Theory and Applications of Categories, Vol. 18, No. 6, 2007, pp. 157–171.

CCD LATTICES IN PRESHEAF CATEGORIES

G. S. H. CRUTTWELL, F. MARMOLEJO AND R. J. WOOD

ABSTRACT. In this paper we give a characterization of constructively completely distributive (CCD) lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, for $\mathbf{C}$ a small category with pullbacks.

1. Introduction

Recall that an ordered set L is a sup lattice if and only if the down-segment embedding of L into its complete lattice of down-sets, $\downarrow : L \to DL$, has a left adjoint $\bigvee : DL \to L$. It is well-known that a sup lattice L is a locale, meaning that binary meet distributes over arbitrary sup, if and only if $\bigvee : DL \to L$ preserves binary meets. Since [Fawcett and Wood, 1990] a complete lattice has been said to be constructively completely distributive (CCD) if $\bigvee : DL \to L$ preserves *all* infima, equivalently, if $\bigvee : DL \to L$ has itself a left adjoint. Classically, (CCD) is equivalent to ordinary complete distributivity (CD). In fact, it was shown in [Fawcett and Wood, 1990] that

$$(\mathrm{AC}) \iff ((\mathrm{CD}) \iff (\mathrm{CCD}))$$

and thus, (CCD) is an appropriate version of complete distributivity in an elementary topos. It is therefore natural to seek characterizations of (CCD) lattices in familiar toposes.

We envision our present results, though not our proof techniques, in the tradition of [Joyal and Tierney, 1984], wherein $\mathbf{sup}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$, the category of sup lattices and sup-preserving arrows with respect to the topos of presheaves $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, for $\mathbf{C}$ small with pullbacks, is given in terms of functors $L : \mathbf{C}^{\mathrm{op}} \to \mathbf{sup}$. Here $\mathbf{sup}$ is the category of sup-lattices and sup-preserving arrows with respect to the base topos $\mathbf{set}$. We recall the statement of their result (Proposition 1, Section 2, Chapter VI, op. cit.):

> A sup-lattice L in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ is a functor $L : \mathbf{C}^{\mathrm{op}} \to \mathbf{sup}$ such that for every $f : B \to C$ in $\mathbf{C}$, $Lf : LC \to LB$ has a left adjoint $\bigvee_f : LB \to LC$, and these

The second author acknowledges the financial support received from DGAPA of UNAM that made his sabbatical stay at Dalhousie University possible, and also the hospitality of this latter institution. The third author thanks NSERC Canada for financial support.

Received by the editors 2006-10-09 and, in revised form, 2007-03-27.

Transmitted by Susan Niefield. Published on 2007-03-29.

2000 Mathematics Subject Classification: 18B35, 06D10, 06B23.

Key words and phrases: Constructive complete distributivity, presheaves, Beck-Chevalley condition, Frobenius reciprocity, change of base.

adjunctions satisfy the Beck-Chevalley condition with respect to pullbacks. An arrow $\alpha : L \to M$ in $\mathbf{sup}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$ is a natural transformation between such functors such that, for each $f : B \to C$ in $\mathbf{C}$,

$$\begin{tikzcd} LB \arrow[r, "\alpha B"] \arrow[d, "\bigvee_f"'] & MB \arrow[d, "\bigvee_f"] \\ LC \arrow[r, "\alpha C"'] & MC, \end{tikzcd}$$

commutes.

Observe too that because the $Lf : LC \to LB$, for $f : B \to C$ in $\mathbf{C}$, are shown to take values in $\mathbf{sup}$, it follows that for each f there is also an adjunction $Lf \dashv \bigwedge_f$ (and these adjunctions necessarily satisfy the Beck-Chevalley condition too).

Thus, we also assume throughout that $\mathbf{C}$ has pullbacks and we characterize CCD lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ in terms of functors $L : \mathbf{C}^{\mathrm{op}} \to \mathbf{ccd}$, where $\mathbf{ccd}$ is the category of constructively completely distributive (CCD) lattices, with respect to $\mathbf{set}$, and functions that preserve both suprema and infima. The characterization is surprising in the appearance of a *right* adjoint for each $\bigwedge_f$. While an ordered object L in a topos has a left adjoint, $\bigvee$, for $\downarrow\, : L \to DL$ if and only if it has a right adjoint, $\bigwedge$, for $\uparrow\, : L \to UL$, where UL is the complete lattice of up-subobjects of L ordered by reverse inclusion, this symmetry does not extend to further adjoints. For $\bigwedge : UL \to L$ to have a right adjoint is for L^{op} to be (CCD), a condition which in terms of L is called ($^{\mathrm{op}}$CCD). In [Rosebrugh and Wood, 1991] it was shown that

$$(\mathrm{BLN}) \iff ((\mathrm{CCD}) \iff ({}^{\mathrm{op}}\mathrm{CCD}))$$

where (BLN) is the Boolean axiom for the topos under consideration. Of course (BLN) does not generally hold for the toposes $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$.

These extra adjunctions do not suffice. A generalization of Frobenius reciprocity is needed to construct left adjoints to the components $\bigvee_C : DL(C) \to LC$ of the natural transformation $\bigvee : DL \to L$ and a further distributivity condition is required to show that such left adjoints constitute the components of a natural transformation $L \to DL$.

This characterization was begun by Cruttwell in his thesis [G. S. H. Cruttwell, 2005]. There we find a generalization of the Frobenius reciprocity law called *wide Frobenius reciprocity*. As explained there, the requirement needed to express this condition is for $\mathbf{C}$ to have wide pullbacks. It is noted there that this is a very strong condition: for $\mathbf{C}$ small with a terminal object, the existence of wide pullbacks implies that $\mathbf{C}$ is a poset. More importantly, the characterization theorem, even in this case, is incomplete, as the example $\mathbf{2} \to \mathbf{1}$ in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, for $\mathbf{C} = \{0 \to 1\}$, shows (see below).

We duly give a new version of this condition, that we call *complete Frobenius reciprocity*, which does not require wide pullbacks, and add a further condition, along with the afore-mentioned right adjoints to the $\bigwedge_f$, which enable us to characterize CCD lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$.

