{a}} \in T^2 M.$$

We then obtain that

$$\begin{aligned} \overline{\sum_{a \in Ol_1 M} B(a)a} &= \sum_{x \in M} \sum_{a \in Ol_1 M} B(a)a(x)\delta_x \\ &= \sum_{a \in Ol_1 M} B(a)(\sum_{x \in M} a(x)\delta_x) \\ &= \sum_{a \in Ol_1 M} B(a)\overline{a} \\ &= \int_{\alpha \in TM} \alpha(-)\mathrm{d}\beta(\alpha). \end{aligned}$$

We also have that for all $A \in 2^M$

$$\begin{aligned}
TI_M(\beta)(A) &= \beta(I_M^{-1}(A)) \\
&= \sum_{a \in Ol_1 M} B(a)\delta^{c}_{\overline{a}}(I_M^{-1}(A)) \\
&= \sum_{x \in M} \sum_{\substack{a \in Ol_1 M \\ I_M(\overline{a}) = x}} B(a)\delta_{\overline{a}}(I_M^{-1}(A)) \\
&= \sum_{x \in M} \sum_{\substack{a \in Ol_1 M \\ I_M(\overline{a}) = x}} B(a)\delta_x(A) \\
&= \sum_{x \in M} \Big(\sum_{\substack{a \in Ol_1 M \\ E^{I_M}(a) = x}} B(a)\Big)\delta_x(A).
\end{aligned}$$

Hence

$$\begin{aligned}
E^{I_M}\Big(\sum_{a \in Ol_1 M} B(a)a\Big) &= I_M\Big(\overline{\sum_{a \in Ol_1 M} B(a)a}\Big) \\
&= I_M\left(\int_{\alpha \in TM} \alpha(-)\mathrm{d}\beta(\alpha)\right) \\
&= I_M(TI_M(\beta)) \\
&= I_M\Big(\sum_{x \in M} \sum_{\substack{a \in Ol_1 M \\ E^{I_M}(a) = x}} B(a)\delta_x\Big) \\
&= I_M\Big(\overline{\sum_{x \in M} \Big(\sum_{\substack{a \in Ol_1 M \\ E^{I_M}(a) = x}} B(a)\Big)\delta^f_x}\Big) \\
&= E^{I_M}\left(\sum_{x \in M} \Big(\sum_{\substack{a \in Ol_1 M \\ E^{I_M}(a) = x}} B(a)\Big)\delta^f_x \right).
\end{aligned}$$

To finish the proof, we have to check that for an **SC**–morphism $f : (M, I_M) \longrightarrow (N, I_N)$, automatically $f : (M, E^{I_M}) \longrightarrow (N, E^{I_N})$ is a **TC**–morphism. That is, from the commutation of the diagram

$$\begin{tikzcd}
TM \arrow[r, "I_M"] \arrow[d, "Tf"'] & M \arrow[d, "f"] \\
TN \arrow[r, "I_N"'] & N
\end{tikzcd}$$

we have to show the commutation of

$$\begin{tikzcd} Ol_1M \arrow[r, "E^{I_M}"] \arrow[d, "Ol_1f"'] & M \arrow[d, "f"] \\ Ol_1N \arrow[r, "E^{I_N}"'] & N \end{tikzcd}.$$

Take $a \in Ol_1M$. Then

$$\begin{aligned}(f \circ E^{I_M})(a) &= (f \circ I_M)(\overline{a}) \\ &= I_N(Tf(\overline{a})) \\ &= I_N(\overline{Ol_1f(a)}) \\ &= (E^{I_N} \circ Ol_1f)(a),\end{aligned}$$

where the last but one equality is true because, as one easily verifies, we have $Tf(\overline{a}) = \overline{Ol_1f(a)}$. ■

4.4. Corollary. *If (M, I_M) and (N, I_N) are isomorphic spaces of charges then (M, E_{I_M}) and (N, E_{I_N}) are isomorphic totally convex structures.*

From the above remark, in combination with the following theorem, it is noted that the **SC** theory is of strictly stronger nature than the theory of totally convex modules.

