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PULLBACK AND FINITE COPRODUCT PRESERVING FUNCTORS BETWEEN CATEGORIES OF PERMUTATION REPRESENTATIONS: CORRIGENDUM AND ADDENDUM

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ABSTRACT. Francisco Marmolejo pointed out a mistake in the statement of Proposition 4.4 in our paper [PS]. The mistaken version is used later in that paper. Our purpose here is to correct the error by providing an explicit description of the finite coproduct completion of the dual of the category of connected G-sets. The description uses the distinguished morphisms of a factorization system on the category of G-sets.

1. Introduction

We are grateful to Francisco Marmolejo for pointing out a few minor errors in our paper [PS] but, more importantly, he pointed out a serious error which the present paper corrects by addending new material. First we attend to the minor corrections.

On page 771 we say that Lindner [Li] characterizes Mackey functors as coproduct preserving functors. In fact, he characterizes them in terms of finite product preserving functors. We should have made it clear that, in our case, the functors go between additive categories so that finite products are finite coproducts and there is no contradiction.

In the proof of Theorem 2.1 in the middle of page 774, the object K should not be the equalizer of **all** endomorphisms of P, rather only of those endomorphisms e for which $T(e)(p) = p$.

At the bottom of page 775, in the description of $\mathbf{Fam}(\mathscr{C}^{\mathrm{op}})$, the family $f = (f_i)_{i\in I}$ should consist of morphisms $f_i : D_{\xi(i)} \longrightarrow C_i$; the C and D were interchanged.

The serious error starts innocently enough as an apparent typographical mistake in Proposition 4.4 where $\mathbf{Fam}(\mathscr{C}_G^{\mathrm{op}})$ should be $\mathbf{Fam}(\mathscr{C}_G)$. The Proof following Proposition 4.4 is a correct proof of the equivalence $G\text{-set}_{\mathit{fin}} \simeq \mathbf{Fam}(\mathscr{C}_G)$. However, to obtain Corollary 4.5, the mistaken version of Proposition 4.4 is used. Strangely, many months earlier while writing our direct calculations on the Bouc Theorem and before preparing the "slick" version, we knew that this Corollary 4.5 could not be true.

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To correct the error, we require a nice description of the finite coproduct completion $\mathbf{Fam}(\mathscr{C}_G^{\mathrm{op}})$ of the dual of the category of connected G-sets. This nice category will replace G-set$_{fin}$ in Corollary 4.5 and have repercussions for Corollary 4.7, Theorem 5.1 and Theorem 5.3. To describe the category, we will use the morphisms of a factorization system on G-**set**. We are grateful to Alexei Davydov for valuable discussions on this material.

2. A factorization for G-morphisms

For any G-set X, we have the set $X/G = \{C \subseteq X : C \text{ is an orbit of } X\}$ of connected sub-G-sets of X. We have the function $orb : X \longrightarrow X/G$ taking each element $x \in X$ to its orbit $orb(x) = \{gx : g \in G\}$. Each G-morphism $f : X \longrightarrow Y$ induces a direct image function $f/G : X/G \longrightarrow Y/G$ defined by $(f/G)(C) = f_*(C)$.

A G-morphism $f : X \longrightarrow Y$ is said to be *slash inverted* when $f/G : X/G \longrightarrow Y/G$ is a bijection.

2.1. Proposition. *A G-morphism $f : X \longrightarrow Y$ is slash inverted if and only if it is surjective and $f(x_1) = f(x_2)$ implies $orb(x_1) = orb(x_2)$.*

Proof. Suppose f is slash inverted. For each $y \in Y$ there exists $x \in X$ with $f_*(orb(x)) = orb(y)$. So $f(x) = gy$ for some $g \in G$. It follows that $y = f(g^{-1}x)$, so f is surjective. If $f(x_1) = f(x_2)$ then $(f/G)(orb(x_1)) = (f/G)(orb(x_2))$; so $orb(x_1) = orb(x_2)$. For the converse, take $orb(y) \in Y/G$. Then $y = f(x)$ for some $x \in X$ and so $orb(f(x)) = orb(y)$. Also, if $orb(f(x_1) = orb(f(x_2))$, then $f(x_1) = gf(x_2) = f(gx_2)$ for some $g \in G$. So $orb(x_1) = orb(gx_2) = orb(x_2)$. ■

A G-morphism $f : X \longrightarrow Y$ is said to be *orbit injective* when $orb(x_1) = orb(x_2)$ and $f(x_1) = f(x_2)$ imply $x_1 = x_2$. Orbit injective morphisms were considered by Bouc [Bo].