2. Comparison transformation for evaluation

We will write L generically for an object of $\mathbf{ord}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$. The object of down-subobjects of L is then $DL\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{set}$ given, for $C \in \mathbf{C}$, by

$$DL(C) = \{F \hookrightarrow L \times \mathbf{C}(-,C) | (b,f) \in FB \text{ and } b' \leq b \text{ imply } (b',f) \in FB\}$$

while, for $f\colon B \to C$ in $\mathbf{C}$, $DL(f)$ is given by pulling back along $L \times \mathbf{C}(-,f)$. The arrow $\downarrow\, : L \to DL$ is given by

$$\downarrow_C c\,(B) = \{(b,f) | b \leq Lf(c)\}$$

for c in $L(C)$.

Any left exact functor $\Gamma\colon \mathcal{E} \to \mathcal{S}$ between toposes gives rise to a natural transformation $\gamma\colon \Gamma D_{\mathcal{E}} \to D_{\mathcal{S}}\Gamma$, where the D's are considered to be functorial via inverse image. For L in $\mathbf{ord}(\mathcal{E})$, γ_L classifies the order ideal $\Gamma DL \to \Gamma L$ obtained by applying Γ to the order ideal $\downarrow^+\, : DL \to L$ which is right adjoint to the map $\downarrow\, : L \to DL$ considered as an order ideal. (Note that for x in L and S in DL, one has $x \downarrow^+ S$ if and only if $x \in S$.) This natural transformation was introduced in [Rosebrugh and Wood, 1992] where it was called the 'logical comparison transformation'. In the case at hand, and taking $\Gamma\colon \mathbf{set}^{\mathbf{C}^{\mathrm{op}}} \to \mathbf{set}$ to be evaluation at C, we have $\gamma = \gamma_L : DL(C) \to D(LC)$, where $D(LC)$ is the set of down-sets of the ordered set LC. It is not hard to show that, for $F \in DL(C)$,

$$\gamma(F) = \{c \in LC | (c, 1_C) \in FC\}$$

2.1. Lemma. *The component* $\gamma\colon DL(C) \to D(LC)$ *has a left adjoint* γ^* *and a right adjoint* γ_*.

Proof. It is not hard to see that the left adjoint γ^* is given by

$$\gamma^*(S)(B) = \{(b,f) | \text{there is } c \in S \text{ with } b \leq Lf(c)\}$$

for any $S \in D(LC)$, and that the right adjoint is given by

$$\gamma_*(S)(B) = \{(b,f) | \text{for every } s\colon C \to B \text{ with } fs = 1_C, Ls(b) \in S\}.$$

Observe that $\gamma^* \dashv \gamma \dashv \gamma_*$ is a fully faithful adjoint string. ∎

We will need the following property of γ^*:

2.2. Lemma. $\gamma^* \downarrow_{LC} = \downarrow_C$.

Proof. For $c \in LC$ and $B \in \mathbf{C}$ we have

$$\begin{aligned}
(\gamma^* \downarrow_{LC} (c))(B) &= \{(b,f) | b \leq Lf(c') \text{ for some } c' \in \downarrow_{LC} (c)\} \\
&= \{(b,f) | b \leq Lf(c') \text{ for some } c' \leq c\} \\
&= \{(b,f) | b \leq Lf(c)\} = \downarrow_C (c)(B).
\end{aligned}$$

∎

For $f : B \to C$ in $\mathbf{C}$, the left adjoint $\bigvee_f^{DL}$ of $DL(f)$ is given by the image

$$\begin{tikzcd} G \arrow[r, hook] \arrow[d] & L \times \mathbf{C}(-,B) \arrow[d, "L\times\mathbf{C}(-,f)"] \\ \bigvee_f^{DL}(G) \arrow[r, hook] & L \times \mathbf{C}(-,C) \end{tikzcd}$$

for any $G \in DL(B)$. In what follows, the elements in the image of

$$LB \xrightarrow{\downarrow_B} DL(B) \xrightarrow{\bigvee_f^{DL}} DL(C)$$

will become important, so we give them a name:

2.3. Definition. *For $f : B \to C$ in $\mathbf{C}$ and $b \in LB$, we define*

$$\downarrow_C (b,f) = \bigvee_f^{DL}(\downarrow_B b).$$

Therefore

$$\downarrow_C (b,f)(A) = \{(a,h) | \text{there is } g : A \to B \text{ such that } h = fg \text{ and } a \le Lg(b)\}$$

Observe that this extends our existing notation, in the sense that $\downarrow_B b = \downarrow_B (b, 1_B)$. The main properties that we need of these objects are given by the following lemma and corollary:

2.4. Lemma. *For $f : B \to C$ in $\mathbf{C}$ and $b \in LB$, $\downarrow_C (b,f)$ is the smallest element $F \in DL(C)$ such that $(b,f) \in FB$.*

Proof. Clearly $(b,f) \in (\downarrow_C (b,f))(B)$. Assume now that $F \in DL(C)$ and that $(b,f) \in FB$. If $(y,k) \in (\downarrow_C (b,f))(Y)$, then we can find $s : Y \to B$ such that $k = fs$ and $y \le Ls(b)$. Since $(b,f) \in FB$, we have $(Ls(b),k) = (Ls(b),fs) \in FY$. This together with $y \le Ls(b)$ imply that $(y,k) \in FY$. ∎

2.5. Corollary. *For any $C \in \mathbf{C}$ and $F \in DL(C)$,*

$$F = \bigcup\{\downarrow_C (b,f) | B \in \mathbf{C}, (b,f) \in FB\}.$$

∎

We turn now to the possibility of a left adjoint to γ^*.