4.5. Theorem. *The categories* **SC** *and* **TC** *(with their canonical forgetful functors) are* not *concretely isomorphic.*

Proof. It is well–known from the general theory of monads (see e.g. [3, 6]) that the assignment $FS := (TS, \mu_S)$ $(S \in |\mathbf{Set}|)$ defines a functor $F : \mathbf{Set} \longrightarrow \mathbf{SC}$ that is left adjoint to $V : \mathbf{SC} \longrightarrow \mathbf{Set}$. Now suppose **SC** and **TC** were concretely isomorphic. Since adjunctions are determined up to natural isomorphism, this would imply that the underlying sets of the free **TC**–object on $\mathbb{R}$ (i.e. $\widehat{O}l_1\mathbb{R}$) and the free **SC**–object on $\mathbb{R}$ (i.e. $F\mathbb{R}$) would have the same cardinality. Now on the one hand we see that

$$\#Ol_1\mathbb{R} = \#l_1\mathbb{R} \leq \#(\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}) = \#\mathbb{R}.$$

On the other hand, we can define for every ultrafilter $\mathcal{U}$ on $\mathbb{R}$ a charge $\alpha_{\mathcal{U}}$ that is an element of $T\mathbb{R}$ by

$$\alpha_{\mathcal{U}}(A) := \begin{cases} 1 & \text{if } A \in \mathcal{U}, \\ 0 & \text{otherwise,} \end{cases}$$

and it is easy to see that $\alpha_{\mathcal{U}} \neq \alpha_{\mathcal{V}}$ if $\mathcal{U} \neq \mathcal{V}$. If $\mathbb{R}$ is equipped with the discrete topology, it therefore follows from [2], Theorem 9.2, that

$$\#T\mathbb{R} = \#O\mathrm{ba}(\mathbb{R}, 2^{\mathbb{R}}) \geq \#\beta(\mathbb{R}) = \#2^{2^{\mathbb{R}}} > \#2^{\mathbb{R}}.$$

Hence $\#Ol_1\mathbb{R} < \#T\mathbb{R}$, yielding a contradiction. ■

It is always nice to have the dual of a normed space represented by a concrete Banach space. This representation puts, often in a canonical way, an **SC**–structure on the closed unit ball of that Banach space. If we apply the forgetful functor E, we see from the commutation of diagram (15) that we then recover the pointwise **TC**–structure on the closed unit ball of the Banach space. As an example we reconsider the representation theorem 2.1 in this context.

4.6. THEOREM. *The pair* $(O\mathrm{ba}(S,\mathcal{F}), \int_{O\mathrm{ba}(S,\mathcal{F})})$ *is an* **SC**–*object and*

$$\gamma_{(S,\mathcal{F})} : (O\mathrm{ba}(S,\mathcal{F}), \int_{O\mathrm{ba}(S,\mathcal{F})}) \longrightarrow \widehat{C}B(S,\mathcal{F})$$

is an **SC**–*isomorphism.*

PROOF. We only have to show that $\gamma_{(S,\mathcal{F})}$ is an **SC**–morphism, that is, we have to show the commutation of the diagram

$$\begin{tikzcd}
TO\mathrm{ba}(S,\mathcal{F}) \arrow[r, "\int_{O\mathrm{ba}(S,\mathcal{F})}"] \arrow[d, "T\gamma_{(S,\mathcal{F})}"'] & O\mathrm{ba}(S,\mathcal{F}) \arrow[d, "\gamma_{(S,\mathcal{F})}"] \\
TCB(S,\mathcal{F}) \arrow[r, "I_{CB(S,\mathcal{F})}"'] & CB(S,\mathcal{F}).
\end{tikzcd}$$