2.2. Proposition. *The slash inverted and orbit injective G-morphisms form a factorization system (in the sense of [FK]) on the category of G-sets.*

Proof. To factor a G-morphism $f : X \longrightarrow Y$, construct the G-set $S = \sum_{C \in X/G} f_*(C)$ and define G-morphisms $u : X \longrightarrow S$ and $v : S \longrightarrow Y$ by

$$u(x) = f(x) \in f_*(orb(x)) \text{ and } v(y \in f_*(C)) = y.$$

Then $f = v \circ u$ while u is slash inverted and v is orbit injective.

The only other non-obvious thing remaining to prove is the diagonal fill-in property. For this, suppose $k \circ u = v \circ h$ where u is slash inverted and v is orbit injective.

If $u(x_1) = u(x_2)$ then $orb(x_1) = orb(x_2)$, so $orb(h(x_1)) = orb(h(x_2))$. Yet we also have $v(h(x_1)) = k(u(x_1)) = k(u(x_2)) = v(h(x_2))$. Since v is orbit injective, we deduce that $h(x_1) = h(x_2)$.

Since u is surjective, for each $s \in S$ there is an $x \in X$ with $u(x) = s$. By the last paragraph, the value $h(x)$ is independent of the choice of x. So we obtain a function r by defining $r(s) = h(x)$. Clearly r is a G-morphism with $r \circ u = h$ and $v \circ r = k$; and r is unique since u is surjective. ■

This factorization system has a special property.

2.3. PROPOSITION. *The pullback of a slash inverted G-morphism along an orbit injective G-morphism is slash inverted.*

PROOF. Suppose the G-morphisms $u : X \longrightarrow S$ and $v : Y \longrightarrow S$ are slash inverted and orbit injective, respectively. Let P be the pullback of u and v with projections $p : P \longrightarrow X$ and $q : P \longrightarrow Y$. We claim that q is slash inverted. It is clearly surjective so suppose that $q(x_1, y_1) = q(x_2, y_2)$ where $u(x_1) = v(y_1)$ and $u(x_2) = v(y_2)$. So $y_1 = y_2$ and $u(x_1) = u(x_2)$. Since u is slash inverted, $orb(x_1) = orb(x_2)$; so there exists $g \in G$ with $x_2 = gx_1$. Since y_1 and gy_1 are in the same orbit, the calculation

$$v(gy_1) = gv(y_1) = gu(x_1) = u(gx_1) = u(x_2) = v(y_2) = v(y_1)$$

implies that $gy_1 = y_1 = y_2$. So $g(x_1, y_1) = (x_2, y_2)$, which implies that (x_1, y_1) and (x_2, y_2) are in the same orbit. ∎

3. A new category of G-sets

For a finite group G, we write G-$\mathbf{set}_{\mathit{fin}}$ for the category of finite G-sets and G-morphisms. We write $\mathscr{C}_G$ for (a skeleton of) the category of connected finite G-sets and all G-morphisms between them. There is also the category $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ whose objects are finite G-sets and whose morphisms are isomorphism classes of spans between finite G-sets. All these categories are important for the study of Mackey functors.

However, we now wish to introduce another category $\mathscr{B}_G$ whose objects are all the finite G-sets. In fact, $\mathscr{B}_G$ is the subcategory of $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ whose morphisms are the isomorphism classes of those spans $(u, S, v) : X \longrightarrow Y$ in which $u : S \longrightarrow X$ is slash inverted and $v : S \longrightarrow Y$ is orbit injective. It follows from Proposition 2.2 and Proposition 2.3 that these particular spans are closed under span composition.