2.6. Proposition. *Assume that LC is complete. The map γ^* has a left adjoint $\gamma_!$ if and only if for every $f : B \to C$ in $\mathbf{C}$, Lf has a left adjoint $\bigvee_f$.*

PROOF. Assume that for every $f : B \to C$ we have $\bigvee_f \dashv Lf$. Define

$$\gamma_!(F) = \{c \in LC | c \leq \bigvee_f(b) \text{ for some } B \in \mathbf{C} \text{ and some } (b, f) \in FB\}.$$

Then

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{F \subseteq \gamma^*(S)}{\text{for every } B,\quad FB \subseteq \gamma^*(S)(B)}}{\text{for every } B,\quad \text{if } (b,f) \in FB, \text{ then there is } c \in S \text{ with } b \leq Lf(c)}}{\text{for every } B,\quad \text{if } (b,f) \in FB, \text{ then there is } c \in S \text{ with } \bigvee_f(b) \leq c}}{\text{for every } B,\quad \text{if } (b,f) \in FB, \text{ then } \bigvee_f(b) \in S}}{\gamma_!(F) \subseteq S,}$$

where we use the adjunction $\bigvee_f \dashv Lf$ from the third to the fourth line, and the fact that S is downclosed on the next one.

Assume now that $\gamma_! \dashv \gamma^*$ and take $f : B \to C$ in $\mathbf{C}$. Define $\bigvee_f$ as the composite

$$LB \xrightarrow{\downarrow_B} DL(B) \xrightarrow{\bigvee_f^{DL}} DL(C) \xrightarrow{\gamma_!} D(LC) \xrightarrow{\bigvee_{LC}} LC,$$

where $\bigvee_{LC} \dashv \downarrow_{LC} : LC \to D(LC)$. For $b \in LB$ and $c \in LC$ we have

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{\bigvee_f(b) \leq c}{\bigvee_{LC}\gamma_!\bigvee_f^{DL} \downarrow_B b \leq c}}{\bigvee_f^{DL} \downarrow_B b \leq \gamma^* \downarrow_{LC} (c)}}{\downarrow_C (b,f) \leq \downarrow_C (c)}}{(b,f) \in \downarrow_C (c)(B)}}{b \leq Lf(c),}$$

where we have applied first the definition of $\bigvee_f$, then the adjunction $\bigvee_{LC}\gamma_! \dashv \gamma^* \downarrow_{LC}$, the definition of $\downarrow_C (b,f)$, Lemma 2.2 and Lemma 2.4. ∎

2.7. LEMMA. *If LC is a sup lattice and $\gamma_!$ exists, then $\bigvee_{LC}\gamma_! \dashv \downarrow_C$. In particular, if L is a sup lattice in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, then $\bigvee_C = \bigvee_{LC}\gamma_!$.*

PROOF. We have $\bigvee_{LC} \dashv \downarrow_{LC}$ and $\gamma_! \dashv \gamma^*$, thus $\bigvee_{LC}\gamma_! \dashv \gamma^* \downarrow_{LC} = \downarrow_C$, by Lemma 2.2. ∎

2.8. REMARK. When $L \in \sup(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$, the formula for $\bigvee_f$ given in the proof of Proposition 2.6 becomes

$$LB \xrightarrow{\downarrow_B} DL(B) \xrightarrow{\bigvee_f^{DL}} DL(C) \xrightarrow{\bigvee_C} LC.$$

3. A new perspective on an old theorem

We use the adjoints from the previous section to give a different proof of Proposition 1, Section 2, Chapter VI of [Joyal and Tierney, 1984]:

3.1. Theorem. *An object $L \in \mathbf{sup}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$ is a functor $L\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{sup}(\mathbf{set})$ such that for every $f\colon B \to C$ in $\mathbf{C}$, Lf has a left adjoint $\bigvee_f \dashv Lf$, and the Beck-Chevalley condition with respect to pullbacks is satisfied for these adjoints.*

We begin by assuming $L \in \mathbf{sup}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$ and deriving the conditions. First of all we observe that when L is a sup lattice, we have

$$\begin{tikzcd} LC \arrow[r, shift right, "\downarrow_C"'] \arrow[r, phantom, "\perp"] \arrow[rr, bend right, "\downarrow_{LC}"'] & DL(C) \arrow[l, shift right, "\bigvee_C"'] \arrow[r, shift right, "\gamma"'] \arrow[r, phantom, "\perp"] & D(LC), \arrow[l, shift right, "\gamma^*"'] \end{tikzcd} \tag{1}$$

We know from [Rosebrugh and Wood, 1992] (or directly) that $\gamma \downarrow_C = \downarrow_{LC}$. Thus $\bigvee_C \gamma^*$ is a left adjoint to $\downarrow_{LC}$, and LC turns out to be a sup lattice.

It is easy to see that for every $f\colon B \to C$ in $\mathbf{C}$, the square on the left in the diagram

$$\begin{tikzcd} D(LC) \arrow[r, "\gamma^*"'] \arrow[d, "D(Lf)"'] \arrow[rr, bend left, "\bigvee_{LC}"] & DL(C) \arrow[r, "\bigvee_C"'] \arrow[d, "DL(f)"'] & LC \arrow[d, "Lf"] \\ D(LB) \arrow[r, "\gamma^*"] \arrow[rr, bend right, "\bigvee_{LB}"'] & DL(B) \arrow[r, "\bigvee_B"] & LB \end{tikzcd}$$

commutes. Since the square on the right also commutes, we have that Lf preserves sups or, equivalently, that Lf has a right adjoint $Lf \dashv \bigwedge_f$. Thus, $L\colon \mathbf{C}^{\mathrm{op}} \to \mathbf{sup}(\mathbf{set})$ is a functor. A dual argument with upclosed sets instead of downclosed sets, and the general remark that every sup lattice is an inf lattice, gives us that Lf has a left adjoint $\bigvee_f \dashv Lf$.