Take $\beta \in TO\mathrm{ba}(S,\mathcal{F})$ and $a \in B(S,\mathcal{F})$. Then on the one hand we have that

$$\begin{aligned}
& \left(I_{CB(S,\mathcal{F})}(T\gamma_{(S,\mathcal{F})}(\beta))\right)(a) && (16)\\
= & \left(I_{CB(S,\mathcal{F})}(\beta_{\gamma_{(S,\mathcal{F})}})\right)(a) && (17)\\
= & \int_{\varphi\in CB(S,\mathcal{F})} \mathrm{ev}(-,a)(\varphi)\mathrm{d}\beta_{\gamma_{(S,\mathcal{F})}}(\varphi) && (18)\\
= & \int_{\alpha\in TO\mathrm{ba}(S,\mathcal{F})} \mathrm{ev}(-,a)\circ\gamma_{(S,\mathcal{F})}(\alpha)\mathrm{d}\beta(\alpha) && (19)\\
= & \int_{\alpha\in TO\mathrm{ba}(S,\mathcal{F})} \left(\int_{s\in S} a(s)\mathrm{d}\alpha(s)\right)\mathrm{d}\beta(\alpha). && (20)
\end{aligned}$$

Calculating the other way around the diagram, we obtain that

$$\begin{aligned}
& \left((\gamma_{(S,\mathcal{F})}\circ\int_{O\mathrm{ba}(S,\mathcal{F})})(\beta)\right)(a) && (21)\\
= & \left(\gamma_{(S,\mathcal{F})}(\int_{\alpha\in O\mathrm{ba}(S,\mathcal{F})}\alpha(-)\mathrm{d}\beta(\alpha))\right)(a) && (22)\\
= & \int_{s\in S} a(s)\mathrm{d}\left(\int_{\alpha\in O\mathrm{ba}(S,\mathcal{F})}\alpha(-)\mathrm{d}\beta(\alpha)\right)(s). && (23)
\end{aligned}$$

We are therefore done if we show that (20) and (23) are equal. Let $(a_n)_n$ be a sequence of $\mathcal{F}$–simple functions, converging uniformly to a. We write $a_n = \sum_{i=1}^{m_n} a_i^n 1_{A_i^n}$, with all $a_i^n \in \mathbb{R}$ and all $A_i^n \in \mathcal{F}$ such that all $\{A_1^n, \ldots, A_{m_n}^n\}$ are a partition of S. Then

$$\begin{aligned}
& \int_{s\in S} a(s)\mathrm{d}\left(\int_{\alpha\in O\mathrm{ba}(S,\mathcal{F})} \alpha(-)\mathrm{d}\beta(\alpha)\right)(s) \\
= \; & \lim_{n\to\infty} \int_{s\in S} a_n(s)\mathrm{d}\left(\int_{\alpha\in O\mathrm{ba}(S,\mathcal{F})} \alpha(-)\mathrm{d}\beta(\alpha)\right)(s) \\
= \; & \lim_{n\to\infty} \sum_{i=1}^{m_n} a_i^n \int_{\alpha\in O\mathrm{ba}(S,\mathcal{F})} \alpha(A_i^n)\mathrm{d}\beta(\alpha) \\
= \; & \lim_{n\to\infty} \int_{\alpha\in O\mathrm{ba}(S,\mathcal{F})} \left(\sum_{i=1}^{m_n} a_i^n \alpha(A_i^n)\right) \mathrm{d}\beta(\alpha) \\
= \; & \lim_{n\to\infty} \int_{\alpha\in O\mathrm{ba}(S,\mathcal{F})} \left(\int_{s\in S} a_n(s)\mathrm{d}\alpha(s)\right) \mathrm{d}\beta(\alpha) \\
= \; & \int_{\alpha\in O\mathrm{ba}(S,\mathcal{F})} \left(\int_{s\in S} a_n(s)\mathrm{d}\alpha(s)\right) \mathrm{d}\beta(\alpha),
\end{aligned}$$

where the last step is valid because the sequence $(\alpha \longmapsto \int_{s\in S} a_n(s)\mathrm{d}\alpha(s))_n$ is uniformly convergent to $\alpha \longmapsto \int_{s\in S} a(s)\mathrm{d}\alpha(s)$. ∎

That Theorem 4.6 is indeed a strengthening of the Riesz–type representation theorem 2.1 is easily seen now since the latter can be obtained as a simple corollary using the commutative diagram 15.

4.7. Corollary. $(\mathrm{ba}(S,\mathcal{F}), \|\ \|)$ *and* $L_C B(S,\mathcal{F})$ *are isomorphic Banach spaces.*

This is of course not surprising since Corollary 4.7 has served as a starting point for our theory.

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