3.1. PROPOSITION. *The subcategory $\mathscr{B}_G$ of $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ is closed under finite coproducts.*

PROOF. The coproduct in $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ is that of G-$\mathbf{set}_{\mathit{fin}}$, namely, disjoint union. For G-sets X and Y, the coprojections $X + Y$ in $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ are the spans

$$(1_X, X, \mathrm{copr}_1) : X \longrightarrow X + Y \text{ and } (1_Y, Y, \mathrm{copr}_2) : Y \longrightarrow X + Y$$

which clearly yield morphisms in $\mathscr{B}_G$. Moreover, if $(u, S, v) : X \longrightarrow Z$ and $(h, T, k) : Y \longrightarrow Z$ are spans with u and h slash inverted and with v and k orbit injective then $u + h : S + T \longrightarrow X + Y$ is slash inverted and $[v, k] : S + T \longrightarrow Z$ orbit injective. So the induced span $(u + h, S + T, [v, k]) : X + Y \longrightarrow Z$ yields a morphism in $\mathscr{B}_G$. ∎

3.2. Remark. While the coproduct in $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ is also the product (since $\mathbf{Spn}(G\text{-}\mathbf{set}_{\mathit{fin}})$ is self dual), this is no longer true in $\mathscr{B}_G$.

There is a functor $\mathscr{C}_G^{\mathrm{op}} \longrightarrow \mathscr{B}_G$ taking each connected G-set to itself and each G-morphism $f : C \longrightarrow D$ between connected G-sets to the isomorphism class of the span $(f, C, 1_C) : D \longrightarrow C$; clearly f must be slash inverted since C and D each have one orbit. Using the universal property of $\mathbf{Fam}(\mathscr{C}_G^{\mathrm{op}})$ and Proposition 3.1, we obtain a finite coproduct preserving functor $\Sigma : \mathbf{Fam}(\mathscr{C}_G^{\mathrm{op}}) \longrightarrow \mathscr{B}_G$ extending $\mathscr{C}_G^{\mathrm{op}} \longrightarrow \mathscr{B}_G$.

3.3. Theorem. *The functor $\Sigma : \mathbf{Fam}(\mathscr{C}_G^{\mathrm{op}}) \longrightarrow \mathscr{B}_G$ is an equivalence of categories.*

Proof. Each object of $\mathscr{B}_G$ is a coproduct of connected G-sets so Σ is certainly essentially surjective on objects. To prove Σ fully faithful we need to use the description of $\mathbf{Fam}(\mathscr{C}_G^{\mathrm{op}})$ from the bottom of page 775 of [PS] as corrected above. The objects are finite families $(C_i)_{i \in I}$ of connected G-sets and a morphism $(\xi, f) : (C_i)_{i \in I} \longrightarrow (D_j)_{j \in J}$ consists of a function $\xi : I \longrightarrow J$ and a family $f = (f_i)_{i \in I}$ of morphisms $f_i : D_{\xi(i)} \longrightarrow C_i$. The functor Σ takes the morphism $(\xi, f) : (C_i)_{i \in I} \longrightarrow (D_j)_{j \in J}$ to the isomorphism class of the span $(u, S, v) : X \longrightarrow Y$ where $X = \sum_{i \in I} C_i$, $Y = \sum_{j \in J} D_j$, $S = \sum_{i \in I} D_{\xi(i)}$,

$$u \circ \mathrm{copr}_i = \mathrm{copr}_i \circ f_i \text{ and } v \circ \mathrm{copr}_i = \mathrm{copr}_{\xi(i)}.$$

Notice that u induces a bijection $S/G \longrightarrow X/G$ and that both of these sets are isomorphic to I. It is then clear that u is slash inverted and v is orbit injective. Yet this process can be inverted as follows. Given any span $(u, S, v) : X \longrightarrow Y$ for the same X and Y, with u slash inverted and v orbit injective, the direct image $v_* u^*(C_i)$ of the inverse image $u^*(C_i)$ of C_i must be an orbit $D_{\xi(i)}$ of Y. This defines a function ξ while f_i is the composite of the restriction of u to $u^*(C_i)$ with the inverse of the isomorphism $u^*(C_i) \cong v_* u^*(C_i)$ induced by v. ∎

The correct version of Corollary 4.5 of [PS] is

3.4. Corollary. *There is an equivalence*

$$\mathscr{B}_G \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{\mathit{fin}}, \mathbf{set}_{\mathit{fin}})$$

taking the left G-set C to the functor

$$\sum_{w \in C/G} G\text{-}\mathbf{set}_{\mathit{fin}}(C_w, -)$$

where C_w is the orbit w as a sub-G-set of C.