We are thus left with the Beck-Chevalley condition. For this we need:

3.2. Lemma. *For any $L \in \mathbf{ord}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$, the adjunctions $\bigvee_f^{DL} \dashv DL(f)\colon DL(C) \to DL(B)$ with $f\colon B \to C$ in $\mathbf{C}$, satisfy the Beck-Chevalley condition.*

Proof. Take a pullback diagram

$$\begin{tikzcd} P \arrow[r, "r"] \arrow[d, "q"'] & B \arrow[d, "f"] \\ Z \arrow[r, "h"'] & C \end{tikzcd} \tag{2}$$

in $\mathbf{C}$. Thus, the inner square in

$$\begin{tikzcd}
DL(r)(F) \arrow[rrr] \arrow[dddd, dashed] \arrow[dr, hook] & & & F \arrow[dddd] \arrow[dl, hook'] \\
& L \times \mathbf{C}(-,P) \arrow[r, "L\times\mathbf{C}(-,r)"] \arrow[d, "L\times\mathbf{C}(-,q)"'] & L \times \mathbf{C}(-,B) \arrow[d, "L\times\mathbf{C}(-,f)"] & \\
& L \times \mathbf{C}(-,Z) \arrow[r, "L\times\mathbf{C}(-,h)"'] & L \times \mathbf{C}(-,C) & \\
& & & \\
DL(h)\bigvee_f^{DL}(F) \arrow[rrr] \arrow[uur, hook] & & & \bigvee_f^{DL}(F) \arrow[uul, hook']
\end{tikzcd}$$

is a pullback. A general $F \in DL(B)$ is depicted on the upper right corner of the previous diagram. Now $\bigvee_f^{DL}(F)$ is calculated as the image of F under $L \times \mathbf{C}(-,f)$, as shown on the right of the diagram. Now, $DL(h)\bigvee_f^{DL}(F)$ is the pullback along $L \times \mathbf{C}(-,h)$ as shown on the bottom of the diagram. On the other hand, we calculate $DL(r)(F)$ as the pullback along $L\times\mathbf{C}(-,r)$. Thus, to calculate $\bigvee_q^{DL} DL(r)(F)$ we have to calculate the image of this along $L \times \mathbf{C}(-,q)$. But since we are dealing here with a regular category, we have that the image is given by the dotted arrow, that is $\bigvee_q^{DL} DL(r)(F) = DL(h)\bigvee_f^{DL}(F)$. Thus, the diagram

$$\begin{tikzcd}
DL(B) \arrow[r, "\bigvee_f^{DL}"] \arrow[d, "DL(r)"'] & DL(C) \arrow[d, "DL(h)"] \\
DL(P) \arrow[r, "\bigvee_q^{DL}"'] & DL(Z)
\end{tikzcd}$$

commutes. This is the Beck-Chevalley condition. ∎

Assume again that $L \in \mathbf{sup}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$. To show Beck-Chevalley in this case, take a pullback as in (2) and consider the diagram

$$\begin{tikzcd}
LB \arrow[rrr, "\bigvee_f"] \arrow[ddd, "Lr"'] & & & LC \arrow[ddd, "Lh"] \\
& DL(B) \arrow[ul, "\bigvee_B"'] \arrow[r, "\bigvee_f^{DL}"] \arrow[d, "DL(r)"'] & DL(C) \arrow[ur, "\bigvee_C"] \arrow[d, "DL(h)"] & \\
& DL(P) \arrow[dl, "\bigvee_P"] \arrow[r, "\bigvee_q^{DL}"'] & DL(Z) \arrow[dr, "\bigvee_Z"'] & \\
LP \arrow[rrr, "\bigvee_q"'] & & & LZ.
\end{tikzcd}$$

Observe that the right and left faces commute by the naturality of $\bigvee\colon DL \to L$. The commutativity of the top is equivalent, by taking right adjoints, to $\downarrow_B Lf = DL(f) \downarrow_C$, which is just the naturality of $\downarrow : L \to DL$. The bottom commutes for the same reason. The middle square commutes by the previous lemma. We thus deduce the commutativity of the exterior square by recalling that $\bigvee_B \downarrow_B = 1_{LB}$.

A morphism $\alpha : L \to M$ in $\mathbf{sup}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$ is a natural transformation $\alpha : L \to M$ such that the diagram

$$\begin{tikzcd} DL \arrow[r, "D\alpha"] \arrow[d, "\bigvee"'] & DM \arrow[d, "\bigvee"] \\ L \arrow[r, "\alpha"'] & M \end{tikzcd} \tag{3}$$

commutes.

3.3. Lemma. *If $\alpha : L \to M$ is a morphism in $\mathbf{sup}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$, then $\alpha : L \to M$ is a natural transformation in $\mathbf{sup}(\mathbf{set})^{\mathbf{C}^{\mathrm{op}}}$ such that the diagram*

$$\begin{tikzcd} LB \arrow[r, "\alpha B"] \arrow[d, "\bigvee_f"'] & MB \arrow[d, "\bigvee_f"] \\ LC \arrow[r, "\alpha C"'] & MC \end{tikzcd}$$

commutes for every $f : B \to C$ in $\mathbf{C}$.

Proof. Since α is a morphism of sup lattices, it has a right adjoint α_*. Thus for every $C \in \mathbf{C}$, $\alpha C \dashv \alpha_* C$, and α turns out to be a natural transformation in $\mathbf{sup}(\mathbf{set})^{\mathbf{C}^{\mathrm{op}}}$. Furthermore, the naturality of α_* produces the commutative diagram

$$\begin{tikzcd} MC \arrow[r, "\alpha_* C"] \arrow[d, "Mf"'] & LC \arrow[d, "Lf"] \\ MB \arrow[r, "\alpha_* B"'] & LB, \end{tikzcd}$$

for every $f : B \to C$. By taking left adjoints we obtain the commutativity of the square in the statement of the lemma. ∎

Assume now that we have a functor $L : \mathbf{C}^{\mathrm{op}} \to \mathbf{sup}(\mathbf{set})$ with the property that for every $f : B \to C$ in $\mathbf{C}$, Lf has a left adjoint $\bigvee_f \dashv Lf$, and that these adjunctions satisfy the Beck-Chevalley condition.

Lemma 2.7 gives us a left adjoint to $\downarrow_C : LC \to DL(C)$, namely $\bigvee_{LC} \gamma_!$ (the existence of $\gamma_!$ is guaranteed by Proposition 2.6, whereas $\bigvee_{LC}$ exists since L takes values in $\mathbf{sup}(\mathbf{set})$). We are thus left with the task of showing that these left adjoints fit together to form a natural transformation, that is, we need to show the commutativity of the exterior rectangle in the diagram

$$\begin{tikzcd} DL(C) \arrow[r, "\gamma_!"] \arrow[d, "DL(f)"'] & D(LC) \arrow[r, "\bigvee_{LC}"] \arrow[d, "D(Lf)"'] & LC \arrow[d, "Lf"] \\ DL(B) \arrow[r, "\gamma_!"'] & D(LB) \arrow[r, "\bigvee_{LB}"'] & LB \end{tikzcd}$$

for every $f : B \to C$ in $\mathbf{C}$. Since Lf has a right adjoint (recall that L takes values in $\mathbf{sup}(\mathbf{set})$), we have that the right square in the above diagram commutes, so it suffices to show that the left square commutes. This is taken care of in:

3.4. Lemma. *Let $L \in \mathbf{ord}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$. Assume that LC is a sup lattice for every $C \in \mathbf{C}$, that for every $f : B \to C$ in $\mathbf{C}$, Lf has a left adjoint, and that these adjoints satisfy the Beck-Chevalley condition. Then the diagram*

$$\begin{tikzcd} DL(C) \arrow[r, "\gamma_!"] \arrow[d, "DL(f)"'] & D(LC) \arrow[d, "D(Lf)"] \\ DL(B) \arrow[r, "\gamma_!"'] & D(LB) \end{tikzcd}$$

commutes.

Proof. Observe that $\gamma_!$ exists by Proposition 2.6. For $\gamma_! DL(f)(F) \subseteq D(Lf)\gamma_!(F)$ it suffices to show that, if $(w, fk) \in FW$, then $\bigvee_k w \in D(Lf)\gamma_!(F)$. But the condition $(w, fk) \in FW$ implies that $Lf(\bigvee_{fk} w) \in D(Lf)\gamma_!(F)$, and $\bigvee_k w \leq Lf \bigvee_f \bigvee_k w = Lf \bigvee_{fk} w$, using the unit of $\bigvee_f \dashv Lf$. Thus $\bigvee_k w \in D(Lf)\gamma_!(F)$.

To prove the other containment is suffices to show that for any $(z, h) \in FZ$, $Z \in \mathbf{C}$, we have $Lf(\bigvee_h z) \in \gamma_!(D(Lf)F)$. Take the pullback

$$\begin{tikzcd} P \arrow[r, "d"] \arrow[d, "e"'] & B \arrow[d, "f"] \\ Z \arrow[r, "h"'] & C. \end{tikzcd}$$

Since $(z, h) \in FZ$, we have $(Le(z), fd) = (Le(z), he) \in FP$. But by the Beck-Chevalley condition we also have $Lf(\bigvee_h z) = \bigvee_d Le(z)$. Thus $Lf(\bigvee_h z) \in \gamma_!(DL(f)F)$. ∎

To complete the picture, we have to deal with morphisms.

3.5. Lemma. *If L and M are sup lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$, and $\alpha : L \to M$ is a natural transformation in $\mathbf{sup}(\mathbf{set})^{\mathbf{C}^{\mathrm{op}}}$ with the property that for every $f : B \to C$ in $\mathbf{C}$ the diagram*

$$\begin{tikzcd} LB \arrow[r, "\alpha B"] \arrow[d, "\vee_f"'] & MB \arrow[d, "\vee_f"] \\ LC \arrow[r, "\alpha C"'] & MC \end{tikzcd}$$

commutes, then $\alpha : L \to M$ is a morphism in $\mathbf{sup}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$.

Proof. Since α takes values in $\mathbf{sup}(\mathbf{set})$, for every $C \in \mathbf{C}$, αC has a right adjoint $\alpha_* C$. The naturality of α_* is obtained by taking right adjoints on the commutative square on the statement of the lemma. Thus $\alpha \dashv \alpha_*$, and α is a morphism of sup lattices. ∎

4. An extra right adjoint for CCD lattices

Take $L \in \mathbf{sup}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$. According to the theorem from [Joyal and Tierney, 1984] mentioned in the Introduction and for which we gave an alternative proof in Theorem 3.1, this is equivalent to the conditions: LC is a sup lattice for every $C \in \mathbf{C}$, for every $f : B \to C$ we have adjunctions $\bigvee_f \dashv Lf \dashv \bigwedge_f$, and the adjunctions $\bigvee_f \dashv Lf$ satisfy the Beck-Chevalley condition. If $L \in \mathbf{ccd}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$, then for every $C \in \mathbf{C}$ we have the diagram

$$\begin{tikzcd}[column sep=large]
LC \arrow[r, shift left=4, "\Downarrow_C"] \arrow[r, shift right=4, "\downarrow_C"'] \arrow[rr, bend right, "\downarrow"'] & DL(C) \arrow[l, "\bigvee_C \ \perp"'] \arrow[r, shift left=4, "\gamma_!"] \arrow[r, shift right=4, "\gamma"'] & D(LC) \arrow[l, "\gamma^* \perp"'] \arrow[l, phantom, shift right=3, "\perp"] \arrow[ll, phantom, shift right=3, "\perp"]
\end{tikzcd} \tag{4}$$

where $\Downarrow_C$ is the instance at C of $\Downarrow \dashv \bigvee : DL \to L$, and, as before, $\gamma \downarrow_C = \downarrow_{LC} : LC \to D(LC)$. Thus we have

4.1. Lemma. *If $L \in \mathbf{ccd}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$, then for every $C \in \mathbf{C}$, LC is a CCD lattice.*

Proof. It follows immediately that supremum on downsets for LC is given by $\bigvee_C \gamma^*$. Thus the defining left adjoint for LC to be CCD is provided by $\gamma_! \Downarrow_C$. ∎

We can obtain from (4) a right adjoint to $\bigwedge_f$, for every f in $\mathbf{C}$. To do that we paste together the diagram of Lemma 3.4 with diagram (4), assuming $L \in \mathbf{ccd}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$, to obtain, for $f : B \to C$ in $\mathbf{C}$, the commutativity of the diagram

$$\begin{tikzcd}
LC \arrow[r, "\Downarrow_{LC}"] \arrow[d, "Lf"'] & D(LC) \arrow[d, "D(Lf)"] \\
LB \arrow[r, "\Downarrow_{LB}"'] & D(LB),
\end{tikzcd}$$

where $\Downarrow_{LC} \dashv \bigvee_{LC} : D(LC) \to LC$. If we take right adjoints, we have the commutativity of the diagram

$$\begin{tikzcd}
LC & D(LC) \arrow[l, "\bigvee_{LC}"'] \\
LB \arrow[u, "\bigwedge_f"] & D(LB). \arrow[l, "\bigvee_{LB}"] \arrow[u, "D(\bigwedge_f)"']
\end{tikzcd}$$