The correct version of Corollary 4.7 is

3.5. Corollary. *There is an equivalence*

$$\mathscr{B}_{G^{\mathrm{op}}} \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{\mathit{fin}}, \mathbf{set}_{\mathit{fin}}), \quad A \longmapsto A \circ_G -,$$

which on morphisms, takes a span (u, S, v) from A to B with u slash inverting and v orbit injective, to the natural transformation whose component at X is the function $A \circ_G X \longrightarrow B \circ_G X$ taking $[a, x]$ to $[v(s), x]$ where $u(s) = a$.

4. The bicategory of finite groups

The definition of the bicategory **Bouc** in Section 5 of [PS] must be modified in the light of the above corrections to Corollaries 4.5 and 4.7.

For any finite monoid H and any category $\mathscr{X}$ with finite coproducts, there is a monad $H\cdot$ on $\mathscr{X}$ whose underlying endofunctor is defined by $H \cdot X = \sum_H X$ (the coproduct of H copies of X); the unit and multiplication are induced in the obvious way by the unit and multiplication of H. The category $\mathscr{X}^{H\cdot}$ of Eilenberg-Moore algebras for the monad is none other than the functor category $[H, \mathscr{X}]$ where H is regarded as a category with one object and with morphisms the elements of H. We are interested in the particular case where H is a finite group and $\mathscr{X} = \mathscr{B}_{G^{\mathrm{op}}}$.

Define $\mathbf{Bouc}(G, H)$ to be the category obtained as the pullback of the inclusion of $\mathscr{B}_{G^{\mathrm{op}}}$ in $\mathbf{Spn}(G^{\mathrm{op}}\text{-}\mathbf{set}_{\mathit{fin}})$ along the forgetful functor $\mathbf{Spn}(G^{\mathrm{op}} \times H\text{-}\mathbf{set}_{\mathit{fin}}) \longrightarrow \mathbf{Spn}(G^{\mathrm{op}}\text{-}\mathbf{set}_{\mathit{fin}})$. That is, $\mathbf{Bouc}(G, H)$ is the subcategory of $\mathbf{Spn}(G^{\mathrm{op}} \times H\text{-}\mathbf{set}_{\mathit{fin}})$ consisting of all the objects yet, as morphisms, only the isomorphism classes of spans (u, S, v) in $G^{\mathrm{op}} \times H\text{-}\mathbf{set}_{\mathit{fin}}$ for which u is slash inverted and v is orbit injective as G-morphisms.

There is an isomorphism of categories $\Gamma : \mathbf{Bouc}(G, H) \longrightarrow [H, \mathscr{B}_{G^{\mathrm{op}}}]$ defined as follows. For each $G^{\mathrm{op}} \times H$-set A, the left action by H provides injective right G-morphisms $h : A \longrightarrow A$ for all $h \in H$; so the isomorphism class of the span $(1_A, A, h) : A \longrightarrow A$ is a morphism in $\mathscr{B}_{G^{\mathrm{op}}}$. So the right G-set underlying A becomes a left H-object ΓA in $\mathscr{B}_{G^{\mathrm{op}}}$. Conversely, each left H-object X in $\mathscr{B}_{G^{\mathrm{op}}}$ has, for each $h \in H$, an invertible morphism $[u_h, M_h, v_h] : X \longrightarrow X$ in $\mathscr{B}_{G^{\mathrm{op}}}$; it follows that u_h and v_h are invertible and so $[u_h, M_h, v_h] = [1_X, X, w_h]$ where the $w_h : X \longrightarrow X$ define a left H-action on X making it a $G^{\mathrm{op}} \times H$-set A with $\Gamma A = X$. That Γ is fully faithful should now be obvious.

The correct form of Theorem 5.1 of [PS] is:

4.1. Theorem. *There is an equivalence of categories*

$$\mathbf{Bouc}(G, H) \simeq \mathbf{CopPb}(G\text{-}\mathbf{set}_{\mathit{fin}}, H\text{-}\mathbf{set}_{\mathit{fin}}), \quad A \longmapsto A \circ_G -.$$

The correct form of the first sentence of Theorem 5.3 of [PS] is:

4.2. Theorem. *There is a bicategory* **Bouc** *whose objects are finite groups, whose hom-categories are the categories*

$$\mathbf{Bouc}(G, H)$$

and whose composition functors are

$$\mathbf{Bouc}(G, H) \times \mathbf{Bouc}(K, G) \longrightarrow \mathbf{Bouc}(K, H), \quad (A, B) \longmapsto A \circ_G B.$$

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