This means that $\bigwedge_f$ preserves sups, thus it has a right adjoint. We have shown:

4.2. Proposition. *If $L \in \mathbf{ccd}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$, then for any $f : B \to C$ in $\mathbf{C}$ we have an adjoint string $\bigvee_f \dashv Lf \dashv \bigwedge_f \dashv \Uparrow_f$.* ∎

5. Complete Frobenius Reciprocity

In this section we assume that $L \in \mathbf{sup}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$. We modify the condition called wide Frobenius reciprocity in [G. S. H. Cruttwell, 2005] so as not to need wide pullbacks in $\mathbf{C}$, and we show that the condition is closely related to every $\bigvee_C : DL(C) \to LC$ having a left adjoint. The condition is

5.1. Definition. *L satisfies complete Frobenius reciprocity at $C \in \mathbf{C}$ if for any family of the form*

$$\langle (a_{ij}, f_{ij}) \rangle_{i \in I, j \in J_i}$$

with $f_{ij} : A_{ij} \to C$ and $a_{ij} \in LA_{ij}$, we have

$$\bigvee \{ \textstyle\bigwedge_i c_i | \langle c_i \rangle_{i \in I} \subseteq LC \text{ is such that for every } i \text{ there is } j \in J_i \text{ with } c_i \leq \bigvee_{f_{ij}} a_{ij} \}$$
$$=$$
$$\bigvee \{ \textstyle\bigvee_h z | \text{for every } i \in I \text{ there are } j \in J_i \text{ and } \begin{array}{c} Z \\ {\scriptstyle s}\downarrow \\ A_{ij} \end{array} \text{ with } h = f_{ij}s \text{ and } z \leq Ls(a_{ij}) \}.$$

5.2. Remark. If we take an element $\bigvee_h z$ as in the second term of the previous definition, and we consider the constant family $\langle \bigvee_h z \rangle_{i \in I}$, we have that for every i there are $j \in J_i$ and $s : Z \to A_{ij}$ such that $h = f_{ij}s$ and $z \leq Ls(a_{ij})$. Thus $\bigvee_h z \leq \bigvee_h Ls(a_{ij}) = \bigvee_{f_{ij}} \bigvee_s Ls(a_{ij}) \leq \bigvee_{f_{ij}} a_{ij}$, using the counit of the adjunction $\bigvee_s \dashv Ls$. Thus, the second term in the definition is always less than or equal to the first.

5.3. Proposition. *Assume that $C \in \mathbf{C}$ is such that LC is a CCD lattice. Then L satisfies complete Frobenius reciprocity at C if and only if $\bigvee_C : DL(C) \to LC$ has a left adjoint.*

Proof. Given any family $\langle F_i \rangle_{i \in I}$ in $DL(C)$, we have that

$$\textstyle\bigvee_C (\bigcap_i F_i) = \bigvee_{Z \in \mathbf{C}} \bigvee_{(z,h) \in (\bigcap_i F_i)(Z)} \bigvee_h z.$$

On the other hand, define $S_i = \{ c \in LC | c \leq \bigvee_h z \text{ for some } Z \in \mathbf{C} \text{ and } (z, h) \in F_i Z \}$. We have that $S_i \in D(LC)$. If we use the fact that LC is a CCD lattice, we have

$$\begin{aligned} \textstyle\bigwedge_i \bigvee_C F_i &= \textstyle\bigwedge_i \bigvee_{Z \in \mathbf{C}} \bigvee_{(z,h) \in F_i Z} \bigvee_h z = \bigwedge_i \bigvee_{LC} S_i \\ &= \textstyle\bigvee_{\langle c_i \rangle_i \in \prod_i S_i} \bigwedge_i c_i. \end{aligned}$$

Assume now that $\bigvee_C : DL(C) \to LC$ has a left adjoint, and that $\langle (a_{ij}, f_{ij}) \rangle_{ij}$, $i \in I$, $j \in J_i$ is a family as in Definition 5.1. For every $i \in I$ define $F_i = \bigcup_{j \in J_i} \downarrow_C (a_{ij}, f_{ij})$ (recall Definition 2.3). Since $\bigvee_C$ preserves infs, we have that $\bigvee_C (\bigcap_i F_i) = \bigwedge_i \bigvee_C F_i$. Observe that, in this case, the condition $(z, h) \in \left(\bigcap_i F_i\right)(Z)$ is exactly the one expressed on the second term in Definition 5.1. So all we have to do prove complete Frobenius reciprocity is to take a family $\langle c_i \rangle_{i \in I}$ that satisfies the condition expressed in the first term of Definition 5.1, and show that $\langle c_i\rangle_{i\in I} \in \prod_i S_i$. But for every $i \in I$ there is a $j \in J_i$ such that $c_i \leq \bigvee_{f_{ij}} a_{ij}$. So all we have to do is to take $s = 1_{A_{ij}}$ and $(z, h) = (a_{ij}, f_{ij})$.

Assume now that L satisfies the condition of Definition 5.1. As always, we have $\bigvee_C(\bigcap_i F_i) \leq \bigwedge_i \bigvee_C F_i$. Thus, to show that $\bigvee_C$ preserves infs, it suffices to show that for every $\langle c_i\rangle_i \in \prod_i S_i$ we have

$$\bigwedge_i c_i \leq \bigvee_{Z\in\mathbf{C}} \bigvee_{(z,h)\in(\bigcap_i F_i)(Z)} \bigvee_h z.$$

According to Corollary 2.5, we can express every F_i in the form $F_i = \bigcup_{j\in J_i} \downarrow_C (a_{ij}, f_{ij})$. We thus have that

$$S_i = \{\bigvee_h z | Z \in \mathbf{C}, (z, h) \in \bigcup_{j\in J_i} \downarrow_C (a_{ij}, f_{ij})(Z)\}^{\downarrow}.$$

If $\langle c_i\rangle_i \in \prod_i S_i$, then for every i we can find $j \in J_i$, $s : Z \to C$, with $h = f_{ij}s$, $z \in LZ$ such that $z \leq Ls(a_{ij})$ and $c \leq \bigvee_h z$. Then

$$c_i \leq \bigvee_h z = \bigvee_h Ls(a_{ij}) = \bigvee_{f_{ij}} \bigvee_s Ls(a_{ij}) \leq \bigvee_{f_{ij}} a_{ij},$$

using the counit of the adjunction $\bigvee_s \dashv Ls$. The family $\langle c_i\rangle_i$ satisfies the condition on the first term of Definition 5.1, and thus we have

$$\begin{aligned} \bigwedge_i c_i \quad \leq \quad \bigvee\{\bigvee_h z | &\text{for every } i \in I \text{ there are } j \in J_i \text{ and} \\ &s : Z \to A_{ij} \text{ such that } h = f_{ij}s \text{ and } z \leq Ls(a_{ij})\} \end{aligned}$$

As before, the right hand side is precisely $\bigvee_C\big(\bigcap_i F_i\big)$. Thus $\bigvee_C$ has a left adjoint ∎

5.4. Example. Even assuming that $L \in \mathbf{sup}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$ satisfies complete Frobenius reciprocity at every $C \in \mathbf{C}$, and that every LC is CCD, there is no reason why the corresponding left adjoints of the $\bigvee_C : DL(C) \to LC$ would form a natural transformation $L \to DL$. To see this, take $\mathbf{C} = \{0 \to 1\}$, and take as L the only function $! : \mathbf{2} \to \mathbf{1}$ seen as an object of $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. Since $! \dashv \top$ and $\top$ does not have a right adjoint, Proposition 4.2 tells us that L is not a CCD lattice. However, it is not hard to see that it does satisfy complete Frobenius reciprocity. Thus, in this case the left adjoint of $\bigvee_0 : DL(0) \to L0$ and the left adjoint of $\bigvee_1 : DL(1) \to L1$ cannot form a natural transformation $L \to DL$ (because it would be a left adjoint to $\bigvee : DL \to L$, which we know to not exist).

6. Characterization of CCD's in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$

In this section we prove a characterization theorem for CCD lattices in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ by adding a condition that guarantees that the left adjoints to the $\bigvee_C$'s match together to form a left adjoint to $\bigvee : DL \to L$. We consider the category $\mathbf{ccd}(\mathbf{set})$ whose objects are CCD lattices, and whose arrows are those order functions that have both, a left adjoint and a right adjoint. Given $L \in \mathbf{sup}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$ we observe that for any $f : B \to C$ in $\mathbf{C}$, the map $\bigwedge_f^{DL} : DL(B) \to DL(C)$ is given by

$$\bigwedge_f^{DL}(G)(Z) = \{(z,h) | (Le(z), d) \in GP \text{ with } \begin{array}{ccc} P & \xrightarrow{d} & B \\ {\scriptstyle e}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[h]{} & C \end{array} \text{ a pullback}\}.$$

6.1. Theorem. *A CCD lattice in* $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$ *is a functor* $L : \mathbf{C}^{\mathrm{op}} \to \mathbf{ccd}(\mathbf{set})$ *such that*

i) *The adjunctions* $\bigvee_f \dashv Lf$, $f \in \mathbf{C}$, *satisfy Beck-Chevalley.*

ii) *L satisfies complete Frobenius reciprocity at every $C \in \mathbf{C}$.*

iii) *For every $f : B \to C$ in $\mathbf{C}$, there is an extra right adjoint $\bigwedge_f \dashv \Uparrow_f$.*

iv) *For every $A \xrightarrow{g} B \xrightarrow{f} C$ in $\mathbf{C}$, and any $a \in LA$,*

$$\bigwedge_f \bigvee_g(a) \le \bigvee\{\bigvee_h \bigwedge_e Ls(a) | \begin{tikzcd} & \cdot \arrow[d, "g"] \\ \cdot \arrow[ur, "s"] \arrow[r, "d"] \arrow[d, "e"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "h"'] & \cdot \end{tikzcd} \text{ in } \mathbf{C} \text{ with the square a pullback}\}.$$

Proof. Assume L is a CCD lattice in $\mathbf{set}^{\mathbf{C}^{\mathrm{op}}}$. Then Lemma 4.1 tells us that LC is a CCD lattice for every $C \in \mathbf{C}$. Since L is, in particular, a sup lattice, we have that $Lf : LC \to LB$ has left and right adjoints $\bigvee_f \dashv Lf \dashv \bigwedge_f$ for every $f : B \to C$ in $\mathbf{C}$, and i) is also satisfied. Proposition 5.3 gives us ii) and Proposition 4.2 gives us iii). We are thus left with condition iv). Take $a \in LA$ and f, g as shown in iv). Let $c \in LC$ be the sup shown on the right hand side of the inequality in iv). We want to show that $\bigwedge_f^{DL}(\downarrow_B (a,g)) \le \downarrow_C c$. But $(z,h) \in \bigwedge_f^{DL}(\downarrow_B (a,g))(Z)$ when there is an $s : P \to A$ such that $Le(z) \le Ls(a)$ and $d = gs$, with

$$\begin{tikzcd} P \arrow[r, "d"] \arrow[d, "e"'] & B \arrow[d, "f"] \\ Z \arrow[r, "h"'] & C \end{tikzcd}$$

a pullback diagram. But then $z \le \bigwedge_e Ls(a)$, which implies that $\bigvee_h z \le \bigvee_h \bigwedge_e Ls(a) \le c$. Thus $z \le Lh(c)$. That is $(z,h) \in \downarrow_C c$. Since L is a CCD lattice, we have that $\Downarrow \dashv \bigvee : DL \to L$. Since the square

$$\begin{tikzcd} LC \arrow[r, "\Downarrow_C"] \arrow[d, "Lf"'] & DL(C) \arrow[d, "DL(f)"] \\ LB \arrow[r, "\Downarrow_B"'] & DL(B) \end{tikzcd}$$

commutes, we obtain, by taking the right adjoints of the right adjoints of the arrows shown, that

$$\begin{tikzcd} LC \arrow[r, "\downarrow_C"] \arrow[d, "\Uparrow_f"'] & DL(C) \arrow[d, "\Uparrow_f^{DL}"] \\ LB \arrow[r, "\downarrow_B"'] & DL(B) \end{tikzcd}$$

commutes. We have that $\bigwedge_f^{DL}(\downarrow_C\ c) = \bigcup\{G|\bigwedge_f^{DL}(G) \leq\downarrow_C\ c\}$, and since $\bigwedge_f^{DL}(\downarrow_B (a,g)) \leq\downarrow_C\ c$, we have that $\downarrow_B\ (a,g) \leq\downarrow_B\Uparrow_f\ (c)$. In particular $a \leq Lg \Uparrow_f (c)$, which is equivalent to $\bigwedge_f\bigvee_g a \leq c$, which is what we wanted.

Assume now that $L : \mathbf{C}^{\mathrm{op}} \to \mathbf{ccd}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$ is such that conditions i) to iv) are satisfied. Clearly L is a sup lattice and Proposition 5.1 tells us that for every $C \in \mathbf{C}$ we have an adjunction $\Phi_C \dashv \bigvee_C : DL(C) \to LC$. We need to show that these left adjoints form a natural transformation. So take $f : B \to C$ in $\mathbf{C}$. Since $Lf\bigvee_C = \bigvee_B DL(f)$ we immediately obtain $\Phi_B Lf \leq DL(f)\Phi_C$. Thus we are left with showing the other inequality $DL(f)\Phi_C \leq \Phi_B Lf$. Observe that by taking right adjoints twice, the inequality we need to prove is $\Uparrow_f^{DL}\downarrow_C\leq\downarrow_B\Uparrow_f$. Now, for any $c \in LC$ we have

$$\Uparrow_f^{DL}\downarrow_C c = \bigcup\{G|\bigwedge_f^{DL}(G) \leq\downarrow_C c\} = \bigcup\{\downarrow_B (a,g)|\bigwedge_f^{DL}(\downarrow_B (a,g)) \leq\downarrow_C c\},$$

according to corollary 2.5. But the condition $\bigwedge_f^{DL}(\downarrow_B\ (a,g)) \leq\downarrow_C c$ means that for any commutative diagram of the form

$$\begin{tikzcd} & A \arrow[d, "g"] \\ P \arrow[ur, "s"] \arrow[r, "d"'] \arrow[d, "e"'] & B \arrow[d, "f"] \\ Z \arrow[r, "h"'] & C, \end{tikzcd}$$

with the square a pullback, if $Le(z) \leq Ls(a)$ then $z \leq Lh(c)$. Since, in particular, $Le\bigwedge_e Ls(a) \leq Ls(a)$ (counit of the adjunction $Le \dashv \bigwedge_e$), we have $\bigwedge_e Ls(a) \leq Lh(c)$, or $\bigvee_h\bigwedge_e Ls(a) \leq c$. If we take the sup of all the possible elements on the left hand side and apply iv), we end up with $\bigwedge_f\bigvee_g a \leq c$, or $a \leq Lg \Uparrow_f (c)$. This is equivalent to $\downarrow_B (a,g) \leq\downarrow_B\Uparrow_f (c)$. ∎

The question of the arrows in $\mathbf{ccd}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$ is quickly settled (see Lemmas 3.3 and 3.5). If $L, M \in \mathbf{ccd}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$, a morphism $\alpha : L \to M$ is one that has a left as well as a right adjoint. These are characterized as

6.2. Proposition. *A morphism $\alpha : L \to M$ in $\mathbf{ccd}(\mathbf{set}^{\mathbf{C}^{\mathrm{op}}})$ is a natural transformation $\alpha : L \to M : \mathbf{C}^{\mathrm{op}} \to \mathbf{ccd}(\mathbf{set})$ such for every $f : B \to C$ in $\mathbf{C}$, the squares*

$$\begin{array}{ccc} LB & \xrightarrow{\alpha B} & MB \\ {\scriptstyle\bigvee_f}\downarrow & & \downarrow{\scriptstyle\bigvee_f} \\ LC & \xrightarrow[\alpha C]{} & MC, \end{array} \qquad \begin{array}{ccc} LB & \xrightarrow{\alpha B} & MB \\ {\scriptstyle\bigwedge_f}\downarrow & & \downarrow{\scriptstyle\bigwedge_f} \\ LC & \xrightarrow[\alpha C]{} & MC \end{array}$$

are commutative.

References

G. S. H. Cruttwell. A study of CCD lattices in a functor category. Master's degree thesis. Dalhousie University 2005.

B. Fawcett and R. J. Wood. Constructive complete distributivity I. Math. Proc. Cambridge Philos. Soc. 107 (1990), 81–89.

A. Joyal and M. Tierney. An extension of the Galois theory of Grothendieck. Mem. Amer. Math. Soc. 51 (1984), no. 309,

R. Rosebrugh and R. J. Wood. Constructive complete distributivity II. Math. Proc. Cambridge Philos. Soc. 110 (1991), 245–249.

R. Rosebrugh and R. J. Wood. Constructive complete distributivity III. Canad. Math. Bull. Vol 35 (1992) no. 4 537-547.

R. Rosebrugh and R. J. Wood. Constructive complete distributivity IV. Appl. Categ. Structures 2 (1994) no.2, 119–144.

Department of Mathematics and Statistics, Dalhousie University
Chase Building, Halifax, Nova Scotia, Canada B3H 3J5

Instituto de Matemáticas, Universidad Nacional Autónoma de México
Área de la investigación científica, Circuito Exterior, Ciudad Universitaria
Coyoacán 04510, México, D.F. México

Department of Mathematics and Statistics, Dalhousie University
Chase Building, Halifax, Nova Scotia, Canada B3H 3J5

Email: `cruttw@mathstat.dal.ca`
`quico@matem.unam.mx`
`rjwood@mathstat.dal.ca`

This article may be accessed at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/html/volumes/18/6/18-06.{dvi,ps,pdf}`